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ABSTRACT

The structure of an extremely strong magnetohydrodynamic shock is discussed in the limit of no particle collisions. It is tentatively concluded that the shock transition takes place through the mechanism of a strong electric field produced by charge separation. The pressure in the shocked plasma is due primarily to a very high electron temperature. The ions, on the other hand, undergo an irreversible temperature change of only 3.

* Work was performed under auspices of the U.S. Atomic Energy Commission.

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I. INTRODUCTION

The following description of a shock wave in ionized matter with an included magnetic field is meant to describe some of the features of a particular limit where the scattering mean free path is extremely long compared to either an electron or ion Larmor radius. This particular limit is applicable to the collisions of interstellar gas clouds and to certain problems in the controlled thermonuclear program.

The method of approach is not intended to be rigorous, (the Boltzmann treatment of the problem is probably formidable) but instead the deductions from certain assumptions are examined in terms of the conservation laws of energy, momentum, flux, mass, and charge. It is hoped that the physical insight to the problem that can be gained from this examination will be helpful in understanding the more rigorous solutions when and if available.

In ordinary gas hydrodynamics one of the principal features of the strong shock solution is that any sound wave behind the shock can catch up to the shock, and that the shock in turn travels faster than sound speed ahead of it. This feature gives rise to stability and governs a qualitative argument concerning the structure. If the speed of sound is greater behind a propagating wave transition than ahead, then any perturbation behind the the transition tends to catch up to the disturbance but can not travel ahead. In turn the shock overtakes any propagating disturbance ahead, transforms it through the shock, and by the previous argument maintains the forward propagating fraction again at the shock front. The result of the process is that the shock front becomes as steep a transition as possible; (we will discuss "possible" later) for suppose the shock were wider than "possible", then the sound wave associated with the more gradual transition would catch up, thereby altering the more gradual structure. The limiting steepness or thickness of the shock front is determined by a characteristic dimension of the process that permits the change of state of the gas in

* This report is based upon a talk given at the American Physical Society meeting, fluid dynamics section, Pasadena, California, March 1956.

accordance with the dynamical restrictions of conservation of energy, momentum, mass (magnetic flux and charge). In ordinary gas dynamics this dimension is the scattering mean free path, but it is assumed here without proof that if a process existed that could change the state of the gas in accordance with the required conservation laws (Hugoniot relations) over a distance smaller than the collision mean free path, that then the strong shock would develop based upon this process rather than upon the one characterized by the larger dimension.

It is the object of this paper to discuss such a process for an ionized gas. The sound speed argument will still apply provided Alfvén speed is interpreted for sound speed ahead and behind the shock.

It first must be made plausible that in a plasma with an internal magnetic field, large discontinuities can take place within dimensions of the order of an electron Larmor radius. In ordinary hydrodynamics, the sharpest discontinuity corresponds to a shock which is a transition existing over dimensions of a number of collision mean free paths. If, however, in a plasma, the collision mean free path is long compared to the electron Larmor radius in the magnetic field, then the forces of charge separation can cause a large discontinuity within the comparatively short distance of the electron radius of curvature.

To understand the preponderance of the forces of charge separation, imagine a group of ions and electrons moving across a magnetic field. This

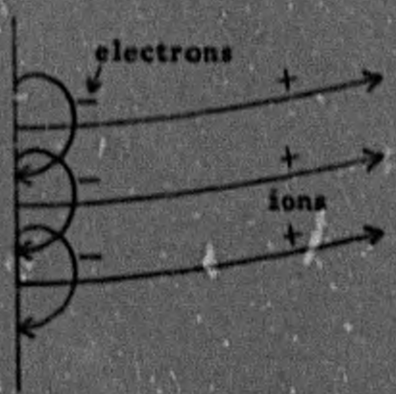

Fig. 1. Charge separation.

will immediately give rise to a charge separation, (as in Fig. 1) because the electrons have their velocity direction changed in the magnetic field in a distance short compared to the ions. If the electrons are held in one position relative to the field and the ions move ahead, the electric field of charge separation can be sufficiently large to govern the dynamics of the more massive ions for a wide variety of problems. The space charge electric field can be calculated assuming that either the velocity or the energy of the electrons and ions is equal. It is evident that assuming equal velocity will result in a smaller electron Larmor radius with a consequently sharper discontinuity. However, a self-consistent treatment of the one boundary-layer problem that has been calculated[1] showed that the electric field of charge separation rapidly causes the electron energy to equal the ion energy. The latter condition is also true when the thermal velocity is large compared to the ordered velocity (i.e., weak shock). Equal energy will therefore be assumed.

Assume that the charge on all particles is unity. Let

M = mass of the ion
m = mass of the electron
u_i = velocity of ions
u_e = velocity of electrons
d = distance of charge separation
N = number density of electrons or ions in neutral plasma
H = magnetic field.

The condition to be investigated is whether the potential between a charge separation layer one electron Larmor radius thick is greater than or equal to the ion kinetic energy. This is equivalent to inquiring whether the space charge separation electric field can reverse the trajectory of an ion.

The potential energy per particle of space charge separation over a distance d is

$$V = 4\pi Ne^2 d^2 . \tag{1}$$

The distance of charge separation equals an electron Larmor radius:

$$d = \frac{mcu_e}{eH} = \frac{mcu_i\sqrt{M/m}}{eH} . \tag{2}$$

The potential V must be equal to or greater than the ion kinetic energy.

$$\frac{1}{2} M(u_i)^2 \leq V = 4\pi Ne^2 \frac{m^2c^2(u_i)^2(M/m)}{e^2H^2} . \tag{3}$$

1. M. Rosenbluth, LASL report LA-1850, Sept. 14, 1954; and Proceedings of Magneto-Hydrodynamics Conference, Lockheed, December 28, 1956.

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$$\frac{H^2}{8\pi} \leq N\,mc^2 \,, \tag{4}$$

Therefore the space charge separation field can govern the ion motion provided the magnetic field energy density is less than the electron rest mass density. This condition is met for a wide range of plasmas.

The character of the shock transition will be investigated within the limits of

1. No collisions.
2. Extremely strong shock limit (i.e., the pressure in the shocked region is many times - 50 or greater - the pressure in the unshocked region)
3. The shock transition takes place within a space charge separation layer.

For these conditions the ions can only be acted upon by the charge separation force, namely an average electric field E. (There are no collisions and the ion Larmor radius is large compared to the transition region.) Therefore all the ions will receive the same average momentum impulse because each ion experiences the same electric field. The statistical charge fluctuation within a cube one electron Larmor radius on a side is small so that the electric field any one ion experiences is close to the average. This implies that if the ion thermal velocity is small compared to the velocity imparted by the shock, (i.e., strong shock condition) then the ions will all have a uniform directed velocity behind the shock. Since this process is reversible, there is no change in entropy.

The electrons will be accelerated through the shock gaining some fraction of the potential of the charge separation layer. The electric field in the layer is in a direction such as to push the ions in the direction of the shock while accelerating the electrons back through it. This is the only possible direction of the electric field. The electrons will move in circular orbits in the magnetic field of the shocked region and will tend to be coherent. That is, since the phase of all electrons going through the shock layer at a given time is identical, then the phase in the shocked region will be a coherent function of position as well as time, and as a result, the electron density and velocity functions will oscillate with a large amplitude. Such a strongly oscillating function does not lend itself to a continuous shock solution.[2]

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2. This was first recognized and pointed out by Marshal Rosenbluth and Conrad Longmire of Los Alamos.

A dissipative or randomizing process is needed in order that the electron kinetic energy behave like a constant pressure function. Fortunately - for ease of solution - the electron phases are rapidly randomized within a very short distance (an electron Larmor radius) behind the shock. As will be discussed later, this occurs provided the shock pressure is less than 160 times the unshocked pressure. This represents a very strong shock indeed and not many physical problems are likely to extend into regions of greater pressure ratios.

Therefore, a derivation of the hydrodynamic quantities will be made on the basis of the assumption that the electron pressure behind the shock is a uniform, constant function. Afterwards, with the use of the velocity relationships the randomization of the electron distribution will be examined.

If a piston of pressure P_o pushes against an infinite plasma of pressure P_1, then a disturbance will propagate into the plasma (see Fig. 2). In general, there will be a perturbed and an unperturbed region. The separation between these two regions is assumed discontinuous and is called the shock. The state of the plasma in both regions must be uniform and time- and space-independent except for the shock transformation. Otherwise a steady-state solution cannot run to infinity. The piston pushing on

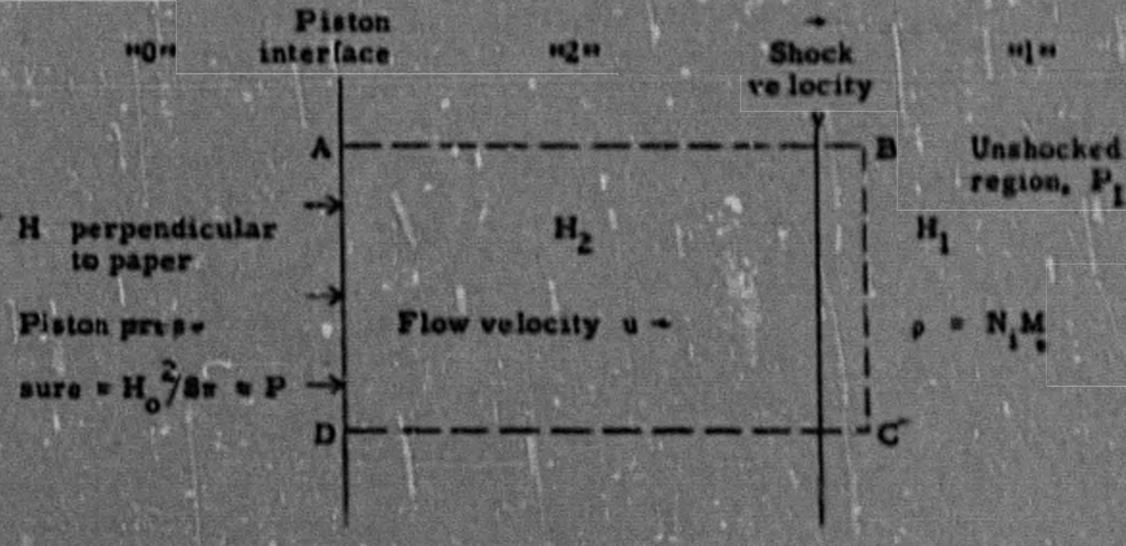

Fig. 2. Shock nomenclature.

the plasma is the vacuum field H_o ; $P_o{}^2/8\pi$. The shocked region has a magnetic field H_2 in it and a uniform ion velocity u. Because of space charge conservation the electrons have also a flow velocity u, otherwise, a small space-charge separation will give rise to a large electric field (as discussed earlier) and the ion velocity would be changed. This is contrary to the assumption of uniform mass flow behind the shock. Therefore the electron flow, or drift velocity, must be the same as the ion velocity. Similarly the magnetic field lines in the shocked region "2" must move with the mass flow, otherwise there would again be charge separation with its resultant inconsistency. This concept of uniform flow of magnetic flux lines implies that the magnetic field is locked to the fluid and undergoes exactly the same compressions as the fluid.

In order to elaborate this further, consider the flux within the boundary ABCD in Fig. 2. By symmetry the electric field along the segment AB must exactly cancel that in the opposite direction along CD (i.e., the shock is plane parallel). There can be no electric field along the segment BC, because this is stationary in the unshocked region and no perturbation can have reached it. The remaining segment DA is moving with the piston interface. Since the piston can be a perfectly conducting membrane - e.g., solid metal - no electric field can exist parallel to the surface. Therefore the curl of E around the path ABCD is zero and the included magnetic flux is conserved. The evident corollary is that a given area of magnetic flux in the unshocked region is transformed as the same flux in the shocked region, but compressed by exactly the fluid compression. This implies that the conservation of mass and flux are equivalent.

The shock velocity is v, and the unshocked region has a magnetic field H_1, density NM, and temperature T_1.

The conservation laws of the shock are

<u>1. Conservation of mass:</u> This is equivalent to the conservation of magnetic flux. If the compression from region "1" to "2" is η, then

$$\rho_2 = \eta \rho \quad \text{and} \quad H_2 = \eta H_1 \tag{5}$$

The compression can be written in terms of the velocities,

$$\eta = \frac{v}{v - u} \; . \tag{6}$$

2. Conservation of energy: The rate of work done by the piston must equal the rate of work done across the shock. The velocity of the piston is the same as the flow velocity u, and the pressure driving the piston is the same as the pressure in the shocked region "2". (If the pressures were not equal, the piston would accelerate or decelerate, thereby contradicting the assumption of uniform steady-state flow.)

This gives rise to the relationship

$$u\left(\frac{H_o^2}{8\pi}\right) = u\left[\frac{H_2^2}{8\pi} + nNK(T_{2e} + T_{2i})\right] \tag{7}$$

$$= v\left[NK(T_{2e} - T_{1e} + T_{2i} - T_{1i}) + (M + m)\frac{Nu^2}{2} + \frac{H_2^2}{8\pi} - \frac{H_1^2}{8\pi}\right] .$$

The pressure in the shocked region is the sum of the magnetic field pressure $H_2^2/8\pi$ and the particle pressure. The particle pressure is primarily due to the electron temperature T_{2e} because the ion temperature T_{2i} is assumed small. All the ions are accelerated uniformly through the shock transition due to the average electric field, whereas the electron energy gained through the shock is randomized. This randomization of the electron energy takes place in two dimensions only since it is a phase broadening. (It will be discussed in more detail later.) The restriction of two dimensions implies that the electron pressure and energy density are equal. Three degrees of freedom would give rise to the standard relationship that $p = \frac{2NKT}{3}$. It is just this restriction to the dimensionality of the electron gas that results in the hydrodynamic behavior of γ equaling 2 (ratio of specific heat at constant pressure to that at constant volume).

The rate of work done across the shock is the velocity of the shock v times the change in energy of a unit volume in the unshocked region transposed to the shocked region. The ions and electrons change both their thermal and kinetic energy across the shock. Since the pressure in the unshocked region is small compared to the shocked region, $T_{1e} = T_{1i} \to 0$; also $T_{2i} - T_{1i} \to 0$, so that the thermal energy term

$$NK(T_{2e} - T_{1e} + T_{2i} - T_{1i}) \to NKT_{2e} .$$

Similarly the mass of the electron can be neglected compared to the ion, so that the kinetic energy term becomes

$$(M + m)\frac{Nu^2}{2} \to \frac{NMu^2}{2}$$

The rate of doing work in compressing the magnetic field from a value H_1 to H_2 is the velocity of compression times the difference in energy of $H_1^2/8\pi$ at volume "1" and $H_2^2/8\pi$ at volume "2", or

$$\frac{dE}{dt} = v\left[\frac{-H_2^2}{8\pi\eta} - \frac{H_1^2}{8\pi}\right] . \tag{8}$$

Note that this is not simply the energy difference per unit volume, but instead the energy difference associated with a unit of flux.

3. Conservation of momentum: The pressure equals the time rate of change of momentum.

$$\frac{H_o^2}{8\pi} = \frac{(H_2)^2}{8\pi} + \eta NK(T_{2e} + T_{2i}) = NK(T_{1e} + T_{1i}) + \frac{H_1^2}{8\pi} + NMuv \tag{9}$$

Again the temperatures T_{2i}, T_{1e}, T_{1i} are small. The mass is contained all in the ions so that the momentum is NMu. The rate of change is proportional to the shock speed v.

If $(T_{1e} + T_{1i}) = \epsilon_1$ and $T_{2i} = \epsilon_2$ then using the conservation of flux and mass the equations become

$$u\left[\frac{\eta^2 H_1^2}{8\pi} + \eta NK(T_2 + \epsilon_2)\right] = v\left[NK(T_2 + \epsilon_2 - \epsilon_1) + \frac{NMu^2}{2} + \frac{H_1^2}{8\pi}(\eta - 1)\right] \tag{10}$$

and

$$\frac{\eta^2 H_1^2}{8\pi} + \eta NK(T_2 + \epsilon_2) = NK\epsilon_1 + \frac{H_1^2}{8\pi} NMuv \tag{11}$$

or

$$\eta NK(T_2 + \epsilon_2) = \frac{H_1^2}{8\pi}(1 - \eta^2) + NK\epsilon_1 + NMuv \tag{12}$$

substituting into (10), and using $\eta = v/(v - u)$

$$u\left(\frac{H_1^2}{8\pi} + NK\epsilon_1 + NMuv\right) = v\left[\frac{v - u}{v}\left(\frac{H_1^2}{8\pi}(1 - \eta^2) + (1 - \eta)NK\epsilon_1 + NMuv\right) + \frac{NMu^2}{2} + \frac{H_1^2}{8\pi}(\eta - 1)\right] \tag{13}$$

Then by rearranging one gets

$$\frac{u}{2} = v - u - \frac{1}{NMv}\left(\frac{H_1^2}{8\pi} + NK\epsilon_1\right) + \frac{1}{NMu}\left[\frac{H_1^2}{8\pi}\left(-\frac{u}{v}\right) - \frac{u}{v}NK\epsilon_1\right] \qquad (14)$$

or

$$\frac{3u}{2} = v - \frac{1}{NMv}\left[\frac{H_1^2}{8\pi} + NK\epsilon_1\right] . \qquad (15)$$

Therefore

$$\eta = \frac{v}{v - \frac{2}{3}v + \frac{2}{3NMv}\left[\frac{H_1^2}{8\pi} + NK\epsilon_1\right]} \qquad (16)$$

or

$$\eta = 3\left[1 - \frac{\frac{H_1^2}{8\pi} + NK\epsilon_1}{NM\frac{v^2}{2}}\right] \qquad (17)$$

Since $\frac{H_1^2}{8\pi} + NK\epsilon_1$ is the pressure in region "1", P_1, then by Eqs. (12) and (15) the compression becomes

$$\eta \cong 3\left[1 - \frac{P_1}{\frac{3}{4}(P_2 - P_1)}\right] \qquad (18)$$

However, in the strong shock limit $P_2 \gg P_1$, so that the compression approaches 3. This can be derived more simply from the strong shock result of ordinary hydrodynamics in which

$$\eta \to \frac{\gamma + 1}{\gamma - 1} . \qquad (19)$$

For a two-dimensional system, $\gamma = 2$, so that the compression has the limiting value 3.

Equation (9) implies that the energy per electron is equal to the kinetic energy per ion

$$\frac{H_o^2}{8\pi} \cong \eta NKT_e = NMuv$$

Let $\eta = 3$, $v \sim \frac{3u}{2}$, by (15) therefore

$$\frac{H_o^2}{8\pi} = 3\,NKTe = N\,M\,\frac{3u^2}{2} \qquad (20)$$

or

$$KT_e = M\frac{u^2}{2} . \qquad (21)$$

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II. ION HEATING

Since the ions everywhere move with the same velocity as the magnetic field lines, the magnetic field itself cannot do work on the ions. Similarly, the mean free paths are long so that scattering cannot accelerate the ions across the shock front. Therefore, the impulse given to the ions must come from an electric field existing within the shock front, parallel to the motion of the shock front. To first order the impulse given to each ion will be a constant so that the random velocity of the ions in the moving frame will be the same as the random velocities in the stationary frame. Since the temperature is proportional to the mean square velocities in the fluid frame, it is evident that the ions do not undergo a temperature change comparable to their change in kinetic energy. There is, however, a second-order effect which increases the ion temperature after passing through the shock front. If every ion received exactly the same impulse through the shock, then the change in temperature would be zero. However, an ion with initial velocity Δu directed towards the shock spends a shorter time in the electric field and hence receives a smaller impulse. The converse is true for an ion with initial velocity directed with the shock. This separation of the "sheep from the goats" results in an increase in the random velocity and hence temperature increase in shocked region.

A solution to the heating can be obtained by considering the change in energy of the ions when measured in the moving frame of the shock. If the electric field is $E(y)$ where y is measured in the moving frame of the shock, $\int_{-\infty}^{\infty} E(y)\,dy = V$ where V is a constant, or the voltage across the shock. The integration is across the shock front, so that the limits imply a large distance from one side to the other of the shock. The change in energy of an ion measured in the shock frame will be a constant independent of the initial velocity of the ions, because the ion falls through a constant potential V.

The change in energy in the shock frame can be written

$$V = \frac{M}{2}\left[v^2 - (v - u)^2\right] . \tag{22}$$

Taking the strong shock limit and Eq. (15) gives $v = 3u/2$. Therefore

$$V = \frac{M}{2}\left[\left(\frac{3u}{2}\right)^2 - \left(\frac{3u}{2} - u\right)^2\right] = Mu^2 \tag{23}$$

If an ion has an initial velocity $\pm \Delta u$, then conservation of energy in the shock frame gives a final laboratory velocity $u \pm x$ by the relation

$$V = Mu^2 = \frac{M}{2}\left[\left(\frac{3u}{2} \pm \Delta u\right)^2 - \left(\frac{3u}{2} - u \mp x\right)^2\right] \tag{24}$$

Solving for x gives

$$x^2 \mp ux \mp 3u\,\Delta u - \Delta u^2 = 0 \; . \tag{25}$$

Therefore

$$x = \frac{u \pm \sqrt{u^2 \mp 12u\,\Delta u + 4\Delta u^2}}{2} \tag{26}$$

If $\frac{\Delta u}{u} \to 0$, i.e., the strong shock limit, then

$$x \simeq \pm 3\Delta u \; . \tag{27}$$

The ion temperature ratio will be

$$\frac{T_2}{T_1} = \frac{x^2}{(\Delta u)^2} = 9 \; . \tag{28}$$

The passage of a shock therefore multiplies the ion temperature by 9. If the ions were compressed adiabatically by a factor of 3 instead of undergoing a change of state through the shock front, the increase of temperature would be 3 also. An adiabatic compression of a two-dimensional gas causes an increase in temperature proportional to the compression. The increased heating of the ions by the shock process over and above the adiabatic ratio represents an irreversible increase in entropy. Therefore in a cycle of shock followed by adiabatic expansion the temperature of the ions will be increased by a factor of 3 per cycle.

III. THICKNESS OF THE SHOCK FRONT

Equations (21) and (22) indicate something of the thickness of the shock front, namely, that it must be approximately an electron Larmor radius thick. Suppose it were thin compared to an electron Larmor radius. Then an electron would fall through the field $E(y)$ in a linear path and the energy change in the moving frame y would be

$$V = \int E(y)\,dy = \frac{m}{2}\left[v^2 - (v - u_e)^2\right] \tag{29}$$

where v = shock velocity

u_e = electron velocity in laboratory frame in the shocked region.

From equation (21)

$$KT_2 = \frac{mu_e^2}{2} = \frac{Mu^2}{2}$$

or

$$u_e = \sqrt{\frac{M}{m}}\,u.$$

Since u_e is much greater than u or v, Eq. (29) becomes

$$V = \int E(y)\,dy \cong \frac{m}{2}\,u_e^2 = KT_e \quad . \tag{30}$$

This is equivalent to saying that the shock frame is essentially at rest insofar as electrons are concerned, and that a free electron would gain the full potential energy of the layer. However, Eq. (22) indicates that the potential of the layer must be

$$V = Mu^2 \quad .$$

By equation (21) the electrons must pick up only half of this energy, namely

$$\frac{Mu^2}{2} \quad .$$

There are two ways of understanding how the electrons can pick up less energy than the full potential of the layer. The simplest concept is that the layer is more than a Larmor radius thick. Then the fraction of the potential that the electrons experience will be less than the full potential, roughly inversely proportional to the thickness of the layer measured in electron Larmor radii. Another concept of the processes for the electron to gain less than the full energy of the layer concerns the orbit of the electrons as they undergo multiple collisions with a receding infinitely thin layer (see Fig. 3). An electron will in general be reflected from the reverse side of an infinitely thin layer because the kinetic energy of the electron relative to the receding layer is less than the full potential of the layer. Rosenbluth and Longmire have shown in closed form that the electron loses exactly 1/2 its energy after the multiple collisions with the layer. The integration of the path is complicated. The same result can be seen from the fact that the slow receding of the layer away from the electron guiding

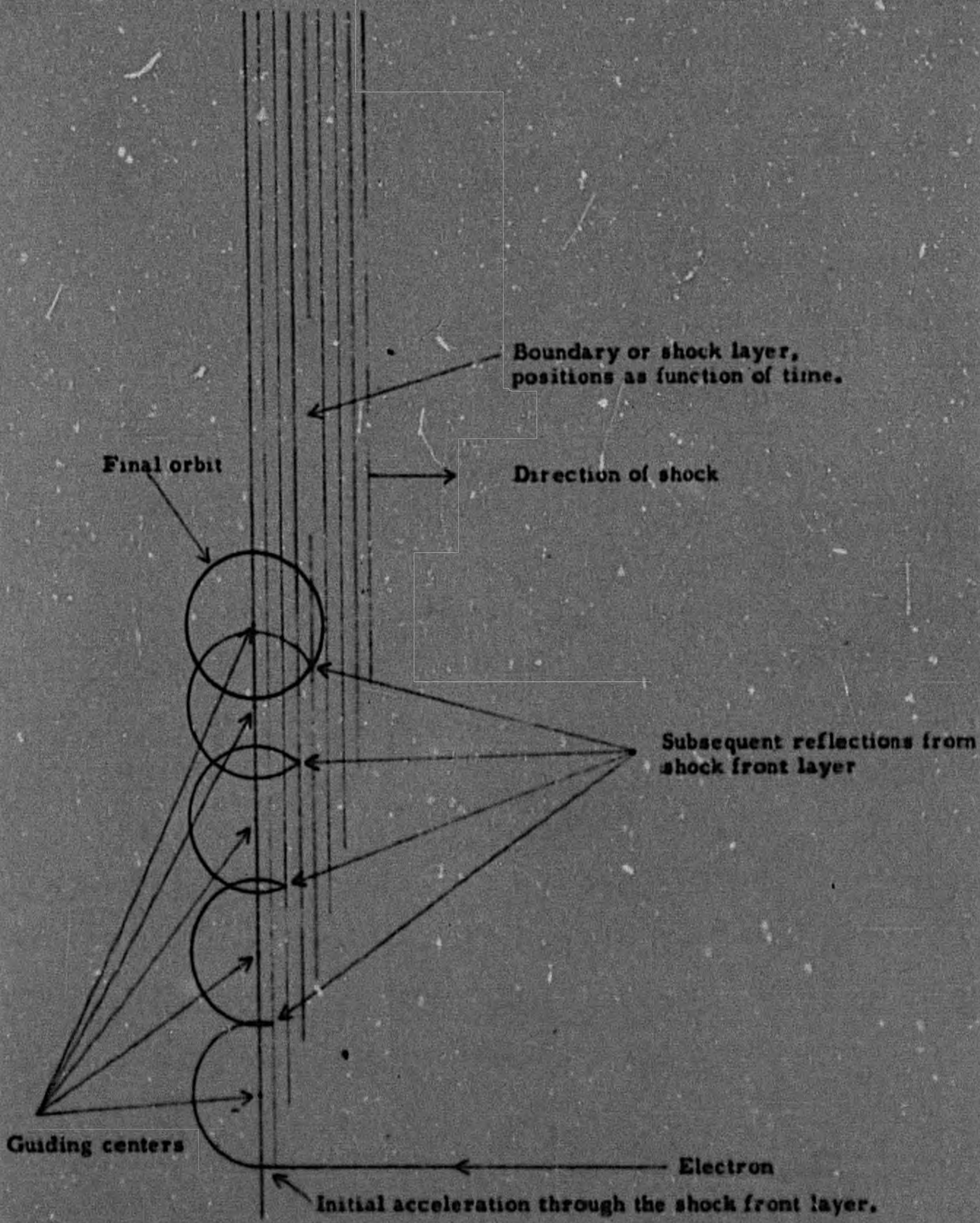

Fig. 3. Electron orbit after accelerating through infinitely thin layer and subsequent bounces from the rear side of the layer. The orbits are represented in the shocked, or fluid frame of velocity u.

center behaves like an adiabatic expansion of an electron gas. Since the phase space volume doubles (from one-half orbit to full orbit) the kinetic energy must be reduced by 1/2. This factor of 1/2 identically satisfies conditions (21) and (22) for the final kinetic energy of the electrons and potential of the layer. However, the electron orbit behind the layer is not the equilibrium orbit in the fluid. The drift motion parallel to the shock front causes a large current and consequent magnetic field change, so that the original assumption of the layer being infinitely thin is inconsistent. It must remain for a self-consistent calculation of the charge and current densities of the layer to determine its actual structure and thickness. The above considerations do indicate that the layer must be at most as thick as a couple of electron Larmor radii and possibly much thinner. The property of randomizing the electron phases does not depend upon the thickness of the layer.

IV. ELECTRON RANDOMIZATION

The derivation of the shock hydrodynamics was based upon the assumption that the electron pressure term NKT was a constant independent of position or time in the shocked fluid. As pointed out earlier a single-orbit picture would give a strongly oscillating current and charge density oscillation and we must look for the processes that tend to randomize the phases of the individual electrons that make up such oscillations.

Let us first consider the structure of such an oscillation. In the shocked region the relative phases of electrons should be preserved independent of the thickness of the layer, provided statistical density fluctuations within the layer are small. As can be seen from Fig. 3, the position (x) of the guiding center of an electron measured in the fluid from behind the shock depends only upon the position of the shock at the time when the electron was accelerated through it. The phase is similarly determined at that instant. Therefore the phase is a function of the position of the guiding centers. Fig. 4 shows the electron distribution for a group of electrons whose phase shifts $\pi/2$ radians for every diameter shift of the guiding centers.

It is evident that the charge separation and current oscillation repeats for a wavelength equal to $8r_e$. The principal wavelength is the distance between guiding centers that corresponds to 2π change in

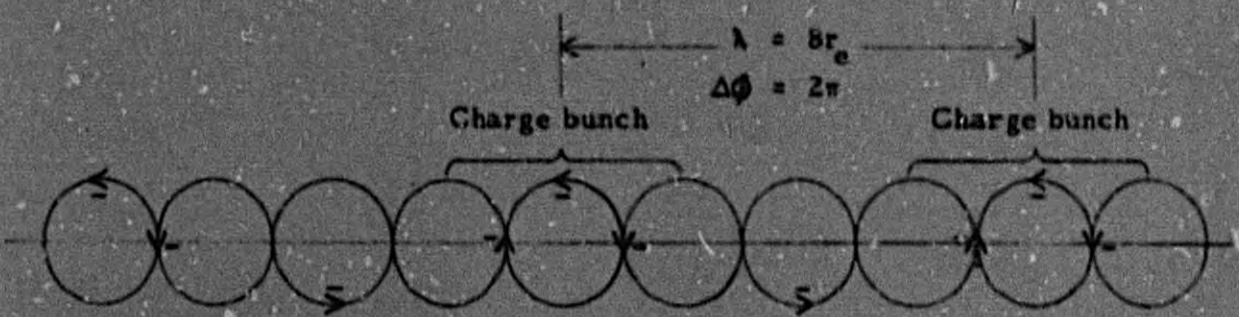

Fig. 4. Electron distribution for a hypothetical phase relation $\Delta\phi = \pi/2$ for Δx = electron orbit diameter = $2r_e$.

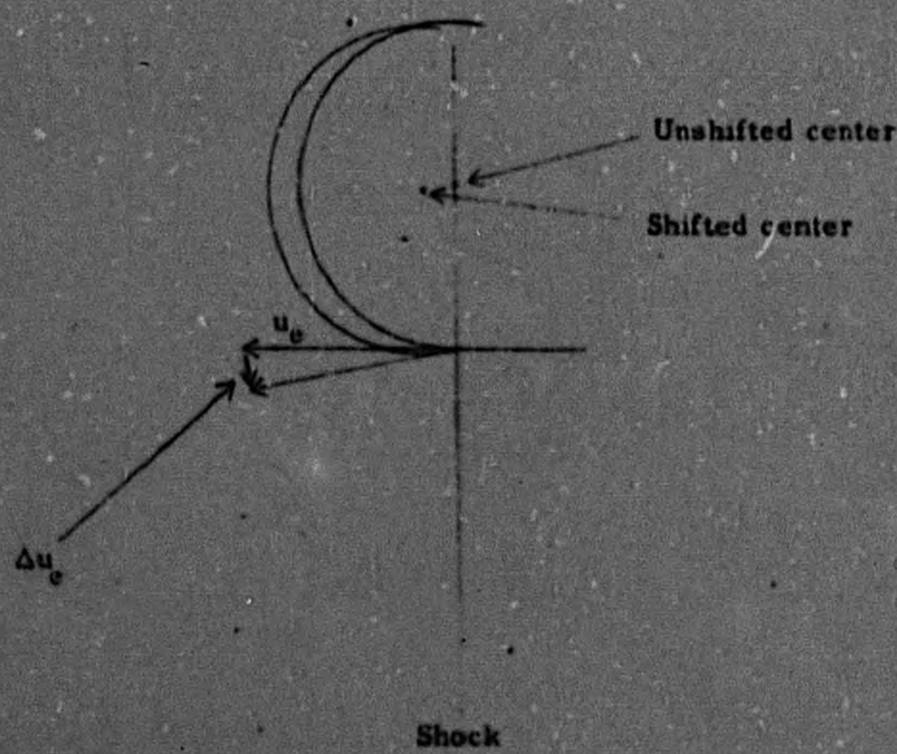

Fig. 5. Shift in guiding center due to initial transverse velocity.

electron phase.

In the case of the shock, let

r_e = electron orbit radius

λ = wavelength of oscillations

u_e = electron velocity

v = shock velocity

τ = time for electron to go 2π radians.

Therefore

$$\tau = \frac{2\pi r_e}{u_e} \quad (31)$$

$$\lambda = v\tau \quad (32)$$

however

$$v = \frac{3}{2} u = \frac{3}{2} u_2 \sqrt{\frac{m}{M}} \quad . \quad (33)$$

Therefore

$$\lambda = 3\pi r_e \sqrt{\frac{m}{M}} \quad . \quad (34)$$

If the mass ratio is that for protons and electrons, then the repeat wavelength of the electron oscillation is 0.222 r_e. In other words, one electron must contribute coherently to the phase oscillation over 9 wavelengths within its orbit diameter. This is a high degree of coherence, and it is not surprising that small perturbations can average it out.

Let

y = the position of an electron measured in the fluid or shocked frame,

x = position of guiding centers in the same frame.

Then

$$y = x\, r_e \sin(\omega t + \frac{2\pi x}{\lambda}) \quad (35)$$

$$\dot{y} = \omega\, r_e \cos(\omega t + \frac{2\pi x}{\lambda}) \quad (36)$$

The average velocity at a point $\langle \dot{y}(y, t) \rangle$ determines the amplitude of the plasma oscillation. It is the vector sum of the velocities of all the electrons at y, so that we must solve for $\langle \dot{y} \rangle$ as a function of y. This will be a multivalued solution and in general it will include electron phases from $\Delta x = 2r_e$, so that the number of phases contributing will be approximately $\frac{2r_e}{\lambda} \approx 9$ in the case of protons. The electron distribution at a point will therefore be composed of a sum over 9 circular oscillations of an electron. It is plausible, and can be made rigorous, that if the phases of all the electrons are randomized by $2\pi/9$ radians, then the coherent fluctuations will be averaged out. This is equivalent to the statement that if the phase of all electrons contributing to the charge density at a point is random with a width 2π, then the coherent amplitude becomes negligibly small.

If it is required that the oscillations die out within one electron Larmor radius from the shock front, then either the electron phases must be randomized by $2\pi/9$ radians on passage through the shock front, or they must undergo scatterings that accumulate to $2\pi/9$ radians in the time the guiding center has moved $2r_e$ from the shock front.

Randomizing through the shock front occurs due to an initial random transverse velocity from an initial temperature T_{1e}. The effect of this initial temperature is to shift the guiding center relative to the shock front (Fig. 5).

The shift in the guiding center Δx to first order is

$$\Delta x = r_e \frac{\Delta u_e}{u_e} . \qquad (37)$$

Δu_e is the initial random velocity distribution

$$\frac{\Delta u_e}{u_e} = \sqrt{\frac{T_1}{T_2}} . \qquad (38)$$

The shift in guiding centers needed to average out the oscillation is $\pm \frac{\lambda}{4}$. Therefore for damping the oscillation $\Delta x \geq \frac{\lambda}{4}$. Or

$$r_e \sqrt{\frac{T_1}{T_2}} \geq \frac{3\pi}{4} r_e \sqrt{\frac{m}{M}} \qquad (39)$$

or

$$T_1 \geq \left(\frac{3\pi}{4}\right)^2 \frac{m}{M} T_2 = \frac{T_2}{326} \text{ for protons.} \qquad (40)$$

Since $P_1 \cong \frac{NKT_1}{2}$, and $P_2 \cong NKT_2$, then it is evident that condition (40) is satisfied up to a shock pressure ratio $P_2/P_1 = 160$

which is a very strong shock indeed. If the shock pressure ratio is stronger still, then the averaging of phases will take a longer time, and the oscillation will die out further behind the shock front. This residual oscillation will be still further damped by small angle collisions.

Since even a small angle collision can give a cumulatively large phase-angle shift, the effectively large coulomb scattering cross sections for very small angle collisions cannot be neglected in this instance even though it has been for the shock structure itself.

A collision that changes the direction of an electron by $\Delta\theta$ will change the guiding center position by $r_e \Delta\theta = \Delta x$.

However, for damping within a distance of an electron Larmor radius behind the shock

$$\Delta x \geq \frac{\lambda}{4} = \frac{3\pi}{4} \cdot r_e \sqrt{\frac{m}{M}} \tag{41}$$

therefore

$$\Delta\theta \geq \frac{\lambda}{4r_e} = 0.055 \text{ radians for protons.} \tag{42}$$

The Rutherford scattering cross section behaves as $\sigma \sim 1/\theta^4$ so that the effective scattering cross section will be 10^3 greater than for scattering one radian. The path length over which this scattering must occur is larger than a Larmor orbit by the number of phases in $2r_e$ of guiding center space - namely 2×9 phases. Therefore the effective probability of scattering is increased by 2×10^5 greater than the probability of one scattering in a Larmor orbit. From two standpoints, then, it is expected that the electron coherent oscillations will die out rapidly and that the pressure term becomes a constant NKT.

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