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Jack L. Uretsky

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Jack L. Uretsky

Physics Department, Massachusetts Institute of Technology
and
Massachusetts Institute of Technology Digital Computer Lab†

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ABSTRACT

Scattering cross sections have been computed for zero-energy neutrons incident upon a square well of spheroidal symmetry, depth 42 Mev, and volume equal to that of a sphere of radius $1.45 \times 10^{-13} A^{1/3}$ cm. For small distortions, the resonance that, in the spherical case, occurs near mass number 150 at first shifts slowly in the direction of higher mass numbers and then splits. When the ratio of the interfocal distance to the major axis of the spheroid is 0.4, one finds resonances at mass numbers of 135 and 160. The effect of including absorption is estimated by assuming that each resonance has the same shape as it would for a spherical well. The net effect upon the cross section is to give a very broad peaking over the rare earth region, with a maximum near mass number 170.

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† Present address, University of California, Radiation Laboratory, Berkeley, California. The author desires to express his appreciation to the A. E. C. for supporting the terminal portion of this project.

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In its original form the optical model of the scattering of low-energy neutrons by nuclei[1] was based upon the assumption that the neutrons were scattered by a spherical square-well potential which contained a small ($\sim$ 3%) imaginary part. It was found that such a potential gave a reasonable account of the variation of the scattering cross sections with mass number over much of the periodic table. However, a rather marked disagreement between theory and experiment was found in the predicted location and shape of the 4S resonance. In particular, the curve of $\overline{\Gamma}_n/\overline{D}$ in the vicinity of this resonance was found to be too high and too narrow (see Fig. 7 of Reference 1). The agreement was only slightly improved by increasing the imaginary part of the scattering potential from 3% to 5% of the well depth.

Feshbach, Porter, and Weisskopf observed that the region of the periodic table in which the disagreement occurs ($155 \leq A \leq 200$) is just the region that is characterised by the presence of very distorted nuclei. They speculated that if the scattering potential of the optical model were allowed to take on a nonspherical shape in this region of elements the 4S resonance might take on a more desirable character.

This paper deals with calculations performed in order to estimate the effect of including variations of nuclear shape with mass number in the optical model. The calculations were carried out for zero-energy neutrons scattered by a real square well of spheroidal symmetry. The value of $\overline{\Gamma}_n/\overline{D}$ was then estimated by assuming that each resonance of the spheroidal potential had the same shape as a complex spherical square-well resonance.[2]

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[1] Feshbach, Porter, and Weisskopf, Phys. Rev. 96, 448 (1954).

[2] I am indebted to Professor F. Villars for suggdsting this approximation.

We assume that a scattered neutron sees a potential that is constant over a prolate spheroidal region of volume v and eccentricity ϵ, and vanishes everywhere else. For a nucleus of mass number A we take

$$v = \frac{4}{3}\pi R^3 = \frac{4\pi}{3}\left[1.45 \times 10^{-13}\right]^3 A \text{ cm.}$$

The depth of the potential is 42 Mev. If we define the eccentricity of the spheroid as the ratio of the focal distance to the length of the major axis we find that, up to terms in ϵ^4, the scattering cross section for a zero-energy neutron is just

$$\sigma_0 = 4\pi R^2 \left| 1 - \Gamma \right|^2 .$$

Here Γ is a complicated function which, in the limit of spherical symmetry, becomes tan KR /KR, where K is the wave number of the neutron when it is inside the nuclear boundary. Figure 1 shows the variation of with mass number, in the vicinity of the 4S resonance, for different values of the eccentricity.

One observes from Fig. 1 that the effect of increasing eccentricity is first to shift the resonance point in the direction of increasing mass numbers and then to split the single resonance into a pair. Calculations for distortions larger than those shown seem to indicate that additional splittings will occur at the larger distortions.

It should be clear that the splitting of resonances does provide a possible mechanism for bringing the optical model into closer accord with experiment. This would come about because the inclusion of an imaginary term in the scattering potential would be expected to "smear" the two peaks of the companion pair of a split resonance into each other, thus providing for a very broad peaking over the entire resonance region.

One can put this last statement into a more quantitative form by assuming that each resonance of the distorted potential has the same shape as a spherical square-well resonance at the same mass number. In detail, we suppose that the cross section in the vicinity of a split resonance may be written

$$\sigma = 4\pi R^2 \left| 1 - \tan X\delta_1/X\delta_1 - \tan X\delta_2/X\delta_2 \right|^2 ,$$

$$X^2 \equiv (2m/\hbar^2)V_0R^2(1 + i\,\zeta).$$

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[3] J.L. Uretsky, Ph.D. Thesis, Mass. Inst. of Tech., 1956 (unpublished). It is shown here that $\epsilon = 0.55$ would be a better choice; however, the calculation of Γ was not considered reliable for $\epsilon > 0.4$. It is considered that the approximations used here suffice to give a qualitative indication of the effect of distortion.

In the last expression V_0 is the depth of the potential well, which is taken to be 42 Mev. The constants δ_1 and δ_2 are chosen to make the tangent functions infinite (in the limit of vanishing ζ) at the resonance points of the spheroidal well. If we assumed that an eccentricity of 0.4 is a reasonable average in the rare earth region,[3] we would choose the δ's to match the top curve of Fig. 1. When this has been done, it is a simple matter to calculate $\overline{\Gamma}_n/D$ in order to obtain the curves of Figs. 2 and 3.

As the figures indicate, the calculation was carried out for two different values of ζ, and the results are compared with the corresponding results for the spherical square well. In each case a "transition line" has been sketched to remind us that we must take into account a smooth variation from spherical to spheroidal shape in the vicinity of mass number 150.

In Fig. 2 we note that for a ζ of 0.03 the "smearing" effect from the resonance at mass number 135 is negligible. It follows that the main effect of the distortion is to provide a slight shift in the position of the resonance. It seems clear that there is little improvement in the theory for this case.

The situation is quite different when ζ is increased to 0.06. There is now a marked "smearing" effect, which is apparent in the asymmetry of the resonance at mass number 165 (solid curve of Fig. 3). The inclusion of a "transition line" joining the curves corresponding to distorted and undistorted scatterers leads one to the prediction of a double peak in the resultant curve of $\overline{\Gamma}_n/D$. It appears that this prediction is qualitatively fulfilled by the experimental data.[4] One might remark that a detailed quantitative agreement should not be expected, both because of the nature of the model and because of the approximations that have been incorporated into the calculation.

It seems reasonable to conclude that low-energy neutron scattering from the rare earth elements is describable in terms of an optical model, provided that distortions in nuclear shape are taken into account. The parameters describing the scattering potential are just those which give a good fit at lower mass numbers, except that the imaginary part of the potential should be increased by a factor of approximately two.

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[4] The experimental data are taken from Carter, Harvey, Hughes, and Pilcher, Phys. Rev. 96, 114 (1954) (circles); and Schwartz, Pilcher, Hughes and Zimmerman, Bull. Amer. Phys. Soc., Series II, 1, 347 (1956) (triangles). I am indebted to Dr. Kirk McVoy for bringing the latter data to my attention.

I wish to thank Professor Victor Weisskopf, who suggested the problem, Professors Phillip M. Morse, Herman Feshbach, Felix Villars, and David Peaslee, and Dr. Warren Heckrotte for interesting discussions. The computational work could not have been accomplished without the invaluable assistance of Dr. F. J. Corbato and the staff of the MIT Whirlwind computer.

APPENDIX

We use the spheroidal coordinates ξ, η, ϕ and eigenfunctions $S_{m\ell}$, $je_{m\ell}$, $he_{m\ell}$ as defined by Morse and H. Feshbach.[5,2] The boundary of the scatterer is at ξ_0, and the interfocal distance is 2d. The dimensionless wave-number parameters are defined by

$$H^2 = K^2 d^2 , \qquad \xi < \xi_0 .$$

$$h^2 = k^2 d^2 \to 0, \qquad \xi > \xi_0 .$$

Where k and K are, respectively, the wave numbers inside and outside the scatterer.

The wave function outside the scatterer is expanded in spheroidal harmonics and, in the limit of zero energy, is

$$\Psi_0 = \sum_{\ell=0}' P_\ell(\eta) \left\{ a_\ell he_{0\ell}(0, \xi) + \delta_{0\ell} \right\},$$

while inside the spheroid we have

$$\Psi_i = \sum_{\lambda=0}^{\infty} A_\lambda S_{0\lambda}(H, \eta)\, je_{0\lambda}(H, \xi) .$$

Application of the usual continuity conditions leads to the infinite set of equations

$$\sum_{\ell=0}' A_\ell d_\lambda (H \,|\, 0\, \ell) \left\{ je_{0\ell}(H, \xi_0) - \left[Q_\lambda(\xi_0) \Big/ \frac{\partial}{\partial \xi_0} Q_\lambda(\xi_0) \right] \frac{\partial}{\partial \xi_0} je_{0\ell}(h, \xi_0) \right\} = \delta_{0\ell} . \qquad (1)$$

The important step in the derivation of the last equation is the observation that we have

$$\lim_{h \to 0} he_{0\ell}(h, \xi) \sim h^{-\ell-1} Q_\ell(\xi) ,$$

where $Q_\ell(\xi)$ is the irregular Legendre function. The primed summation symbol in Eqs. (1) indicates that only even values of ℓ are to be included.

By considering the asymptotic form of the spheroidal functions[5] one can easily show that the scattering cross section is, in the zero-energy limit, just

$$\sigma_0 = 4\pi d^2 \quad Q_0(\xi_0) \quad ^2 \quad 1 - \quad ^2$$

$$\approx 4\pi R^2 \quad 1 - \quad ^2 + O(\epsilon^4) ,$$

where we define

$$= \quad _{\ell=0} \; A_\ell d_0(H \quad 0\ell) \; je_{0\ell} \, (H, \xi_0) .$$

It follows that the cross section is determined once Eqs. (1) have been solved for the A_λ .

The solutions were approximated by considering the finite set of equations obtained from Eqs. (1) by restricting ℓ , λ to the range

$$0 \leq \ell , \lambda \leq 20.$$

The MIT Whirlwind computer was used to generate the matrix elements, solve the equations, and compute the value of Γ. It was found that this range of ℓ , λ was sufficient for good convergence for values of ϵ less than about 0.4.

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[5] P.M. Morse and H. Feshbach, Methods of Theoretical Physics (McGraw-Hill, New York, 1953), p. 1576.

FIGURE CAPTIONS

Fig. 1. The resonance parameter Γ vs nuclear mass number for various prolate distortions. The vertical line shows the position of the 4S resonance of a real square well.

Fig. 2. The reduced width at 1 ev versus nuclear mass number for a spherical and a prolate scatterer, each with 3% imaginary part, V_0 = - 42 Mev, r_0 = 1.45. The transition line is included as a rough guess at the nature of the curve if distortion were allowed to vary smoothly with mass number.

Fig. 3. Same as Fig. 2 except that the potential is given 6% imaginary part.

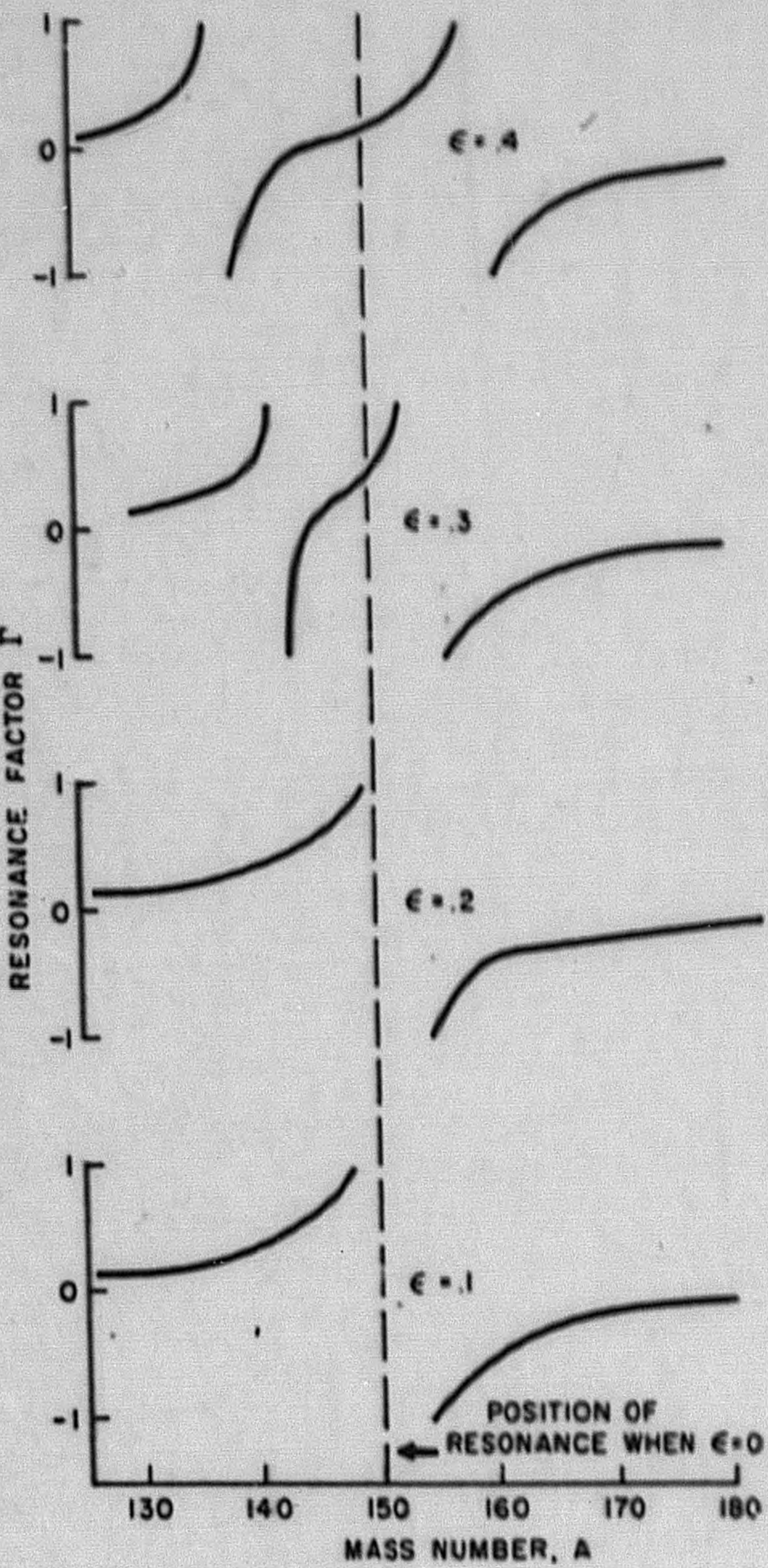

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FIG 2

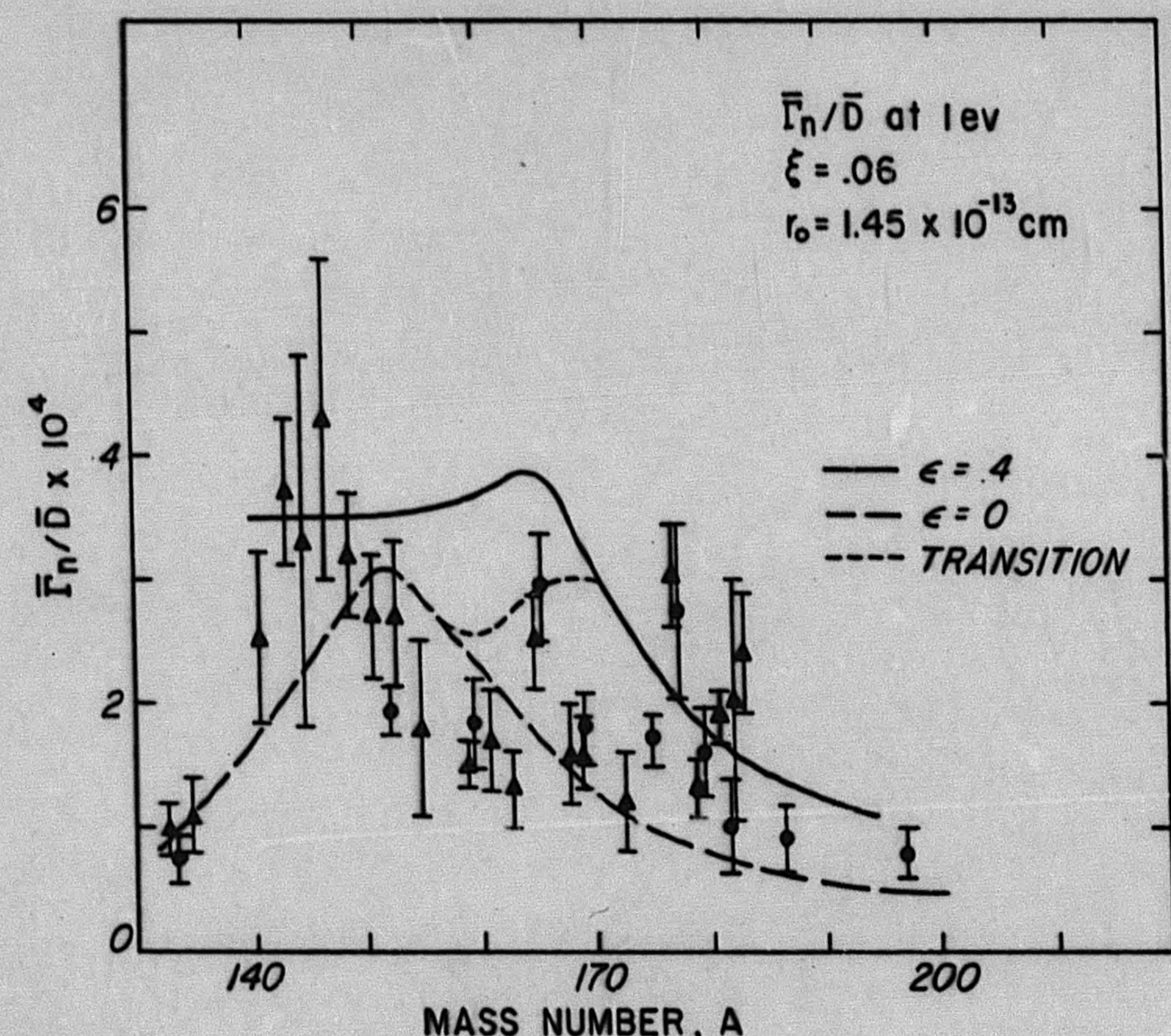

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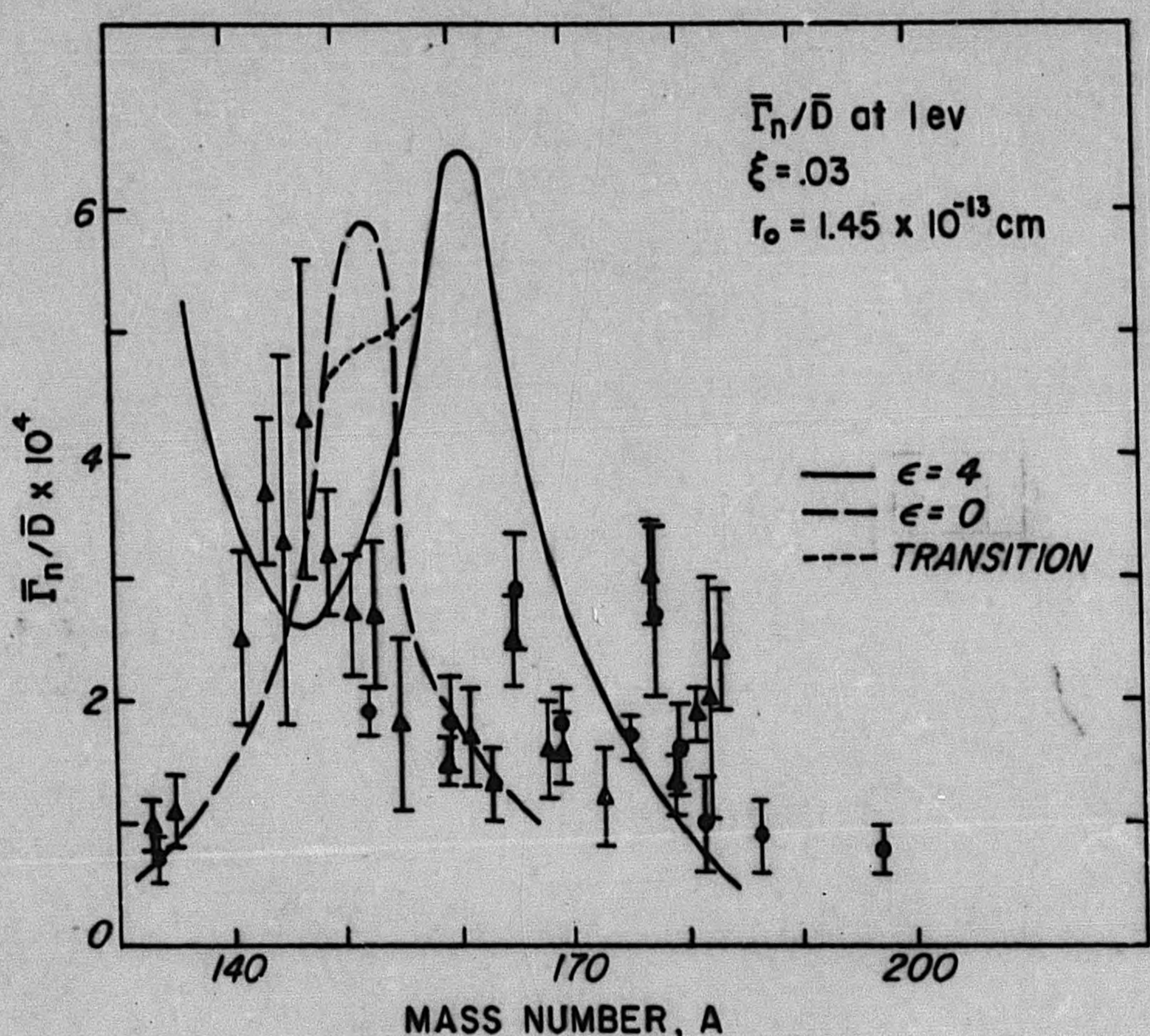
FIG 3



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