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Input Impedance of Antennas in High Frequency Cavities

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Abstract

This report presents models and measurements of antenna input impedance in resonant cavities at high frequencies. The behavior of input impedance is useful in determining the transmission and reception characteristics of an antenna (as well as the transmission characteristics of certain apertures). Results are presented for both the case where the cavity is undermoded (modes with separate and discrete spectra) as well as the overmoded case (modes with overlapping spectra). A modal series is constructed and analyzed to determine the impedance statistical distribution. Both an electrically small dipole as well as an electrically longer resonant dipole and wall mounted monopole are analyzed. Measurements in a large mode stirred chamber cavity are compared with calculations. Finally a method based on power arguments is given, yielding simple formulas for the impedance distribution.

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1 SUMMARY

In this report models are constructed for the input impedance Z_{in} of a linear antenna in an electrically large cavity. The behavior of the impedance and its extreme values are useful in determining the transmission and reception characteristics of an antenna (as well as the transmission characteristics of certain apertures) and practical bounds for these quantities. The electrically short center driven dipole is treated first by means of a modal series for the cavity field. The statistical properties of this high frequency cavity field are introduced from which distributions for the antenna input impedance are extracted by means of Monte Carlo simulation and asymptotic analysis. These are compared to measurements in a mode stirred chamber. It is then shown how these results apply to an electrically longer resonant dipole and a wall mounted monopole antenna. The known enhancement of the field near the cavity wall is shown to correspond to the behavior of the field correlation function, which is used in the treatment of the monopole antenna. Finally, a simplified approach using conservation of power is carried out to yield practically useful formulas for the impedance distributions and extreme values.

1.1 Electrically Short Antenna

Using a modal series for the cavity field, with time dependence $e^{-i\omega t}$, the input impedance of an electrically short dipole aligned with the z axis is

$$Z_{in} \sim R - iX + R_{rad} \sum_n \left(\frac{2\pi Q}{k^3 V} \right) \frac{(i\omega^2/Q) \omega^2/\omega_n^2}{\omega^2(1+i/Q) - \omega_n^2} 3A_{nz}^2(\underline{r})$$

where R_{rad} is the antenna radiation resistance in free space, which for a short dipole of length $2h$, with triangular current distribution, is

$$R_{rad} \sim \frac{\eta_0}{6\pi} (kh)^2$$

The quantity $\eta_0 \approx 120\pi$ ohms is the impedance of free space and $k = \omega/c$ is the wavenumber (c is the vacuum velocity of light). The quantity $Z = R - iX$ is the local impedance of the antenna, consisting of the ohmic resistance R and local reactance X (this includes the quasistatic part of the cavity summation), which is approximately capacitive $1/(\omega C)$ for a short dipole

$$C \sim \frac{2\pi\epsilon_0 h}{\Omega_e}$$

where the expansion parameter is

$$\Omega_e = \Omega - 2(1 + \ln 2)$$

the antenna fatness parameter is $\Omega = 2\ln(2h/a)$, and a is the dipole radius. The cavity volume is V , its quality factor is Q , and the cavity vector potential eigenfunctions are $\underline{A}_n$, normalized such that

$$\frac{1}{V} \int_V \underline{A}_n \cdot \underline{A}_n dV = 1$$

1.2 Statistics of Cavity Field

This report is concerned with electrically large, complex cavities, for which a statistical description of the modes becomes applicable. The cavity eigenvalues (resonant frequencies) ω_n have spacings that can be described by a slowly varying mean $\langle \Delta\omega_n \rangle$ times a random variable s

$$\Delta\omega_n = \omega_{n+1} - \omega_n = \langle \Delta\omega_n \rangle s$$

where the asymptotic formula for the mean is

$$\langle \Delta\omega_n \rangle \sim \pi^2 c^3 / (V \omega_n^2), \omega_n \rightarrow \infty$$

The probability density function for the normalized spacing s is Poisson (exponential) when the cavity geometry is simple (e.g., separable, where eigenvalue degeneracy occurs frequently)

$$f_s^P(s) = e^{-s}, 0 < s < \infty$$

and is Rayleigh (Wigner) when the cavity is complex

$$f_s^W(s) = \frac{\pi}{2} s e^{-s^2 \pi/4}, 0 < s < \infty$$

Complex geometry is typical of applications and thus the Rayleigh spacing is more frequently encountered. Constant spacing

$$s \equiv 1$$

is also useful to study because it gives similar results for the impedance as does the Rayleigh spacing, but is simple enough that asymptotic analysis of the modal series can be carried out.

The cavity eigenfunctions are taken to be isotropic (all three components have similar statistics) with Gaussian density

$$\zeta = \sqrt{3} A_{nz}$$

$$f_\zeta(\zeta) = \frac{1}{\sqrt{2\pi}} e^{-\zeta^2/2}$$

which follows the chosen normalization

$$3 \int_{-\infty}^{\infty} A_{nz}^2 f_{A_{nz}}(A_{nz}) dA_{nz} = \int_{-\infty}^{\infty} \zeta^2 f_\zeta(\zeta) d\zeta = 1$$

An argument in support of the Gaussian nature of these eigenfunctions, relates to the ray description of these modes, where the ray contributions to the modal field at an observation point consist of many separate returns from the complex cavity boundary that are uncorrelated. Experiments on cavities with smooth

walls have shown that deviations from this simple density do arise and can be included as contributions corresponding to periodic ray trajectories

The correlation function for the eigenfunction components is different than that for scalar wavefunctions and is given by

$$\rho_z(z_1, z_2) = \frac{\langle A_{nz}(z_1) A_{nz}(z_2) \rangle}{\sqrt{\langle A_{nz}^2(z_1) \rangle \langle A_{nz}^2(z_2) \rangle}} = \frac{3}{2} \left(1 + \frac{1}{k_n^2} \frac{\partial^2}{\partial z_1^2} \right) \frac{\sin[k_n(z_1 - z_2)]}{k_n(z_1 - z_2)}$$

where $k_n = \omega_n/c$.

1.3 Experiments and Simulations

The parameter that describes the degree of spectral overlap is

$$\alpha = k^3 V / (2\pi Q)$$

This parameter is the ratio of the energy stored in the cavity modes over a narrow spectral bandwidth (containing many complete modes) to the same energy if the modal amplitudes are fixed at the average peak level. If the cavity is undermoded (separate discrete modal spectra) $\alpha \ll 1$. If the cavity is overmoded (many overlapping modes) $\alpha \gg 1$. Figures 1 through 3 show Smith charts for the input impedance of near resonant monopoles in the wall of a mode stirred chamber for the undermoded through overmoded range.

The mode stirred chamber (37 ft x 23 ft x 13 ft) has a volume of $V \approx 313 \text{ m}^3$. The cavity is not simply a rectangular box, since a mode stirrer was present in the chamber, but was not moved during the frequency sweeps that generated the data. The quality factor of the chamber was determined to be $Q \approx 80,000$ by examining the 3 dB width of isolated modes at 220 MHz; at 920 MHz it was estimated from the 220 MHz value, by the scaling $\sqrt{\omega}$, to be approximately $Q \approx 165,000$; at 15 GHz it was taken as the experimental value 1,280,000. The antennas were near resonant wall mounted monopoles. The dimensions of the monopoles were $2a \approx 0.102 \text{ in}$, $h \approx 12.953 \text{ in}$ at 220 MHz, $h \approx 2.97 \text{ in}$ at 920 MHz, and $2a \approx 1.51 \text{ mm}$ with $h \approx 4.325 \text{ mm}$ at 15 GHz.

Figures 4 and 5 illustrate the behavior of the spectra for the two limits of α . Figures 6 and 7 show comparisons of the input resistance of these monopoles with Monte Carlo simulation of the series

$$z_{in} = r_{in} - ix_{in} = (Z_{in} - Z) / R_{rad} \approx \left(\frac{2\pi Q}{k^3 V} \right) \sum_{n_r} \frac{i\omega / (2Q)}{\omega (1 + i / (2Q)) - \omega_n} 3A_{nz}^2(\underline{r})$$

where we have approximated the summand since we are including only those modes near the observation range of ω values captured in the figures (the range of included modes n_r contains a range of ω_n that is slightly larger than the ω range so that negligible error is incurred in this approximation). The simulations were done with all three types of eigenvalue spacings. The agreement with the experimental results is good; the Rayleigh and uniform spacing results are in slightly better agreement with measurements than the Poisson spacing.

The near resonant monopoles in the experiment had nearly zero free space reactance (except the 920 MHz antenna which had the experimentally determined value $j8.5 \text{ ohms}$). The experimentally determined free space value of the radiation resistances were 44 ohms at 220 MHz and 46 ohms at 920 MHz. The frequency span was 10 MHz with 4800 frequency points in the 220 MHz experiment; the simulations used 200 modes with 1000 frequency points. The frequency span was 1 MHz with 801 frequency points in the 920 MHz experiment; the simulations used 400 modes with 1000 frequency points. The frequency span was 10 MHz also with 801 frequency points in the 15 GHz experiment.

1.4 Asymptotic Behaviors

Using the modal series it can be shown that the frequency average (taken over a narrow band, but including many complete modal spectra) of the normalized input impedance approaches unity

$$\langle z_{in} \rangle_{\omega} = \frac{1}{\omega_+ - \omega_-} \int_{\omega_-}^{\omega_+} z_{in} d\omega \rightarrow 1$$

It can also be shown in the overmoded limit that by replacing the modal sum by an integral (inserting $d\omega_n / \langle \Delta\omega_n \rangle$ in the summand) the impedance approaches unity

$$z_{in} \rightarrow 1, \alpha \rightarrow \infty$$

In the next two subsubsections the uniformly spaced modal series is used to estimate extreme behaviors of the impedance.

1.4.1 Uniform Spacing, Single Mode

The undermoded limit $\alpha \ll 1$ has separated discrete spectra. The largest values of the input resistance and reactance in this region occur when ω is near a resonance. Thus we can consider a single mode of the series (the closest mode to the observation frequency) and estimate the extreme statistics by regarding $\omega - \omega_n$ to be a random variable with uniform density between $\pm \langle \Delta\omega_n \rangle / 2$ (for typically used uniform frequency sampling). The distribution function this gives is

$$F(r_{in}) = 1 - \int_{r_{in}}^{\infty} f(r) dr \sim 1 + \frac{\alpha}{\pi} \sqrt{r_{in}\alpha / (2\pi)} e^{-r_{in}\alpha/4} [K_0(r_{in}\alpha/4) - K_1(r_{in}\alpha/4)], r_{in} \gg \alpha, \alpha \ll 1$$

where the identity $\int_0^{\infty} e^{-\nu x} \frac{dx}{\sqrt{x(x+a)}} = e^{\nu a/2} K_0(\nu a/2)$ has been used and $K_0(x)$, $K_1(x)$ are the modified Bessel functions. Figures 8 and 9 show this result (long dashed curves) compared to measurements and Monte Carlo simulations of the uniformly spaced modal series (α is too large in Figure 9 for this result to be valid over any substantial range of r_{in}). Over most of the valid range this can be simplified to

$$F(r_{in}) \approx 1 - \frac{2}{\pi} \sqrt{\frac{2\alpha}{\pi r_{in}}}, 1/\alpha \gg r_{in} \gg \alpha$$

However near the upper limit of r_{in} , the corresponding density function exhibits exponential behavior that allows one to establish practical upper bounds for the resistance values

$$f(r_{in}) \approx \frac{e^{-r_{in}\alpha/2}}{\pi r_{in}} \alpha, r_{in}\alpha \gg 1$$

The normalized reactance also exhibits exponential behavior near the upper limit

$$f(x_{in}) \approx \frac{\alpha\sqrt{2}}{\pi |x_{in}|} e^{-|x_{in}|\alpha}, |x_{in}|\alpha \gg 1$$

which shows the extreme reactance magnitude is approximately half the extreme resistance.

The number of independent samples in a frequency sweep is dependent on the number of modes spanned. For example at 220 MHz there are only 141 modes in the frequency sweep even though there are many more frequencies sampled. If, for $\alpha \ll 1$, the frequency sweep is sufficiently fine to resolve the spectral peaks (over-sampling in frequency) then the density function of the peaks is of interest. Thus, near the upper limit of r_{in} we can set $\omega = \omega_n$ and find the single mode density function for the peak values (the square of a Gaussian random variable)

$$f(r_{in}) \approx \sqrt{\frac{\alpha}{2\pi r_{in}}} e^{-r_{in}\alpha/2}, r_{in}\alpha \gg 1$$

The exponential behavior here is the same as the preceding result. The distribution function corresponding to this density function is $F(r_{in}) \approx \text{erf}\left(\sqrt{r_{in}\alpha/2}\right)$, where $\text{erf}(x)$ is the error function, and the number of independent samples corresponds to the number of modes contained in the frequency sweep.

1.4.2 Uniform Spacing, Between Modes

The smallest values of the input resistance for $\alpha \ll 1$ occur when ω is between modes. Taking the frequency to be exactly between modes of the uniformly spaced series, the modal terms can be taken in pairs about the observation frequency, each pair having a simple exponential distribution. The infinite summation requires an infinite sequence of convolutions to be performed to obtain the density function. Using the Laplace transform to convert the convolutions to an infinite product, and using the identity $\prod_{k=1}^{\infty} \left[1 - \frac{x^2}{k^2 - a^2}\right] = \frac{a}{\sqrt{a^2 + x^2}} \frac{\sin(\pi\sqrt{a^2 + x^2})}{\sin(\pi a)}$ to evaluate the product, the density can be found by inverse transform of the resulting function. The integral of the density function is thus the distribution

$$F(r_{in}) = F_{\chi}\left(\frac{\pi^2 r_{in}}{8\alpha}\right)$$

where from the residue method

$$F_{\chi}(\chi) = 1 - e^{-\chi^4 \alpha^2 / \pi^2} \cosh(\alpha) \frac{4}{\pi} \sum_{m=0}^{\infty} (-1)^m \frac{2m+1}{(2m+1)^2 + 4\alpha^2 / \pi^2} e^{-(2m+1)^2 \chi}$$

An alternative representation for the density function, that converges rapidly for $\chi \rightarrow 0$, can also be obtained from the inverse transform using the identity $\frac{a}{2\sqrt{\pi}t^3} e^{-a^2/(4t)} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} e^{-a\sqrt{s}} ds$ (although it is difficult to integrate to obtain the distribution function) $f(r_{in}) = f_{\chi}(\pi^2 r_{in} / (8\alpha)) \pi^2 / (8\alpha)$

$$f_{\chi}(\chi) = e^{-\chi^4 \alpha^2 / \pi^2} \cosh(\alpha) \sqrt{\frac{\pi}{\chi^3}} \sum_{m=0}^{\infty} (-1)^m (m+1/2) e^{-(m+1/2)^2 \pi^2 / (4\chi)}$$

Figures 8 and 9 show this distribution function (short dashed curves) compared to measurements and Monte Carlo Simulations (this result describes the entire distribution in Figure 9 since the placement of ω is not critical when the modes are overlapping and α is of order unity). Using the second representation we see that the density function exhibits exponential decay for very small r_{in}

$$f(r_{in}) \sim \cosh(\alpha) \sqrt{\frac{2\alpha}{\pi r_{in}^3}} e^{-\alpha(r_{in}+1/r_{in})/2}, r_{in} \ll \alpha$$

which again allows one to establish practical lower bounds for the input resistance. For $\alpha \ll 1$, the $\alpha r_{in}/2$ term in the exponential can be dropped, and this density can be integrated to give the distribution function $F(r_{in}) \approx 2\text{erfc}\left[\sqrt{\alpha/(2r_{in})}\right]$, where $\text{erfc}(x)$ is the complementary error function. The number of independent samples, when we are over-sampling in frequency, is again the number of modes spanned in the frequency sweep.

For α of order unity, the first representation can be used to give

$$f(r_{in}) \sim \frac{\pi}{2\alpha} \cosh(\alpha) e^{-r_{in}[(2\alpha)/\pi + \pi/(2\alpha)]\pi/4}, r_{in} \rightarrow \infty, \alpha = O(1)$$

showing the exponential decay for large r_{in} . If we take the overmoded limit $\alpha \gg 1$, from the second representation we find that the normalized input resistance is Gaussian distributed about the mean of unity

$$f(\rho_r) \sim \sqrt{\frac{\alpha}{2\pi}} e^{-\alpha\rho_r^2/2}, r_{in} = 1 + \rho_r, \alpha \rightarrow \infty$$

1.4.3 Overmoded Limit

By the central limit theorem we expect both components of the impedance to become Gaussian distributed in the overmoded limit $\alpha \gg 1$ since many modes are equally contributing to the modal series. Finding the variance of both components thus allows us to write

$$z_{in} \sim 1 + r_0\zeta - ix_0\zeta', \alpha \gg 1$$

where ζ and ζ' are independent, normalized, zero mean Gaussians, and the standard deviations are found to be

$$r_0 = x_0 = 1/\sqrt{\alpha}$$

1.5 Electrically Longer Antenna

The preceding analytical and simulation results were based on the assumption of an electrically short dipole, but the experiments were conducted using near resonant monopole antennas. This conflicting situation will be resolved in the present and next subsections. The integro-differential equation for a center driven linear antenna inside a cavity can be written as

$$-V_0\delta(z) = E_z$$

$$\approx -\frac{i\omega}{\varepsilon_0 V} \sum_n \frac{\omega^2(1+i/Q)/\omega_n^2}{\omega^2(1+i/Q) - \omega_n^2} A_{nz}(z) \int_{-h}^h A_{nz}(z') I(z') dz'$$

$$+\frac{i\omega\mu_0}{4\pi}\left[\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)\int_{-h}^h\frac{I(z')dz'}{\sqrt{a^2+(z-z')^2}}-I(z)\right]$$

where the second term is the local quasistatic contribution. The antenna current distribution $I(z)$ is the unknown. The antenna is assumed to be thin and thus the local quasistatic term can be thought of as having the transmission line form

$$\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)\int_{-h}^h\frac{I(z')dz'}{\sqrt{a^2+(z-z')^2}}\approx\Omega_e\left(1+\frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right)I(z)$$

This term in addition to the boundary conditions

$$I(\pm h)=0$$

play a dominant role in determining the distribution of current (at least up to the first resonance). The leading term of the current can thus be taken as

$$I(z)\approx I_0\sin k(h-|z|)$$

The impedance is then found by using this current, and the integro-differential representation for the electric field, in the stationary (first order corrections to the current do not contribute) EMF representation

$$Z_{in}=-\frac{1}{I^2(0)}\int_V\underline{E}\cdot\underline{J}dV$$

Noting that the integral of a Gaussian random process is a Gaussian random variable we find

$$Z_{in}\approx R-iX+\sum_n\left(\frac{\eta_0k^2}{2\pi}U_n^{tot}\right)\left(\frac{2\pi Q}{k^3V}\right)\frac{(i\omega^2/Q)\omega^2/\omega_n^2}{\omega^2(1+i/Q)-\omega_n^2}3A_{nz}^2$$

where again the antenna ohmic resistance is R and the local reactance is (we are ignoring quasistatic images in the cavity walls)

$$X\approx-\frac{\eta_0}{2\pi}\Omega_e\cot(kh)$$

$$+\frac{\eta_0}{2\pi}\frac{1}{\sin^2(kh)}\left[2\text{Si}(kh)+\sin(2kh)\left\{2\text{Cin}(kh)-\text{Cin}(2kh)-\frac{3}{4}\right\}-\cos(2kh)\{\text{Si}(2kh)-2\text{Si}(kh)\}-\frac{1}{2}kh\right]$$

and $\text{Si}(x)$, $\text{Cin}(x)$ are the sine and cosine integrals. The variance of the stochastic integral appearing in the impedance representation is

$$U_n^{tot} = \frac{1}{\sin^2(kh)} \int_{-h}^h \sin k(h - |z|) \int_{-h}^h \frac{3}{2} \left(1 + \frac{1}{k_n^2} \frac{\partial^2}{\partial z^2} \right) \frac{\sin k_n(z - z')}{k_n(z - z')} \sin k(h - |z'|) dz' dz$$

A small error is made (mostly in the reactance) if we set $k_n \rightarrow k$ in U_n^{tot} for all values of α (this approximation is consistent with the previously discussed truncation of the series in the range of the resonant modes). If k_n is retained in U_n^{tot} , it can be shown in the overmoded limit $\alpha \gg 1$, that the correct total antenna reactance $X_{in} = -\frac{\eta_0}{2\pi} \Omega_e \cot(kh) + \frac{\eta_0}{4\pi} [2 \text{Si}(2kh) + \sin(2kh) \{-2 + 2\text{Cin}(2kh) - \text{Cin}(4kh)\} - \cos(2kh) \{\text{Si}(4kh) - 2\text{Si}(2kh)\}] / \sin^2(kh)$ is produced rather than the value X , that is obtained when the approximation $k_n \rightarrow k$ is invoked; at low frequencies these expressions become the same; even for $kh = \pi/2$, where the dominant leading term of the reactance vanishes, the error is $X \approx 64$ ohms versus the correct $X_{in} \approx 43$ ohms. The result of the replacement $k_n \rightarrow k$ is

$$Z_{in} \approx R - iX + R_{rad} \sum_n \left(\frac{2\pi Q}{k^3 V} \right) \frac{(i\omega^2/Q) \omega^2/\omega_n^2}{\omega^2(1 + i/Q) - \omega_n^2} 3A_{nz}^2$$

surprisingly the same result we had previously for the short antenna, except that the radiation resistance R_{rad} is now the correct free space value for the electrically longer antenna given by

$$(4\pi/\eta_0) \sin^2(kh) R_{rad} = 2\text{Cin}(2kh) + \sin 2kh [\text{Si}(4kh) - 2\text{Si}(2kh)] - \cos 2kh [\text{Cin}(4kh) - 2\text{Cin}(2kh)]$$

Thus the quantity $z_{in} = (Z_{in} - Z)/R_{rad}$ from the electrically short antenna theory is approximately the same for electrically longer antennas.

1.6 Monopole Antenna and Wall Behavior

A previous paper has shown that the correlation dyad for the field is proportional to the imaginary part of the dyadic Green's function. Thus in a local vicinity of the cavity boundary (near the wall mounted monopole antenna) at $z = 0$, we can use the half space dyadic Green's function to obtain the correlation function transition near the cavity wall. The result is

$$R_{zz}^h(z_1, z_2) = \frac{3}{2} \left(1 + \frac{1}{k_n^2} \frac{\partial^2}{\partial z_1^2} \right) \left[\frac{\sin k_n(z_1 - z_2)}{k_n(z_1 - z_2)} + \frac{\sin k_n(z_1 + z_2)}{k_n(z_1 + z_2)} \right]$$

Using this correlation function, it is easy to show that the impedance of the wall mounted monopole is half that of the dipole. Thus, again the quantity $z_{in} = (Z_{in} - Z)/R_{rad}$ for the monopole is the same as for the dipole (assuming R_{rad} is taken to be the monopole free space radiation resistance), and the comparisons with experiment, made above, are justified.

It is interesting that the known 3 dB wall enhancement of the normal electric field, and its transition into the cavity volume are represented by this half space correlation function

$$\frac{\langle |E_{norm}|^2 \rangle}{\langle |E_i|^2 \rangle} = 1 + \frac{3}{2} \left(1 + \frac{\partial^2}{\partial u^2} \right) \frac{\sin u}{u} \Big|_{u=2kz} = R_{zz}^h(z, z)$$

Figures 10 through 12 show a mode stirred chamber experiment and results verifying the presence of this wall enhancement in the undermoded region. The normal electric field distribution on the wall is 3 dB higher than in the volume of the cavity (this is borne out for the field as a function of frequency, and approximately for the field at the resonant mode frequencies).

1.7 Power Balance

Now that the usefulness of the electrically short antenna theory has been demonstrated, we return to the electrically short antenna and develop a practically useful simplified model.

We break up the field at the antenna E_z into the sum of a reflected part E_z^{ref} and a part E_z^{rad} radiated as if in free space. The impedance components of the short dipole are correspondingly broken into the sums

$$R_{in} = R_{rad} + R_{wall}$$

and

$$X_{in} = X + X_{wall}$$

The quantity R_{rad} is the free space radiation resistance associated with the field E_z^{rad} and X is the local reactance associated with the quasistatic part of the field E_z^{rad} . The quantities R_{wall} and X_{wall} are associated with the reflected field from the cavity wall E_z^{ref} . The wall impedance $Z_{wall} = R_{wall} - iX_{wall}$ can be written in terms of the received voltage at the dipole due to the reflected field $V_{ref} \sim -hE_z^{ref}$

$$Z_{wall} = V_{ref}/I(0) = -hE_z^{ref}/I(0) = \frac{-E_z^{ref}}{\sqrt{\langle |E_z|^2 \rangle_V}} h\sqrt{U/(3\varepsilon_0/2)}$$

where the received voltage has been determined from the effective height (the positive reference of the voltage is on the positive z arm of the antenna) of the short dipole, and the mean energy density in the cavity is

$$U = \frac{3}{2}\varepsilon_0 \langle |E_z|^2 \rangle_V$$

where the subscript V denotes volume average. Now using the definition of cavity quality factor

$$Q = \frac{\omega V U}{P_{in}}$$

with the average power into the antenna (the dissipated power) given by

$$P_{in} = \frac{1}{2} R_{in} |I(0)|^2$$

we obtain

$$Z_{wall} = \frac{-E_z^{ref}}{\sqrt{\langle |E_z|^2 \rangle_V}} \sqrt{\frac{Q h^2 R_{in} \eta_0}{3kV}}$$

Identifying the radiation resistance of a short dipole in free space R_{rad} , using the definition of α , and using lower case impedances to denote quantities $r_{wall} = R_{wall}/R_{rad}$ and $x_{wall} = X_{wall}/R_{rad}$ (note that this scaled

reactance is really the same as x_{in} , since x_{in} was defined with the local reactance subtracted out), we finally obtain

$$r_{wall} = r_{in} - 1 = \tau \sqrt{r_{in}/\alpha}$$

$$x_{wall} = \zeta \sqrt{r_{in}/\alpha}$$

where

$$\tau = \frac{-\text{Re}(E_z^{ref})}{\sqrt{\langle |E_z|^2 \rangle_V}}$$

$$\zeta = \frac{\text{Im}(E_z^{ref})}{\sqrt{\langle |E_z|^2 \rangle_V}}$$

1.7.1 Extreme Values

The quantities τ and ζ describe the fluctuation of the real and imaginary parts of the reflected field at the antenna location normalized by the mean cavity field. For the present we assume τ and ζ have normalized Gaussian densities with zero mean (this assumption is refined in the next subsection). To obtain an extreme value curve for the impedance variation we could take these random variables to be fixed at the three sigma point $M_0 = 3$ of the underlying real and imaginary Gaussian distributions. It is interesting to note that if the cavity field is viewed as **a three dimensional standing wave in the frequency range of the fundamental cavity modes**, then **the maximum-to-mean-ratio of the field is eight-to-one**, corresponding to the value $M_0 = 2\sqrt{2}$; a value that is **not very different from the three sigma value $M_0 = 3$; these extreme results may therefore be useful at lower frequencies than anticipated.** Replacing τ and ζ with this value and solving the quadratic equation gives

$$r_{in} = 1 + \frac{1}{2\alpha} M_0^2 \cos^2 \varphi \pm \sqrt{\left(1 + \frac{1}{2\alpha} M_0^2 \cos^2 \varphi\right)^2 - 1}$$

$$x_{wall} = \pm M_0 \sin \varphi \sqrt{r_{in}/\alpha}$$

The dashed circles in Figures 1 through 3 are plots of these results. These extreme circles provide a reasonable containment of the experimental impedance variations. The radiation resistance of the 15 GHz monopole was taken as the nominal 36 ohm value.

The extreme values of the real and imaginary parts on this circle can be easily found as

$$1 + \frac{1}{2\alpha} M_0^2 - \sqrt{\left(1 + \frac{1}{2\alpha} M_0^2\right)^2 - 1} < r_{in} < 1 + \frac{1}{2\alpha} M_0^2 + \sqrt{\left(1 + \frac{1}{2\alpha} M_0^2\right)^2 - 1}$$

$$|x_{wall}| < \sqrt{\left(1 + \frac{1}{2\alpha} M_0^2\right)^2 - 1}$$

The highly undermoded limit is $1/(2 + M_0^2/\alpha) < r_{in} < (2 + M_0^2/\alpha)$ and $2|x_{wall}| < 2 + M_0^2/\alpha$. The highly overmoded limit is $1 - M_0/\sqrt{\alpha} < r_{in} < 1 + M_0/\sqrt{\alpha}$ and $|x_{wall}| < M_0/\sqrt{\alpha}$, thus giving $r_{in} \rightarrow 1$ and $x_{wall} \rightarrow 0$.

1.7.2 Density

The distribution of input resistance generated by the power balance results, using the normalized Gaussian assumption for the normalized reflected field, is shown as the dotted curves in Figures 13 and 14. The extreme values are representative but the midrange distribution is not even close to the experimental or simulation results. Using the modal series field representation we can generate the actual distributions for τ and ζ , from which we construct more accurate density function approximations. **Taking $I(0)$ to be real and positive** (this choice is to be noted when interpreting the real and imaginary parts of the reflected field) we find

$$\frac{-E_z^{ref}}{\sqrt{\langle |E_z|^2 \rangle_V}} \approx i \frac{\sum_{n_r} \frac{3A_{nz}^2}{\pi(\omega - \omega_n)/\langle \Delta\omega_n \rangle + i\alpha}}{\sqrt{\sum_{n_r} \frac{3A_{nz}^2}{|\pi(\omega - \omega_n)/\langle \Delta\omega_n \rangle + i\alpha|^2}}} - \frac{1}{\sqrt{\sum_{n_r} \frac{3A_{nz}^2}{|\pi(\omega - \omega_n)/\langle \Delta\omega_n \rangle + i\alpha|^2}}}$$

The first term, which corresponds to the total normalized field at the antenna, has a positive real part. The second term, which corresponds to the normalized radiated field at the antenna, is negative real. Note that the local quasistatic normalized field has been subtracted from each term in the difference. In the undermoded limit $\alpha \ll 1$ the first term is imaginary except in the narrow frequency band about the resonances. The real part is thus skewed toward negative values. Thus we try taking the asymmetric Gaussian

$$f(\tau) \approx \frac{p(\alpha)}{\sqrt{2\pi}} e^{-\tau^2/2}, \quad 0 < \tau < \infty$$

$$\approx \frac{2 - p(\alpha)}{\sqrt{2\pi}} e^{-\tau^2/2}, \quad -\infty < \tau < 0$$

as a fit to the density function of the real part of the normalized scattered field. If we apply the result that $\langle r_{in} \rangle_\omega = \langle r_{in} \rangle_\tau \rightarrow 1$ we can determine the function of α as

$$p(\alpha) = 1 - 1/\left[2\sqrt{2\alpha/\pi} + e^{2\alpha} \operatorname{erfc}(\sqrt{2\alpha})\right]$$

The function $p(\alpha)$ approaches 2α as $\alpha \rightarrow 0$ and approaches 1 as $\alpha \rightarrow \infty$. Figures 15 and 16 show a comparison of the distributions for the real and imaginary parts of the normalized reflected field obtained from Monte Carlo simulation of the modal series representation (solid curves) and the asymmetric (short dashed curves) and symmetric (dotted curves) Gaussian distributions

$$F(\tau) \approx 1 - \frac{1}{2}p(\alpha) \operatorname{erfc}\left(\tau/\sqrt{2}\right), \quad 0 < \tau < \infty$$

$$\approx \left[1 - \frac{1}{2}p(\alpha)\right] \operatorname{erfc}\left(-\tau/\sqrt{2}\right), \quad -\infty < \tau < 0$$

$$F(\zeta) = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\zeta/\sqrt{2}\right), \quad -\infty < \zeta < \infty$$

Figure 15 used 500 modes; 100 modes at each end of the interval are beyond the sampled frequency range. Figure 16 used 1000 modes; 200 modes at each end of the interval are beyond the sampled frequency range. The agreement is reasonably good. The “kink” discrepancy in Figure 15 is caused by the discontinuity of the density function at $\tau = 0$. The single mode approximate distribution $F(r_{in})$ can be transformed by means of the quadratic relation $r_{in} = (\sqrt{\tau^2 + 4\alpha} + \tau)^2 / (4\alpha)$ to a distribution of normalized real scattered field, for small values of τ in the undermoded limit

$$F(\tau) \approx 1 - \frac{4\alpha}{\pi} \sqrt{\frac{2}{\pi}} / \left(\sqrt{\tau^2 + 4\alpha} + \tau\right), \quad |\tau| \ll 1, \quad \alpha \ll 1$$

This simple approximate distribution indicates how the “kink” should be interpolated as shown by the long dashed curve in Figure 15. The short dashed curves in Figures 13 and 14 show the improvement in the power balance distributions by use of this asymmetric Gaussian distribution (representing the limited bandwidth of the resonances). The long dashed curve in Figure 13 shows the single mode approximate distribution at the “kink” discrepancy.

The exponential decays of the density functions extracted in the asymptotic analyses are all reproduced by the power balance results. The exponential decays of the density functions extracted in the asymptotic analyses are all reproduced by the power balance results. One might be tempted to use the asymmetrical Gaussian distribution to refine the extreme curves of the preceding subsection, instead of basing these on the symmetrical three sigma point $M_0 = 3$. However, when the distributions are over-sampled in frequency, such that the resonances are fully resolved (for example the 220 MHz data), the extreme values must be determined from the confidence levels associated with the number of independent modes contained within the frequency sweep, as discussed in the above asymptotic behavior subsection. Thus, the use of the symmetrical estimate, is appropriate for the extremes, when the data is over-sampled in frequency, but it is not appropriate for the midrange distribution.

1.8 Conclusions

The input impedance of linear antennas inside high Q , electrically large cavities, has been investigated theoretically and experimentally. Monte Carlo simulations based on a modal series representation, with statistical estimates for modal spacing and eigenfunction amplitudes, are found to agree with measurements in a mode stirred chamber cavity. The parameter $\alpha = k^3 V / (2\pi Q)$, equal to the ratio of modal width to modal spacing, determined the magnitude of the impedance variations; the undermoded limit (separated, distinct modal spectra) $\alpha \ll 1$ results in large variations; the overmoded limit (many overlapping modes) $\alpha \gg 1$ results in small variations. Asymptotic analysis of the modal series yields formulas for the extreme values of the impedance. The modal series for an electrically short antenna has been shown to approximately represent

resonant dipoles and wall mounted monopoles, provided the local impedance and free space radiation resistance parameters are appropriately modified; the half space correlation function used for the monopole treatment was shown to represent the known wall 3 dB normal field enhancement. A simplified model based on balance of power gives practically useful simple formulas for the impedance distributions and the extreme values.

2 INTRODUCTION

The input impedance of an antenna is an important quantity in determining antenna response in both transmitting and receiving situations. Normally, in overmoded (overlapping modes) cavities, it is assumed that the antenna input impedance is equal to that of the antenna in free space [1]. This paper explores the variation of the input impedance of linear antennas when the cavity is undermoded (distinct, separate modes), but still operating at high frequencies (high-order modes). The transition to the overmoded region is also examined. A related problem is the radiation of atoms in small cavities [2].

Figure 1 shows the Smith chart obtained from measurements of the input impedance of a near resonant monopole antenna in the wall of a mode stirred chamber operating in the undermoded region (the frequency span of the Smith chart is less than five percent of the center frequency). Figure 2 shows the Smith chart for a near resonant monopole operating near the transition between the overmoded and undermoded regions. Figure 3 shows the Smith chart for a near resonant monopole operating in the overmoded region. The purpose of this paper is to obtain formulas describing the distributions for this impedance variation and to obtain the extremes of the variation.

One use of this impedance information is for the so called “V curve” requirement, which uses a dipole coupling model (which in some respects forms a practical worst case) to place limits on field levels in areas where explosives are stored and handled. Assuming there is an allowed power level for the explosive device load with attached cabling, the coupling model places limits on electric field E that has the form of a “V” on a log-log plot of field versus frequency. The left arm of the “V curve” assumes the dipole is not matched to the load. The right arm of the “V curve” assumes that a transmission line (part of attached cabling) matches the load to the antenna, at least at some discrete frequencies. The power delivered is then $P_{rec} = \frac{1}{2} |V_{oc}|^2 / (4R_{in})$; where V_{oc} is the peak open circuit voltage of the antenna ($V_{oc} = -h_e E_z$ for a z directed linear antenna of effective height h_e) and R_{in} is the real part of the antenna input impedance. Because explosive areas may be enclosed, the variation of the antenna input impedance when in a cavity, versus in free space, is of interest. We see from this expression that the minimum value of R_{in} is a quantity of particular interest.

Another question of interest is whether changes in the reactance of the antenna, as a result of being in a cavity, might allow matching in frequency regions that are typically not possible. A quantity of interest here is the maximum value of $|X_{in}|$. The answers to these questions are also of interest in the dual problem of aperture penetration.

3 MODAL SERIES REPRESENTATION FOR ANTENNA IMPEDANCE

This section formulates the modal series representation for the potentials and fields which are used to determine the antenna input impedance.

3.1 Modal Series Representation for Potentials and Fields

We follow Smythe [3] in setting up the series representation for the antenna input impedance. The fields are found from the potentials by (time dependence $e^{-i\omega t}$ is suppressed)

$$\underline{B} = \nabla \times \underline{A} \quad (1)$$

$$\underline{E} = -\nabla\phi + i\omega\underline{A} \quad (2)$$

The vector potential in the Coulomb gauge $\nabla \cdot \underline{A} = 0$ satisfies the equation

$$(\nabla^2 + k^2) \underline{A} = -\mu_0 \underline{J} - i\omega\mu_0\epsilon_0 \nabla\phi = -\mu_0 \underline{J}_s \quad (3)$$

where the right hand side is the solenoidal part of the current and $k = \omega\sqrt{\mu_0\epsilon_0}$ is the free space wavenumber. However [3] because the vector potential is orthogonal to the gradient of the scalar potential over the cavity volume (with perfectly conducting walls) we can find the vector potential by taking projections of the total current on the eigenmodes. Expanding the potential in the eigenmodes we can write

$$\underline{A} = \sum_n e_n \underline{A}_n \quad (4)$$

where the eigenmodes for the perfectly conducting cavity satisfy

$$(\nabla^2 + k_n^2) \underline{A}_n = 0 \quad (5)$$

and we choose normalization

$$\frac{1}{V} \int_V \underline{A}_n \cdot \underline{A}_m dV = \delta_{nm} \quad (6)$$

The introduction of cavity loss in the form of a wall impedance follows closely the derivation in [4]. The divergence free modes $\underline{A}_n$ correspond to fields $\underline{E}_n$ and $\underline{H}_n$ satisfying the Maxwell equations $\nabla \times \underline{E}_n = i\omega_n\mu_0\underline{H}_n$ and $\nabla \times \underline{H}_n = -i\omega_n\epsilon_0\underline{E}_n$ with $\underline{n} \times \underline{E}_n = 0$, $\underline{n} \cdot \underline{H}_n = 0$ on the boundary S ($\underline{n}$ is the unit normal out of the region V). The fields are expanded as $\underline{E} = -\nabla\phi + \sum_n e_n \underline{E}_n$ and $\underline{H} = \sum_n h_n \underline{H}_n$ where the fields satisfy the Maxwell equations $\nabla \times \underline{E} = i\omega\mu_0\underline{H}$ and $\nabla \times \underline{H} = -i\omega\epsilon_0\underline{E} + \underline{J}$ and the surface impedance boundary condition $\underline{n} \times \underline{E} = -Z_s \underline{n} \times \underline{n} \times \underline{H} = Z_s \underline{H}_t$ (where the subscript t represents tangential part to the surface S). From the first Maxwell equation for the total field and the accompanying impedance boundary condition we obtain

$$\begin{aligned} i\omega\mu_0 h_m \int_V \underline{H}_m \cdot \underline{H}_m dV &= \int_V \underline{H}_m \cdot (\nabla \times \underline{E}) dV \\ &= \int_V (\nabla \times \underline{H}_m) \cdot \underline{E} dV - \oint_S (\underline{H}_m \times \underline{E}) \cdot \underline{n} dS \\ &= -i\omega_m\epsilon_0 \int_V \underline{E}_m \cdot \underline{E} dV + \oint_S \underline{H}_m \cdot (\underline{n} \times \underline{E}) dS \\ &= -i\omega_m\epsilon_0 e_m \int_V \underline{E}_m \cdot \underline{E}_m dV + Z_s \oint_S \underline{H}_m \cdot \underline{H}_m dS \end{aligned}$$

where because the eigenmode is tangential to the surface we can drop the subscript t . From the second Maxwell equation we obtain

$$\begin{aligned}
& -i\omega\varepsilon_0 e_m \int_V \underline{E}_m \cdot \underline{E}_m dV + \int_V \underline{E}_m \cdot \underline{J} dV = \int_V \underline{E}_m \cdot (\nabla \times \underline{H}) dV \\
& = \int_V (\nabla \times \underline{E}_m) \cdot \underline{H} dV - \oint_S (\underline{E}_m \times \underline{H}) \cdot \underline{n} dS \\
& = i\omega_m \mu_0 \int_V \underline{H}_m \cdot \underline{H} dV - \oint_S (\underline{n} \times \underline{E}_m) \cdot \underline{H} dS \\
& = i\omega_m \mu_0 h_m \int_V \underline{H}_m \cdot \underline{H}_m dV
\end{aligned}$$

Eliminating h_m using the previous equation gives

$$(\omega^2 - \omega_m^2) \varepsilon_0 e_m \int_V \underline{E}_m \cdot \underline{E}_m dV - i\omega_m Z_s \oint_S \underline{H}_m \cdot \underline{H} dS = -i\omega \int_V \underline{E}_m \cdot \underline{J} dV$$

Assuming there is no coupling between modes in the surface integral [4] we find

$$\begin{aligned}
iR_{sm} \oint_S \underline{H}_m \cdot \underline{H} dS &= \sum_n i h_n R_{sm} \oint_S \underline{H}_m \cdot \underline{H}_n dS \approx i\omega_m \mu_0 h_m \int_V \underline{H}_m \cdot \underline{H}_m dV \frac{R_{sm} \oint_S \underline{H}_m \cdot \underline{H}_m dS}{\omega_m \mu_0 \int_V \underline{H}_m \cdot \underline{H}_m dV} \\
&= \left[-i\omega\varepsilon_0 e_m \int_V \underline{E}_m \cdot \underline{E}_m dV + \int_V \underline{E}_m \cdot \underline{J} dV \right] / Q_m
\end{aligned}$$

where R_{sm} is the surface resistance at frequency ω_m and $Q_m = \omega_m \mu_0 \int_V \underline{H}_m \cdot \underline{H}_m dV / R_{sm} \oint_S \underline{H}_m \cdot \underline{H}_m dS$ is the quality factor associated with the field distribution of the m th mode. Thus we find

$$\left[\omega^2 + i(1-i) \frac{R_s}{R_{sm}} \frac{\omega\omega_m}{Q_m} - \omega_m^2 \right] e_m \varepsilon_0 \int_V \underline{E}_m \cdot \underline{E}_m dV = \left[-i\omega + \omega_m(1-i) \frac{R_s}{R_{sm} Q_m} \right] \int_V \underline{E}_m \cdot \underline{J} dV$$

where $Z_s = (1-i) R_s$. Now we need to examine the frequency behavior of the loss terms. Obviously Q_m is a function of ω_m not ω ; if the wall impedance were frequency independent the ratio R_s/R_{sm} would be unity. However because the wall impedance does depend on frequency (typically as the square root) we can rewrite this expression as

$$[\omega^2 \{1 + (1+i)/Q\} - \omega_m^2] e_m \varepsilon_0 \int_V \underline{E}_m \cdot \underline{E}_m dV = -i\omega [1 + (1+i)/Q] \int_V \underline{E}_m \cdot \underline{J} dV$$

where

$$Q = \frac{\omega\mu_0 \int_V \underline{H}_m \cdot \underline{H}_m dV}{R_s \oint_S \underline{H}_m \cdot \underline{H}_m dS} \quad (7)$$

is dependent on ω but is nearly independent of ω_m (we will ignore this dependence for the higher order modes from here on). We thus can write the vector potential as

$$\underline{A}(\underline{r}) = -\frac{1}{\varepsilon_0 V} \sum_n \frac{\underline{A}_n(\underline{r}) \int_V \underline{A}_n(\underline{r}') \cdot \underline{J}(\underline{r}') dV'}{\omega^2 (1 + i/Q) - \omega_n^2} \quad (8)$$

where we have dropped the $1/Q$ shifts in frequency.

The scalar potential satisfies Poisson's equation

$$\nabla^2 \phi = -\rho/\varepsilon_0 \quad (9)$$

where the charge density ρ arises from the continuity equation $\nabla \cdot \underline{J} = i\omega\rho$.

3.2 Difference Representation

It is convenient to subtract from the above modal series (8) the quasistatic limit $\omega \rightarrow 0$ and form the difference expansion

$$\underline{A}(\underline{r}) - \underline{A}_s(\underline{r}) = -\frac{1}{\varepsilon_0 V} \sum_n \frac{\omega^2 (1 + i/Q) / \omega_n^2}{\omega^2 (1 + i/Q) - \omega_n^2} \underline{A}_n(\underline{r}) \int_V \underline{A}_n(\underline{r}') \cdot \underline{J}(\underline{r}') dV' \quad (10)$$

This accelerates the convergence of the series. The quasi-static part can be added in separately; in this paper we will take the antenna to be far from the cavity walls and thus ignore quasi-static images in the walls of the cavity (unless the source is mounted on a wall, as in the case of a monopole) and take the quasi-static part to be approximately

$$\begin{aligned} \underline{A}_s(\underline{r}) &= \frac{1}{\varepsilon_0 V} \sum_n \frac{1}{\omega_n^2} \underline{A}_n(\underline{r}) \int_V \underline{A}_n(\underline{r}') \cdot \underline{J}(\underline{r}') dV' \approx \frac{\mu_0}{4\pi} \int_V \frac{\underline{J}_s(\underline{r}') dV'}{|\underline{r} - \underline{r}'|} \\ &\approx \frac{\mu_0}{4\pi} \int_V \underline{J}(\underline{r}') \frac{dV'}{|\underline{r} - \underline{r}'|} - \nabla \nabla \cdot \frac{\mu_0}{8\pi} \int_V \underline{J}(\underline{r}') |\underline{r} - \underline{r}'| dV' \end{aligned} \quad (11)$$

where $\underline{J}_s$ is the solenoidal part of the current and the final form can be found by considering a dipole in free space. This approximation captures the logarithmic singularity of the static contribution when the antenna radius becomes small; if higher order accuracy is desired we can add the difference of the static modal series and this approximation as a correction. Similarly, we take the scalar potential to be approximately

$$\phi(\underline{r}) = \frac{1}{\varepsilon_0 V} \sum_n \frac{1}{\eta_n^2} \phi_n(\underline{r}) \int_V \phi_n(\underline{r}') \rho(\underline{r}') dV' \approx \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho(\underline{r}') dV'}{|\underline{r} - \underline{r}'|} \quad (12)$$

where we have substituted $\phi = \sum_n \varphi_n \phi_n(\underline{r})$, the eigenfunctions satisfy $(\nabla^2 + \eta_n^2) \phi_n = 0$, $\underline{n} \times \nabla \phi_n = 0$ on the boundary, and we use normalization $\int_V \phi_n(\underline{r}) \phi_m(\underline{r}) dV = V \delta_{mn}$.

3.3 Input Impedance of Short Linear Antenna

Now we focus on a center driven linear dipole antenna of length $2h$ and radius a . This section assumes the antenna is electrically short. The input impedance of the antenna can be found from [5]

$$Z_{in} = -\frac{1}{I^2(0)} \int_V \underline{E} \cdot \underline{J} dV \quad (13)$$

The current for the electrically short dipole is linear $\underline{J} = \underline{e}_z I(z) \sim \underline{e}_z I(0) (1 - |z|/h)$ and thus

$$\begin{aligned} Z_{in} &= R - iX + \frac{i\omega}{\varepsilon_0 V I^2(0)} \sum_n \frac{\omega^2 (1 + i/Q) / \omega_n^2}{\omega^2 (1 + i/Q) - \omega_n^2} \int_V \underline{A}_n(\underline{r}) \cdot \underline{J}(\underline{r}) dV \int_V \underline{A}_n(\underline{r}') \cdot \underline{J}(\underline{r}') dV' \\ &= R - iX + \frac{i\omega}{\varepsilon_0 V I^2(0)} \sum_n \frac{\omega^2 (1 + i/Q) / \omega_n^2}{\omega^2 (1 + i/Q) - \omega_n^2} \int_{-h}^h A_{nz}(z) I(z) dz \int_{-h}^h A_{nz}(z') I(z') dz' \\ &\sim \frac{1}{-i\omega C} - i\omega L + R + \frac{i\omega h^2}{\varepsilon_0 V} \sum_n \frac{\omega^2 / \omega_n^2}{\omega^2 (1 + i/Q) - \omega_n^2} A_{nz}^2(\underline{r}) \\ &\sim \frac{1}{-i\omega C} - i\omega L + R + R_{rad} \sum_n \left(\frac{2\pi Q}{k^3 V} \right) \frac{(i\omega^2/Q) \omega^2 / \omega_n^2}{\omega^2 (1 + i/Q) - \omega_n^2} 3A_{nz}^2(\underline{r}) \end{aligned} \quad (14)$$

where $X \sim \omega L - 1/(\omega C)$ is the local antenna reactance, R is the ohmic antenna resistance, and [5]

$$R_{rad} \sim \frac{\eta_0}{6\pi} (kh)^2 \quad (15)$$

is the radiation resistance of a short dipole antenna in free space, $\eta_0 = \sqrt{\mu_0/\varepsilon_0} \approx 120\pi$ ohms is the intrinsic impedance of free space. Note that the capacitance term C results from the scalar potential contribution and the local inductance term L results from the quasistatic vector potential contribution as well as a higher order term in the scalar potential contribution. The leading contribution for a short antenna is the capacitance [5]

$$C \sim \frac{2\pi\varepsilon_0 h}{\Omega_e} \quad (16)$$

where the expansion parameter for the center driven antenna is

$$\Omega_e = \Omega - 2(1 + \ln 2) \quad (17)$$

and the fatness parameter is

$$\Omega = 2 \ln(2h/a) \quad (18)$$

We will study the properties of the series (14) in the next sections.

4 STATISTICAL PROPERTIES

The primary interest in this paper is an electrically large cavity which has an irregular wall (a crude approximation, for example, is a mode stirred chamber); such a cavity has been termed complex [6]. The eigenvalues ω_n and eigenfunctions A_{nz} in (14) are difficult to predict in this frequency range. Exact predictions of these quantities are probably not meaningful anyway, since many factors can perturb their values [7], [8].

4.1 Eigenvalues

Studies in nuclear physics and quantum mechanics on the eigenvalues of bound systems at high energy [9], [10] have shown that the eigenvalues, and particularly the eigenvalue spacings, follow statistical rules in these cases; this has also been experimentally demonstrated for three dimensional electromagnetic cavities [11]. We write the spacing as

$$\Delta\omega_n = \omega_{n+1} - \omega_n = \langle\Delta\omega_n\rangle s \quad (19)$$

where s is a normalized random variable and the mean spacing is [12]

$$\langle\Delta\omega_n\rangle \sim \pi^2 c^3 / (V\omega_n^2), \quad \omega_n \rightarrow \infty \quad (20)$$

Regular, integrable (e.g., separable) [10], geometries exhibit eigenvalue spacings which follow a Poisson (exponential) distribution [11] with density

$$f_s^P(s) = e^{-s}, \quad 0 < s < \infty \quad (21)$$

Complex, nonintegrable, geometries exhibit eigenvalue spacings which follow a Rayleigh (Wigner) distribution [11], [9] with density

$$f_s^W(s) = \frac{\pi}{2} s e^{-s^2 \pi/4}, \quad 0 < s < \infty \quad (22)$$

Notice that degeneracy is more likely with the Poisson distribution than with the Rayleigh distribution. The Poisson spacings result from the assumption of no correlations between eigenvalues [10], [9]. The Rayleigh (Wigner) spacings result from assuming the two point correlation between eigenvalues grows linearly with the spacing and higher order correlations are negligible [9]. Calculations have shown that although a more rigorous treatment of the quantum mechanical problem leads to a more complicated distribution for the spacings, the simple Rayleigh (Wigner) distribution (22) is within a few percent of the one produced by this more rigorous analysis.

Interpolation functions which match both the Poisson and Rayleigh (Wigner) distributions are known [11], [10], however it is not obvious how to choose the additional parameter in this function for a particular cavity geometry. Monte Carlo simulations using each of the two distributions will therefore be given and compared to experimental data in a following section. The Monte Carlo simulations discussed in following sections use the eigenvalue spacing distributions to generate the eigenvalue sequence (higher order correlations between eigenvalues are not included). Simulations will also be given with a uniform spacing distribution

$$s \equiv 1 \quad (23)$$

since this produces similar results, but it also produces a stochastic series that can be more easily analyzed.

4.2 Eigenfunctions

An argument given in the quantum mechanical literature on a particle in a cavity can be found in [10]: “the wavefunction can be conceived as built by superposing the phases from all the classical trajectories which pass through a particular neighborhood; but these phases are uncorrelated because between consecutive returns to the same neighborhood, the trajectory wanders almost randomly around the energy surface; various authors have therefore speculated that the wavefunction is a Gaussian random function of position.” A similar argument has been made for an electromagnetic mode in a complex cavity being composed of many uncorrelated rays [6]. Experiments on cavities with smooth walls (for example, stadium shaped) have shown that deviations from this simple distribution arise from contributions corresponding to periodic ray trajectories [10] and much work has been done on representing these [13], [14]. Nevertheless we will use this same simple assumption here and take the three components of the field (statistical isotropy) to be distributed as

$$\zeta = \sqrt{3}A_{nz} \quad (24)$$

$$f_{\zeta}(\zeta) = \frac{1}{\sqrt{2\pi}} e^{-\zeta^2/2} \quad (25)$$

The factor $\sqrt{3}$ is present to give the eigenfunction normalization [6]

$$3 \int_{-\infty}^{\infty} A_{nz}^2 f_{A_{nz}}(A_{nz}) dA_{nz} = \int_{-\infty}^{\infty} \zeta^2 f_{\zeta}(\zeta) d\zeta = \frac{1}{V} \int_V \underline{A}_n \cdot \underline{A}_n dV = 1 \quad (26)$$

4.3 Correlation Function

The values of A_{nz} at two spatial points are in general correlated. The case of the electrically short dipole (14) has complete correlation since the source and observation points are nearly the same. The correlation along the antenna will become important when we treat the case of a resonant antenna below. Here we are only concerned with a linear antenna but for other types of antenna other components of the correlation dyad become important [15].

The correlation function in the z direction for the z component of the cavity field is given by [1]

$$\rho_z(z_1, z_2) = \frac{\langle E_z(z_1) E_z^*(z_2) \rangle}{\sqrt{\langle |E_z(z_1)|^2 \rangle \langle |E_z(z_2)|^2 \rangle}} = \frac{3}{2} \left(1 + \frac{1}{k^2} \frac{\partial^2}{\partial z_1^2} \right) \frac{\sin[k(z_1 - z_2)]}{k(z_1 - z_2)}$$

Although this was derived in the overmoded region [1], the arguments that have been used to justify the statistical description of the individual modes in the undermoded region mean that the plane wave integral representation for the field used in the derivation should be applicable here as well (although the modal field is real in this case). We must substitute k_n for k at a particular cavity mode

$$\rho_z(z_1, z_2) = \frac{\langle A_{nz}(z_1) A_{nz}(z_2) \rangle}{\sqrt{\langle A_{nz}^2(z_1) \rangle \langle A_{nz}^2(z_2) \rangle}} = \frac{3}{2} \left(1 + \frac{1}{k_n^2} \frac{\partial^2}{\partial z_1^2} \right) \frac{\sin[k_n(z_1 - z_2)]}{k_n(z_1 - z_2)} \quad (27)$$

Note that this correlation coefficient is not the same as that for the scalar wavefunction [10].

5 SIMULATIONS AND EXPERIMENTS

Simulations of the impedance (14) were made using random number generators to obtain eigenvalue spacings and eigenfunction amplitudes.

5.1 Simulation of Mode Series

To simplify the simulation we rewrite (14) as

$$\begin{aligned}
(Z_{in} - Z) / R_{rad} &\sim \sum_n \left(\frac{2\pi Q}{k^3 V} \right) \frac{(i\omega^2/Q) \omega^2 / \omega_n^2}{\omega^2 (1 + i/Q) - \omega_n^2} 3A_{nz}^2(\underline{r}) \\
&\approx \sum_{n_r} \left(\frac{2\pi Q}{k^3 V} \right) \frac{(i\omega^2/Q) \omega^2 / \omega_n^2}{\omega^2 (1 + i/Q) - \omega_n^2} 3A_{nz}^2(\underline{r}) \\
&\quad + \left(\int_0^{\omega - W_{-n}} + \int_{\omega + W_n}^{\infty} \right) \left(\frac{2\pi Q}{k^3 V} \right) \frac{(i\omega^2/Q) \omega^2 / \omega_n^2}{\omega^2 (1 + i/Q) - \omega_n^2} 3A_{nz}^2(\underline{r}) \frac{\Delta n}{\langle \Delta \omega_n \rangle} d\omega_n \\
&\approx \sum_{n_r} \left(\frac{2\pi Q}{k^3 V} \right) \frac{(i\omega^2/Q) \omega^2 / \omega_n^2}{\omega^2 (1 + i/Q) - \omega_n^2} 3A_{nz}^2(\underline{r}) + \frac{i2\omega}{\pi} \left(\int_0^{\omega - W_{-n}} + \int_{\omega + W_n}^{\infty} \right) \frac{d\omega_n}{\omega^2 (1 + i/Q) - \omega_n^2} \\
&\approx \sum_{n_r} \left(\frac{2\pi Q}{k^3 V} \right) \frac{(i\omega^2/Q) \omega^2 / \omega_n^2}{\omega^2 (1 + i/Q) - \omega_n^2} 3A_{nz}^2(\underline{r}) \\
&\quad - \frac{i}{\pi (1 + i/(2Q))} \ln \left[\frac{2\omega + W_n + i\omega/(2Q)}{2\omega - W_{-n} + i\omega/(2Q)} \right] - \frac{i}{\pi (1 + i/(2Q))} \ln \left[\frac{W_{-n} + i\omega/(2Q)}{W_n - i\omega/(2Q)} \right] \\
&\approx \sum_{n_r} \left(\frac{2\pi Q}{k^3 V} \right) \frac{(i\omega^2/Q) \omega^2 / \omega_n^2}{\omega^2 (1 + i/Q) - \omega_n^2} 3A_{nz}^2(\underline{r}) - i \frac{W_n + W_{-n}}{2\pi\omega} + \frac{\omega}{2\pi Q} \left(\frac{1}{W_n} + \frac{1}{W_{-n}} \right) \quad (28)
\end{aligned}$$

where $\Delta n = 1$, $Z = R - iX$ is the local impedance of the antenna, n_r includes a range of frequencies ω_n within $\omega - W_{-n} < \omega_n < \omega + W_n$, and we have assumed that $Q \gg 1$, $W_{\pm n} \gg \omega/(2Q)$, and $2\omega \gg W_{\pm n}$. The final two terms in (28) are typically small corrections. We will also make the following approximation of $\omega_n \rightarrow \omega$ in the finite sum near resonance. After dropping the correction terms, since they do not have appreciable value in the simulations, one has

$$(Z_{in} - Z) / R_{rad} \approx \left(\frac{2\pi Q}{k^3 V} \right) \sum_{n_r} \frac{i\omega/(2Q)}{\omega (1 + i/(2Q)) - \omega_n} 3A_{nz}^2(\underline{r}) \quad (29)$$

The frequency difference in the denominator must be maintained [16]. Monte Carlo simulations of this sum were performed using Rayleigh (Wigner), Poisson (exponential), and uniform spacing for the eigenvalues with Gaussian eigenfunctions. The formula (29) was evaluated at a number of uniformly spaced frequencies to correspond with the experiment. The number of modes in the n_r sum was selected to make $\omega_0 - W_{-n} < \omega_n < \omega_0 + W_n$ cover a range greater than the frequency sweep (where ω_0 is the center of the frequency sweep). The further simplification of fixing the mean eigenvalue spacing $\langle \Delta \omega_n \rangle$ as well as the value of Q , at the value of the center frequency of the frequency sweeps ω_0 , was made in the simulations.

5.2 Undermoded-Overmoded Threshold

The factor appearing in front of the sum in (29) occurs repeatedly and is worth discussing. Suppose we consider the total energy in the modal spectrum (over some specified frequency range that is narrow but includes many individual modes)

$$W_e = \int_{\omega_-}^{\omega_+} \int_V \frac{1}{2} \varepsilon_0 |E_z|^2 dV d\omega$$

The energy in each mode is [3]

$$\begin{aligned} & |e_n|^2 \omega^2 V \frac{1}{2} \varepsilon_0 \int_{\omega_-}^{\omega_+} \frac{(\omega_n^2/Q_n)^2}{(\omega^2 - \omega_n^2)^2 + (\omega_n^2/Q_n)^2} d\omega \\ & \approx |e_n|^2 \omega^2 V \frac{1}{2} \varepsilon_0 \int_0^\infty \frac{(\omega_n^2/Q_n)^2 d\omega}{(\omega^2 - \omega_n^2)^2 + (\omega_n^2/Q_n)^2} \approx \frac{1}{2} \varepsilon_0 |e_n|^2 \omega^2 V \left(\frac{\pi}{2} \omega_n/Q_n \right) \end{aligned}$$

where the integral of the mode over the volume is normalized to the volume, since we are considering a damped mode [3], the quality factor is denoted by Q_n and enters the denominator as $i\omega_n^2/Q_n$, and $|e_n|^2 \omega^2$ is the peak square amplitude of the electric field resonance. Using the mean modal spacing with $k_n = \omega_n/c$, we can write

$$\int_{\omega_-}^{\omega_+} \frac{1}{2} \varepsilon_0 \int_V |E_z|^2 dV d\omega \approx \frac{1}{2} \varepsilon_0 \langle |e_n|^2 \rangle \omega^2 V \left(\frac{\pi}{2} \omega_n/Q_n \right) \frac{(\omega_+ - \omega_-)}{\langle \Delta\omega_n \rangle} \approx \frac{1}{2} \varepsilon_0 \langle |e_n|^2 \rangle \omega^2 V (\omega_+ - \omega_-) \frac{k_n^3 V}{2\pi Q_n}$$

where orthogonality of the modes eliminates cross terms. Thus the final factor

$$\alpha_n = \frac{k_n^3 V}{2\pi Q_n} = \frac{\pi \omega_n/Q_n}{2 \langle \Delta\omega_n \rangle} \quad (30)$$

is the ratio of the actual energy to the energy if the field were constant in frequency at the mean resonance peak level $\int_{\omega_-}^{\omega_+} \frac{1}{2} \varepsilon_0 \int_V |E_z|^2 dV d\omega \approx \frac{1}{2} \varepsilon_0 \langle |e_n|^2 \rangle \omega^2 V (\omega_+ - \omega_-)$. Obviously a value much less than one indicates an undermoded condition whereas a value much greater than one indicates an overmoded condition. Figures 4 and 5 illustrate the two limits.

5.3 Experimental Comparison

A large welded aluminum mode stirred chamber was used as the cavity in the experiments. The chamber has dimensions (37 ft x 23 ft x 13 ft) and a volume of $V \approx 313 \text{ m}^3$. The stirrer was in the chamber, but was not moved during the experiments. The quality factor of the chamber was determined to be $Q \approx 80,000$ by examining the 3 dB width of isolated modes at 220 MHz. The parameter (30) at this frequency, dropping the subscript n

$$\alpha = k^3 V / (2\pi Q) \quad (31)$$

is approximately $\alpha \approx 0.0609$; the chamber is therefore highly undermoded. The experiments were also conducted at higher frequencies including 920 MHz. The quality factor at 920 MHz is not precisely known but was estimated from the 220 MHz value by the scaling $\sqrt{\omega}$ to be approximately $Q \approx 165,000$. The parameter (30) is approximately $\alpha \approx 2.16$; thus modal overlap begins to occur here.

The antenna was taken to be a thin, near-resonant monopole in the wall of the mode stirred chamber. The dimensions of the monopoles were $2a \approx 0.102$ in, $h \approx 12.953$ in at 220 MHz, and $h \approx 2.97$ in at 920 MHz. Monopoles were used since they are the simplest antennas to implement in the chamber, and because both amplitude and phase were desired. The monopoles were operated near resonance to make the local reactance $X \rightarrow 0$ and thus obtain good accuracy on the measurements of the cavity wall effects. The data was normalized by the free space value of the radiation resistance just as in (29). It will be demonstrated in sections below (at least when the current distribution is not significantly perturbed) that electrically longer antennas, including resonant antennas, as well as monopoles in walls, obey approximately the same formula, provided that the free space radiation resistance is appropriately modified for these cases.

Figure 6 shows a comparison of the normalized input resistance of a resonant monopole at 220 MHz with simulations. The frequency span was 10 MHz with 4800 frequency points in the experiment. The simulations used 200 modes with 1000 frequency points. Notice the large variation of the input resistance in the distribution. Also note that the effect of the eigenvalue spacing distribution in the simulations is not large. The experimental agreement with the Rayleigh (Wigner) distribution appears to be the best but it should be noted that there is some variation among realizations of the ensemble. The experimental data was normalized by 44 ohms, since this was the experimental value of the input resistance at 220 MHz in free space (the monopole was slightly above the first resonance). Figure 7 shows a comparison of the normalized input resistance of a resonant monopole at 920 MHz with simulations. The frequency span was 1 MHz with 801 frequency points in the experiment. The simulations used 400 modes with 1000 frequency points. The distribution has been narrowed considerably at this frequency. The agreement with the simulation is again reasonable. The experimental data was normalized by $R_{rad} \approx 46$ ohms since this was the experimental value of the input resistance at 920 MHz in free space.

6 ASYMPTOTIC FORMULAS

The Monte Carlo simulations of the preceding section show reasonable agreement with the experimental results. However because we are particularly interested in the extreme values of the impedance, it is important to obtain an estimate of the behavior in the tail regions of the distribution. The overmoded limit $\alpha \rightarrow \infty$ from (29) gives

$$z_{in} = (Z_{in} - Z) / R_{rad} \sim \left(\frac{2\pi Q}{k^3 V} \right) \int_0^\infty \frac{(i\omega^2/Q) \omega^2/\omega_n^2}{\omega^2(1+i/Q) - \omega_n^2} 3A_{nz}^2(r) \frac{\Delta n}{\langle \Delta \omega_n \rangle} d\omega_n$$

$$\sim \frac{i\omega}{\pi} 2 \int_0^\infty \frac{d\omega_n}{\omega^2(1+i/Q) - \omega_n^2} \approx 1 \quad (32)$$

where we have again dropped the $O(1/Q)$ term. Thus the overmoded limit produces the free space radiation resistance R_{rad} .

The mean input impedance as a function of frequency can also be found. Integration with respect to ω over the interval ω_- to ω_+ gives (the interval is assumed to contain a large number of modes $\omega_+ - \omega_- \gg \langle \Delta \omega_n \rangle \sim \pi^2 c^3 / (V \omega^2)$, is assumed to satisfy $\omega/Q \ll \omega_+ - \omega_-$, but is assumed to be small compared to the center frequency $\omega \gg \omega_+ - \omega_-$)

$$\langle z_{in} \rangle_\omega = \frac{1}{\omega_+ - \omega_-} \int_{\omega_-}^{\omega_+} z_{in} d\omega \sim \sum_n \left(\frac{\pi c^3}{V \omega_n^2} \right) 3A_{nz}^2(r) \frac{i}{\omega_+ - \omega_-} \int_{\omega_-}^{\omega_+} \frac{2\omega d\omega}{\omega^2 - \omega_n^2 + i\omega^2/Q}$$

where the subscript ω on the mean indicates that the expectation is taken with respect to frequency. The integral is carried out as

$$\int_{\omega_-}^{\omega_+} \frac{2\omega d\omega}{\omega^2 - \omega_n^2 + i\omega^2/Q} = \int_{v_-}^{v_+} \frac{dv}{v - v_n + iv/Q} = \frac{1}{1 + i/Q} \ln \left[\frac{v_+ - v_n/(1 + i/Q)}{v_- - v_n/(1 + i/Q)} \right]$$

$$\sim -i\pi + \ln \left| \frac{\omega_+^2 - \omega_n^2}{\omega_-^2 - \omega_n^2} \right|, \quad \omega_- < \omega_n < \omega_+$$

$$\sim \ln \left| \frac{\omega_+^2 - \omega_n^2}{\omega_-^2 - \omega_n^2} \right|, \quad \text{otherwise}$$

where we have assumed that Q is large (we have also approximated Q as independent of ω over this relatively narrow band). Thus we find

$$\langle z_{in} \rangle_\omega \sim \frac{1}{\omega_+ - \omega_-} \sum_{n_\pm} \left(\frac{\pi^2 c^3}{V \omega_n^2} \right) 3A_{nz}^2(r) + \frac{i/\pi}{\omega_+ - \omega_-} \sum_n \left(\frac{\pi^2 c^3}{V \omega_n^2} \right) \ln \left| \frac{\omega_+^2 - \omega_n^2}{\omega_-^2 - \omega_n^2} \right| 3A_{nz}^2(r)$$

where the first sum $n_\pm$ includes only the terms with $\omega_- < \omega_n < \omega_+$. Assuming that $1/\omega_n^2$ is slowly varying in the interval $\omega_- < \omega_n < \omega_+$ we can replace $3A_{nz}^2$ by its unit mean and the sum by an integral

$$\langle r_{in} \rangle_\omega \sim \frac{1}{\omega_+ - \omega_-} \sum_{n_\pm} \left(\frac{\pi^2 c^3}{V \omega_n^2} \right) \sim \frac{1}{\omega_+ - \omega_-} \int_{\omega_-}^{\omega_+} d\omega_n = 1 \quad (33)$$

Assuming that the coefficient of the eigenfunction for the reactance term is slowly varying (the logarithmic singularities are smoother than the poles that we started with as a result of the frequency average) we similarly obtain

$$\langle x_{in} \rangle_\omega \sim \frac{-1/\pi}{\omega_+ - \omega_-} \int_0^\infty \ln \left| \frac{\omega_+^2 - \omega_n^2}{\omega_-^2 - \omega_n^2} \right| d\omega_n = 0 \quad (34)$$

Thus the frequency mean of the normalized impedance approaches unity.

Distributions will now be constructed for the impedance in both the undermoded limit, in the transition region, and the overmoded limit. The distribution for the modal series with Rayleigh (Wigner) spaced eigenvalues appears to be a slightly more accurate representation near the small tail of the input resistance. However the complications introduced by the presence of the random variables in the denominators of the series have prevented much progress in this direction. Therefore we will focus on the uniformly spaced eigenvalue series to obtain an approximate description of the impedance behavior.

6.1 Single Resonant Mode Distribution (uniform spacing, undermoded, impedance large)

The largest values in the distribution arise from frequencies lying near resonant modes. Let us take a single resonant mode and determine the impedance distribution; a single mode will dominate the large impedances in the undermoded case. The impedance contribution for a single mode from (29) is

$$(Z_{in} - Z) / R_{rad} \approx \left(\frac{2\pi Q}{k^3 V} \right) \frac{i\omega / (2Q)}{\omega - \omega_n + i\omega / (2Q)} 3A_{nz}^2(x) = \left(\frac{2\pi Q}{k^3 V} \right) \frac{(\omega - \omega_n) i\omega / (2Q) + \omega^2 / (2Q)^2}{(\omega - \omega_n)^2 + \omega^2 / (2Q)^2} \zeta^2 \quad (35)$$

Using (31) and noting that $\omega / (2Q \langle \Delta\omega_n \rangle) \sim k k_n^2 V / (2\pi^2 Q) \sim \alpha / \pi$, we can write

$$(R_{in} - R) / R_{rad} = r_{in} \approx \frac{\zeta^2 \alpha}{\pi^2 (\omega_n - \omega)^2 / \langle \Delta\omega_n \rangle^2 + \alpha^2} \quad (36)$$

$$(X_{in} - X) / R_{rad} = x_{in} \approx \frac{\zeta^2 \pi (\omega_n - \omega) / \langle \Delta\omega_n \rangle}{\pi^2 (\omega_n - \omega)^2 / \langle \Delta\omega_n \rangle^2 + \alpha^2} \quad (37)$$

Because of the typical uniform frequency spacing we can treat the quantity $\pi (\omega_n - \omega) / \langle \Delta\omega_n \rangle$ as a uniform random variable on $-\pi/2$ to $+\pi/2$ (of course this repeats as ω is swept over many modes). The eigenfunction amplitude ζ can be thought of as a random variable since as ω is swept from mode to mode only that mode with nearest ω_n is being represented. The number of independent samples is limited by the number of modes contained in the frequency sweep.

6.1.1 Input Resistance

The density function for the denominator $D = \pi^2 (\omega_n - \omega)^2 / \langle \Delta\omega_n \rangle^2 + \alpha^2$ of (36) [17], with $\pi (\omega_n - \omega) / \langle \Delta\omega_n \rangle$ being a uniform random variable, is

$$f_D(D) = \frac{1}{\pi \sqrt{D - \alpha^2}}, \quad \alpha^2 < D < \alpha^2 + \pi^2/4$$

The density function for the numerator $N = \zeta^2 \alpha$, with ζ being a Gaussian random variable, is

$$f_N(N) = \frac{1}{\sqrt{2\pi N \alpha}} e^{-N/(2\alpha)}, \quad 0 < N < \infty$$

Thus the density function for the normalized input resistance (assuming the numerator and denominator distributions are independent) is [17]

$$\begin{aligned} f(r_{in}) &= \int_{\alpha^2}^{\alpha^2 + \pi^2/4} D f_N(r_{in} D) f_D(D) dD = \frac{1}{\pi \sqrt{2\pi r_{in} \alpha}} \int_{\alpha^2}^{\alpha^2 + \pi^2/4} e^{-r_{in} D / (2\alpha)} \frac{\sqrt{D} dD}{\sqrt{D - \alpha^2}} \\ &= \frac{1}{\pi \sqrt{2\pi r_{in} \alpha}} \int_0^{\pi^2/4} e^{-r_{in} (u + \alpha^2) / (2\alpha)} \frac{\sqrt{u + \alpha^2}}{\sqrt{u}} du \end{aligned} \quad (38)$$

It is interesting that, based on this single mode density, the average normalized resistance is $\langle r_{in} \rangle_\omega = \int_0^\infty r_{in} f(r_{in}) dr_{in} = \frac{2}{\pi} \arctan\left(\frac{\pi}{2\alpha}\right) \rightarrow 1$ for the undermoded region $\alpha \ll 1$.

Because this single mode distribution is only useful in the undermoded region for larger values of the resistance $r_{in} \gg \alpha$, we can approximate the integral by extending upper limit to infinity and obtain

$$\begin{aligned} f(r_{in}) &\sim \frac{e^{-r_{in}\alpha/2}}{\pi\sqrt{2\pi r_{in}\alpha}} \int_0^\infty e^{-r_{in}u/(2\alpha)} \frac{\sqrt{u+\alpha^2}}{\sqrt{u}} du \\ &= \frac{e^{-r_{in}\alpha/4}}{2\pi\sqrt{2\pi r_{in}\alpha}} \alpha^2 \{K_0(r_{in}\alpha/4) + K_1(r_{in}\alpha/4)\}, \quad r_{in}/\alpha \gg 1 \end{aligned} \quad (39)$$

where we have used the identity [18] $\int_0^\infty e^{-\nu x} \frac{dx}{\sqrt{x(x+a)}} = e^{\nu a/2} K_0(\nu a/2)$. There is a transition at $r_{in}\alpha = O(1)$; if this quantity is small (39) gives

$$f(r_{in}) \approx \frac{1}{\pi r_{in}} \sqrt{\frac{2\alpha}{\pi r_{in}}}, \quad 1/\alpha \gg r_{in} \gg \alpha \quad (40)$$

and if this quantity is large then

$$f(r_{in}) \approx \frac{e^{-r_{in}\alpha/2}}{\pi r_{in}} \alpha, \quad r_{in}\alpha \gg 1 \quad (41)$$

which shows the exponential decay of the density for large r_{in} .

The corresponding distribution function is

$$F(r_{in}) = 1 - \int_{r_{in}}^\infty f(r) dr \sim 1 + \frac{\alpha}{\pi} \sqrt{r_{in}\alpha/(2\pi)} e^{-r_{in}\alpha/4} [K_0(r_{in}\alpha/4) - K_1(r_{in}\alpha/4)], \quad r_{in} \gg \alpha \quad (42)$$

and approximately

$$F(r_{in}) \approx 1 - \frac{2}{\pi} \sqrt{\frac{2\alpha}{\pi r_{in}}}, \quad 1/\alpha \gg r_{in} \gg \alpha \quad (43)$$

Figures 8 and 9 show comparison of the formula (42) with simulation and experiment; the overlapping modes for the case shown in Figure 9 make (42) inaccurate. The approximation (43) represents the distribution over a large portion of its range and would agree with the data shown in these figures. It is interesting that the median value $F(r_{in}) = 1/2$ from the approximation (43) gives $r_{in} \approx \alpha 32/\pi^3 \approx O(\alpha)$ for $\alpha \ll 1$.

The number of independent samples in a frequency sweep is dependent on the number of modes spanned. For example at 220 MHz there are only 141 modes in the frequency sweep even though there are many more frequencies sampled. If, for $\alpha \ll 1$, the frequency sweep is sufficiently fine to resolve the spectral peaks (over-sampling in frequency) then the density function of the peaks is of interest. Thus, near the upper limit of r_{in} we can set $\omega = \omega_n$ and find the single mode density function for the peak values (the square of a Gaussian random variable)

$$f(r_{in}) \approx \sqrt{\frac{\alpha}{2\pi r_{in}}} e^{-r_{in}\alpha/2}, \quad r_{in}\alpha \gg 1 \quad (44)$$

The exponential behavior in (41) is the same as (44). The distribution function corresponding to (44) is

$$F(r_{in}) \approx \text{erf}\left(\sqrt{r_{in}\alpha/2}\right) \quad (45)$$

where $\text{erf}(x)$ is the error function, and the number of independent samples corresponds to the number of modes contained in the frequency sweep.

6.1.2 Input Reactance

The density function for the denominator $D = \pi(\omega_n - \omega) / \langle \Delta\omega_n \rangle + \alpha^2 / \{\pi(\omega_n - \omega) / \langle \Delta\omega_n \rangle\}$ in this case (assuming $\alpha < \pi/2$) is

$$\begin{aligned} f_D(D) &= \frac{1}{\pi\sqrt{1-4\alpha^2/D^2}}, \frac{\pi}{2}(1+4\alpha^2/\pi^2) > |D| > 2\alpha \\ &= \frac{1}{2\pi} \left(\frac{1}{\sqrt{1-4\alpha^2/D^2}} - 1 \right), \infty > |D| > \frac{\pi}{2}(1+4\alpha^2/\pi^2) \end{aligned}$$

The density function of the numerator $N = \zeta^2$ in this case is

$$f_N(N) = \frac{1}{\sqrt{2\pi N}} e^{-N/2}, 0 < N < \infty$$

Thus the density function of the normalized reactance is

$$\begin{aligned} f(x_{in}) &= \int_{2\alpha}^{\infty} D f_N(|x_{in}|D) f_D(D) dD \\ &= \frac{(\alpha/\pi)^{3/2}}{\sqrt{|x_{in}|}} \int_1^{\infty} e^{-\alpha|x_{in}|u} \left(\frac{u}{\sqrt{u^2-1}} - 1 \right) \sqrt{u} du + \frac{(\alpha/\pi)^{3/2}}{\sqrt{|x_{in}|}} \int_1^{\{\pi/(2\alpha)+2\alpha/\pi\}/2} e^{-\alpha|x_{in}|u} \left(\frac{u}{\sqrt{u^2-1}} + 1 \right) \sqrt{u} du \end{aligned} \quad (46)$$

The mean is zero but the mean absolute value is $\langle |x_{in}| \rangle_{\omega} = 2 \int_0^{\infty} x_{in} f(x_{in}) dx_{in} = \frac{1}{\pi} \text{arccosh}[\{\pi/(2\alpha) + (2\alpha)/\pi\}/2] + \frac{1}{\pi} \ln\{\pi/(2\alpha) + (2\alpha)/\pi\} \sim \frac{2}{\pi} \ln\{\pi/(2\alpha)\}$, $\alpha \rightarrow 0$. Assuming that $|x_{in}| \gg 1$ we can extend the upper limit of the second integral to simplify to

$$f(x_{in}) \sim 2 \frac{(\alpha/\pi)^{3/2}}{\sqrt{|x_{in}|}} e^{-|x_{in}|\alpha} \int_0^{\infty} e^{-|x_{in}|\alpha v} \sqrt{\frac{v+1}{v+2}} (\sqrt{v} + 1/\sqrt{v}) dv$$

Using the averaging method [19] to fix the value of v in the slowly varying square roots at $3/(2|x_{in}|\alpha)$ in the first term and $1/(2|x_{in}|\alpha)$ in the second term, we find

$$f(x_{in}) \approx \frac{1}{\pi|x_{in}|^2} e^{-|x_{in}|\alpha} \sqrt{\frac{3+2|x_{in}|\alpha}{3+4|x_{in}|\alpha}} + 2 \frac{\alpha}{\pi|x_{in}|} e^{-|x_{in}|\alpha} \sqrt{\frac{1+2|x_{in}|\alpha}{1+4|x_{in}|\alpha}}, |x_{in}| \gg 1$$

The additional limit of $|x_{in}| \alpha$ large gives

$$f(x_{in}) \approx \frac{\alpha\sqrt{2}}{\pi|x_{in}|} e^{-|x_{in}|\alpha}, |x_{in}|\alpha \gg 1 \quad (47)$$

The exponential decay constant shows that the reactance magnitude achieves approximately half the extreme value of the resistance (41). It should be remembered that the reactance and resistance are not independent.

6.2 Between Mode Distribution (uniform spacing, undermoded, small resistance) or (modes overlapping, general resistance)

The distribution for small values of the input resistance occurs when the observation frequencies are between modes. Let the observation frequency be exactly between the uniformly spaced resonant frequencies ω_n of the modal series and determine the conditional density function for this case. The number of independent samples would again be the number of modes spanned in the frequency sweep when we are over-sampling in frequency (and thus resolving the frequency behavior). Then we can write

$$\begin{aligned} (R_{in} - R) / R_{rad} = r_{in} &= \sum_{n_r} \frac{3(A_{nz})^2 \alpha}{\pi^2 (\omega_n - \omega)^2 / \langle \Delta \omega_n \rangle^2 + \alpha^2} \\ &\approx \sum_{n=-\infty}^{\infty} \frac{3(A_{n+n_r z})^2 \alpha}{\pi^2 (n - 1/2)^2 + \alpha^2} = \sum_{n=1}^{\infty} \frac{\frac{1}{2} [3(A_{n+n_r z})^2 + 3(A_{1-n+n_r z})^2] 2\alpha}{\pi^2 (n - 1/2)^2 + \alpha^2} \end{aligned} \quad (48)$$

Using (24) and (25) it can be easily shown that the density function for $\xi = \frac{1}{2} [3(A_{n+n_r z})^2 + 3(A_{1-n+n_r z})^2]$ is the simple exponential

$$f_{\xi}(\xi) = e^{-\xi}, 0 < \xi < \infty$$

Scaling the various distributions we can rewrite (48) in a simple form

$$r_{in} = (8\alpha/\pi^2) \sum_{n=1}^{\infty} \xi_n$$

with

$$f_{\xi_n}(\xi_n) = \left[(2n-1)^2 + 4\alpha^2/\pi^2 \right] e^{-\xi_n [(2n-1)^2 + 4\alpha^2/\pi^2]}, 0 < \xi_n < \infty$$

The density of the summation of these random variables

$$\chi = \sum_{n=1}^{\infty} \xi_n$$

involves an infinite set of convolutions [17]. Use of the Laplace transform

$$\bar{f}_{\xi_n}(s) = \int_0^\infty f_{\xi_n}(\xi_n) e^{-s\xi_n} d\xi_n = \frac{(2n-1)^2 + 4\alpha^2/\pi^2}{(2n-1)^2 + 4\alpha^2/\pi^2 + s}$$

simplifies the computation. Thus the transform of the infinite sequence of convolutions is

$$\bar{f}_\chi(s) = \prod_{n=1}^{\infty} \bar{f}_{\xi_n}(s)$$

The identity [20] $\prod_{k=1}^{\infty} \left[1 - \frac{x^2}{k^2 - a^2}\right] = \frac{a}{\sqrt{a^2 + x^2}} \frac{\sin(\pi\sqrt{a^2 + x^2})}{\sin(\pi a)}$ can be manipulated to give

$$1/\bar{f}_\chi(s) = \prod_{n=1}^{\infty} \left[1 + \frac{s}{(2n-1)^2 + 4\alpha^2/\pi^2}\right] = \frac{\cosh\left(\frac{\pi}{2}\sqrt{4\alpha^2/\pi^2 + s}\right)}{\cosh(\alpha)}$$

The inverse transform is

$$\begin{aligned} f_\chi(\chi) &= \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \bar{f}_\chi(s) e^{s\chi} ds = \cosh(\alpha) \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \operatorname{sech}\left(\frac{\pi}{2}\sqrt{4\alpha^2/\pi^2 + s}\right) e^{s\chi} ds \\ &= e^{-\chi 4\alpha^2/\pi^2} \cosh(\alpha) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \operatorname{sech}\left(\frac{\pi}{2}\sqrt{s}\right) e^{s\chi} ds \end{aligned}$$

where c is greater than -1 . Two convergent series representations can be derived from this formula (one better for $\chi \rightarrow 0$ and one better for $\chi \rightarrow \infty$). Pushing the contour to the right and expanding the hyperbolic function as $\sum_{m=0}^{\infty} (-1)^m e^{-(m+1/2)\pi\sqrt{s}}$ we find

$$f_\chi(\chi) = e^{-\chi 4\alpha^2/\pi^2} \cosh(\alpha) \sqrt{\frac{\pi}{\chi^3}} \sum_{m=0}^{\infty} (-1)^m (m+1/2) e^{-(m+1/2)^2 \pi^2/(4\chi)} \quad (49)$$

where we have used the identity [21] $\frac{a}{2\sqrt{\pi t^3}} e^{-a^2/(4t)} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} e^{-a\sqrt{s}} ds$. The Jacobian theta function [21] $\vartheta_1(z, q) = 2 \sum_{m=0}^{\infty} (-1)^m q^{(m+1/2)^2} \sin(2m+1)z$ can be used to sum this series

$$f_\chi(\chi) = e^{-\chi 4\alpha^2/\pi^2} \cosh(\alpha) \frac{1}{4} \sqrt{\frac{\pi}{\chi^3}} \vartheta_1' \left(0, e^{-\pi^2/(4\chi)}\right)$$

where the prime denotes differentiation with respect to the first argument. Alternatively if we enclose the poles $s = -(2m+1)^2$ on the left we have

$$f_\chi(\chi) = e^{-\chi 4\alpha^2/\pi^2} \cosh(\alpha) \frac{4}{\pi} \sum_{m=0}^{\infty} (-1)^m (2m+1) e^{-(2m+1)^2 \chi} \quad (50)$$

The corresponding distribution function is

$$F_{\chi}(\chi) = 1 - e^{-\chi^4 \alpha^2 / \pi^2} \cosh(\alpha) \frac{4}{\pi} \sum_{m=0}^{\infty} (-1)^m \frac{2m+1}{(2m+1)^2 + 4\alpha^2 / \pi^2} e^{-(2m+1)^2 \chi} \quad (51)$$

The normalized input resistance has the density

$$f(r_{in}) = \frac{\pi^2}{8\alpha} f_{\chi} \left(\frac{\pi^2 r_{in}}{8\alpha} \right) \quad (52)$$

and distribution

$$F(r_{in}) = F_{\chi} \left(\frac{\pi^2 r_{in}}{8\alpha} \right) \quad (53)$$

The short dashed curve in Figure 9 is (51) and (53); thus when the modes are overlapping $\alpha > 1$, restricting the observation frequency to be exactly between modes still generates the entire distribution of the uniformly spaced modal series. It is easy to verify that [20] $F_{\chi}(0) = 0$ in (51); this can be used to show that $\langle \chi \rangle = \int_0^{\infty} \chi f(\chi) d\chi = \cosh(\alpha) \frac{4}{\pi} \sum_{m=0}^{\infty} (-1)^m \frac{2m+1}{[(2m+1)^2 + 4\alpha^2 / \pi^2]^2} = \frac{\pi^2}{8\alpha} \tanh(\alpha)$ or $\langle r_{in} \rangle_{\omega} = \tanh(\alpha) \rightarrow 1$, $\alpha \gg 1$. The limiting density for $r_{in} \rightarrow 0$, from (49) and (52), is

$$f(r_{in}) \sim \cosh(\alpha) \sqrt{\frac{2\alpha}{\pi r_{in}^3}} e^{-\alpha(r_{in} + 1/r_{in})/2} \quad (54)$$

where we see the exponential decay as $r_{in} \rightarrow 0$. The limiting density for $r_{in} \rightarrow \infty$ from (50) and (52) is

$$f(r_{in}) \sim \frac{\pi}{2\alpha} \cosh(\alpha) e^{-r_{in} [(2\alpha)/\pi + \pi/(2\alpha)] \pi/4} \quad (55)$$

from which we see exponential decay as $r_{in} \rightarrow \infty$.

The undermoded limit, $\alpha \ll 1$ and $\alpha r_{in} \ll 1$, allows us to simplify (49) to

$$f_{\chi}(\chi) = \sqrt{\frac{\pi}{\chi^3}} \sum_{m=0}^{\infty} (-1)^m (m+1/2) e^{-(m+1/2)^2 \pi^2 / (4\chi)} \quad (56)$$

(this form ignores the damping term in the denominator of (48)) which has distribution

$$F_{\chi}(\chi) \sim 2 \sum_{m=0}^{\infty} (-1)^m \operatorname{erfc} \left[\frac{\pi}{2} (m+1/2) / \sqrt{\chi} \right] \quad (57)$$

The short dashed curve in Figure 8 is (57) and (53); the results for this small value of α agree with those from (51). The limiting form of the undermoded resistance, from (56) and (52), is

$$f(r_{in}) \sim \sqrt{\frac{2\alpha}{\pi r_{in}^3}} e^{-\alpha/(2r_{in})} \quad (58)$$

The overmoded limit $\alpha \gg 1$ of (49) and (52) gives

$$f(r_{in}) \sim \sqrt{\frac{\alpha}{2\pi r_{in}^3}} e^{-\alpha(r_{in}+1/r_{in}-2)/2}, \quad 0 < r_{in} < \infty$$

Letting $r_{in} = 1 + \rho_r$ we find that

$$f(\rho_r) \sim \sqrt{\frac{\alpha}{2\pi}} e^{-\alpha\rho_r^2/2} \quad (59)$$

Thus the correction ρ_r is Gaussian distributed with variance $1/\alpha$.

6.3 Overmoded Gaussian Impedance in Space

This subsection determines the impedance variation in the overmoded limit $\alpha \gg 1$. In this limit, using the central limit theorem [17], the impedance can be written as

$$z_{in} = (Z_{in} - Z) / R_{rad} \sim 1 + r_0\zeta - ix_0\zeta' \quad (60)$$

where ζ and ζ' are taken as a normalized Gaussians with zero mean and unit variance and are independent. The components are

$$(R_{in} - R) / R_{rad} \sim \text{Re} \left[\left(\frac{2\pi Q}{k^3 V} \right) \sum_n \frac{(i\omega^2/Q) \omega^2/\omega_n^2}{\omega^2(1+i/Q) - \omega_n^2} 3A_{nz}^2(r) \right]$$

$$(X_{in} - X) / R_{rad} \sim -\text{Im} \left[\left(\frac{2\pi Q}{k^3 V} \right) \sum_n \frac{(i\omega^2/Q) \omega^2/\omega_n^2}{\omega^2(1+i/Q) - \omega_n^2} 3A_{nz}^2(r) \right]$$

Subtracting the means gives

$$\Delta(R_{in} - R) / R_{rad} \sim \left(\frac{2\pi\omega^6}{k^3 V Q} \right) \sum_n \frac{1/\omega_n^2}{(\omega^2 - \omega_n^2)^2 + \omega^4/Q^2} \{3A_{nz}^2(r) - 1\} \approx r_0\zeta$$

$$\Delta(X_{in} - X) / R_{rad} \sim -\left(\frac{2\pi\omega^4}{k^3 V} \right) \sum_n \frac{(\omega^2 - \omega_n^2)/\omega_n^2}{(\omega^2 - \omega_n^2)^2 + \omega^4/Q^2} \{3A_{nz}^2(r) - 1\} \approx x_0\zeta'$$

where we assume these are Gaussian by the central limit theorem. Thus taking the variance gives

$$r_0^2 = \left(\frac{2\pi\omega^6}{k^3 V Q} \right)^2 \sum_n \frac{1/\omega_n^2}{(\omega^2 - \omega_n^2)^2 + \omega^4/Q^2} \sum_{n'} \frac{1/\omega_{n'}^2}{(\omega^2 - \omega_{n'}^2)^2 + \omega^4/Q^2} \langle \{3A_{nz}^2(\underline{r}) - 1\} \{3A_{n'z}^2(\underline{r}) - 1\} \rangle$$

$$x_0^2 = \left(\frac{2\pi\omega^4}{k^3 V} \right)^2 \sum_n \frac{(\omega^2 - \omega_n^2)/\omega_n^2}{(\omega^2 - \omega_n^2)^2 + \omega^4/Q^2} \sum_{n'} \frac{(\omega^2 - \omega_{n'}^2)/\omega_{n'}^2}{(\omega^2 - \omega_{n'}^2)^2 + \omega^4/Q^2} \langle \{3A_{nz}^2(\underline{r}) - 1\} \{3A_{n'z}^2(\underline{r}) - 1\} \rangle$$

Using the fact that

$$\langle \{3A_{nz}^2(\underline{r}) - 1\} \{3A_{n'z}^2(\underline{r}) - 1\} \rangle = \delta_{nn'} \langle (\zeta^2 - 1)(\zeta^2 - 1) \rangle = \delta_{nn'} (\langle \zeta^4 \rangle - 1) = 2\delta_{nn'}$$

gives

$$r_0^2 = 2 \left(\frac{2\pi\omega^6}{k^3 V Q} \right)^2 \sum_n \frac{1/\omega_n^4}{[(\omega^2 - \omega_n^2)^2 + \omega^4/Q^2]^2}$$

$$x_0^2 = 2 \left(\frac{2\pi\omega^4}{k^3 V} \right)^2 \sum_n \frac{(\omega^2 - \omega_n^2)^2/\omega_n^4}{[(\omega^2 - \omega_n^2)^2 + \omega^4/Q^2]^2}$$

We can approximate the summations by integrals provided we interpret the integral as being deformed around the pole at the origin

$$r_0^2 \approx \frac{4\omega^9}{k^3 V Q^2} \int_{-\infty}^{\infty} \frac{1}{[(\omega^2 - \omega_n^2)^2 + \omega^4/Q^2]^2} \frac{d\omega_n}{\omega_n^2}$$

$$x_0^2 \approx \frac{4\omega^5}{k^3 V} \int_{-\infty}^{\infty} \frac{(\omega^2 - \omega_n^2)^2}{[(\omega^2 - \omega_n^2)^2 + \omega^4/Q^2]^2} \frac{d\omega_n}{\omega_n^2}$$

The second order pole at the origin does not contribute in either integral. The other second order poles are at $\omega_n^2 = \omega^2 (1 \pm i/Q)$ and thus we can approximately factor the denominator as

$$(\omega^2 - \omega_n^2)^2 + \omega^4/Q^2 \approx \{\omega_n - \omega(1 + i/(2Q))\} \{\omega_n + \omega(1 + i/(2Q))\}$$

$$\{\omega_n - \omega(1 - i/(2Q))\} \{\omega_n + \omega(1 - i/(2Q))\}$$

If we close in the upper half plane we retain residues at the second order poles $\omega_n \approx \omega(1 + i/(2Q))$, $-\omega(1 - i/(2Q))$. The residues are thus

$$\begin{aligned}
r_0^2 &\approx \frac{4\omega^9}{k^3 V Q^2} 2\pi i \frac{-2}{\{\omega_n - \omega(1 - i/(2Q))\}^3 (\omega_n + \omega)^4 \omega_n^2} \Big|_{\omega_n \approx \omega(1+i/(2Q))} \\
&+ \frac{4\omega^9}{k^3 V Q^2} 2\pi i \frac{-2}{\{\omega_n + \omega(1 + i/(2Q))\}^3 (\omega_n - \omega)^4 \omega_n^2} \Big|_{\omega_n \approx -\omega(1-i/(2Q))} \\
x_0^2 &\approx \frac{4\omega^5}{k^3 V} 2\pi i \frac{(\omega_n - \omega) i\omega/Q}{\{\omega_n - \omega(1 - i/(2Q))\}^3 (\omega_n + \omega)^2 \omega_n^2} \Big|_{\omega_n \approx \omega(1+i/(2Q))} \\
&+ \frac{4\omega^5}{k^3 V} 2\pi i \frac{(\omega_n + \omega) i\omega/Q}{\{\omega_n + \omega(1 + i/(2Q))\}^3 (\omega_n - \omega)^2 \omega_n^2} \Big|_{\omega_n \approx -\omega(1-i/(2Q))}
\end{aligned}$$

The result is

$$r_0^2 \approx \frac{2\pi Q}{k^3 V} = 1/\alpha \approx x_0^2 \quad (61)$$

Thus the standard deviations of the Gaussian corrections to the real and imaginary parts of the impedance in the limit $\alpha \rightarrow \infty$ are $1/\sqrt{\alpha}$. The real part agrees with the previous result (59) from the direct modal series evaluated between modes. Note the same result is obtained if the total impedance is used to obtain the variance instead of the difference impedance.

6.4 Overmoded Gaussian Impedance in Frequency

The preceding analysis gives the variance with respect to spatial position of the antenna in the overmoded limit. This subsection estimates the variance with respect to frequency in the overmoded limit $\alpha \gg 1$. In this limit the impedance can again be written as

$$z_{in} = (Z_{in} - Z) / R_{rad} \sim 1 + r_0 \zeta - i x_0 \zeta'$$

where ζ and ζ' are taken as a normalized Gaussians with zero mean and unit variance and are independent (we assume these are Gaussian by the central limit theorem [17] with many modes contributing when $\alpha \gg 1$). We have shown that the impedance has a mean of unity; in the overmoded limit, this mean can be subtracted out as (we can show by conversion of the sum to an integral for $\alpha \gg 1$ that the subtracted term in the braces corresponds asymptotically to the unit mean)

$$z_{in} - 1 \sim \left(\frac{2\pi Q}{k^3 V} \right) \sum_n \frac{(i\omega^2/Q) \omega^2/\omega_n^2}{\omega^2(1+i/Q) - \omega_n^2} \{3A_{nz}^2(r) - 1\}$$

The components are

$$r_{in} - 1 = (R_{in} - R) / R_{rad} \sim \left(\frac{2\pi c^3}{V} \right) \sum_n \frac{\omega^3 / (Q\omega_n^2)}{(\omega^2 - \omega_n^2)^2 + \omega^4 / Q^2} \{3A_{nz}^2(\underline{r}) - 1\} \approx r_0 \zeta$$

$$x_{in} = (X_{in} - X) / R_{rad} \sim - \left(\frac{2\pi c^3}{V} \right) \sum_n \frac{\omega (\omega^2 - \omega_n^2) / \omega_n^2}{(\omega^2 - \omega_n^2)^2 + \omega^4 / Q^2} \{3A_{nz}^2(\underline{r}) - 1\} \approx x_0 \zeta'$$

Taking the square gives

$$(r_{in} - 1)^2 \sim \left(\frac{2\pi c^3}{V} \right)^2 \sum_n \frac{\omega^3 / (Q\omega_n^2)}{(\omega^2 - \omega_n^2)^2 + \omega^4 / Q^2} \{3A_{nz}^2(\underline{r}) - 1\} \sum_{n'} \frac{\omega^3 / (Q\omega_{n'}^2)}{(\omega^2 - \omega_{n'}^2)^2 + \omega^4 / Q^2} \{3A_{n'z}^2(\underline{r}) - 1\}$$

$$\approx r_0^2 \zeta^2$$

$$x_{in}^2 \sim \left(\frac{2\pi c^3}{V} \right)^2 \sum_n \frac{\omega (\omega^2 - \omega_n^2) / \omega_n^2}{(\omega^2 - \omega_n^2)^2 + \omega^4 / Q^2} \{3A_{nz}^2(\underline{r}) - 1\} \sum_{n'} \frac{\omega (\omega^2 - \omega_{n'}^2) / \omega_{n'}^2}{(\omega^2 - \omega_{n'}^2)^2 + \omega^4 / Q^2} \{3A_{n'z}^2(\underline{r}) - 1\}$$

$$\approx x_0^2 \zeta'^2$$

Taking the mean in frequency gives

$$\langle (r_{in} - 1)^2 \rangle_\omega \sim \left(\frac{2\pi c^3}{V} \right)^2 \sum_n \{3A_{nz}^2(\underline{r}) - 1\} / \omega_n^2 \sum_{n'} \{3A_{n'z}^2(\underline{r}) - 1\} / \omega_{n'}^2$$

$$\frac{1}{\omega_+ - \omega_-} \int_{\omega_-}^{\omega_+} \frac{\omega^3 / Q}{(\omega^2 - \omega_n^2)^2 + \omega^4 / Q^2} \frac{\omega^3 / Q}{(\omega^2 - \omega_{n'}^2)^2 + \omega^4 / Q^2} d\omega \approx r_0^2$$

$$\langle x_{in}^2 \rangle_\omega \sim \left(\frac{2\pi c^3}{V} \right)^2 \sum_n \{3A_{nz}^2(\underline{r}) - 1\} / \omega_n^2 \sum_{n'} \{3A_{n'z}^2(\underline{r}) - 1\} / \omega_{n'}^2$$

$$\frac{1}{\omega_+ - \omega_-} \int_{\omega_-}^{\omega_+} \frac{\omega (\omega^2 - \omega_n^2)}{(\omega^2 - \omega_n^2)^2 + \omega^4 / Q^2} \frac{\omega (\omega^2 - \omega_{n'}^2)}{(\omega^2 - \omega_{n'}^2)^2 + \omega^4 / Q^2} d\omega \approx x_0^2$$

Noting the orthogonality of $\{3A_{nz}^2(\underline{r}) - 1\}$ for different n , only the $n' = n$ terms contribute. Thus we can write

$$\begin{aligned}
\int_{\omega_-}^{\omega_+} \left[\frac{\omega^3/Q}{(\omega^2 - \omega_n^2)^2 + \omega^4/Q^2} \right]^2 d\omega &\approx \frac{\omega_n^2}{16Q^2} \int_{\omega_-}^{\omega_+} \frac{d\omega}{\left[(\omega - \omega_n)^2 + \omega_n^2/(2Q)^2 \right]^2} \\
&\approx \frac{Q}{2\omega_n} \int_{2Q(\omega_- - \omega_n)/\omega_n}^{2Q(\omega_+ - \omega_n)/\omega_n} \frac{du}{(u^2 + 1)^2} \approx \frac{Q}{2\omega_n} \int_{\arctan\{2Q(\omega_- - \omega_n)/\omega_n\}}^{\arctan\{2Q(\omega_+ - \omega_n)/\omega_n\}} \cos^2 \theta d\theta \\
&\approx \frac{Q}{4\omega_n} \int_{\arctan\{2Q(\omega_- - \omega_n)/\omega_n\}}^{\arctan\{2Q(\omega_+ - \omega_n)/\omega_n\}} (1 + \cos 2\theta) d\theta \approx \frac{Q}{4\omega_n} \int_{\operatorname{sgn}(\omega_- - \omega_n)\pi/2}^{\operatorname{sgn}(\omega_+ - \omega_n)\pi/2} (1 + \cos 2\theta) d\theta \\
&\approx \frac{\pi Q}{8\omega_n} [\operatorname{sgn}(\omega_+ - \omega_n) - \operatorname{sgn}(\omega_- - \omega_n)] \\
\int_{\omega_-}^{\omega_+} \left[\frac{\omega(\omega^2 - \omega_n^2)}{(\omega^2 - \omega_n^2)^2 + \omega^4/Q^2} \right]^2 d\omega &\approx \frac{1}{4} \int_{\omega_-}^{\omega_+} \left[\frac{(\omega - \omega_n)}{(\omega - \omega_n)^2 + \omega_n^2/(2Q)^2} \right]^2 d\omega \\
&\approx \frac{Q}{2\omega_n} \int_{2Q(\omega_- - \omega_n)/\omega_n}^{2Q(\omega_+ - \omega_n)/\omega_n} \left(\frac{u}{u^2 + 1} \right)^2 du \approx \frac{Q}{2\omega_n} \int_{\arctan\{2Q(\omega_- - \omega_n)/\omega_n\}}^{\arctan\{2Q(\omega_+ - \omega_n)/\omega_n\}} \sin^2 \theta d\theta \\
&\approx \frac{Q}{4\omega_n} \int_{\arctan\{2Q(\omega_- - \omega_n)/\omega_n\}}^{\arctan\{2Q(\omega_+ - \omega_n)/\omega_n\}} (1 - \cos 2\theta) d\theta \approx \frac{Q}{4\omega_n} \int_{\operatorname{sgn}(\omega_- - \omega_n)\pi/2}^{\operatorname{sgn}(\omega_+ - \omega_n)\pi/2} (1 - \cos 2\theta) d\theta \\
&\approx \frac{\pi Q}{8\omega_n} [\operatorname{sgn}(\omega_+ - \omega_n) - \operatorname{sgn}(\omega_- - \omega_n)]
\end{aligned}$$

Using these approximations to the integrals we find

$$\begin{aligned}
\langle (r_{in} - 1)^2 \rangle_\omega &\sim r_0^2 \approx \left(\frac{\pi Q c^3}{V} \right) \frac{\pi^2 c^3}{V} \frac{1}{\omega_+ - \omega_-} \sum_{n_\pm} \{ 3A_{nz}^2(\underline{r}) - 1 \}^2 / \omega_n^5 \\
\langle x_{in}^2 \rangle_\omega &\sim x_0^2 \approx \left(\frac{\pi Q c^3}{V} \right) \frac{\pi^2 c^3}{V} \frac{1}{\omega_+ - \omega_-} \sum_{n_\pm} \{ 3A_{nz}^2(\underline{r}) - 1 \}^2 / \omega_n^5
\end{aligned}$$

where $n_\pm$ covers the range $\omega_- < \omega_n < \omega_+$. Because ω_n in the summand is slowly varying with respect to n we can replace the eigenfunction factor by its mean value

$$\langle \{ 3A_{nz}^2(\underline{r}) - 1 \}^2 \rangle = \langle (\zeta^2 - 1)^2 \rangle = 2$$

and replace the sum by an integral

$$r_0^2 \approx \left(\frac{2\pi Q c^3}{V} \right) \frac{1}{\omega_+ - \omega_-} \int_{\omega_-}^{\omega_+} \frac{d\omega_n}{\omega_n^3} = \left(\frac{2\pi Q c^3}{V} \right) \left(\frac{\omega_+ + \omega_-}{2\omega_+^2 \omega_-^2} \right) \approx x_0^2$$

Approximating the result for a narrow frequency range yields

$$r_0^2 \approx 1/\alpha \approx x_0^2 \quad (62)$$

Thus the standard deviation of both impedance components is $1/\sqrt{\alpha}$ in the overmoded limit.

7 ELECTRICALLY LONGER ANTENNA

This section considers the impedance problem when the antenna is no longer electrically small and the case where the antenna is a monopole on the wall of the cavity.

7.1 Dipole Antenna

The z component of the difference vector potential for the linear antenna of length $2h$ using (10) is

$$A_z(z) - A_{sz}(z) = -\frac{1}{\varepsilon_0 V} \sum_n \frac{\omega^2 (1 + i/Q) / \omega_n^2}{\omega^2 (1 + i/Q) - \omega_n^2} A_{nz}(z) \int_{-h}^h A_{nz}(z') I(z') dz'$$

The local static part from (11) can be written as

$$A_{sz}(z) \approx \frac{\mu_0}{4\pi} \left[\int_{-h}^h I(z') \frac{dz'}{\sqrt{a^2 + (z - z')^2}} - I(z) \right]$$

Similarly we take the scalar potential from (12) to be approximately

$$\phi(z) \approx \frac{1}{4\pi\varepsilon_0} \int_{-h}^h \frac{q(z') dz'}{\sqrt{a^2 + (z - z')^2}} = \frac{1}{i\omega\varepsilon_0 4\pi} \int_{-h}^h \frac{\partial}{\partial z'} I(z') \frac{dz'}{\sqrt{a^2 + (z - z')^2}}$$

where we have replaced the transverse integration with the thin wire kernel, q is the charge per unit length of the antenna, and the continuity equation gives $i\omega q = \frac{\partial}{\partial z} I$. The electric field along the antenna, using the fact that $I(\pm h) = 0$, is thus

$$\begin{aligned} E_z \approx & -\frac{i\omega}{\varepsilon_0 V} \sum_n \frac{\omega^2 (1 + i/Q) / \omega_n^2}{\omega^2 (1 + i/Q) - \omega_n^2} A_{nz}(z) \int_{-h}^h A_{nz}(z') I(z') dz' \\ & + \frac{i\omega\mu_0}{4\pi} \left[\left(1 + \frac{1}{k^2} \frac{\partial^2}{\partial z^2} \right) \int_{-h}^h \frac{I(z') dz'}{\sqrt{a^2 + (z - z')^2}} - I(z) \right] \end{aligned}$$

$$\approx -\frac{i\omega}{\varepsilon_0 V} \sum_n \frac{\omega^2 (1 + i/Q) / \omega_n^2}{\omega^2 (1 + i/Q) - \omega_n^2} A_{nz}(z) \int_{-h}^h A_{nz}(z') I(z') dz' + \frac{i\omega\mu_0}{4\pi} \left[\Omega_e \left(1 + \frac{1}{k^2} \frac{\partial^2}{\partial z^2} \right) I(z) \right] \quad (63)$$

where the final approximate expression results from using the thin wire expansion parameter for the center driven antenna and illustrates the transmission line form for the leading term of the antenna current. The expression (63) can now be set equal to the antenna boundary condition (the center driven antenna)

$$E_z = -V_0 \delta(z) \quad (64)$$

to obtain a stochastic integral equation for the antenna current.

Because in this paper we are only interested in antennas operating up to the first resonance, we expect the current distribution to be determined primarily from the second local term in (63) along with the zero current boundary conditions at the ends of the antenna. Thus the leading solution for the current is the usual

$$I(z) \approx I_0 \sin k(h - |z|) \quad (65)$$

Now the stationary expression for the input impedance (13) and (14) gives

$$Z_{in} = R - iX + \frac{i\omega}{\varepsilon_0 V I^2(0)} \sum_n \frac{\omega^2 (1 + i/Q) / \omega_n^2}{\omega^2 (1 + i/Q) - \omega_n^2} \int_{-h}^h A_{nz}(z) I(z) dz \int_{-h}^h A_{nz}(z') I(z') dz' \quad (66)$$

where the local reactance is

$$\begin{aligned} X &= \frac{1}{I^2(0)} \int_{-h}^h I(z) \frac{\omega\mu_0}{4\pi} \left[\left(1 + \frac{1}{k^2} \frac{\partial^2}{\partial z^2} \right) \int_{-h}^h \frac{I(z') dz'}{\sqrt{a^2 + (z - z')^2}} - I(z) \right] dz \\ &\approx -\frac{\eta_0}{2\pi} \Omega_e \cot(kh) \\ &+ \frac{\eta_0}{2\pi} \frac{1}{\sin^2(kh)} \left[2\text{Si}(kh) + \sin(2kh) \left\{ 2\text{Cin}(kh) - \text{Cin}(2kh) - \frac{3}{4} \right\} - \cos(2kh) \{ \text{Si}(2kh) - 2\text{Si}(kh) \} - \frac{1}{2}kh \right] \end{aligned} \quad (67)$$

Note that at low frequencies

$$X \sim -\frac{\eta_0}{2\pi} \Omega_e \cot(kh) + \frac{\eta_0}{9\pi} kh \sim -\frac{1}{\omega C} + \omega L$$

where C is given by (16) and

$$L \sim \frac{1}{3} \frac{\mu_0 h}{2\pi} \left(\Omega_e + \frac{2}{3} \right) \quad (68)$$

which is the correct low frequency limit. At resonance $kh = \pi/2$

$$X = \frac{\eta_0}{2\pi} \left[\text{Si}(\pi) - \frac{\pi}{4} \right] \approx 64 \text{ ohms} \quad (69)$$

Because $A_{nz}(z)$ is assumed to be a zero mean Gaussian process, the linear combination $\int_{-h}^h A_{nz}(z) I(z) dz$, is a zero mean Gaussian random variable [17] (assuming that the current is deterministic in this approximation). Thus its variance is

$$\begin{aligned} 3 \left\langle \int_{-h}^h A_{nz}(z) I(z) dz \int_{-h}^h A_{nz}(z') I(z') dz' \right\rangle &= \int_{-h}^h I(z) \int_{-h}^h \left\langle \sqrt{3} A_{nz}(z) \sqrt{3} A_{nz}(z') \right\rangle I(z') dz' dz \\ &= \int_{-h}^h I(z) \int_{-h}^h \rho_z(z, z') I(z') dz' dz \\ &= I_0^2 \int_{-h}^h \sin k(h - |z|) \int_{-h}^h \frac{3}{2} \left(1 + \frac{1}{k_n^2} \frac{\partial^2}{\partial z^2} \right) \frac{\sin[k_n(z - z')]}{k_n(z - z')} \sin k(h - |z'|) dz' dz \end{aligned} \quad (70)$$

where ρ_z is the correlation function (27). The approximation made to arrive at the simplified modal series (29) was to replace ω_n by ω (except in the denominator difference) for these high-order modes. This approximation will be made here also; in the overmoded subsection below it is shown that if this approximation is not made, then in the limit $\alpha \rightarrow \infty$, a small dynamic reactance correction is obtained that brings this limit into agreement with the antenna reactance in free space. Thus if we approximate k_n by k in (70) we find

$$\left\langle \int_{-h}^h A_{nz}(z) I(z) dz \int_{-h}^h A_{nz}(z') I(z') dz' \right\rangle \approx I_0^2 \sin^2(kh) \frac{2\pi}{\eta_0 k^2} R_{rad} = I^2(0) \frac{2\pi}{\eta_0 k^2} R_{rad}$$

where R_{rad} is the usual free space radiation resistance of a dipole [5]

$$(4\pi/\eta_0) \sin^2(kh) R_{rad} = 2\text{Cin}(2kh) + \sin 2kh [\text{Si}(4kh) - 2\text{Si}(2kh)] - \cos 2kh [\text{Cin}(4kh) - 2\text{Cin}(2kh)] \quad (71)$$

At resonance $kh = \pi/2$

$$R_{rad} = \frac{\eta_0}{4\pi} \text{Cin}(2\pi) \approx 73 \text{ ohms}$$

Thus using this variance we can write the square of the random variable in (66) as

$$\int_{-h}^h A_{nz}(z) I(z) dz \int_{-h}^h A_{nz}(z') I(z') dz' \approx I^2(0) \frac{2\pi}{\eta_0 k^2} R_{rad} \zeta^2$$

where $\zeta = \sqrt{3}A_{nz}$ is a normalized zero mean Gaussian. The impedance is therefore

$$Z_{in} \approx R - iX + R_{rad} \sum_n \left(\frac{2\pi Q}{k^3 V} \right) \frac{(i\omega^2/Q) \omega^2/\omega_n^2}{\omega^2(1+i/Q) - \omega_n^2} 3A_{nz}^2 \quad (72)$$

which is the same as (14), except that now R_{rad} is not the low frequency value, but is given by (71).

7.2 Overmoded Limit of Electrically Longer Antenna

We now illustrate the overmoded limit of the input impedance. The double integral in the impedance (66) can be written as

$$3 \int_{-h}^h A_{nz}(z) I(z) dz \int_{-h}^h A_{nz}(z') I(z') dz' = I^2(0) U_n^{tot} 3A_{nz}^2 \quad (73)$$

where the variance of the random variable is (we retain the k_n dependence here)

$$U_n^{tot} = \frac{1}{\sin^2(kh)} \int_{-h}^h \sin k(h - |z|) \int_{-h}^h \frac{3}{2} \left(1 + \frac{1}{k_n^2} \frac{\partial^2}{\partial z^2} \right) \frac{\sin k_n(z - z')}{k_n(z - z')} \sin k(h - |z'|) dz' dz$$

Now taking the overmoded limit we can replace the sum by an integral

$$\begin{aligned} Z_{in} &\sim R - iX + \frac{i\omega}{3\varepsilon_0 V} \int_0^\infty \frac{\omega^2(1+i/Q)/\omega_n^2}{\omega^2(1+i/Q) - \omega_n^2} U_n^{tot} 3A_{nz}^2 \frac{\Delta n}{\langle \Delta \omega_n \rangle} d\omega_n \\ &\sim R - iX + \frac{ik^3}{3\varepsilon_0 \pi^2} (1+i/Q) \int_0^\infty \frac{U_n^{tot} d\omega_n}{\omega^2(1+i/Q) - \omega_n^2} \end{aligned} \quad (74)$$

The residue evaluation of this integral requires us to split the sinusoid into complex exponentials (this occurs because of the retention of the k_n dependence); thus the poles at $\omega_n = 0$ and at $\omega_n = \pm\omega(1+i/(2Q))$ are included. We take the contour to be deformed slightly below the pole at the origin (the origin is originally a removable singularity before the slitting of the sinusoid, so it is possible to take it either above or below this pole)

$$\begin{aligned} &\frac{1}{4i} \int_{-\infty}^\infty \left(1 + \frac{1}{k_n^2} \frac{\partial^2}{\partial z^2} \right) \frac{e^{ik_n|z-z'|} - e^{-ik_n|z-z'|}}{\omega^2(1+i/Q) - \omega_n^2} \frac{d\omega_n}{k_n|z-z'|} \\ &= \pi \left\{ 1 + \frac{1}{k^2(1+i/Q)} \frac{\partial^2}{\partial z^2} \right\} \frac{e^{ik(1+i/(2Q))|z-z'|}}{-2\omega(1+i/Q)} \frac{1}{k|z-z'|} + \frac{\pi}{2} \frac{1}{\omega^2(1+i/Q)} \frac{c}{|z-z'|} \\ &+ \frac{1}{4i} \frac{c}{\omega^2(1+i/Q)} \frac{\partial^2}{\partial z^2} \oint_{C_0} \left[\frac{1}{|z-z'|} + ik_n - \frac{1}{2} k_n^2 |z-z'| \right] \left[\frac{1}{k_n^3} + \frac{1}{k^2 k_n(1+i/Q)} \right] dk_n \end{aligned}$$

$$= \frac{\pi}{2} \frac{1}{ck^2(1+i/Q)} \left[\left\{ 1 + \frac{1}{k^2(1+i/Q)} \frac{\partial^2}{\partial z^2} \right\} \frac{1 - e^{ik(1+i/(2Q))|z-z'|}}{|z-z'|} - \frac{1}{2} \frac{\partial^2}{\partial z^2} |z-z'| \right]$$

where C_0 is a small contour encircling the origin. Therefore

$$\begin{aligned} Z_{in} &\sim R - iX + \eta_0 \frac{ik}{4\pi} \frac{1}{\sin^2(kh)} \int_{-h}^h \sin k(h-|z|) \\ &\int_{-h}^h \left[\left\{ 1 + \frac{1}{k^2(1+i/Q)} \frac{\partial^2}{\partial z^2} \right\} \frac{1 - e^{ik(1+i/(2Q))|z-z'|}}{|z-z'|} - \frac{1}{2} \frac{\partial^2}{\partial z^2} |z-z'| \right] \sin k(h-|z'|) dz' dz \\ &\sim R - \frac{i\omega\mu_0}{4\pi} \frac{1}{\sin^2(kh)} \int_{-h}^h \sin k(h-|z|) \\ &\left(1 + \frac{1}{k^2} \frac{\partial^2}{\partial z^2} \right) \int_{-h}^h \left[\frac{e^{ik|z-z'|} - 1}{|z-z'|} + \frac{1}{\sqrt{a^2 + (z-z')^2}} \right] \sin k(h-|z'|) dz' dz \\ &\sim R + R_{rad} + i\eta_0 \frac{\Omega_e}{2\pi} \cot(kh) - iX_{ant} \end{aligned} \quad (75)$$

where we have assumed that $Q \gg 1$ and that $2kh/Q \ll 1$, R_{rad} is given by (71), and

$$\begin{aligned} (4\pi/\eta_0) \sin^2(kh) X_{ant} &= 2 \text{Si}(2kh) + \sin(2kh) \{-2 + 2\text{Cin}(2kh) - \text{Cin}(4kh)\} \\ &- \cos(2kh) \{\text{Si}(4kh) - 2\text{Si}(2kh)\} \end{aligned} \quad (76)$$

This is the free space impedance of the antenna [5]. At resonance $kh = \pi/2$

$$X_{ant} = \frac{\eta_0}{4\pi} \text{Si}(2\pi) \approx 42.5445 \text{ ohms} \quad (77)$$

Thus since the resonant value of the local reactance X is (69), a small error is made by the replacement $k_n \rightarrow k$ in the correlation function in the overmoded limit at resonance (away from resonance, the transmission line Ω_e term is typically dominant). As $kh \rightarrow 0$ we find

$$X_{ant} \sim \frac{\eta_0}{9\pi} kh \quad (78)$$

7.3 Monopole Antenna

It has been shown in the overmoded region [15] that the correlation dyad is proportional to the imaginary part of the dyadic Green's function $R_{ij} = \frac{1}{2} [\langle E_i(\underline{r}_1) E_j^*(\underline{r}_2) \rangle + \langle E_i^*(\underline{r}_1) E_j(\underline{r}_2) \rangle] = (6\pi/k) \text{Im}(\Gamma_{ij})$. This result was used previously to obtain the correlation function in the undermoded region by replacing the wavenumber k by the modal wavenumber k_n . Here we generalize the correlation dyad to a half space by using the half space dyadic Green's function $R_{ij}^h = (6\pi/k) \text{Im}(\Gamma_{ij}^h)$. The diagonal component perpendicular to the boundary then has the form

$$R_{zz}^h(z_1, z_2) = \frac{3}{2} \left(1 + \frac{1}{k_n^2} \frac{\partial^2}{\partial z_1^2} \right) \left[\frac{\sin k_n(z_1 - z_2)}{k_n(z_1 - z_2)} + \frac{\sin k_n(z_1 + z_2)}{k_n(z_1 + z_2)} \right] \quad (79)$$

Modifying (66) to the case of a monopole in the wall $z = 0$ gives

$$Z_{in} = \frac{1}{2} (R - iX) + \frac{i\omega}{\varepsilon_0 V I^2(0)} \sum_n \frac{\omega^2 (1 + i/Q) / \omega_n^2}{\omega^2 (1 + i/Q) - \omega_n^2} \int_0^h A_{nz}(z) I(z) dz \int_0^h A_{nz}(z') I(z') dz'$$

Now the variance is

$$\begin{aligned} 3 \left\langle \int_0^h A_{nz}(z) I(z) dz \int_0^h A_{nz}(z') I(z') dz' \right\rangle &= I_0^2 \int_0^h I(z) \int_0^h R_{zz}^h(z, z') I(z') dz' dz \\ &= I_0^2 \int_0^h \sin k(h - |z|) \int_0^h \frac{3}{2} \left(1 + \frac{1}{k_n^2} \frac{\partial^2}{\partial z^2} \right) \left[\frac{\sin k_n(z - z')}{k_n(z - z')} + \frac{\sin k_n(z + z')}{k_n(z + z')} \right] \sin k(h - |z'|) dz' dz \\ &= \frac{1}{2} I_0^2 \int_{-h}^h \sin k(h - |z|) \int_{-h}^h \frac{3}{2} \left(1 + \frac{1}{k_n^2} \frac{\partial^2}{\partial z^2} \right) \frac{\sin k_n(z - z')}{k_n(z - z')} \sin k(h - |z'|) dz' dz \end{aligned}$$

This is exactly half the dipole result (70) and shows that the monopole radiation resistance (half the dipole radiation resistance) is produced as a factor in front of the series (72).

7.4 Wall Field Experiment

The field at the wall of an overmoded cavity is enhanced by 3 dB relative to the field away from the wall [22]. It is interesting that Dunn's enhancement factor for the normal electric field component near the wall can be written as

$$\frac{\langle |E_{norm}|^2 \rangle}{\langle |E_i|^2 \rangle} = 1 + \frac{3}{2} \left(1 + \frac{\partial^2}{\partial u^2} \right) \frac{\sin u}{u} \Big|_{u=2kz} = R_{zz}^h(z, z) \quad (80)$$

where the correlation function on the right is the limit $z_2 \rightarrow z_1 = z$ of (79).

It is of interest to verify the wall field enhancement in the undermoded region in an electrically large mode stirred chamber. Figure 10 shows a drawing of the experimental setup; short dipole probes ($2h \approx 1.27$

cm) were used near (and perpendicular to) the wall as well as away from the wall in the room volume; high impedance lines isolated the probes. The frequency sweep covered the range 220 MHz to 230 MHz in the mode stirred chamber. The 118.5 inch distance from the wall corresponds to more than two wavelengths at 225 MHz, well into the room volume. The 2.7 inch distance corresponds to approximately one twentieth of a wavelength at this same frequency and thus provides a wall enhanced measurement; this distance is still far enough from the wall compared to the dipole length that we do not expect quasistatic images to significantly perturb the measurement. Figure 11 shows a comparison of the distributions on the wall and in the room volume. The 3 dB enhancement is apparent in the data. Figure 12 shows the comparison when only the resonant mode peaks are used to form the distribution (providing a single mode-type spatial distribution over the volume of the cavity). This data also shows the 3 dB enhancement although it is not as definitive; the mean of the distribution on the wall is 2.8 dB above the room value; there is probably a tendency to pick distinctive peaks (above the noise) which will tend to increase the smaller room data.

8 POWER BALANCE

This section considers a short dipole antenna in a cavity from the point of view of conservation of power. The power into the cavity from the antenna is

$$P_{in} = \frac{1}{2} (R_{rad} + R_{wall}) |I(0)|^2 = \frac{1}{2} R_{in} |I(0)|^2 \quad (81)$$

where R_{wall} is the real part of the impedance resulting from the presence of the cavity wall. Using the definition of the Q (this is the Q of the cavity assuming the dipole absorbs no power, since it can be terminated in an ideal current source to measure the input impedance), one has

$$Q = \frac{\omega V U}{P_{in}} \quad (82)$$

where U is the mean energy density

$$U = \frac{3}{2} \epsilon_0 \langle |E_z|^2 \rangle_V \quad (83)$$

where the subscript V denotes volumetric mean (as we will see below, it is important here to note the type of mean), and spatial isotropy among the field components is assumed. The open circuit voltage at the short dipole resulting from field due to the wall currents is

$$V_{ref} \sim -h E_z^{ref} \quad (84)$$

where h is the physical half height of the short antenna and the positive reference of the voltage is on the positive z arm of the antenna. Thus the dipole impedance due to the wall is

$$Z_{wall} = R_{wall} - iX_{wall} = V_{ref} / I(0) \quad (85)$$

From (85) through (81), multiplying and dividing by $\sqrt{\langle |E_z|^2 \rangle_V}$, we can write

$$\begin{aligned}
Z_{wall} &\approx \frac{-E_z^{ref}}{\sqrt{\langle |E_z|^2 \rangle_V}} h \sqrt{\frac{2U}{3\varepsilon_0 |I(0)|^2}} = \frac{-E_z^{ref}}{\sqrt{\langle |E_z|^2 \rangle_V}} h \sqrt{\frac{2QP_{in}}{3\varepsilon_0 \omega V |I(0)|^2}} = \frac{-E_z^{ref}}{\sqrt{\langle |E_z|^2 \rangle_V}} \sqrt{\frac{Qh^2 R_{in} \eta_0}{3kV}} \\
&= \frac{-E_z^{ref}}{\sqrt{\langle |E_z|^2 \rangle_V}} \sqrt{\left(\frac{2\pi Q}{k^3 V}\right) R_{in} R_{rad}} = \frac{-E_z^{ref}}{\sqrt{\langle |E_z|^2 \rangle_V}} \sqrt{R_{in} R_{rad} / \alpha}
\end{aligned} \tag{86}$$

where we have used (15) for the radiation resistance in free space and again α is given by (31). Taking real and imaginary parts of this equation and normalizing by R_{rad} (with $R_{wall}/R_{rad} = r_{wall}$ and $X_{wall}/R_{rad} = x_{wall}$) gives

$$r_{wall} = r_{in} - 1 = \frac{-\text{Re}(E_z^{ref})}{\sqrt{\langle |E_z|^2 \rangle_V}} \sqrt{r_{in}/\alpha} \tag{87}$$

$$x_{wall} = \frac{\text{Im}(E_z^{ref})}{\sqrt{\langle |E_z|^2 \rangle_V}} \sqrt{r_{in}/\alpha} \tag{88}$$

Note that x_{wall} is the really the same as x_{in} , since x_{in} was defined with the local reactance subtracted out. The coefficients of $\sqrt{r_{in}/\alpha}$ vary with position. The next subsection considers the extreme values of this function. A later subsection constructs an approximate fit to the entire distribution of this function.

8.1 Extreme-Value Approximation

Let us write

$$\frac{-E_z^{ref}}{\sqrt{\langle |E_z|^2 \rangle_V}} = M e^{i\varphi} \tag{89}$$

where M is a random variable, and φ is taken to vary over a large range of phases. Inserting this into (87) and (88) gives

$$r_{in} - 1 = M \cos \varphi \sqrt{r_{in}/\alpha}$$

$$x_{wall} = M \sin \varphi \sqrt{r_{in}/\alpha}$$

Suppose we consider the complex components that give rise to the magnitude M to be Gaussian. To obtain an extreme curve for the impedance variation we could take the three sigma point of the underlying

real and imaginary Gaussian distributions $M_0 = 3$. It is interesting that the view of the field being a **three dimensional standing wave in the frequency range of the fundamental cavity modes, and producing an eight-to-one maximum-to-mean-ratio**, would correspond to $M_0 = 2\sqrt{2}$; a value that is **not very different from the three sigma value; these extreme results may therefore be useful at lower frequencies than anticipated**. Figures 1, 2 and 3 show dashed bounding curves based on this assumption. Solving the quadratic equation gives

$$r_{in} = 1 + \frac{1}{2\alpha} M_0^2 \cos^2 \varphi \pm \sqrt{\left(1 + \frac{1}{2\alpha} M_0^2 \cos^2 \varphi\right)^2 - 1} \quad (90)$$

$$x_{wall} = \pm M_0 \sin \varphi \sqrt{r_{in}/\alpha} \quad (91)$$

with $M_0 = 3$ is used in the Figures. The experimentally determined free space radiation resistance of 44 ohms was used to remove the normalization of these formulas for the plot in Figure 1. The experimentally determined free space radiation resistance of 46 ohms was used to remove the normalization in Figure 2; the experimentally determined local antenna reactance in free space $X = j8.5$ ohms was also added to the impedance (91) in Figure 2. The quality factor of the cavity in Figure 3 was taken as the experimental value 1,280,000 (the parameter $\alpha \approx 1206.7$ in this case); the radiation resistance at resonance 36 ohms was used to remove the normalization from these power balance formulas for the plot in Figure 3; this monopole antenna has dimensions $2a \approx 1.51$ mm and $h \approx 4.325$ mm; the 10 MHz sweep contained 801 frequency points.

The maximum variation of the resistance is obtained by taking $\cos \varphi = \pm 1$ in (90)

$$1 + \frac{1}{2\alpha} M_0^2 - \sqrt{\left(1 + \frac{1}{2\alpha} M_0^2\right)^2 - 1} < r_{in} < 1 + \frac{1}{2\alpha} M_0^2 + \sqrt{\left(1 + \frac{1}{2\alpha} M_0^2\right)^2 - 1} \quad (92)$$

Letting $u = 1 + \frac{1}{2\alpha} M_0^2 \cos^2 \varphi$, $r_{in} < u + \sqrt{u^2 - 1}$, and $x_{wall}^2 < 2 \left(\frac{1}{2\alpha} M_0^2 - u + 1\right) (u + \sqrt{u^2 - 1})$, and maximizing x_{wall}^2 with respect to u , gives $r_{in} = \frac{1}{2\alpha} M_0^2 + 1$, $u = \frac{1}{2} \left[\left(\frac{1}{2\alpha} M_0^2 + 1\right) + 1/\left(\frac{1}{2\alpha} M_0^2 + 1\right)\right]$ and the maximum variation of the reactance

$$|x_{wall}| < \sqrt{\left(1 + \frac{1}{2\alpha} M_0^2\right)^2 - 1} \quad (93)$$

The highly undermoded limit gives $1/(2 + M_0^2/\alpha) < r_{in} < (2 + M_0^2/\alpha)$ and $2|x_{wall}| < 2 + M_0^2/\alpha$. The highly overmoded limit gives $1 - M_0/\sqrt{\alpha} < r_{in} < 1 + M_0/\sqrt{\alpha}$ and $|x_{wall}| < M_0/\sqrt{\alpha}$; thus the limit is $r_{in} \rightarrow 1$ and $x_{wall} \rightarrow 0$.

8.2 Sphere Cavity Canonical Example

The canonical problem of a short dipole at the center of a spherical cavity of radius b with wall impedance $Z_s = R_s(1 - i)$ can be easily solved [23]. It is convenient to use the Lorentz gauge here with incident potential

$$A_z^{inc} = \mu_0 I(0) h \frac{e^{ikr}}{4\pi r} \quad (94)$$

and reflected potential

$$A_z^{ref} = \mu_0 I(0) h \frac{\sin(kr)}{4\pi r} \rho_{ref} \quad (95)$$

where ρ_{ref} is the reflection coefficient. The corresponding fields are

$$H_\varphi^{inc} = I(0) h \frac{e^{ikr}}{4\pi r} (-ik + 1/r) \sin \theta$$

$$-i\omega\varepsilon_0 E_\theta^{inc} = I(0) h \frac{e^{ikr}}{4\pi r} (-k^2 - ik/r + 1/r^2) \sin \theta$$

$$H_\varphi^{ref} = I(0) h \frac{\sin \theta}{4\pi r} [-k \cos(kr) + \sin(kr)/r] \rho_{ref}$$

$$-i\omega\varepsilon_0 E_\theta^{ref} = I(0) h \frac{\sin \theta}{4\pi r} [(-k^2 + 1/r^2) \sin(kr) - k \cos(kr)/r] \rho_{ref}$$

The boundary condition at the wall is

$$E_\theta^{inc} + E_\theta^{ref} = Z_s (H_\varphi^{inc} + H_\varphi^{ref}) \quad (96)$$

This gives

$$-e^{-ikb} \rho_{ref} = \left[1 - 1/(kb)^2 + i/(kb) - (Z_s/\eta_0)(1 + i/(kb)) \right] / D_{sp} \quad (97)$$

where the denominator is

$$D_{sp} = \left(1 - \frac{1}{k^2 b^2} \right) \sin(kb) + \frac{1}{kb} \cos(kb) + i(Z_s/\eta_0) \left\{ \cos(kb) - \frac{1}{kb} \sin(kb) \right\} \quad (98)$$

The impedance is found from the EMF method using the reflected field $E_z^{ref} = \frac{i}{\omega\mu_0\varepsilon_0} \left(\frac{\partial^2}{\partial z^2} + k^2 \right) A_z^{ref}$ and then adding the radiated impedance in free space

$$\begin{aligned} Z_{in} &\sim R - iX + R_{rad} - \frac{1}{I^2(0)} \int_{-h}^h E_z^{ref} I(z) dz \\ &\sim R - iX + R_{rad} (1 - i\rho_{ref}) \end{aligned} \quad (99)$$

where $X \sim 1/(\omega C)$ is the local capacitive reactance (16) and the radiation resistance is (15). The result for the normalized input impedance is

$$r_{in} = (R_{in} - R) / R_{rad} = (R_s / \eta_0) / |D_{sp}|^2 \quad (100)$$

$$\begin{aligned} x_{in} = (X_{in} - X) / R_{rad} = & \left[-\frac{1}{2} \left(1 - \frac{3}{k^2 b^2} + \frac{1}{k^4 b^4} \right) \sin(2kb) - \frac{1}{kb} \left(1 - \frac{1}{k^2 b^2} \right) \cos(2kb) \right. \\ & - (R_s / \eta_0) \left\{ \left(1 - \frac{2}{k^2 b^2} \right) \cos(2kb) - \frac{1}{kb} \left(2 - \frac{1}{k^2 b^2} \right) \sin(2kb) \right\} \\ & \left. + (R_s / \eta_0)^2 \left\{ \left(1 - \frac{1}{k^2 b^2} \right) \sin(2kb) + \frac{2}{kb} \cos(2kb) \right\} \right] / |D_{sp}|^2 \end{aligned} \quad (101)$$

Simplifying for $kb \gg 1$ and taking the limiting cases we find

$$r_{in} \sim R_s / \eta_0, \quad kb \sim (n - 1/2) \pi$$

$$r_{in} \sim \eta_0 / R_s, \quad kb \sim n\pi$$

$$x_{in} \sim \pm \eta_0 / (2R_s), \quad kb \sim n\pi$$

the second two results being near resonance. Note that the quality factor of the TM_{n01} mode is [24] $Q = \eta_0 / (2R_s) [kb / \sin^2(kb) - 1/kb - kb] / (kb)^2$, where the modes $k = k_n$ are the roots of $\tan(kb) = kb / [1 - (kb)^2]$. For large kb this becomes $Q \sim kb\eta_0 / (2R_s)$, and thus

$$\frac{kb}{2Q} < r_{in} < \frac{2Q}{kb}$$

$$|x_{in}| < \frac{Q}{kb}$$

Because the modal spacing in this one dimensional case is $\Delta k = \Delta\omega / c = \pi / b$, the parameter α is

$$\alpha = \frac{\pi}{2} \frac{\omega / Q}{\langle \Delta\omega \rangle} = \frac{kb}{2Q}$$

Thus the power balance extremes reproduce these results if we take $M_0 = 1$ (and $\alpha \ll 1$), meaning there is no statistical fluctuation in this one dimensional example.

8.3 Density Approximation

Let us now compare the distribution of input resistances obtained from the power balance method with the modal series results (29) and experimental results. Figures 13 and 14 show the comparison for the 220 MHz and 920 MHz data. The dotted curve is the result based on (87) with

$$\frac{-\operatorname{Re}(E_z^{ref})}{\sqrt{\langle |E_z|^2 \rangle_V}} \rightarrow \zeta \quad (102)$$

where again ζ is a zero mean, unit variance Gaussian random variable. Although the extreme limits are representative, the midrange-distribution shown in Figure 13 is not even close to the correct result (in particular, the median value is near unity rather than the small value $O(\alpha)$).

Let us further examine the function

$$\frac{E_z^{ref}}{\sqrt{\langle |E_z|^2 \rangle_V}} = \frac{(E_z - E_z^{rad})}{\sqrt{\langle |E_z|^2 \rangle_V}}$$

where E_z^{rad} is the field at the antenna in free space and E_z (in the numerator) is the total field at the antenna. We can subtract the local quasistatic field E_{sz} from each of these two fields. The difference field in the cavity is, from (10),

$$\begin{aligned} E_z(z) - E_{sz}(z) &\approx -\frac{i\omega}{\varepsilon_0 V} \sum_n \frac{\omega^2/\omega_n^2}{\omega^2(1+i/Q) - \omega_n^2} A_{nz}(z) \int_{-h}^h A_{nz}(z') I(z') dz' \\ &\approx -\frac{i\omega h}{3\varepsilon_0 V} I(0) \sum_n \frac{3A_{nz}^2 \omega^2/\omega_n^2}{[\omega^2(1+i/Q) - \omega_n^2]} \end{aligned}$$

The free space impedance for the short dipole $Z_{rad} \approx R_{rad} + R - iX$, using the EMF method, can be written as

$$-\frac{1}{I^2(0)} \int_{-h}^h \{E_z^{rad}(z) - E_{sz}(z)\} I(z) dz = Z_{rad} - Z \approx R_{rad}$$

where Z is the local impedance and the difference field approximately eliminates the reactive contribution (as well as the ohmic loss contribution) for a small dipole. Thus the difference field is approximately

$$E_z^{rad}(z) - E_{sz}(z) \approx -\frac{1}{h} I(0) R_{rad} \approx -I(0) \frac{\eta_0 k^2 h}{6\pi}$$

Thus the volumetric mean square field is (we ignore the local static field here since it contributes little over the entire volume)

$$\begin{aligned}
\langle |E_z|^2 \rangle_V &= \left(\frac{\omega}{\varepsilon_0 V} \right)^2 \sum_n \sum_{n'} \frac{\omega^4}{\omega_n^2 \omega_{n'}^2} \frac{\frac{1}{V} \int_V A_{n'z}(r) A_{nz}(r) dV \int_{-h}^h A_{nz}(z') I(z') dz' \int_{-h}^h A_{n'z}(z') I^*(z') dz'}{[\omega^2(1+i/Q) - \omega_n^2][\omega^2(1-i/Q) - \omega_{n'}^2]} \\
&= \frac{1}{3} \left(\frac{\omega}{\varepsilon_0 V} \right)^2 \sum_n \frac{\omega^4}{\omega_n^4} \frac{\int_{-h}^h A_{nz}(z') I(z') dz' \int_{-h}^h A_{nz}(z') I^*(z') dz'}{[\omega^2(1+i/Q) - \omega_n^2][\omega^2(1-i/Q) - \omega_n^2]} \\
&\approx \left(\frac{\omega}{\varepsilon_0 V} \right)^2 |I(0)|^2 \frac{1}{9} h^2 \sum_n \frac{3A_{nz}^2 \omega^4 / \omega_n^4}{|\omega^2(1+i/Q) - \omega_n^2|^2}
\end{aligned}$$

Note that the volumetric mean denotation is important because the expectation applies only to the observation location, not the source location; thus this quantity is still a random variable. Therefore **taking** $I(0)$ **to be real and positive** we find

$$\begin{aligned}
\frac{-E_z^{ref}}{\sqrt{\langle |E_z|^2 \rangle_V}} &= i \frac{\sum_n \frac{3A_{nz}^2 \omega^2 / \omega_n^2}{[\omega^2(1+i/Q) - \omega_n^2]}}{\sqrt{\sum_n \frac{3A_{nz}^2 \omega^4 / \omega_n^4}{|\omega^2(1+i/Q) - \omega_n^2|^2}}} - \alpha \frac{Q/\omega^2}{\sqrt{\sum_n \frac{3A_{nz}^2 \omega^4 / \omega_n^4}{|\omega^2(1+i/Q) - \omega_n^2|^2}}} \\
&\approx i \frac{\sum_{n_r} \frac{3A_{nz}^2}{\pi(\omega - \omega_n) / \langle \Delta\omega_n \rangle + i\alpha}}{\sqrt{\sum_{n_r} \frac{3A_{nz}^2}{|\pi(\omega - \omega_n) / \langle \Delta\omega_n \rangle + i\alpha|^2}}} - \frac{1}{\sqrt{\sum_{n_r} \frac{3A_{nz}^2}{|\pi(\omega - \omega_n) / \langle \Delta\omega_n \rangle + i\alpha|^2}}} \quad (103)
\end{aligned}$$

The first term is complex with a positive real part, the second term is negative real. Consider the first term when the cavity is highly undermoded ($\alpha \rightarrow 0$). If we drop all modes except that nearest ω

$$i \frac{|\pi(\omega - \omega_n) / \langle \Delta\omega_n \rangle + i\alpha|}{\pi(\omega - \omega_n) / \langle \Delta\omega_n \rangle + i\alpha} \left| \sqrt{3} A_{nz} \right|$$

where, say, $-\pi/2 < \pi(\omega - \omega_n) / \langle \Delta\omega_n \rangle < \pi/2$. If we ignore the narrow resonance region this becomes

$$i \operatorname{sgn}[\pi(\omega - \omega_n) / \langle \Delta\omega_n \rangle] \left| \sqrt{3} A_{nz} \right| \approx -i\zeta \quad (104)$$

where ζ is a zero mean unit variance Gaussian function. We intend to use $-i\zeta$ as an approximation to the imaginary part even when α is not small. We will show some Monte Carlo simulations of (103) to justify this. Thus the reactance in general is

$$x_{wall} = \zeta \sqrt{r_{in}/\alpha} \quad (105)$$

The overmoded limit $\alpha \gg 1$ gives $x_{wall} \sim \zeta / \sqrt{\alpha}$.

Now we return to the highly undermoded behavior of the real part of (103). Near the resonance we see that the real part of the first term is significant and the imaginary part vanishes exactly at resonance; the contribution of the second, negative term, is in general small at resonance. Thus in the undermoded limit we can approximately look at the real part of this quantity as separate positive and negative distributions. The composite of these two distributions for the complete real part is skewed toward the negative for $\alpha \ll 1$, because the positive half only exists near the resonances in a region of $O(\alpha)$ on the frequency interval. The real part of the positive first term is

$$\frac{\alpha}{|\pi(\omega - \omega_n) / \langle \Delta\omega_n \rangle + i\alpha|} \left| \sqrt{3}A_{nz} \right| = W(\omega) |\zeta|$$

The density function for $|\zeta|$ is twice the positive Gaussian density

$$f(|\zeta|) = \sqrt{\frac{2}{\pi}} e^{-\zeta^2/2}, \quad 0 < |\zeta| < \infty$$

We intend to approximate the density function of the real part

$$\tau = \frac{-\text{Re}(E_z^{ref})}{\sqrt{\langle |E_z|^2 \rangle_V}} \quad (106)$$

as the asymmetrical Gaussian

$$\begin{aligned} f(\tau) &\approx \frac{p(\alpha)}{\sqrt{2\pi}} e^{-\tau^2/2}, \quad 0 < \tau < \infty \\ &\approx \frac{2 - p(\alpha)}{\sqrt{2\pi}} e^{-\tau^2/2}, \quad -\infty < \tau < 0 \end{aligned} \quad (107)$$

Note that the extreme approximation $\tau = M_0 = 3$ selected in the preceding subsection still roughly holds for (107), because the form of the exponentials has not been modified.

We must now select the function of α . If we preserve the mean square of the first term (positive contribution) we have

$$\langle W^2 \rangle_\omega = \frac{1}{\langle \Delta\omega_n \rangle} \int_{\omega_n - \langle \Delta\omega_n \rangle/2}^{\omega_n + \langle \Delta\omega_n \rangle/2} \frac{\alpha^2 d\omega}{\pi^2 (\omega - \omega_n)^2 / \langle \Delta\omega_n \rangle^2 + \alpha^2} = \frac{2\alpha}{\pi} \arctan\left(\frac{\pi}{2\alpha}\right) \rightarrow \alpha$$

$$\int_0^\infty f(\tau) \tau^2 d\tau = \frac{1}{2} p(\alpha)$$

Thus we could choose

$$p(\alpha) \rightarrow 2\alpha$$

But we would also like to have a fit function that holds through the transition to $\alpha \rightarrow \infty$, for which we expect the distribution to become symmetrical

$$p(\alpha) \rightarrow 1$$

The solution of the quadratic equation (87) for the input resistance can be written as

$$r_{in} = \left(\sqrt{\tau^2 + 4\alpha} + \tau \right)^2 / (4\alpha)$$

If we take the mean value of this quantity we find

$$\begin{aligned} \langle r_{in} \rangle_\omega &= \int_0^\infty r_{in} f(r_{in}) dr_{in} = \int_{-\infty}^\infty r_{in}(\tau) f(\tau) d\tau \\ &= \frac{p(\alpha)}{4\alpha\sqrt{2\pi}} \int_0^\infty \left(\sqrt{\tau^2 + 4\alpha} + \tau \right)^2 e^{-\tau^2/2} d\tau + \frac{2-p(\alpha)}{4\alpha\sqrt{2\pi}} \int_0^\infty \left(\sqrt{\tau^2 + 4\alpha} - \tau \right)^2 e^{-\tau^2/2} d\tau \\ &= \frac{1}{\alpha\sqrt{2\pi}} \int_0^\infty (\tau^2 + 2\alpha) e^{-\tau^2/2} d\tau - \frac{1-p(\alpha)}{\alpha\sqrt{2\pi}} \int_0^\infty \tau \sqrt{\tau^2 + 4\alpha} e^{-\tau^2/2} d\tau \end{aligned}$$

where we have used the subscript ω on the expectation because the distribution for τ has been constructed using the frequency as the ensemble variable (the eigenfunction amplitudes were also independent random variables). Using the identities [18] $\int_0^\infty e^{-\tau^2/2} d\tau = \sqrt{\frac{\pi}{2}} = \int_0^\infty \tau^2 e^{-\tau^2/2} d\tau$ and $\int_0^\infty \tau \sqrt{\tau^2 + 4\alpha} e^{-\tau^2/2} d\tau = 2\sqrt{\alpha} + \frac{1}{\sqrt{2}} \int_0^\infty e^{-v} \frac{dv}{\sqrt{v+2\alpha}} = 2\sqrt{\alpha} + \sqrt{\frac{\pi}{2}} e^{2\alpha} \text{erfc}(\sqrt{2\alpha})$ gives

$$\langle r_{in} \rangle_\omega = \frac{1}{2\alpha} (1 + 2\alpha) - \frac{1-p(\alpha)}{2\alpha} \left[2\sqrt{2\alpha/\pi} + e^{2\alpha} \text{erfc}(\sqrt{2\alpha}) \right]$$

It has been shown previously (33) that the frequency mean is unity $\langle r_{in} \rangle_\omega \sim 1$ for general values of α ; suppose we enforce $\langle r_{in} \rangle_\omega = 1$ to determine $p(\alpha)$

$$p(\alpha) = 1 - 1/\left[2\sqrt{2\alpha/\pi} + e^{2\alpha} \text{erfc}(\sqrt{2\alpha}) \right] \quad (108)$$

This result reproduces the limiting cases of small and large α discussed above. It also gives $p(0.0609) \approx 0.08756$ and $p(2.1598) \approx 0.61427$. Figures 15 and 16 show a comparison of the distribution obtained from Monte Carlo simulation of (103) with the approximate results (104), (106), and (107), or

$$\begin{aligned} F(\tau) &\approx 1 - \frac{1}{2} p(\alpha) \text{erfc}(\tau/\sqrt{2}), \quad 0 < \tau < \infty \\ &\approx \left[1 - \frac{1}{2} p(\alpha) \right] \text{erfc}(-\tau/\sqrt{2}), \quad -\infty < \tau < 0 \end{aligned} \quad (109)$$

with (108). The agreement is reasonable (again there is some slight variability in the simulations). (Figure 15 used 500 modes; 100 modes at each end of the interval are beyond the sampled frequency range. Figure 16 used 1000 modes; 200 modes at each end of the interval are beyond the sampled frequency range.) The short dashed curves in Figures 13 and 14 show the impedance distributions obtained from (87), i.e.,

$$\tau = (r_{in} - 1) / \sqrt{r_{in}/\alpha} \quad (110)$$

with the distribution of τ given by (109). The asymmetry introduced in the real reflected field distribution does significantly improve the agreement in Figure 13 from the dotted curve. The ‘kink’ discrepancy at $\tau \rightarrow 0$ in Figure 13 is produced by the discontinuity of the asymmetric Gaussian density function at $\tau = 0$. The long dashed curve in this figure is the simple single mode approximate distribution (43) with density (40); this provides a simple alternative to the ‘kink’ discrepancy in the $\alpha \ll 1$ limit.. The distribution (43) can be transformed to the normalized reflected field τ as

$$F(\tau) \approx 1 - \frac{4\alpha}{\pi} \sqrt{\frac{2}{\pi}} / \left(\sqrt{\tau^2 + 4\alpha} + \tau \right), \quad |\tau| \ll 1, \quad \alpha \ll 1 \quad (111)$$

which is illustrated as the long dashed curve in Figure 15. The corresponding density function is

$$f(\tau) \approx \frac{4\alpha}{\pi} \sqrt{\frac{2}{\pi}} \frac{1}{\left(\sqrt{\tau^2 + 4\alpha} + \tau \right) \sqrt{\tau^2 + 4\alpha}}, \quad |\tau| \ll 1, \quad \alpha \ll 1 \quad (112)$$

Note that the power balance reactance is

$$x_{wall} = \zeta \frac{1}{2\alpha} \left(\sqrt{\tau^2 + 4\alpha} + \tau \right) \quad (113)$$

It is interesting to note that the exponential decay in the asymptotic forms (41) $O(e^{-r_{in}\alpha/2})$ and (58) $O(e^{-\alpha/(2r_{in})})$ for $\alpha \ll 1$, are reproduced in these power balance fit densities. In addition, the asymptotic form (47) $O(e^{-|x_{wall}|\alpha})$ for $\alpha \ll 1$ is reproduced in the power balance fit density; this can be shown by expanding (113) as $\tau\zeta/\alpha$, $\tau \gg 2\sqrt{\alpha}$, using the formula for the density of a product of random variables [25], and performing an asymptotic integration. Note that the real and imaginary normalized reflected fields are orthogonal $\langle \tau\zeta \rangle \approx 0$; we approximate them as independent in this power balance calculation. The limits (59), (60), and (61) for $\alpha \gg 1$ are also identical to the power balance limits for the resistance and reactance.

The exponential decays of the density functions extracted in the asymptotic analyses are thus all reproduced by the power balance results. One might be tempted to use the asymmetrical Gaussian distribution (109) to refine the extreme curves (90) and (91), instead of basing these on the symmetrical three sigma point $M_0 = 3$. However, when the distributions are over-sampled in frequency, such that the resonances are fully resolved (for example the 220 MHz data), the extreme values must be determined from the confidence levels associated with the number of independent modes contained within the frequency sweep, as discussed in (45). Thus, the use of the symmetrical estimate, is appropriate for the extremes, when the data is over-sampled in frequency, but it is not appropriate for the midrange distribution.

9 CONCLUSIONS

The variation of the input impedance with frequency for dipole and monopole antennas inside high Q , electrically large cavities, has been investigated. The magnitude of the variation is inversely related to

the mode density of the cavity and can be quantitatively assessed in terms of the simple parameter $\alpha = k^3 V / (2\pi Q)$, equal to the ratio of modal width to modal spacing. Large variations occur in the undermoded limit $\alpha \ll 1$ (separated, distinct modal spectra) and small variations occur in the overmoded limit $\alpha \gg 1$ (many overlapping modes).

Using statistical estimates for modal spacing and eigenfunction amplitudes, a modal series for the input impedance of a small antenna was constructed. Comparisons with experimental data in a large mode stirred chamber indicate reasonable agreement with Monte Carlo simulations. Analytical formulas have been derived for the extreme values of the distribution (making the approximation of uniformly spaced modes) which indicate exponential decay of the probability density at both extremes of the impedance distribution. This allows limits (from a practical point of view, bounding values) of the impedances to be defined.

Extension of the modal series to resonant antennas and wall mounted monopoles has been made. It is shown that the correlation function for a half space reproduces the 3 dB field enhancement on the wall of the cavity which has been discussed elsewhere in the overmoded region. A wall field enhancement experiment was conducted in the undermoded region confirming the 3 dB increase.

The impedance was reexamined from the point of view of power balance. Simple formulas were derived for the impedance extremes. A semi-empirical modification of the power balance formula gives an impedance distribution in reasonable agreement with the modal series results. These simple power balance formulas provide a practically useful model for the impedance of linear antennas in electrically large cavities.

References

- [1] David A. Hill, *Linear Dipole Response in a Reverberation Chamber*, IEEE Transactions on Electromagnetic Compatibility, Vol. 41, No. 4, November 1999.
- [2] Paul R. Berman (editor), **Cavity Quantum Electrodynamics**, Academic Press, Inc., Boston, 1994.
- [3] William R. Smythe, **Static and Dynamic Electricity**, Hemisphere Pub. Corp., New York, 1989, Sections 13.30, 13.32, and 13.17.
- [4] Robert E. Collin, **Field Theory of Guided Waves**, IEEE Press, New York, 1991, Sections 5.9 and 5.10.
- [5] Edward C. Jordan and Keith G. Balmain, **Electromagnetic Waves and Radiating Systems**, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1968, Sections 14.08, 14.09, 10.05, 10.07, 10.08, 14.05, and 14.06.
- [6] T. H. Lehman, *A Statistical Theory of Electromagnetic Fields in Complex Cavities*, AFWL Interaction Note 494, May 1993.
- [7] R. H. Price, H. T. Davis, and E. P. Wenaas, *Determination of the statistical distribution of electromagnetic-field amplitudes in complex cavities*, Physical Review E, vol. 48, No. 6, December 1993.
- [8] Richard K. Chang and Anthony J. Campillo (editors), **Optical Processes in Microcavities**, World Scientific Pub. Co., New Jersey, 1996, pp389-426.
- [9] Mandon Lal Mehta, **Random Matrices**, Academic Press, Inc., San Diego, 1991, Chapter 1.
- [10] Martin C. Gutzwiller, **Chaos in Classical and Quantum Mechanics**, Springer-Verlag, Inc., New York, 1990, Chapter 16 (Sections 16.2, 16.3, 16.4, 16.7, 16.8), Chapter 15 (Sections 15.5 and 15.6).

- [11] S. Deus, P. M. Koch, and L. Sirko, *Statistical properties of the eigenfrequency distribution of three-dimensional microwave cavities*, Physical Review E., Vol. 52, No. 1, July 1995.
- [12] R. Balian and C. Bloch, *Distribution of Eigenfrequencies for the Wave Equation in a Finite Domain. II. Electromagnetic Field. Riemannian Spaces*, Annals of Physics: 64, 271-307, 1971.
- [13] L. E. Reichl, **The Transition to Chaos**, Springer-Verlag, Inc., New York, 1992, Chapter 8.
- [14] Matthias Brack and Rajat K. Bhaduri, **Semiclassical Physics**, Addison Wesley Pub. Co., Reading Mass., 1997.
- [15] L. K. Warne and K. S. H. Lee, *Some Remarks on Antenna Response in a Reverberation Chamber*, submitted to IEEE Transactions on Electromagnetic Compatibility.
- [16] H. Gerald Hudson, Larry K. Warne, Stephen L. Stronach, Ted Lehman, Marvin Morris, and Kelvin Lee, *Statistical Characterization of the Sandia National Laboratories Mode-Stirred Chamber*, Sandia National Laboratories Electromagnetic Test Report, December 1998.
- [17] Athanasios Papoulis, **Probability, Random Variables, and Stochastic Processes**, McGraw-Hill Book Co., New York, 1965, pp. 126-127, 196-197, 189, 266-267, Sections 8-4, 14-1.
- [18] I. S. Gradshteyn and I. M. Ryzhik, **Table of Integrals, Series, and Products**, Academic Press, New York, 1965, pp. 316.
- [19] Kenneth C. Chen and Larry K. Warne, *Improved asymptotic expansions of time domain antenna current*, Radio Science, Vol. 26, No. 5, pp. 1205-1208, Sept.-Oct. 1991.
- [20] L. B. W. Jolley, **Summation of Series**, Dover Pub., Inc., New York, 1961, pp. 196-197, 104.
- [21] Milton Abramowitz and Irene A. Stegun, **Handbook of Mathematical Functions**, Dover Pub., Inc., New York, 1972, pp. 1026, 576.
- [22] John M. Dunn, *Local, High-Frequency Analysis of the Fields in a Mode-Stirred Chamber*, IEEE Transactions on Electromagnetic Compatibility, Vol. 32, No. 1, Feb. 1990.
- [23] L. K. Warne, *Matched Receiving Antenna Inside a Cavity Resonator*, Sandia National Laboratories Internal Report, January 29, 1991, and *Matched Receiving Antenna Inside a Cavity Resonator: Practical Interpretations*, Sandia National Laboratories Internal Report, February 1, 1991.
- [24] Simon Ramo, John R. Whinnery, and Theodore Van Duzer, **Fields and Waves in Communication Electronics**, John Wiley & Sons, Inc., New York, 1965, Section 10.10.
- [25] Harold J. Larson and Bruno O. Shubert, **Probabilistic Models in Engineering Sciences, Volume I**, John Wiley & Sons, Inc., New York, 1979, p 244.

Figure 1: Fifty ohm Smith chart for input impedance of monopole at 220 MHz ($\alpha \approx 0.0609$) with 10 MHz sweep. “Bounding” power balance result comparison. Time dependence on the experimental Smith charts is $e^{j\omega t}$.

Figure 2: Fifty ohm Smith chart of input impedance of monopole at 920 MHz ($\alpha \approx 2.16$) with 1 MHz sweep. “Bounding” power balance result comparison.

Figure 3: Fifty ohm Smith chart of input impedance of monopole at 15 GHz ($\alpha \approx 1206.7$) with 10 MHz sweep. “Bounding” power balance result comparison.

Figure 4: Energy spectra appearance when the cavity is overmoded $\alpha \gg 1$.

Figure 5: Energy spectra appearance when the cavity is undermoded $\alpha \ll 1$.

Figure 6: Normalized input resistance distribution from simulations and experiment at 220 MHz ($\alpha \approx 0.0609$).

Figure 7: Normalized input resistance distribution from simulations and experiment at 920 MHz ($\alpha \approx 2.16$).

Figure 8: Comparison of asymptotic formulas, simulation, and experiment at 220 MHz with 10 MHz sweep.

Figure 9: Comparison of asymptotic formulas, simulation, and experiment at 920 MHz with 1 MHz sweep.

Figure 10: Drawing of 3 dB wall enhancement field measurement at 220 MHz using dipole probe.

Figure 11: Electric field distribution from two dipole probes, one 2.7 inches from wall and one 118.5 inches from wall, showing 3 dB wall enhancement.

Figure 12: Electric field distribution from resonance peaks for both probes showing 3 dB wall enhancement.

Figure 13: Normalized input resistance distribution from simulation, power balance (the bandwidth modification curve uses the asymmetric reflected field distribution) and experiment at 220 MHz.

Figure 14: Normalized input resistance distribution from simulation, power balance (the bandwidth modification curve uses the asymmetric reflected field distribution) and experiment at 920 MHz.

Figure 15: Normalized reflected field from simulation and simple fit at $\alpha \approx 0.0609$ (220 MHz).

Figure 16: Normalized reflected field from simulation and simple fit at $\alpha \approx 2.16$ (920 MHz).

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