A METHOD OF SOLVING THE TWO-END-POINT PROBLEM FOR SECOND-ORDER DIFFERENTIAL EQUATIONS

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October 1956
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Consider the following differential equation

\[ \frac{d^2y}{dx^2} - \alpha^2 y = f(x) \]  

subject to the boundary conditions that \( y(0) \) and \( y(X) \) are given. The values of \( x \) are assumed to be in the range

\[ 0 \leq x \leq X \]

Let this range be divided into \( N \) equal intervals of length \( \Delta x \) and let

\[ y(n \Delta x) = y_n \]
\[ f(n \Delta x) = f_n \]

The differential equation (1) goes over into the following difference equation.

\[ \Delta x^{-2} (y_{n+1} - 2y_n + y_{n-1}) - \alpha^2 y_n = f_n \]

subject to the boundary conditions that \( y_0 \) and \( y_N \) are given.

Equation (4) may be solved by letting \( i = \frac{1}{2} \).

*Work performed under the auspices of U.S. Atomic Energy Commission.*
(5) \[ y = Au + w \]

where \( w \) is an arbitrary solution of equation (4) subject only to the condition that \( w_0 = y_0 \) and \( u \) is a solution of the homogeneous equation

\[
\Delta x^2 (u_{n+1} - 2u_n + u_{n-1}) - a^2 u_n = 0
\]

The constant \( A \) is determined from the equation

\[
y_N = A u_N + w_N
\]

One problem which may arise can be seen by examining the differential form of equation (6)

\[
d^2u/dx^2 - a^2 u = 0
\]

The solution of equation (8) subject to the boundary condition that \( u(0) = 0 \) is

\[
u = B \sinhx
\]

where \( B \) is an arbitrary constant. Thus the difference, \( y - w \), can grow exponentially which implies a loss of accuracy in the final solution. In other words, both \( u \) and \( w \) can be large compared to \( y \) so that \( y \) is found by taking the difference of two numbers close to each other resulting in a loss of significant figures.

This loss of accuracy may be remedied in the following manner: Suppose that

\[
\begin{align*}
(10a) & \quad |w_n^j - f_n| \leq \beta & n < j \\
(10b) & \quad |w_n^j - f_n| > \beta & n = j
\end{align*}
\]
where \( w^1 \) is a solution of equation (4) subject to the condition that \( w^0_o = \gamma_o \) and \( \beta \) is a predetermined constant depending on the accuracy desired. Define the constant \( A^{i+1} \) from the equation

\[
(11) \quad f_j = A^{i+1} u_j + w^j
\]

A new solution, \( w^{i+1} \), to equation (4) may now be found.

\[
(12a) \quad w^{i+1}_n = A^{i+1} u_n + w^i_n \\
(12b) \quad w^{i+1}_n = (2 + \alpha^2 \Delta x^2) w^{i+1}_{n-1} - w^{i+1}_{n-2} + \Delta x^2 f_n 
\]

\( w^{i+1} \) is generated recursively according to equation (12b) until condition (10b) is satisfied for some larger value of \( j \) or until \( n = N \) at which time \( w^{i+2} \) is obtained from \( w^{i+1} \) in the same manner in which \( w^{i+1} \) was obtained from \( w^i \).

The point of the method is to keep \( w \) the same order of magnitude as \( y \). As outlined here, it assumes that \( y \) and \( f \) are the same order of magnitude.

The method has been tested on an IBM 650 with the aid of Robert Pexton. In the cases examined, \( N = 30 \) and \( \beta = 10, 10^2, 10^3, \) and \( 10^4 \). \( y \) and \( f \) were of order of magnitude unity. Eight significant figures were carried. The results were satisfactory although four figures were lost for the largest value of \( \beta \).

Reference