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ABSTRACT

This paper introduces new octant-range, composite-type Gauss and mid-point rule angular quadrature formulas for neutron and photon transport computations. The octant-range quadratures are the three-dimensional (3-D) natural analogue of the one-dimensional (1-D) double-Gauss quadrature most suitable for slab problems with a high degree of heterogeneity. The octant-range quadratures are applicable to a wide range of problems, and are especially suitable for problems with material discontinuities at interfaces and corners where it is advantageous to split the unit sphere of possible directions of particle motion into eight distinct octants. The composite-type Gauss and mid-point rule angular quadratures are designed for shielding problems dominated by particle streaming gaps, as well as certain shielding problems dominated by voids, such as dry cell storage problems.

The octant-range quadratures are generalizations to quadruple-range quadrature formulas for two-dimensional (2-D) rectangular x-y geometries introduced by the present author in the 1970s. A mathematical restriction on the previous quadruple-range quadratures is also relaxed in the present paper. As a result, improved accuracy is achieved with fewer angular directions, which leads to significant savings in computing cost. The octant-range quadratures presented include new quadruple-range quadratures for cylindrical geometries.

A generalization to octant-range quadratures is also introduced in order to allow for discontinuities at material interfaces for two- and three-dimensional transport problems which can be modeled with 60-degree triangular or hexagonal mesh subdivisions in the x-y plane.

1. Introduction

The discrete ordinates method, which is the basis for major production transport programs, approximates integrals of the scattering term of the particle transport equation over all (infinitely many) possible directions of particle motion, by an angular quadrature summation over a finite set of discrete angular directions. As a consequence, the number of angular directions chosen, as well as their distribution over the unit sphere, governs overall accuracy. In addition, for localized source problems and for problems with voids and gaps, artificial flux fluctuations from ray effects often arise, especially in situations with low in-group scattering to total cross section ratios. Partial remedies for ray effects are available [7]. One way to ameliorate ray effect fluctuations is to increase the number of angular directions and to tailor the choice of angular quadratures to the class of problems being addressed [1, 6]. The quadruple-range angular quadratures developed in [1] were demonstrated to be more effective in ameliorating ray effect fluctuations for rectangular x-y problems than standard [8] S_N type quadratures having the same number of ordinates. Experience has shown that tailoring the angular quadrature directions to the class of problems being solved can considerably reduce the artificial flux fluctuations as well as increase accuracy.

For convenience, the angular direction of particle motion Ω will be represented in terms of (a) direction cosines $\Omega = (\Omega_x, \Omega_y, \Omega_z)$ with respect to an x-y-z rectangular coordinate system, or (b) the polar angle θ as measured with respect to the z-axis and the azimuthal angle ϕ as measured with respect to the x-axis, $\Omega = (\theta, \phi)$, $\Omega = (\mu, \phi)$, $\mu = \cos \theta$. The direction cosines are given in terms of the polar and azimuthal angles by the expressions

$$\Omega_x = \sin \theta \cos \phi, \quad \Omega_y = \sin \theta \sin \phi, \quad \Omega_z = \cos \theta. \quad (1)$$

Ninety-degree rotations, including measuring the polar angle with respect to the x-axis or y-axis where appropriate, may result in new quadrature formulas. Quadratures based on allowing for 90-degree rotations will not be discussed further and will be viewed as implicit.

Section 2 of this paper presents general expressions for representing the angular dependence of the neutron or photon flux. Sections 3, 4 and part of Section 6 below develop quadratures based on approximating the angular flux and the flux moments of the transport equation by polynomials in the direction cosines $\Omega_x, \Omega_y, \Omega_z$. Section 5 and part of Section 6

apply composite type quadratures to integrals over $\mu = \cos \theta$ and over ϕ . For brevity, all of the quadratures discussed below will be developed for the principal octant of the unit sphere

$$\Omega_x \geq 0, \quad \Omega_y \geq 0, \quad \Omega_z \geq 0, \quad (2)$$

or for part of this octant for problems with sixty-degree symmetry. The angular quadratures developed below can be extended to the entire unit sphere by rotations and reflections.

Note that for convenience for problems with 2-D symmetries, quadratures developed for the principal octant of the unit sphere in Eq. (2) will be referred to interchangeably as octant-range (OR), or quadruple-range (QR) quadratures.

Angular quadratures which do not explicitly allow for discontinuities at material interfaces are known to include the familiar Chebyshev-Gauss quadrature for integrating over the azimuthal direction ϕ (uniform distribution in ϕ) and single-Gauss (Gauss-Legendre) quadrature for integrating over the cosine of the polar direction θ ($\mu = \cos \theta$). Highly accurate tables for these quadratures are available [10]. These quadratures were referred to in [1] as single-range quadratures. The work in [1] also developed double-range quadratures which split the unit sphere into two hemispheres. A well known double-range quadrature is the double-Gauss quadrature for the polar directions. Single-range and double-range quadratures [1] have useful applications especially for certain deep penetration problems. Further development of variations to the single-range and double-range quadratures for different geometries will not be pursued in the present paper.

2. Angular Dependence

The angular flux for the neutron and the photon particle transport equation is a function of three spatial variables, two angular directions, and energy

$$\psi(\underline{x}, \Omega, E) \equiv \psi(\Omega) \equiv \psi(\mu, \phi) \equiv \psi(\theta, \phi) \equiv \psi(\Omega_x, \Omega_y, \Omega_z). \quad (3)$$

The spatial and energy variables are suppressed in parts of Eq. (3) and will be suppressed below for brevity. Note that the direction cosines in Eq. (3) satisfy the trigonometric relationship

$$\Omega_x^2 + \Omega_y^2 + \Omega_z^2 = 1, \quad (4)$$

and thus the three angular direction cosines in Eq. (3) are coupled.

The treatment in [1] established that the vector flux representation for particle transport in x-y geometry has the functional relationship

$$\psi(\Omega) \equiv \psi(\Omega_x, \Omega_y), \quad (5)$$

and does not depend explicitly on Ω_z . Equation (5) follows from: (i) the fact that two-dimensional symmetry with respect to the x-y plane implies that the angular flux (and all flux moments) must be an even function in Ω_z (i.e., must be a function of Ω_z^2), and (ii) dependence of such functions on Ω_z^2 can be eliminated based on Eq. (4). Likewise, it can be shown that for r-z problems with cylindrical symmetry

$$\psi(\Omega) \equiv \psi(\Omega_r, \Omega_z), \quad (6)$$

where Ω_r is the direction cosine measured with respect to the r-axis. Explicit dependence on the three direction cosines in three-dimensional geometry is still as expressed in Eq. (3)

$$\psi(\Omega) \equiv \psi(\Omega_x, \Omega_y, \Omega_z). \quad (7)$$

The functional dependence on direction cosines as expressed in Eqs. (5)-(7) also holds for flux moments. Thus, as is established in [1], transport problems in x-y geometry involve evaluating integrals of the general form

$$\int_{\{\Omega\}} f(\Omega_x, \Omega_y) d\Omega, \quad (8)$$

where the range of integration $\{\Omega\}$ stands for the unit sphere of angular directions. The previous work derived angular quadrature formulas based on the reasonable assumption that $f(\Omega_x, \Omega_y)$ in Eq. (8) can be well approximated by polynomials or by piece-wise polynomials in Ω_x and Ω_y . Quadrature formulas in [1] are given in the form

$$\int_{\{\Omega\}} f(\Omega_x, \Omega_y) d\Omega \approx \sum_{m=1}^N w_m f(\Omega_x^m, \Omega_y^m). \quad (9)$$

Formulas for the angular ordinates Ω_x^m, Ω_y^m and corresponding quadrature weights w_m in Eq. (9) were derived in [1] such that, for each value of N, the approximation in Eq. (9) is rendered exact for the integral in Eq. (8) for polynomials or piecewise polynomials of as high a degree in Ω_x, Ω_y as possible. Quadruple-range (QR) quadratures suitable for rectangular x-y geometry were derived in [1] by restricting the range of the integrals in Eq. (9) to one quadrant of the unit sphere.

Of particular interest here are quadrature formulas applicable to each octant of the unit sphere of the form

$$\int_{\{\Omega\}/8} f(\Omega) d\Omega \approx \sum_{m=1}^M w_m f(\Omega^m) = \sum_{m=1}^M w_m f(\Omega_x^m, \Omega_y^m, \Omega_z^m), \quad (10)$$

where $\{\Omega\}/8$ stands for an octant of the unit sphere. The quadrature formulas based on Eq. (10) will be referred to as octant-range (OR) quadratures. Note that for rectangular geometry with symmetry with respect to the x-y plane, the quadruple-range quadratures developed in [1] can also be viewed as octant-range quadratures.

For convenience and to allow for reflecting boundary conditions for rectangular subdivisions in x-y geometry, it is customary to choose principal directions $\Omega_x^m > 0$, $\Omega_y^m > 0$ for $m = 1, \dots, M$, and to construct a total of $N=4M$ angular directions of the form

$$\begin{aligned} \Omega^m &= (\Omega_x^m, \Omega_y^m), & \Omega^{m+M} &= (-\Omega_x^m, \Omega_y^m), \\ \Omega^{m+2M} &= (-\Omega_x^m, -\Omega_y^m), & \Omega^{m+3M} &= (\Omega_x^m, -\Omega_y^m), \\ \Omega_x^m &> 0, & \Omega_y^m &> 0, \quad m = 1, \dots, M. \end{aligned} \quad (11a)$$

The associated quadrature weights are also required to satisfy

$$w_m = w_{m+M} = w_{m+2M} = w_{m+3M}. \quad (11b)$$

It is also helpful for x-y and x-y-z problems to reorder the angles in the principal octant such that Ω^m and Ω^{M+1-m} are symmetric pairs of the form

$$\Omega^m = (\Omega_x^m, \Omega_y^m), \quad \Omega^{M+1-m} = (\Omega_y^m, \Omega_x^m), \quad m = 1, \dots, M. \quad (11c)$$

Equation (11c) is particularly useful for problems with 90-degree rotational boundary conditions.

The octant-range quadratures for r-z and x-y-z problems can be extended to the unit sphere in a similar manner. Details will be omitted for brevity.

3. Polynomials in Direction Cosines

The main assumption in this section is that the angular integrands of functions of the form $f(\Omega)$ such as given in Eqs. (5)-(10), which arise in the discrete ordinates transport computations, can be well approximated by polynomials in the direction cosines. This assumption leads to "series of integrals" of the following form over the principal octant of the unit sphere:

(a) **For Rectangular x-y Geometry**: The integrals corresponding to Eqs. (8) and (9) include the following typical terms in the expansion

$$\begin{aligned} \int_{\Omega/8} (\Omega_x)^l (\Omega_y)^m d\Omega &= \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \sin\theta (\sin\theta \cos\phi)^l (\sin\theta \sin\phi)^m \\ &= \left\{ \int_0^{\pi/2} d\phi \cos^l\phi \sin^m\phi \right\} \times \left\{ \int_0^{\pi/2} d\theta \sin\theta \sin^{l+m}\theta \right\}, \\ &\quad \ell + m \leq L, \end{aligned} \quad (12)$$

where L represents the order of the polynomial approximation of the angular flux.

(b) **For Cylindrical r-z Geometry**: The integrals corresponding to Eqs. (6) and (10) include the following typical terms in the expansion

$$\begin{aligned} \int_{\Omega/8} (\Omega_r)^l (\Omega_z)^m d\Omega &= \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \sin\theta (\sin\theta \cos\phi)^l (\cos\theta)^m \\ &= \left\{ \int_0^{\pi/2} d\phi \cos^l\phi \right\} \times \left\{ \int_0^{\pi/2} d\theta \sin\theta \sin^l\theta \cos^m\theta \right\}, \\ &\quad \ell + m \leq L. \end{aligned} \quad (13)$$

(c) **For Rectangular x-y-z Geometry**: The integrals corresponding to Eqs. (7) and (10) include the following typical terms in the expansion

$$\begin{aligned} \int_{\Omega/8} (\Omega_x)^l (\Omega_y)^m (\Omega_z)^n d\Omega &= \int_0^{\pi/2} d\phi \int_0^{\pi/2} d\theta \sin\theta (\sin\theta \cos\phi)^l (\sin\theta \sin\phi)^m (\cos\theta)^n \\ &= \left\{ \int_0^{\pi/2} d\phi \cos^l\phi \sin^m\phi \right\} \times \left\{ \int_0^{\pi/2} d\theta \sin\theta \sin^{l+m}\theta \cos^n\theta \right\}, \\ &\quad \ell + m + n \leq L. \end{aligned} \quad (14)$$

Note that each of the integrals under consideration in Eqs. (12)-(14) is split into a product of two integrals, one over the azimuthal direction and one over the polar direction. In particular, the product integrals in Eqs. (12)-(14) involve four distinct functional forms:

- (i) Integrals over the azimuthal directions in rectangular 2-D and 3-D geometries given by the identity

$$\int_0^{\pi/2} d\phi \cos^l \phi \sin^m \phi = \Gamma\left(\frac{l+1}{2}\right) \Gamma\left(\frac{m+1}{2}\right) / 2\Gamma\left(\frac{l+m+2}{2}\right), \quad (15)$$

where $\Gamma(\nu)$ is the well known Gamma-function,

- (ii) Integrals over the polar angles in 2-D r-z cylindrical and 3-D x-y-z rectangular geometries given by

$$\int_0^{\pi/2} d\theta \sin \theta \sin^l \theta \cos^m \theta = \Gamma\left(\frac{l+2}{2}\right) \Gamma\left(\frac{m+1}{2}\right) / 2\Gamma\left(\frac{l+m+3}{2}\right), \quad (16)$$

- (iii) Integrals over the polar angles in 2-D x-y rectangular geometries given by

$$\int_0^{\pi/2} d\theta \sin \theta \sin^j \theta = \int_0^1 x^j \frac{x dx}{(1-x^2)^{1/2}} = \sqrt{\pi} \Gamma\left(\frac{j+2}{2}\right) / 2\Gamma\left(\frac{j+3}{2}\right), \quad (17)$$

and

- (iv) Integrals over the azimuthal angles in 2-D r-z cylindrical geometries given by

$$\int_0^{\pi/2} d\phi \cos^j \phi = \int_0^1 x^j \frac{dx}{(1-x^2)^{1/2}} = \sqrt{\pi} \Gamma\left(\frac{j+1}{2}\right) / 2\Gamma\left(\frac{j+2}{2}\right). \quad (18)$$

Equations (15)-(18) form the basis for octant-range quadratures.

4. Gauss Type Quadratures for Rectangular x-y, r-z and x-y-z Geometries

Separate quadratures will be presented here for integrals over the azimuthal and polar directions involved in transport approximations. Three such quadratures matching Eq. (10) are

for integrals along the azimuthal direction corresponding in part to Eqs. (15) and (18), and two are for integrals along the polar directions corresponding to Eqs. (16) and (17).

(a) The octant-range quadrature sought for the azimuthal direction is of the form

$$\int_0^{\pi/2} f(\cos\phi, \sin\phi) d\phi \approx \sum_{i=1}^M w_i f(\cos\phi_i, \sin\phi_i). \quad (19)$$

Three sets of angular directions ϕ_i and weights w_i were determined in [1] based on the requirement that the approximation in Eq. (19) be exact for integrating polynomials in Eq. (15) of as high a degree in ℓ and m as possible. To maintain numerical accuracy, the published work in [1] was restricted to $N \leq 7$. This quadrature was referred to as quadruple-range quadrature since it corresponds to subdividing the range of $\phi \in [0, 2\pi]$ into four equal parts. Of particular interest here is the quadruple-range quadrature for the azimuthal direction $\phi \in [0, \pi/2]$ which is symmetric with respect to $\phi = \pi/4$, the 45-degree angle, and thus suitable for implementing 90 degree rotational boundary conditions. This quadrature satisfies the following requirements

$$\phi_{M+1-i} \equiv \frac{\pi}{2} - \phi_i, \quad \phi_i \leq \pi/4, \quad i \leq (M+1)/2, \quad (20a)$$

$$w_{M+1-i} \equiv w_i, \quad i \leq (M+1)/2. \quad (20b)$$

The advantage of symmetry in Eqs. (20a) and (20b) is that it results in a factor of 2 reduction in the number of unknown coordinates and weights ϕ_i, w_i that need to be computed for Eq. (19) for each fixed N . Table I extends the results in [1, p. 308, Table III] for this quadrature from $N \leq 7$ to $N = 22$. Details of the derivation follow below.

It can be shown based on trigonometric identities that for the symmetric case corresponding to Eqs. (20a, b), the integrands in Eq. (15) are not all independent and that it suffices to consider only the following integrands

$$\sin^{4k}\phi, \quad \sin^{2k+1}\phi, \quad \sin^{4k+1}\phi \cos\phi, \quad k = 0, 1, \dots \quad (21)$$

Substituting the terms from Eq. (21) into Eq. (15) and using identities of the Gamma-functions indicate that the moment equations of interest are of the following forms

$$\int_0^{\pi/2} \sin^{4k+1} \phi \cos \phi d\phi = \frac{1}{4k+2}, \quad (22a)$$

$$\int_0^{\pi/2} \sin^{2k+1} \phi d\phi = \frac{2^k k!}{1.3.5...(2k+1)}, \quad (22b)$$

$$\int_0^{\pi/2} \sin^{4k} \phi d\phi = \frac{1.3.5...(4k-1)}{2^{2k+1} (2k)!}. \quad (22c)$$

Equations (22a-c) are used in place of Eq. (15) to develop the desired quadratures. In general, the resulting quadrature for a given N will be of order

$$L = N - 1, \quad (\ell + m \leq L). \quad (23)$$

For convenience, the coordinates

$$x_i \equiv \sin \phi_i, \quad (1 - x_i^2)^{1/2} = \cos \phi_i, \quad (24)$$

will be used in place of ϕ_i . Now, applying the quadrature formulas of Eq. (19) to the functions in Eqs. (21) and (22) and using symmetry yields a set of "nonlinear" equations in the unknowns. For small values of N, this set of nonlinear equations can be solved efficiently by Newton iterations. This solution method, though simple, is known to be subject to round-off errors and numerical instabilities, and its computation requires a high degree of precision arithmetic [2]. In order to avoid numerical difficulties, the work in [1] was limited to $N \leq 7$. In the present paper, the order of quadratures for azimuthal directions is increased to $N \leq 22$, in an attempt to improve accuracy of transport computations for a large class of problems.

The main reason for the numerical instability is the fact that the terms involved in the system of equations combine various powers of x_i with some ordinates x_i close to zero and others close to 1. Accuracy in this case requires increased precision arithmetic with increased orders of the approximations. A new computer program was written by the author to generate quadratures of degrees up to $N = 22$, utilizing double precision arithmetic on Cray computers. The program monitors the loss of significant figures. Computations with more than double precision are needed for quadratures of order higher than 22. Better methods are being

investigated by Walter Gautschi at Purdue University and by others and are needed in order to increase accuracy.

(b) **For sixty-degree symmetry in x-y and x-y-z geometries**, the integral over the azimuthal angles in Eq. (19) is replaced with

$$\int_0^{\pi/3} f(\cos \phi, \sin \phi) d\phi \approx \sum_{i=1}^N w_i f(\cos \phi_i, \sin \phi_i). \quad (25)$$

The coordinates and weights sought here are such that the restriction of the function in Eq. (25) to polynomials in cosine and sine of the azimuthal angles leads to approximations

$$\int_0^{\pi/3} d\phi \cos^l \phi \sin^m \phi \approx \sum_{i=1}^N w_i \cos^l \phi_i \sin^m \phi_i, \quad (26)$$

which are exact for polynomials in the cosine and sine functions of as high a degree as possible.

Note that the polynomials in Eq. (26) are not all independent, and are related by the trigonometric identity

$$\sin^2 \phi + \cos^2 \phi = 1. \quad (27)$$

It can be shown that it is sufficient to consider requiring the quadrature approximation in Eq. (26) to be exact for one of the following sets of independent integrands

$$\sin^j \phi, \quad j = 0, \dots, N-1, \quad (28)$$

$$(\sin^j \phi, \sin^j \phi \cos \phi, \quad j = 0, \dots, N-1), \quad (29)$$

and

$$(1, \sin^j \phi), \quad (\sin^{j-2} \phi \cos \phi, \sin^j \phi, \quad j = 2, \dots, N). \quad (30)$$

The set in Eq. (28) is valid only if we restrict the corresponding coordinates and weights in Eq. (26) to be symmetric with respect to $\phi = \pi/6$, the 30-degree angle. Such a restriction is desirable in order to allow for reflecting as well as for 60-degree rotational boundary conditions. Sets of independent polynomials in the sine and cosine functions other than in Eq. (28)-(30) also exist.

Now, applying the quadrature formula Eq. (26) to one of the sets of integrands in Eqs. (28)-(30), and requiring the results to coincide with the exact analytic values of the integrals on the left hand side of Eq. (26) yield a system of nonlinear equations in the unknown coordinates and weights (ϕ_i, w_i) . It is more convenient to use the coordinate

$$x = \sin \phi, \quad \sqrt{1-x^2} = \cos \phi, \quad (31)$$

in place of ϕ . The system of nonlinear equations can be solved efficiently by Newton iterations. The computed coordinates and weights corresponding to Eqs. (26) and (28), which are symmetric with respect to 30-degree angles, are given in Table II. Computed coordinates and weights corresponding to the independent integrands in Eqs. (29) and (30) are omitted for brevity.

(c) The octant-range quadrature sought for the polar direction in 2-D r-z cylindrical and 3-D x-y-z rectangular geometries is of the form

$$\int_0^{\pi/2} f(\sin \theta, \cos \theta) \sin \theta \, d\theta \approx \sum_{j=1}^N w_j^k f(\sin \theta_j, \cos \theta_j). \quad (32)$$

Equation (32) is similar to Eq. (19), except for the extra sine function in the Jacobian of the integral on the left-hand side. The task now is to find coordinates and weights which render the approximation in Eq. (32) exact for polynomials as in Eq. (16) of as high a degree as possible. Again, as in Eqs. (25)-(30), the integrals involved in Eq. (16) are not all independent for various values of ℓ and m . The two sets of independent integrands given in Eqs. (29) and (30) hold here also for f in Eq. (32). It is also convenient to transform the trigonometric functions and use

$$x = \cos \theta, \quad \sqrt{1-x^2} = \sin \theta, \quad (33)$$

in place of θ . Applying the quadrature formula in Eq. (32) to integrate the set of functions in Eq. (29) or in Eq. (30), and utilizing the analytic expressions in Eq. (16) yield, for each fixed value of N , a system of $2N$ nonlinear equations in $2N$ unknown ordinates and weights for the right-hand side of Eq. (32). Accurate coordinates and weights are given in Tables III and IV.

(d) The octant-range quadrature sought for the polar direction in 2-D x-y rectangular geometry is of the form

$$\int_0^{\pi/2} f(\sin \theta) \sin \theta d\theta = \int_0^1 f(x) \frac{x dx}{(1-x^2)^{1/2}} \approx \sum_{j=1}^N w_j^k f(\sin \theta_j) = \sum_{j=1}^N w_j^k f(x_j), \quad (34)$$

where

$$x \equiv \sin \theta. \quad (35)$$

The quadrature sought is then of the form

$$\int_0^1 f(x) \omega(x) dx \approx \sum_{i=1}^N w_i f(x_i), \quad \omega(x) = x/(1-x^2)^{1/2}, \quad x_i = \sin \theta_i. \quad (36)$$

The corresponding optimal Gauss-Christoffel coordinates and weight for Eq. (36) with $N \leq 7$ were constructed and are tabulated to high accuracy in [1] based on Ref. [2]. The main idea is to determine x_i and w_i ($w_i \neq \omega(x_i)$) such that the approximation in Eq. (36) is rendered exact for all polynomials

$$f(x) = x^j, \quad j = 0, \dots, 2N-1. \quad (37)$$

Table V provides an extension of the results [1, p. 307, Table II] to $N=10$.

(e) The octant-range quadrature sought for the azimuthal direction in 2-D r-z

cylindrical geometry is of the form

$$\int_0^{\pi/2} f(\cos \phi) d\phi = \int_0^1 f(x) \frac{dx}{(1-x^2)^{1/2}} \approx \sum_{j=1}^N w_j^k f(\cos \phi_j) = \sum_{j=1}^N w_j^k f(x_j), \quad (38)$$

where

$$x \equiv \cos \phi. \quad (39)$$

The quadrature sought is then of the form

$$\int_0^1 f(x) \omega(x) dx \approx \sum_{i=1}^N w_i f(x_i), \quad \omega(x) = 1/(1-x^2)^{1/2}, \quad x_i = \cos \phi_i. \quad (40)$$

The optimal Gauss-Christoffel coordinates and weight for Eq. (40) can be constructed based on the computer program in [2], and are provided in Table VI.

5. Mid-Point Rule and Gauss Composite-Type Angular Quadratures

The Newton-Raphson iterative solution method presented in [2] and used to generate the quadratures presented in the previous section, while very simple, requires high precision arithmetic. Consequently the order of the quadratures generated is limited for accuracy considerations. Certain problems with pronounced streaming paths and large voids require very high order standard and octant type quadratures. Considerable computational economy can be realized by tailoring the quadrature for such applications. Samples of specialized quadratures are presented below.

(a) Variable Mid-Point Rule Quadrature for the Azimuthal Direction

Application of the discrete ordinates method to the shield design for a dry cell storage transport problem dominated by regions with very large voids initially exhibited large ray effect artificial flux fluctuations which appeared to dominate around the x-axis and the y-axis. The quadruple-range quadratures discussed in the previous section improve the situation. However, quadruple-range quadratures of order much higher than $N = 22$ are needed to ensure accuracy and mitigate ray effects. Since such higher order quadratures are not available at the present time, the appropriateness of variable mid-point rule quadratures for the azimuthal direction was investigated.

A simple computer program was constructed to generate mid-point rule quadrature ordinates and weights over an arbitrary user-specified interval $[C0, C1]$ based on mesh subdivisions

$$\Delta X(\ell + k) = A_i * \Delta X(\ell + k - 1), \quad k = 1, \dots, N_i, \quad i=1, 2, 3, \quad (41)$$

where $\ell = 0$ and $\Delta X(-1) = \Delta X(1)/A_1$, for $i = 1$, and $\ell =$ number of preceding mesh subdivisions for $i > 1$. The user has the option of specifying up to three values, A_1 , A_2 , and A_3 , for the multiplicative factor A_i in Eq. (41) and the corresponding desired numbers (N_1 , N_2 , N_3) of intervals corresponding to each of the choices $A_i = A_1$, $A_i = A_2$, and $A_i = A_3$.

In the present application, ΔX refers to an azimuthal mesh subdivision $\Delta\phi$. The interval

$$[C0, C1] = [0, \pi/2], \quad (42)$$

is used. The following choices of numbers of mesh subdivisions and corresponding multiplicative factors to be applied in these mesh subdivisions were tested

$$(N1, A1, N2, A2, N3, A3) = (N1, A, N-2N1, 1.0, N1, 1.0/A), \quad (43)$$

with

$$N1 = 2 \text{ to } 8, \quad N = 30 \text{ to } 60, \quad A = 1.5 \text{ to } 2.0. \quad (44)$$

Specific choices of N , $N1$, and A in Eqs. (43) and (44) are quoted in later sections below. The main idea of this quadrature is to (a) crowd intervals close to the x-axis and close to the y-axis to correct for streaming along these directions, (b) allow for these intervals to increase smoothly in size away from the axes, and (c) have a uniform distribution of intervals around the 45-degree angle.

The quadrature for the interval $[0, \pi]$ is generated from that for $\phi \in [0, \pi/2]$ by reflections around the x-axis and y-axis.

Note in particular from the right-hand sides of Eqs. (42) and (43) that the quadrature generated here is symmetric with respect to $\phi = \pi/4$ and the plane $y = x$. Thus the quadrature is also symmetric in x and y and is invariant under 90 degree rotations around the z -axis. The mid-point rule program generates the desired quadrature for a specified set of intervals and multiplication factors $(N1, A1, N2, A2, N3, A3)$ by first constructing the last interval based on the formula

$$\begin{aligned} \Delta X_N = & (C1 - C0) / \{ (A1^{N1} - 1.0) / (A1 - 1.0) \\ & + A1^{N1-1} [(A2^{N2+1} - A2) / (A2 - 1.0) \\ & + A2^{N2} (A3^{N3+1} - A3) / (A3 - 1.0)] \}. \end{aligned} \quad (45)$$

Given ΔX_N , the mid-point rule program proceeds to generate all the azimuthal mesh intervals based on Eq. (41) and specified values of N and A given by Eq. (43).

(b) Variable Mid-Point Rule Quadrature for the Polar Direction

Note that quadratures for the polar directions are referred to as cone quadratures. Here, the mid-point rule quadrature is applied to integration of functions of cosine of the polar angle $\mu = \cos \theta$ over the unit interval

$$\mu \in [C0, C1] = [0, 1]. \quad (46)$$

The program discussed in the previous subsection also applies here with $\Delta X = \Delta\mu$, and makes it possible to divide the unit interval in Eq. (46) into three subregions by specifying

$$(N1, A1, N2, A2, N3, A3), \quad (47)$$

where $(N1, A1)$, $(N2, A2)$ and $(N3, A3)$ stand respectively for the desired number of angular subdivisions in each of the subregions and the corresponding multiplication factor relating them, as in Eq. (41). Note that $(N1, A1)$ stands for the interval close to $C1=1.0$, $(N2, A2)$ stands for the middle interval, and $(N3, A3)$ stands for the interval close to $C0=0$. The size of the smallest angular mesh is given by Eqs. (45) and (46). The program allows for collapsing the three subregions into two subregions by setting $N3=0$, or into a single subregion by setting $N2=N3=0$. For these cases, the minimum mesh is given from Eq. (45) respectively by

$$\Delta\mu_N = (C1 - C0) / \{ (A1^{N1} - 1.0) / (A1 - 1.0) + A1^{N1-1} (A2^{N2+1} - A2) / (A2 - 1.0) \}, \quad N = N1 + N2, \quad (48)$$

$$\Delta\mu_N = (C1 - C0) (A1 - 1.0) / (A1^{N1} - 1.0), \quad N = N1. \quad (49)$$

A sample of mid-point rule (MP) quadratures which proved to be useful for certain gamma ray penetration problems is given in Table VII. This MP quadrature corresponding to Eq. (47) is denoted by

$$\text{MP } N-i \quad (N1, A1, N2, A2, N3, A3), \quad (50)$$

where MP denotes mid-point rule, $N=N1+N2+N3$ denotes the total number of angular mesh cones, and $-i$ is a designation to distinguish quadratures with the same number of mid-point rule cones but differing distributions of cones.

(c) Composite-Type Angular Quadratures for the Polar Direction

This subsection summarizes details pertinent to the combinations of mid-point rule and Gauss quadratures which have been most successful for certain gamma ray penetration calculations.

The presence of discontinuities introduced by cylindrical penetrations passing through bulk shields must be taken into account. Standard angular quadratures when applied to gamma penetration problems tend to produce unacceptable results with ray effects which are not physically meaningful. Standard Single-Gauss quadratures with a very large number of angular directions mitigate these ray effects; however, use of such quadratures results in excessive computer time and memory requirements. Many specially tailored quadratures have been developed and tested. Several of these also mitigate ray effects while requiring significantly smaller numbers of angular directions. A few of these specially tailored quadratures have been applied by Kal Shure and the present author to the analyses of several gamma ray penetration experiments. The calculated values are in reasonable agreement with experiment. The quadratures introduced here are appropriate for integration along the polar directions in cylindrical and 3-D geometries.

The main difficulty with gaps and nozzles in shield design is that standard quadratures do not include enough angular directions for streaming along such penetrations. This appears to be particularly important and noticeable along directions parallel to the z-axis, or lying in the x-y plane. Single-Gauss quadrature for the polar directions is to be preferred to double-Gauss for penetrations parallel to the z-axis since, for the same number of ordinates, it has cones closer to the z-axis. Double-Gauss has more cones close to the x-y plane and is to be preferred to single-Gauss for problems where streaming along directions in the x-y plane are important.

Single-Gauss quadrature is known [5, 10] to be mathematically optimal for sufficiently smooth functions over the interval $[-1, 1]$. Composite-Gauss, where the interval under consideration is subdivided into two or more sub-intervals and where single-Gauss is mapped into each of these sub-intervals, can be expected to be optimal for piecewise continuous functions which are expected to be smooth functions only over the separate sub-intervals [10]. Double-Gauss is a special case of composite-Gauss where the functions being integrated are

expected to be sufficiently smooth over each sub-interval $[-1, 0]$ and $[0, 1]$, with a possible discontinuity in behavior around the origin.

The idea here is to try to improve on single-Gauss and double-Gauss by introducing composite type quadratures which, for the same number of angular directions, possess directions closer to the z-axis, closer to the x-y plane, or closer to both directions than is possible with single-Gauss, double-Gauss or composite-Gauss. Here, single-Gauss or double-Gauss is mapped into a section of the interval $[-1, 1]$ or interval $[0, 1]$, and new subdivisions of increasing or decreasing size are introduced on either side of this section. Three such quadratures, which proved to be beneficial are introduced below.

(i) MG Quadrature: This quadrature is a composite mid-point rule and Gauss quadrature over a user specified interval $(C0, C1)$, or over the default interval $(0, 1)$. The interval is subdivided into three sub-intervals $(C0, CM)$, (CM, CP) , $(CP, C1)$. The user specifies (a) mid-point rule subdivisions $(N1, A1, N2, A2)$ for the interval $(CP, C1)$; such subdivisions being governed by Eq. (48) [in place of Eqs. (45)-(47)], (b) a Gauss quadrature over the middle polar interval (CM, CP) with a total of $2NG$ Gauss-type cones, and (c) mid-point rule subdivisions in the last interval $(C0, CM)$ with $N4$ polar cones. The mid-point rule in the last interval $(C0, CM)$ closest to the x-y plane is governed by Eq. (49) with $N = N1 = N4$ and $A = A1 = A4$. Sample selections of MG quadratures are given in Table VII and are denoted by

$$MG \ N-i \quad (N1, A1, N2, A2 ; NG ; N4, A4 ; C0, CM, CP, C1), \quad (51)$$

where the various terms in Eq. (51) are as defined above.

(ii) MMG Quadrature: This is also a composite mid-point rule and Gauss quadrature over a user specified interval $(C0, C1)$, or over the default interval $(0, 1)$. It differs from MG in two respects: (a) the mid-point rule interval $(CP, C1)$ closest to the z-axis ($\mu = 1$) is subdivided into three parts as governed by Eqs. (45) and (47) rather than into only two parts as in Eq. (48), and (b) the interval $[C0, CM]$ is eliminated. Sample selections of MMG quadratures are given in Table VII and are denoted by

$$MMG \ N-i \quad (N1, A1, N2, A2, N3, A3 ; NG ; C0, CP, C1), \quad (52)$$

where the various terms in Eq. (52) are as defined above.

(iii) MHG Quadrature: This quadrature is a modification of the MG quadrature where the two intervals (C0, CM) and (CM, CP) adjacent to $\mu = 0$ are combined into a single interval and single-Gauss quadrature is mapped into the interval $[-CP, CP]$. This quadrature is referred to as composite mid-point half-Gauss, and is suitable when transport along the z-direction dominates and transport in the x-y plane is less important. Sample selections of MHG quadratures are given in Table VII and are denoted by

$$\text{MHG } N-i \quad (N1,A1, N2,A2; NG; CP) . \quad (53)$$

The expression in Eq. (53) implicitly assumes that $C0 = 0$ and $C1 = 1$.

Table VIII lists polar ordinates and corresponding weights for a subset of the quadratures in Table VII favored in the present study.

6. Combined Polar and Azimuthal Quadratures

The work in [1] developed compatible product angular quadratures that possess the same order of mathematical accuracy for the separate polar and azimuthal integrals. An example based on Eqs. (19) for the azimuthal direction, (34) for the polar direction and (12) for the product is given by

$$\begin{aligned} \int_{(\Omega)} f(\Omega_x, \Omega_y) d\Omega &= \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta f(\sin\theta \cos\phi, \sin\theta \sin\phi) \\ &\approx \sum_{i=1}^{4M} \sum_{j=1}^N w_i^l w_j^K f(\sin\theta_j \cos\phi_i, \sin\theta_j \sin\phi_i) . \end{aligned} \quad (54)$$

The distinct weights in Eq. (54) correspond to the distinct azimuthal and polar quadratures.

Compatible angular quadrature formulas derived in [1] were demonstrated to be superior to previous standard quadratures. The question is: can we do better? Compatibility requires the selection of identical numbers of azimuthal discrete directions on each of the polar cones; and this selection appears to be wasteful. Unfortunately for transport problems, the coupling of the spatial and angular discretizations and the effect of the numerous known singularities are not clearly understood. Thus, as was concluded in [1], the most suitable choices of angular quadrature for certain classes of transport problems will have to be based on actual numerical

experiments. Numerical experiments were conducted to evaluate the merits of differing quadratures and to seek improved selections for representative practical applications. It was found empirically that the use of product quadratures and the requirement of compatibility in [1] are too restrictive. The same or higher accuracy can be achieved with fewer, well positioned angular directions. Thus, the requirement of complete separability of azimuthal and polar directions [1] is relaxed.

The procedure adopted for constructing octant-range quadratures is as follows:

Algorithm 1.

- (i) Select a quadrature for the polar direction. The polar quadrature selected is viewed as defining distinct angular cones and associated weights. The polar weights should sum to one (1.0).
- (ii) Select possibly independent azimuthal quadratures for the distinct polar angular cones. While totally different azimuthal quadratures can be selected for the different polar cones, in practice it is convenient to select one type of quadrature, but vary the order (the number of discrete azimuthal angles or azimuthal intervals on the different polar cones).
- (iii) Associate with each resulting quadrature ordinate a weight equal to the product of the polar weight corresponding to the polar angle, and the azimuthal weight corresponding to the azimuthal direction and the azimuthal quadrature selected.

The concept embodied in Algorithm 1 is not new [4, 6, 8]. Mixtures of new and old well known polar and azimuthal quadratures were combined and tested.

It cannot be overemphasized that most of the new quadrature combinations applied below were arrived at by trial and error and by several numerical tests. Numerous other choices of angular mesh subdivisions, some based on what a priori seemed to be physically reasonable, resulted in exaggerated ray effects. The quadratures presented and applied in this study are not claimed to be optimal.

(a) Combined Polar and Azimuthal Quadratures for General x-y Problems

Experience with x-y transport problems suggests that these problems are more sensitive to the choice of azimuthal directions than polar directions. Since for two dimensional x-y geometries, particles moving close to the z-axis are not expected to contribute as much to transport as particles moving close to the x-y plane, it is advantageous to vary the number of azimuthal directions on separate polar cones and to select more azimuthal directions on cones closer to the x-y plane and less on cones closer to the z-axis. The advantage of varying the distribution of azimuthal angles on the various polar cones as was advocated by Carlson and others [4, 6, 8] is confirmed in the present study.

In the late 1960s, Gelbard and Shure [6] compared numerical results obtained with transport computations using standard S_{16} angular quadratures and a variety of other quadratures, against benchmark computations based on a very high order P_{19} spherical harmonics approximation. They concluded that the choice of 4 double-Gauss cones with 10 angles per octant on the top cone, 11 angles per octant on the second cone, 12 on the third cone, and 14 on the fourth and bottom cone, is appropriate for x-y problems. They intuitively preferred 4 double-Gauss cones to 4 single-Gauss cones, because the double-Gauss cones are closer to the x-y plane than the corresponding single-Gauss cones. Their four-cone quadrature was demonstrably superior to the standard eight cone S_{16} quadrature and was close enough to their benchmark to justify its adoption in shield design and analysis.

The present paper also confirms previous findings in [6] and demonstrates that it is sufficient to consider only 4 polar cone directions. However, improved selection of cones and angular distribution on different cones is provided below.

Four different choices of angular quadratures for the polar direction are considered:

- (i) SG - Single-Gauss quadrature in $\mu = \cos \theta$, θ the polar angle,
- (ii) DG - Double-Gauss quadrature in μ ,
- (iii) U - Uniform distribution of μ in the open interval (0,1), and
- (iv) QR - Quadruple-Range quadrature in the polar angle θ .

The first three are standard quadrature formulas. As far as the polar direction for rectangular x-y problems is concerned, in fact the choice $N=4$ is appropriate, and $N=10$ is viewed as providing a benchmark. Increasing N beyond 10 is not expected to change the results.

Three choices of azimuthal angular directions are considered in combination with the above choices for polar directions:

- (i) U - Uniform distribution of azimuthal angles on the open interval $\phi \in (0, \pi/2)$,
- (ii) QR - Quadruple-Range quadrature in the azimuthal angle ϕ , and
- (iii) MP - Variable mid-point rule quadrature in the azimuthal direction.

The azimuthal distribution however may be different on different polar cones.

The choice of quadrature U corresponding to uniform distribution of discrete directions in the azimuthal angle ϕ over the interval of the unit circle $\phi \in [0, 2\pi]$ is a Chebyshev-Gauss quadrature and is known to be mathematically optimal for functions which are continuous in ϕ over that interval.

The combined polar and azimuthal quadratures tested in the ray effect problem described below, and for other realistic problems are defined in Table IX. The order in which the non-negative integers i, j, k , and l appear in Table IX for the quadratures $sgijkl, \dots, qijkl$ is not important. It is to be understood that, if these integers are reordered in increasing magnitude, then the first and smallest integer would stand for the number of azimuthal angles/octant on the top cone (cone closest to the z-axis), the second integer would stand for the number of azimuthal angles/octant on the second cone from the z-axis, the third integer would stand for the number of azimuthal angles/octant on the third cone from the z-axis, and the fourth and largest integer would stand for the number of azimuthal angles/octant on the bottom cone which is closest to the x-y plane.

In addition to the angular quadratures listed in Table IX above, tests were carried out with analogous $uijkl$ angular quadratures based on polar cones uniformly distributed in $\mu = \cos \theta$, where θ is the polar angle measured from the z-axis. The uniform cone quadratures were experimentally found not to be competitive and are not considered further in this study.

(b) Combined Polar and Azimuthal Quadratures for x-y Problems with Large Voids

For certain problems, such as shielding problems for dry-cell storage, QR quadratures of very high order are needed, and are not currently available. To remedy the situation, quadratures for the azimuthal angle ϕ based on the mid-point rule are generated for variable interval subdivisions in $[0, \pi/2]$ and are designed to emphasize particle streaming along directions close to the x-axis and y-axis. Clearly the uniformly distributed Chebyshev-Gauss quadrature is a special case of a mid-point rule quadrature. The specialized quadratures described here (See Section 5a) lead to higher computational economy.

Initially the dg13,12,11,10 angular quadrature with uniform Chebyshev-Gauss distribution of azimuthal angles proposed in [6] was extended by doubling and tripling the number of angles per cone to get

$$dgijkl = dg26,24,22,20, \quad i+j+k+l = 92 \text{ angles/cone/octant}, \quad (55a)$$

$$dgijkl = dg39,36,33,30, \quad i+j+k+l = 138 \text{ angles/cone/octant}. \quad (55b)$$

In addition to the above, several new quadratures based on Eqs. (41) - (45) were developed. The new quadratures of interest here were initially based on 4 DG cones and respectively $N=30, 34, 38$, and 42 azimuthal angles on the first and top polar cone, and on the second, third, and fourth cones from the z-axis. These quadratures have a total of 144 angles/octant. The distribution of angles was based on the requirements in Eq. (43) that the first N_1 angular intervals increase successively by a factor $A_i = A$ as in Eq. (41), that the next $N-2N_1$ intervals be uniform, and that the last N_1 intervals decrease successively by a factor of A (increase by a factor of $1/A$). The following notation is applied here to label these quadratures

$$(DG4, N, \Delta N; N_1, A) = 4 \text{ DG cones with Variable Mid-PT Rule in } \phi, \quad (56)$$

where

- N = The number of azimuthal angles/octant on bottom cone closest to the x-y plane,
 ΔN = Decrease in the number of azimuthal angles/octant between successive polar cones from the bottom cone closest to the x-y plane to top cone closest to the z-axis,

- N1 = The number of the first set of azimuthal angular intervals per octant to be increased on each of the polar cones, and the number of the last set of azimuthal angular intervals per octant to be decreased on each of the polar cones,
- A = The factor by which the first N1 azimuthal intervals on each polar cone increase, and by which the last N1 azimuthal intervals decrease.

The quadratures of Eq. (30) tested in this study include

$$(DG4, N, \Delta N; N1, A) = (DG4, 42, 4; 6, 2.0), \quad (57)$$

$$(DG4, N, \Delta N; N1, A) = (DG4, 42, 4; 4, 1.5), \quad (58)$$

$$(DG4, N, \Delta N; N1, A) = (DG4, 42, 4; 6, 1.5), \quad (59)$$

$$(DG4, N, \Delta N; N1, A) = (DG4, 42, 4; 8, 1.5). \quad (60)$$

In addition to the 4 DG cone-based quadratures given above, the following 4 QR cones in μ and mid-point rule in ϕ quadrature was also tested

$$(QR4, N, \Delta N; N1, A) = (QR4, 42, 4; 6, 1.5). \quad (61)$$

Again, each of the quadratures in Eq. (57)-(61) has 144 angles/octant. The selection of 4 SG polar cones in place of the 4 DG or 4 QR polar cones is omitted for brevity.

(c) Combined Polar and Azimuthal Quadratures for Other Geometries

For simplicity, the transport program used for this study restricts the azimuthal quadrature for r-z geometry to a uniform distribution in ϕ , and this is applied in all of the r-z numerical experiments reported below. Only three azimuthal angular intervals per cone per octant were used for the numerical examples. Evidence and experience indicate that a higher number of azimuthal intervals, and new quadruple type quadratures shown in Table VI, would provide an improvement. Experiments with more azimuthal directions as applied to a number of practical applications were conducted, but are omitted here for brevity. Several angular quadrature formulas exist for the polar integration of the transport equation in r-z geometry, including the octant-range and composite Gauss and mid-point rule quadratures introduced in Sections 3-5 above. Several of these were tested and the results are reported below.

A combination of the octant-range or quadruple-range quadratures along the z-direction, and the Gauss-type quadratures introduced in Sections 4c and 4d above, would be suitable for 2-D and 3-D problems which can be described with 60-degree mesh subdivisions along the x-y plane. Quadratures based on either the mid-point rule or composite mid-point rule and Gauss would also be useful for problems with severe streaming paths due to gaps and voids.

Several useful combinations of known SG, DG, Uniform, and the new octant-range (OR) and composite mid-point rule and Gauss type quadratures for the azimuthal and polar directions appropriate for 3-D problems can be deduced from this study. However, examples, applications and further discussion of quadratures for 3-D and for problems in 60-degree geometry are beyond the scope of the present paper.

(d) Plots of Sample Angular Quadratures

Figure 1 shows for visual comparison the ordinates for:

- Figure 1a. Standard S_{16} quadrature,
- Figure 1b. Gelbard and Shure $dg_{10,11,12,13} = dg_{13,12,11,10}$ quadrature [See Table IX for notation],
- Figures 1c-d. dg_{461214} and sg_{461214} quadratures respectively with 4 DG and 4 SG polar cones, and respectively 4, 6, 12, and 14 uniformly distributed azimuthal angles/octant on top, second, third, and bottom cones (total 36 angles/octant each),
- Figure 1e. q_{461214} quadrature with 4 QR polar cones and with respectively 4, 6, 12, and 14 QR azimuthal angles/octant on top, second, third, and bottom cones (total 36 angles/octant),
- Figure 1f. MG26/v of Table VIII of mid-point Gauss 26 polar cones per octant quadrature. This quadrature has respectively 6 uniformly distributed azimuthal angles per cone per octant on each of the top sixteen cones (closest to the z-axis) and 7 uniformly distributed azimuthal angles per cone per octant on each of the bottom ten cones (closest to the x-y-plane). This is an example of a high order angular quadrature suitable for r-z problem. Note that cylindrical r-z problems require one in-coming and one out-going direction on each cone along the r-z plane [the r-z plane is shown in the figure as the x-z plane].

Figure 1g shows the subtle differences between the four cone double-Gauss dg461214, single-Gauss sg461214 and quadruple-range q461214 quadratures. The figure shows the location of the cones (projection on the z-axis) and the weights associated with each cone. What matters for x-y problems is not only the locations of the various cones, but also the weights associated with these cones. The QR quadrature has much less weight associated with the cone closest to the z-axis than for the SG quadrature; the SG quadrature in turn has less weight associated with the cone closest to the z-axis than for the DG quadrature. The QR quadrature has much more weight associated with the cone closest to the x-y-plane than for the SG quadrature; the SG quadrature in turn has more weight associated with the cone closest to the x-y-plane than for the DG quadrature. Since for x-y problems, transport along the z-direction is not as important as transport along x-y, Figure 1g suggests that for such problems, quadruple range distribution of polar cones is to be preferred to single-Gauss which in turn is to be preferred to double-Gauss. Our numerical tests indicate that this is indeed the case.

7. Numerical Comparison

As mentioned in [1], the most suitable choices of angular quadratures for certain classes of transport problems will have to be based on actual numerical experiments. This paper summarizes a detailed comparison of various quadratures for three problems:

- (i) A ray effect benchmark problem designed by Lathop [7] and extensively used to determine quadratures suitable for rectangular x-y transport applications [1],
- (ii) A model problem of a localized square source in a large void compartment designed to determine quadratures suitable for dry cell storage applications, and for problems dominated by transport parallel to the x- and y- axes in two dimensional rectangular geometry, and
- (iii) Quadratures for proper treatment of material discontinuities, gaps and nozzles in two dimensional cylindrical geometries.

The new quadratures developed for 60-degree geometry and for 3-D problems have not been extensively tested to date. No further comments on these quadratures will be provided in this paper.

(a) Numerical Results for Benchmark x-y Ray Effect Problem

The various angular quadratures introduced for x-y problems above were applied to a benchmark source problem designed by Lathrop [7] to study the "ray effect," and extensively utilized in [1] to compare different angular quadrature formulas. This problem is a one-energy-group neutron transport benchmark ray effect problem shown in Figure 2. The spatial domain is divided into 30x30 equally spaced mesh intervals $\Delta x = \Delta y = 2/30$. The diamond difference discrete-ordinates approximation was applied to solve this problem for various choices of angular quadratures and for two choices of scattering cross sections: $c = \sigma_s/\sigma_t = 1/3$, and $c = \sigma_s/\sigma_t = 1/10$, c being the number of secondaries per collision. The smaller the $c = \sigma_s/\sigma_t$ ratio, the more pronounced are the ray effect artificial flux fluctuations. It is hard to clearly distinguish between certain quadratures for the case $c = \sigma_s/\sigma_t = 1/3$, in contrast with the situation for $c = \sigma_s/\sigma_t = 1/10$. Consequently, all the numerical results for $c = \sigma_s/\sigma_t = 1/3$ are omitted for brevity.

The net neutron fluxes for the various choices of angular quadratures are compared at the centers of mesh cells and primarily at the centers of the right boundary mesh cells ($x = 1.967$) and/or at the centers of the top boundary mesh cells ($y = 1.967$). All problems were solved with a convergence criterion

$$\frac{1}{1-ER} \max_i \frac{|\Phi_i^n - \Phi_i^{n-1}|}{\Phi_i^n} \leq EPS = 10^{-3}, \quad (62)$$

where EPS is a user defined error measure (fixed in the present study to $EPS=10^{-3}$), Φ_i^n is the net particle density (scalar flux) evaluated at mesh cell i at iteration n and where ER is the error reduction achieved at iteration step n given by the ratio of the L-2 norms

$$ER = \|\Phi^n - \Phi^{n-1}\|_2 / \|\Phi^{n-1} - \Phi^{n-2}\|_2, \quad \|u\|_2 = \left(\sum_i u_i^2 \right)^{1/2}. \quad (63)$$

The factor $1/(1-ER)$ is used in Eq. (62) to avoid false termination of the iterative solution process for slowly converging problems when the error reduction ER between successive steps is close to 1 (The error reduction ER is a positive number [by definition] which provides an estimate of the rate of convergence).

A small sample of the flux results is shown in Figures 3-6, and in Table X. The findings are summarized below.

Figures 3 and 4 show representative net flux contour plots and traverse plots at the last row ($y = 1.967$) or last column ($x = 1.967$) of mesh cells. These figures indicate the following:

(i) Figures 3a-c indicate that the standard S_{16} and comparable quadratures with 36 angles/octant, such as the dg16 angular quadrature described in Table IX, lead to relatively large artificial flux fluctuations for the benchmark ray effect problem. Figure 3d shows that the fluctuations increase but their magnitudes decrease when the dg10111213 quadrature of Gelbard and Shure [6] with 46 angles/octant is used.

(ii) Figures 4a-d show that artificial flux fluctuations are substantially reduced with new quadratures such as s461214 with 36 angles/octant and s2468 with only 20 angles/octant.

The results in Figures 3 and 4 are qualitative and benchmarks are needed to assess accuracy. New quadratures dg4x20, sg4x20, d4x20, s4x20, q4x20 each having 4 cones and 20 angles/cone/octant for a total of 80 angles/octant, and s10x20, q10x20 each having 10 cones and 20 angles/cone/octant for a total of 200 angles/octant (see the definitions in Table IX), were developed and tested for their suitability as benchmarks. Single Gauss with 10 cones is essentially a P_{19} approximation along the polar z-direction, and this high order approximation is viewed to be a benchmark. In fact, the differences in answers start to get smaller and smaller for x-y problems as the number of cones around the z-axis is increased. The different quadratures having 10 cones and twenty angles per cone per octant produced essentially identical answers, as was expected. Omitted results indicate the following:

(iii) The artificial flux fluctuations that are evident in the flux plots in most of Figures 3 and 4 are virtually non-existent when the s10x20 or q10x20 quadratures are used. Artificial flux fluctuations are also not evident when the q4x20, s4x20, or d4x20 quadratures are used (omitted results are much smoother than in Figures 4a and 4b). However, small artificial flux fluctuations still persist for the sg4x20 and dg4x20 where the 20 azimuthal angles are uniformly distributed on each of 4 single-Gauss and double-Gauss cones. This presents clear evidence that the QR distribution of azimuthal angles is vastly superior to the uniform Chebyshev-Gauss type distribution.

Omitted figures suggest that there is little difference between the results for the s10x20 and q10x20 quadratures, and between these 10 cone quadratures and the q4x20, s4x20 and d4x20 quadratures which have only 4 cones. To quantify the distinction between these and other quadratures, ratios of net fluxes were obtained and were plotted along the last row or column of the problem geometry. These last row or column locations are the furthest away from the source where differences typically have their maximum values. The ranges of deviation of edge fluxes corresponding to various quadratures from edge fluxes corresponding to benchmark s10x20 and q10x20 quadratures are listed in Table X. The following observations and conclusions can be drawn based on Table X, Figures 5 and 6, and other results omitted for brevity:

(iv) Figure 5 shows ratios of boundary fluxes obtained with the q10x20, q4x20, s4x20 and d4x20 quadratures divided by boundary fluxes obtained with the s10x20 quadrature. This figure shows that the ratio of results for the q10x20 and s10x20 quadratures is essentially one everywhere. Furthermore, the ratio of the results for the q4x20 quadrature divided by those for the s10x20 quadrature varies between 1.0 and 1.0005 everywhere. These results confirm that either of the s10x20 and q10x20 quadratures, which produce virtually identical net fluxes, can be applied as a benchmark for our quadrature study. Furthermore, the use of 4 cones in the q4x20 quadrature leads to less than 0.05% difference from results obtained with the 10 cone benchmark quadratures. This is within solution convergence.

(v) Figure 5 further indicates that edge fluxes obtained with the s4x20 quadrature differ from benchmark results by about 0.14%. Likewise, edge fluxes obtained with the d4x20 quadrature differ from benchmark results by between 0.0% and 0.33%. In summary Figure 5 suggests that (a) the use of QR polar cone-based quadrature, as in the q4x20 quadrature, is to be preferred to the use of SG polar cone-based quadrature, as in the s4x20 quadrature. Further, this figure suggests that the use of the SG polar cone-based quadrature is to be preferred to the use of DG polar cone-based quadrature as in the d4x20 quadrature. This result is consistent with the comments on Figure 1g made in Section 6d above.

(vi) Figure 6 provides the ratios of the boundary fluxes obtained with the s4x20, d4x20, sg4x20, and dg4x20 quadratures divided by the corresponding results obtained with the benchmark s10x20 quadrature. The sg4x20 and dg4x20 quadratures result in fluctuations as was observed in item (iii) above. These fluctuations vary between -1.3% and 0.9% and Figure 6

again confirms that the use of QR azimuthal angles as in the s4x20 and d4x20 quadratures is superior to the use of uniformly distributed azimuthal angles as in the sg4x20 and dg4x20 quadratures.

Table X and other omitted results indicate the following:

(vii) Edge flux results corresponding to the standard S_{16} quadrature and the similar S_{16} -type, sg16 and dg16 quadratures have errors up to $\pm 8\%$.

(viii) The q461214 quadrature which has 4 QR cones is superior to the s461214 quadrature which has 4 SG cones, and the latter is superior to the d461214 quadrature which has 4 DG cones. Further, the sg461214 quadrature is definitely better than the dg461214 quadrature. Moreover, the latter two quadratures which have a uniform (Chebyshev-Gauss) distribution of azimuthal angles, are not competitive with the corresponding quadratures based on the QR distribution of azimuthal angles on each cone. The ordinates for the dg461214, sg461214 and q461214 quadratures are shown in Figures 1c-e for visual comparison.

(ix) Among all the quadratures with 36 angle/octant considered in the present study, the q461214 quadrature yields the best results for the ray effect benchmark problem. The ordinates and weights for this quadrature are listed in Table XI. This quadrature should also be used for nuclear design and shielding application to rectangular x-y transport problems, in place of the dg10111213 quadrature with 46 angles/octant recommended in [6].

(x) Among the quadratures having 20 angles/octant, the q2468 quadrature is best. Furthermore, this quadrature is superior to the standard S_{16} quadrature which has 36 angles/octant. The ordinates and weights for this quadrature are listed in Table XII.

(xi) Among the qijkl quadratures having 46 angles/octant, the q481620 and q681420 quadratures yield the best results for the benchmark ray effect problem. These quadratures can be constructed from Tables I, V, and Algorithm 1, and are suitable for certain applications which require high order quadratures.

In summary, the QR quadratures for the azimuthal direction are superior to the standard Chebyshev-Gauss quadratures. Furthermore, for the polar direction, our numerical tests suggest that in most situations: (i) the choice of single-Gauss cones is better than double-Gauss cones, (ii) the choice of QR polar cones is comparable for low order quadrature to the choice of SG polar cones, and (iii) QR or OR polar cone-based quadratures yield better results than SG polar quadratures when high order formulas are applied. As was discussed in Section 6d, the reason for the relatively poor behavior of the DG quadrature for x-y problems is most likely due to the fact that, while double-Gauss has a bottom cone much closer to the x-y plane than the bottom SG or QR cones, the weight associated with the bottom DG cone is very small compared to that of the corresponding SG and QR cones. In other words, it is not only how close a cone is to the x-y plane that dominates and governs the transport in rectangular geometry, but what is more important is the combination of the relative position of a cone with respect to the x-y plane, and the weight associated with that position.

(b) Numerical Results for Benchmark x-y Dry-Cell Storage Problem

A model one-energy group P_3 -Scattering dry-cell storage x-y transport problem was set up by Dr. Kin Y. Cheung of Bettis, and was kindly provided by him to carry out the present quadrature study. The problem configuration is shown in part in Figure 7a and is comprised of a localized uniform source placed in a very large air gap surrounded by a shield. The shield has a gap in it for access and a shield cover is placed at the end of the gap. The primary goal here is to accurately determine the neutron flux shape outside the gap. The flux of course can be used to determine dose rate and other parameters of interest to shield designers. For this quadrature study, the scale used for the view in Figures 7d,e is expanded along the y-direction for ease of distinguishing flux shapes in the region of interest corresponding to different angular quadrature formulas.

Figure 7 and other omitted results indicate the following:

- (i) Figure 7b shows that the Gelbard and Shure [6] angular quadrature dg10111213 (=dg13121110, see Table IX and the description in fourth paragraph on page 21) with 4 DG cones and a total of 46 angles/octant, leads to unacceptable ray effect flux fluctuations for the

benchmark dry-cell storage problem. Exaggerated artificial flux fluctuations were also obtained with the standard S_{16} angular quadrature.

(ii) Figures 7d,e show clearly that the choice of quadruple-range distribution of azimuthal angles is considerably superior to the choice of uniform Chebyshev-Gauss distribution of azimuthal angles. Figures 7c,d indicate that the dg4x20 with 80 angles/octant quadrature leads to large artificial fluctuations inside the access gap, and that the fluctuations are more pronounced for directions nearly parallel to either the x-axis or y-axis. Figure 7e shows that the use of quadruple-range q4x20 with 80 angles/octant quadrature with quadruple-range distribution of azimuthal angles considerably reduces the artificial flux fluctuations in the region of interest. Omitted results show that there is little or no difference between the d4x20, s4x20, and q4x20 quadratures. Flux fluctuations still persist, which suggests the need for higher order quadratures for this dry-cell storage problem.

(iii) Figure 7f and omitted results show that the artificial ray effect flux fluctuations are considerably dampened in the top region of interest when applying the variable mid-point rule quadratures with 144 angles/octant. The best results are obtained with the quadratures specified in Eqs. (59) and (61). Quadratures corresponding to Eqs. (55a,b) produce unacceptable results.

(c) Numerical Results for Streaming Through Cylindrical Gaps

1. Application to a Cylindrical Penetration

This section provides a summary of analyses of experimental gamma ray results through cylindrical penetrations in typical secondary shields using the discrete ordinates method. Calculated results obtained with suitably tailored angular quadratures were found to be in reasonably good agreement with experiment [9]. For brevity, only results for a single fast energy group and a low energy group will be presented.

Standard single-Gauss quadratures with a very large number of angular directions was shown to mitigate ray effect artificial flux fluctuations and to lead to high accuracy. However, increasing the order of the quadrature for the gamma penetration problems of interest is in general prohibitive in terms of computer storage and run time requirements. Therefore, the approach taken here is to generate new suitable angular quadratures, which would improve the

accuracy of the flux solution and cut the overall computational cost. Many of the specially tailored quadratures introduced in Table VII were tested. Several of these quadratures also mitigate ray effects while requiring significantly smaller numbers of angular directions.

Of particular interest here is a set of experimental results for gamma ray transmission through cylindrical penetrations in shields available in [9, p. 296] for ^{16}N gamma rays (6.1 MeV) penetrating gaps of different diameters in a 4-in. thick lead shield. For brevity, only results for the 2-in. diameter gap are used here. The experimental results in [9, p. 296, Fig. 8.20], as seen in Figure 8 are presented as traverses across the gap, of the ratio of dose rate with the gap to without the gap, for several distances behind the shield. The gamma ray dose rates were measured with an anthracene scintillation counter.

The problem geometry is shown in Figures 9 and 10. The source is a disk described as water containing ^{16}N which emits 6.1 MeV gamma rays. One of the first quadratures attempted was the double-Gauss quadrature with 16 polar cones and 8 uniformly distributed azimuthal ordinates per cone per octant (DG 16/8). As seen in Figure 9, the corresponding isoflux contours for energy group 1 ($E = 6.1$ MeV) in which the ^{16}N source energy is confined, are not physically meaningful in that ray effects are evident. Consequently, single-Gauss quadratures that contain increasingly large numbers of cones but approximately the same total number of ordinates per octant have been tried. Isoflux contours for group 1 over the central portion of the configuration are also seen as insets in Figure 9 for the two single-Gauss quadratures SG 20/6 and SG 40/3. The SG 20/6 results exhibit ray effects. The SG 40/3 results for group 1 appear to be physically meaningful. Reasonable agreement was observed between experiment and calculation at radial locations away from the axis, which indicates that with a large number of ordinates per octant, the discrete ordinates method can provide reasonable results through penetrations in a shield.

The main results for this sub-section are summarized in Figures 9 to 12. Figures 10a, b, and c exhibit the isoflux contours for energy group 1 respectively for the present problem with the following choices of angular quadrature: (a) single-Gauss SG 20/3, (b) double-Gauss DG 20/3, and (c) octant-range QR 20/3. (Note that for problems with 2-D symmetry, octant-range can also be referred to as quadruple range, i.e. QR 20/3 = OR 20/3). The addition of "/3" in the designation of the quadratures here and below for the quadratures given in Table VII, denotes three uniformly distributed azimuthal angles on each of the polar cones. Specifically the QR 20/3 designation stands for 20 QR cones in the polar direction and 3 azimuthal angles per octant on

each polar cone. Each of Figures 10a-c exhibit ray effects near the axis of the cylindrical gap. In particular these figures support the following conclusions:

(i) Single-Gauss quadrature is to be preferred to double-Gauss quadrature for this gamma penetration problem since, for the same number of ordinates, single-Gauss quadrature has angular ordinates closer to the z-axis direction of physical interest than double-Gauss quadrature.

(ii) For the same number of angles, octant-range quadratures in general result in less ray effects than do single-Gauss and double-Gauss quadratures.

The new octant-range quadratures listed in Tables III and IV, while not adequate for gamma penetrations, should prove to be more useful for some neutron problems.

Ray effects seen in Figures 10a-c for the present penetration problem with angular quadratures of 20 cones and 3 angles per cone per octant were considerably magnified when the quadrature was based on the mid-point rule or the trapezoidal rule with uniform angular subdivisions along the polar direction. Numerical tests for energy group 1 indicate that quadratures based on variable mesh mid-point (MP) rule and on a combined mid-point rule and Gauss (MG) are the most versatile and most suitable. Quadratures based on the trapezoidal rule or on combinations of the trapezoidal rule and Gauss quadrature give results very close to those based on the mid-point (MP) rule and are omitted here for brevity. Quadratures based on parabolic rule, Simpson's rule and composite Gauss are not competitive and were not pursued further. Quadratures based on high order polynomials and Laguerre quadrature lead to exaggerated ray effects and are not at all suitable for gamma penetration problems.

It was not clear a priori what a suitable angular mesh subdivision is, but it became apparent from initial numerical tests that any abrupt change in angular mesh subdivisions could result in exaggerated ray effects. Lobatto and double-Radau quadratures taken from standard tables [10] with one angular direction parallel to the positive z-axis were also unacceptable. Evidence also indicates that the choice of suitable angular quadrature may be problem dependent.

Figures 10d-f show isoflux contours for group 1 respectively for the following choices of mid-point or composite mid-point rule and Gauss angular quadratures: (a) MG 16/3-1,

(b) MP 16/3-1, and (c) MG 12/3-1. These and other quadratures are identified in Table VII. The "/3" is added to the designation of the quadrature to denote three uniformly distributed azimuthal subdivisions on each of the polar cones.

Note from Figures 10a-f that for energy group 1 of the 2-in. diameter penetration problem:

(i) The MG 16/3-1 and MP 16/3-1 quadratures of Table VII mitigate ray effects and result in comparable accuracy as the higher order (factor of 2.5) SG 40/3 quadrature (See also Figure 9). These two new 16/3 quadratures are considerably superior to SG 20/3 and DG 20/3 quadratures for the problem under consideration.

(ii) While the two new 16/3 quadratures used in Figures 10d,e give high accuracy for the 2-in. diameter penetration problem, Figure 10f shows that reasonable accuracy can be obtained with MG 12/3-1 quadrature.

For neutron problems, experience suggests that much lower order standard quadratures can be used for the lower energy groups. This does not seem to be the case for gamma problems. Figures 11a and 11b show the computed flux for a low gamma energy group 16 ($E = 0.10$ Mev) with the uniform 12/3 quadrature and the improvement that would result when the new specialized MG 12/3-1 quadrature is used. In both of these cases, the fluxes in the higher energy groups were all computed with the new MG 12/3-1 angular quadrature. Based on numerous numerical experiments, it is concluded that a specially tailored angular quadrature is needed here for computational efficiency and for accuracy of flux shapes, not only for the source energy group but for lower energy groups.

2. Application to a Ring Penetration

A model problem was constructed by replacing the 2-in. diameter gap in the 4-in. thick lead shield of Figures 8-11 by a 2-in. wide annular gap and by extending the water disk containing the source radially beyond this annulus. Figures 12a,b show the cross-section of this cylindrical configuration and show gamma ray isoflux contours for the source energy group 1 ($E = 6.1$ Mev) for two quadrature selections.

Again, it was found necessary to use the high order SG 40/3 angular quadrature to mitigate ray effects. Figures. 12a, b show that the new MG 12/3-1 quadrature is as effective for

the present problem as the standard SG 40/3 quadrature which has considerably more angular directions. Based on several tests for this and other problems, it was also observed that the angular quadrature cannot be substantially reduced for the lower energy gamma groups without compromising the accuracy of the flux shape.

8. Summary

Low order angular quadrature formulas which allow for possible discontinuities at material interfaces in 2-D rectangular x-y geometries were previously developed and demonstrated to be effective in ameliorating artificial ray effect flux fluctuations, and in improving accuracy [1]. The present paper extends the quadruple-range quadratures developed for azimuthal integration from order $N=7$ in previous work to $N=22$. This paper also generalizes the development in [1] to apply to 3-D rectangular and 2-D cylindrical problems, and also to 2-D and 3-D problems in 60-degree equilateral triangular or regular hexagonal geometries. Separate quadrature formulas are developed and tabulated for the polar and azimuthal integrations of the scattering term of the particle transport equation. Octant-range quadratures can be constructed by selecting a single quadrature for the polar integration, and allowing possibly different quadrature formulas for the azimuthal integration along each of the separate discrete levels (cones) of the polar quadrature. As a consequence, improved accuracy was achieved with far fewer quadrature points as compared to previous work [1], resulting in considerable computational economy.

Among the angular quadratures developed and tested for 2-D x-y problems, the q461214 quadruple-range quadrature with 36 angles/octant was found to be superior to S_{16} and other quadratures with the same number of discrete angular directions. This new quadrature is given in Table XI. It is suitable for general x-y problems that arise in nuclear and shield design and analysis.

The need to tailor the angular quadratures to the class of transport problems under consideration is recognized. Mid-point rule and composite mid-point rule and Gauss quadratures are developed and shown to be effective for shielding problems dominated by streaming through cylindrical gaps and large void regions. New quadrature formulas have been applied to the analysis of experimental results for gamma ray transmission through cylindrical penetrations in shields.

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Polar Cones and Azimuthal Angles on One Octant of the Unit Sphere

FIGURE 1e. $q_{4,6,12,14}$ Quadrature.

Polar Cones and Azimuthal Angles on One Octant of the Unit Sphere

FIGURE 1f. MG 26/v Quadrature.

FIGURE 1g. Weights Associated with Four Cone Quadruple-Range, Single and Double Gauss Quadratures

FIGURE 2. Benchmark Ray Effect Problem.

(a)

(b)

(c)

(d)

FIGURES 3a-d. Edge Flux and Flux Contours for Problem 1 of Figure 2 Corresponding to (a)-(b) dg_{16} Quadrature of 8 Double-Gauss (DG) Cones and S_n -Type Distribution of Angles, (c) Standard S_{16} Quadrature, and (d) 4 DG Cones and Respectively 13, 12, 11, 10 Uniform Angles/Octant from Bottom Cone (closest to x-y-plane) to Top Cone (closest to z-axis).

(a)

(b)

(c)

(d)

FIGURES 4a-d. Edge Flux and Flux Contours for Problem 1 with Sample New Quadruple-Range 4 SG Cone Quadratures (a)-(b) s461214 with 36 Angles/Octant, and (c)-(d) s2468 with 20 Angles/Octant.

FIGURE 5. Edge $X=1.967$ Flux Ratios of (1) Curve 1 q_{10x20} with 10 QR Cones 20 QR Angles/Cone/Octant, (2) Curve 2 q_{4x20} with 4 QR Cones 20 QR Angles/Cone/Octant, (3) Curve 3 s_{4x20} 4 SG Cones 20 QR Angles/Cone/Octant, and (4) Curve 4 d_{4x20} 4 DG Cones 20 QR Angles/Cone/Octant, Each Flux Divided by Benchmark Flux s_{10x20} with 10 QR Cones 20 QR Angles/Cone/Octant.

FIGURE 6. Edge $X=1.967$ Flux Ratios of (1) Curve 1 s_{4x20} 4SG Cones 20 QR Angles/Cone/Octant, (2) Curve 2 d_{4x20} with 4 DG Cones 20 QR Angles/Cone/Octant, (3) Curve 3 sg_{4x20} 4 SG Cones 20 Uniform Angles/Cone/Octant, and (4) Curve 4 dg_{4x20} 4 DG Cones 20 Angles/Cone/Octant, Each Flux Divided by Benchmark Flux s_{10x20} with 10 SG Cones 20 QR Angles/Cone/Octant.

FIGURES 7a-f. Geometry and Group 1 Flux Contours for Model Dry-Cell Storage Problem. (a) Complete Problem Geometry, (b) Flux Contours for the dg10111213 Quadrature (Table IX & Ref. [6]), (c)-(d) Flux Contours and Y-Expanded Section of their Top Part for the dg4x20 Quadrature [4 DG Cones, 20 Uniformly Distributed Angles/Cone/Octant], (e) Expanded Top Part of Flux Contours for the d4x20 Quadrature [4 DG Cones but with 20 Quadruple-Range Azimuthal Angles/Cone/Octant], and (f) Flux Contours Corresponding to 4 QR Cones and Variable Mid-Point Rule for Azimuthal Angle (QR4, 42, 4, 6, 1.5) with 144 Angles/Octant Quadrature (See Eq. (61)).

FIGURE 8. Relative Gamma Dose Rate.

FIGURE 9. Group 1 ($\bar{E} = 6.1$ Mev) Gamma Ray Isoflux Contours for a 2 in. Diameter Penetration in 4 in. Thick Lead Shield with DG 16/8, SG 20/6, and SG 40/3 Quadratures.

^aThe cylindrical axis is at the right boundary and the r-axis points to the left in this Figure and in Figures 10-12 below.

6.1 Mev Source in Water
(a) Group 1 SG 20/3 (240 Angles)

6.1 Mev Source in Water
(b) Group 1 DG 20/3 (240 Angles)

6.1 Mev Source in Water
(c) Group 1 QR 20/3 (240 Angles)

6.1 Mev Source in Water
(d) Group 1 MG 16/3-1 (192 Angles)

6.1 Mev Source in Water
(e) Group 1 MP 16/3-1 (192 Angles)

6.1 Mev Source in Water
(f) Group 1 MG 12/3-1 (144 Angles)

FIGURE 10. Gamma Ray Isoflux Contours for Standard and New Angular Quadratures for a 2-in. Diameter Penetration through 4-in. Lead at the Source Energy Group 1 ($E = 6.1$ Mev).

(a) Group 16 Uniform 12/3 (144 Angles)

(b) Group 16 MG 12/3-1 (144 Angles)

FIGURE 11. Gamma Ray Isoflux Contours for Old and New Angular Quadratures for 2-in. Diameter Penetration through 4-in. Lead at a Low Energy Group 16 ($\bar{E} = 0.1$ Mev).

(a) Group 1 SG 40/3 (480 Angles)

(b) Group 1 MG 12/3-1 (144 Angles)

FIGURE 12. Gamma Ray Isoflux Contours for the Ring Penetration.

Table I.

Quadratures for Azimuthal Integration in Rectangular 2-D and 3-D Geometries

QR Quadratures for Integral of $F(\cos \phi, \sin \phi)$, $0 < \phi < \pi/2$

$\omega(x) = 1$, $X(l)$, $W(l)$ symmetric W.R.T. $\phi = \pi/4$

$X(l) = \sin \phi(l)$ or $X(l) = \cos \phi(l)$

$X(l)$	$W(l)$	$X(l)$	$W(l)$
N = 1		N = 8	
7.07106781187E-01	1.57079632679E+00	3.24869788463E-02	8.26105878919E-02
N = 2		1.63951256940E-01	1.78010819682E-01
3.28861319306E-01	7.85398163397E-01	3.70037938547E-01	2.45365013149E-01
9.44378225429E-01	7.85398163397E-01	6.00618666309E-01	2.79411742675E-01
N = 3		7.99535626274E-01	2.79411742675E-01
1.79705750550E-01	4.41492408805E-01	9.29016643573E-01	2.45365013149E-01
7.07106781187E-01	6.87811509185E-01	9.86468441131E-01	1.78010819682E-01
9.83720409069E-01	4.41492408805E-01	9.99472158795E-01	8.26105878919E-02
N = 4		N = 9	
1.11919417802E-01	2.79049509269E-01	2.60886177621E-02	6.64555994585E-02
4.99008815360E-01	5.06348654128E-01	1.32824656907E-01	1.45296593486E-01
8.66596908714E-01	5.06348654128E-01	3.05325234799E-01	2.05144486413E-01
9.93717285710E-01	2.79049509269E-01	5.10423681971E-01	2.41615952092E-01
N = 5		7.07106781187E-01	2.53771063895E-01
7.60839341001E-02	1.91320724703E-01	8.59923057537E-01	2.41615952092E-01
3.59915255224E-01	3.74855576614E-01	9.52248129951E-01	2.05144486413E-01
7.07106781187E-01	4.38443724162E-01	9.91139551485E-01	1.45296593486E-01
9.32984999374E-01	3.74855576614E-01	9.99659634087E-01	6.64555994585E-02
9.97101416593E-01	1.91320724703E-01	N = 10	
N = 6		2.14063664984E-02	5.45988342772E-02
5.49823833982E-02	1.38991227102E-01	1.09678208079E-01	1.20668935327E-01
2.68645060598E-01	2.84984797595E-01	2.55406277071E-01	1.73450649536E-01
5.65871339345E-01	3.61422138701E-01	4.35996579025E-01	2.09351736725E-01
8.24493558076E-01	3.61422138701E-01	6.21758783706E-01	2.27328007532E-01
9.63239238931E-01	2.84984797595E-01	7.83208793927E-01	2.27328007532E-01
9.98487324665E-01	1.38991227102E-01	8.99948322449E-01	2.09351736725E-01
N = 7		9.66833819037E-01	1.73450649536E-01
4.15504673952E-02	1.05403262712E-01	9.93967147683E-01	1.20668935327E-01
2.07031268158E-01	2.22573419243E-01	9.99770857484E-01	5.45988342772E-02
4.54812160314E-01	2.96659466192E-01		
7.07106781187E-01	3.21524030502E-01		
8.90587389778E-01	2.96659466192E-01		
9.78334326294E-01	2.22573419243E-01		
9.99136406433E-01	1.05403262712E-01		

Table I (Continued₁).

Quadratures for Azimuthal Integration in Rectangular 2-D and 3-D Geometries

QR Quadratures for Integral of $F(\cos \phi, \sin \phi)$, $0 < \phi < \pi/2$ $\omega(x) = 1$, $X(l)$, $W(l)$ symmetric W.R.T. $\phi = \pi/4$ $X(l) = \sin \phi(l)$ or $X(l) = \cos \phi(l)$

$X(l)$	$W(l)$	$X(l)$	$W(l)$
N = 11		N = 14	
1.78781812798E-02	4.56447136792E-02	1.12866459910E-02	2.88696539886E-02
9.20337735274E-02	1.01713727266E-01	5.86080282593E-02	6.53481659572E-02
2.16363520647E-01	1.48226563396E-01	1.40159297798E-01	9.77815191598E-02
3.75037367368E-01	1.82261253182E-01	2.49653542335E-01	1.24592860221E-01
5.46424732468E-01	2.02761156967E-01	3.78014529413E-01	1.44949862586E-01
7.07106781187E-01	2.09581497814E-01	5.14104408362E-01	1.58538699832E-01
8.37508215928E-01	2.02761156967E-01	6.46095281689E-01	1.65317401653E-01
9.27009694166E-01	1.82261253182E-01	7.63256763467E-01	1.65317401653E-01
9.76312873485E-01	1.48226563396E-01	8.57727612534E-01	1.58538699832E-01
9.95755886013E-01	1.01713727266E-01	9.25799662752E-01	1.44949862586E-01
9.99840172545E-01	4.56447136792E-02	9.68335225425E-01	1.24592860221E-01
N = 12		9.90128966974E-01	9.77815191598E-02
1.51542763381E-02	3.87199557399E-02	9.98281072155E-01	6.53481659572E-02
7.82933178482E-02	8.68389882482E-02	9.99936303783E-01	2.88696539886E-02
1.85385682416E-01	1.27922016108E-01	N = 15	
3.25051390048E-01	1.59596288800E-01	9.88545479889E-03	2.52957570834E-02
4.81334101720E-01	1.80853870486E-01	5.14258427096E-02	5.74524794603E-02
6.35937834851E-01	1.91467044015E-01	1.23417452644E-01	8.64651177202E-02
7.71740286758E-01	1.91467044015E-01	2.21052006416E-01	1.11038832659E-01
8.76537211145E-01	1.80853870486E-01	3.37331340616E-01	1.30436640770E-01
9.45696353925E-01	1.59596288800E-01	4.63490786372E-01	1.44335592135E-01
9.82665837788E-01	1.27922016108E-01	5.89865690428E-01	1.52659571621E-01
9.96930366866E-01	8.68389882482E-02	7.07106781187E-01	1.55428343897E-01
9.99885167361E-01	3.87199557399E-02	8.07501372913E-01	1.52659571621E-01
N = 13		8.86101738486E-01	1.44335592135E-01
1.30078426034E-02	3.32559161972E-02	9.41385981751E-01	1.30436640770E-01
6.73938148498E-02	7.49665892004E-02	9.75262021438E-01	1.11038832659E-01
1.60465382602E-01	1.11389848018E-01	9.92354841971E-01	8.64651177202E-02
2.83850626570E-01	1.40595290324E-01	9.98676815943E-01	5.74524794603E-02
4.25605349674E-01	1.61656324749E-01	9.99951137698E-01	2.52957570834E-02
5.71479597861E-01	1.74288947541E-01		
7.07106781187E-01	1.78490494733E-01		
8.20616274045E-01	1.74288947541E-01		
9.04908882888E-01	1.61656324749E-01		
9.58868511213E-01	1.40595290324E-01		
9.87041468727E-01	1.11389848018E-01		
9.97726452351E-01	7.49665892004E-02		
9.99915394436E-01	3.32559161972E-02		

Table I (Continued₂).**Quadratures for Azimuthal Integration in Rectangular 2-D and 3-D Geometries**QR Quadratures for Integral of $F(\cos \phi, \sin \phi)$, $0 < \phi < \pi/2$ $\omega(x) = 1$, $X(l)$, $W(l)$ symmetric W.R.T. $\phi = \pi/4$ $X(l) = \sin \phi(l)$ or $X(l) = \cos \phi(l)$

$X(l)$	$W(l)$	$X(l)$	$W(l)$
N = 16		N = 18	
8.72965710742E-03	2.23456344014E-02	6.95206352986E-03	1.78046165668E-02
4.54813865221E-02	5.08947121298E-02	3.63035575806E-02	4.07286438944E-02
1.09465718192E-01	7.69654619574E-02	8.77589835173E-02	6.20554963558E-02
1.96944891481E-01	9.94883219743E-02	1.58962438822E-01	8.10416378981E-02
3.02448274604E-01	1.17816372561E-01	2.46443244015E-01	9.71946843126E-02
4.19021114815E-01	1.31631849523E-01	3.45703234725E-01	1.10230341949E-01
5.38777013625E-01	1.40833201160E-01	4.51447239310E-01	1.20021162996E-01
6.53724513962E-01	1.45422609690E-01	5.57970321081E-01	1.26535909639E-01
7.56732621105E-01	1.45422609690E-01	6.59662846613E-01	1.29785669785E-01
8.42448413607E-01	1.40833201160E-01	7.51561660011E-01	1.29785669785E-01
9.07976489420E-01	1.31631849523E-01	8.29860904485E-01	1.26535909639E-01
9.53165799423E-01	1.17816372561E-01	8.92297814701E-01	1.20021162996E-01
9.80414560132E-01	9.94883219743E-02	9.38343899378E-01	1.10230341949E-01
9.93990571656E-01	7.69654619574E-02	9.69157225366E-01	9.71946843126E-02
9.98965186320E-01	5.08947121298E-02	9.87284631220E-01	8.10416378981E-02
9.99961895817E-01	2.23456344014E-02	9.96141737311E-01	6.20554963558E-02
N = 17		N = 19	
7.76516751178E-03	1.98823343503E-02	6.25951928174E-03	1.60342230649E-02
4.05069844047E-02	4.53909784761E-02	3.27165178199E-02	3.67417542903E-02
9.77254732972E-02	6.89213492840E-02	7.92219146827E-02	5.61476310279E-02
1.76472506876E-01	8.95844718130E-02	1.43872133508E-01	7.36262534663E-02
2.72416454329E-01	1.06815271243E-01	2.23857123544E-01	8.87486446168E-02
3.79989298340E-01	1.20310163196E-01	3.15511709051E-01	1.01251710769E-01
4.92742032328E-01	1.29950917520E-01	4.14465834283E-01	1.11004006334E-01
6.03908913692E-01	1.35721804820E-01	5.15911914917E-01	1.17961656788E-01
7.07106781187E-01	1.37641745392E-01	6.14970011895E-01	1.22126237364E-01
7.97053338218E-01	1.35721804820E-01	7.07106781187E-01	1.23512091351E-01
8.70175436092E-01	1.29950917520E-01	7.88550495828E-01	1.22126237364E-01
9.24990882737E-01	1.20310163196E-01	8.56641638053E-01	1.17961656788E-01
9.62179440339E-01	1.06815271243E-01	9.10064872530E-01	1.11004006334E-01
9.84305569585E-01	8.95844718130E-02	9.48921683519E-01	1.01251710769E-01
9.95213410214E-01	6.89213492840E-02	9.74621971966E-01	8.87486446168E-02
9.99179255296E-01	4.53909784761E-02	9.89596285967E-01	7.36262534663E-02
9.99969850632E-01	1.98823343503E-02	9.96857004908E-01	5.61476310279E-02
		9.99464671443E-01	3.67417542903E-02
		9.99980409017E-01	1.60342230649E-02

Table I (Continued₃).

Quadratures for Azimuthal Integration in Rectangular 2-D and 3-D Geometries

QR Quadratures for Integral of $F(\cos \phi, \sin \phi)$, $0 < \phi < \pi/2$ $\omega(x) = 1$, $X(l)$, $W(l)$ symmetric W.R.T. $\phi = \pi/4$ $X(l) = \sin \phi(l)$ or $X(l) = \cos \phi(l)$

$X(l)$	$W(l)$	$X(l)$	$W(l)$
N = 20		N = 22	
5.67509348883E-03	1.45393921491E-02	4.15241892965E-03	1.03702865159E-02
2.96820847851E-02	3.33596736792E-02	2.14943983335E-02	2.52171295699E-02
7.19670984270E-02	5.10954081947E-02	5.51949788462E-02	4.17522214778E-02
1.30959454476E-01	6.72144228938E-02	1.04398716322E-01	5.70465053218E-02
2.04344253359E-01	8.13447311863E-02	1.67765936458E-01	7.04500290301E-02
2.89096892987E-01	9.32502149372E-02	2.43263988423E-01	8.38347578417E-02
3.81584702597E-01	1.02806017860E-01	3.28167683038E-01	9.27746889748E-02
4.77753713079E-01	1.09965085574E-01	4.19138577496E-01	1.03667304767E-01
5.73392143001E-01	1.14724634716E-01	5.12304337326E-01	1.05523198981E-01
6.64444346259E-01	1.17098582207E-01	6.03869595918E-01	1.18754396046E-01
7.47337748762E-01	1.17098582207E-01	6.90103150166E-01	7.60076448718E-02
8.19281056991E-01	1.14724634716E-01	7.23711021148E-01	7.60076448718E-02
8.78493818783E-01	1.09965085574E-01	7.97083126860E-01	1.18754396046E-01
9.24333876229E-01	1.02806017860E-01	8.58803974115E-01	1.05523198981E-01
9.57299841463E-01	9.32502149372E-02	9.07922272474E-01	1.03667304767E-01
9.78899088834E-01	8.13447311863E-02	9.44619485195E-01	9.27746889748E-02
9.91387725001E-01	6.72144228938E-02	9.69960118735E-01	8.38347578417E-02
9.97407006565E-01	5.10954081947E-02	9.85826856281E-01	7.04500290301E-02
9.99559389853E-01	3.33596736792E-02	9.94535523765E-01	5.70465053218E-02
9.99983896527E-01	1.45393921491E-02	9.98475595250E-01	4.17522214778E-02
N = 21		9.99768968732E-01	2.52171295699E-02
4.88519874076E-03	1.25272984288E-02	9.99991378671E-01	1.03702865159E-02
2.56466982317E-02	2.89516129387E-02		
6.25820489998E-02	4.48810132722E-02		
1.14852883153E-01	5.99002608724E-02		
1.80994199785E-01	7.36350186708E-02		
2.58831435583E-01	8.56919279330E-02		
3.45501744902E-01	9.59071266252E-02		
4.37547785645E-01	1.03872159085E-01		
5.30993475247E-01	1.09613498442E-01		
6.22088626086E-01	1.13243736140E-01		
7.07106781187E-01	1.14349021980E-01		
7.82946831716E-01	1.13243736140E-01		
8.47375907874E-01	1.09613498442E-01		
8.99195159728E-01	1.03872159085E-01		
9.38418107386E-01	9.59071266252E-02		
9.65922506185E-01	8.56919279330E-02		
9.83484163393E-01	7.36350186708E-02		
9.93382512042E-01	5.99002608724E-02		
9.98039822423E-01	4.48810132722E-02		
9.99671069337E-01	2.89516129387E-02		
9.99988067345E-01	1.25272984288E-02		

Table II.

Quadratures for Azimuthal Integration for 60-Degree 2-D and 3-D GeometriesQuadratures for Integral of $F(\cos \phi, \sin \phi)$, $0 < \phi < \pi/3$ $\omega(x) = 1$, $X(l)$, $W(l)$ symmetric W.R.T. $\phi = \pi/6$ $X(l) = \sin \phi(l)$

$X(l)$	$W(l)$	$X(l)$	$W(l)$
N = 1		N = 8	
5.000000000000E-01	1.047197551197E+00	2.116672098947E-02	5.390107340290E-02
N = 2		1.077099836896E-01	1.174304272251E-01
2.204006147023E-01	5.235987755983E-01	2.477090514247E-01	1.639751319491E-01
7.345290438490E-01	5.235987755983E-01	4.154392350651E-01	1.882921430213E-01
N = 3		5.800352183627E-01	1.882921430213E-01
1.188092437644E-01	2.924116647494E-01	7.151807353951E-01	1.639751319491E-01
5.000000000000E-01	4.623742216978E-01	8.071321855047E-01	1.174304272251E-01
8.004868088793E-01	2.924116647494E-01	8.552480188407E-01	5.390107340290E-02
N = 4		N = 9	
7.354227601392E-02	1.838497154485E-01	1.698285735925E-02	4.331226811706E-02
3.398788689682E-01	3.397490601498E-01	8.703193341794E-02	9.559158344061E-02
6.445308018178E-01	3.397490601498E-01	2.029609959216E-01	1.366103099539E-01
8.269091559844E-01	1.838497154485E-01	3.478562161136E-01	1.624588129993E-01
N = 5		5.000000000000E-01	1.712516021748E-01
4.982032841984E-02	1.255822153599E-01	6.380121851196E-01	1.624588129993E-01
2.409362189663E-01	2.503197051077E-01	7.465201657187E-01	1.366103099539E-01
5.000000000000E-01	2.953937102613E-01	8.192233229517E-01	9.559158344061E-02
7.200450098628E-01	2.503197051077E-01	8.574090776818E-01	4.331226811706E-02
8.400398061104E-01	1.255822153599E-01	N = 10	
N = 6		1.392514911501E-02	3.555346831518E-02
3.592160637118E-02	9.099314515949E-02	7.172693116054E-02	7.921580710316E-02
1.781621914139E-01	1.893984328322E-01	1.689412473260E-01	1.151356098927E-01
3.891369611432E-01	2.432071976067E-01	2.940715544806E-01	1.403725037969E-01
6.031967224199E-01	2.432071976067E-01	4.317157191967E-01	1.533213864903E-01
7.630888844983E-01	1.893984328322E-01	5.653054737378E-01	1.533213864903E-01
8.475056771860E-01	9.099314515949E-02	6.806969336235E-01	1.403725037969E-01
N = 7		7.691066377574E-01	1.151356098927E-01
2.710338013868E-02	6.887180234150E-02	8.279313222656E-01	7.921580710316E-02
1.365227021037E-01	1.473177959530E-01	8.589788597593E-01	3.555346831518E-02
3.076535162238E-01	1.989870692498E-01		
5.000000000000E-01	2.168442161080E-01		
6.701950774400E-01	1.989870692498E-01		
7.896554070650E-01	1.473177959530E-01		
8.521555670845E-01	6.887180234150E-02		

Table III.

Quadratures for Polar Integration in 2-D r-z Cylindrical and 3-D x-y-z Geometries
 Quadratures for Eqs. (32), (33) and (29) for Integral of $F(\cos \theta, \sin \theta) \sin \theta$, $0 < \theta < \pi/2$
 $x = \Omega_z = \mu = \cos \theta$, $0 < x < 1$, $X(l) = \Omega_z(l) = \mu(l) = \cos \theta(l) = \text{Polar cone}$, $W(l) = \text{Weight}$

$X(l) = \Omega_z(l)$	$W(l)$	$X(l) = \Omega_z(l)$	$W(l)$
N = 1		N = 8	
5.000000000000E-01	1.000000000000E+00	9.974844457303E-01	8.277780745262E-03
N = 2		9.742944627776E-01	4.326791144837E-02
7.982729412451E-01	5.208426001509E-01	9.018615803811E-01	1.047726601051E-01
1.757782425530E-01	4.791573998491E-01	7.627790500196E-01	1.719034873910E-01
N = 3		5.659574238030E-01	2.156073600345E-01
9.604390836138E-01	1.282524669588E-01	3.467247058825E-01	2.144524571862E-01
6.615024775419E-01	4.640438146423E-01	1.535856195092E-01	1.642796073823E-01
1.713381113210E-01	4.077037183989E-01	3.048866714027E-02	7.743873570727E-02
N = 4		N = 9	
9.769561194437E-01	7.269057186878E-02	9.982161817257E-01	5.882119987511E-03
8.051994082788E-01	2.806802479853E-01	9.815882273556E-01	3.123725564588E-02
4.457120898319E-01	4.003322151570E-01	9.284963289747E-01	7.783531485279E-02
9.966839481780E-02	2.462969649889E-01	8.227726791358E-01	1.336215797672E-01
N = 5		6.646101786573E-01	1.794138048344E-01
9.870778730285E-01	4.123608227714E-02	4.731051384152E-01	1.980462893247E-01
8.844381854250E-01	1.754439852907E-01	2.804305521065E-01	1.813274987347E-01
6.364110809488E-01	3.079049644863E-01	1.214062791342E-01	1.320076216895E-01
3.131009013529E-01	3.112236010034E-01	2.380032245409E-02	6.062851516327E-02
6.534383959838E-02	1.641913669425E-01	N = 10	
N = 6		9.986639055230E-01	4.415121713275E-03
9.918215612988E-01	2.634533226383E-02	9.860750451867E-01	2.379670388264E-02
9.237808174098E-01	1.196798657034E-01	9.451105882941E-01	6.070437867352E-02
7.447375783988E-01	2.347451787526E-01	8.612251010103E-01	1.077231714704E-01
4.761170699091E-01	2.859569374033E-01	7.308125378206E-01	1.512900248206E-01
2.094840871456E-01	2.301623654333E-01	5.644760247698E-01	1.775375618285E-01
3.999870329916E-02	1.031103204436E-01	3.845843738168E-01	1.775190939431E-01
N = 7		2.186411377282E-01	1.501640282962E-01
9.967708173219E-01	1.073323962524E-02	9.117714390244E-02	1.022057012729E-01
9.659413748128E-01	5.820660832396E-02	1.740246862438E-02	4.464421409895E-02
8.680139332684E-01	1.413124945354E-01		
6.838923780710E-01	2.223456544585E-01		
4.401712181656E-01	2.536885188209E-01		
2.017791242770E-01	2.105189089080E-01		
4.080476764395E-02	1.031945753280E-01		

Table III (Continued₁).**Quadratures for Polar Integration in 2-D r-z Cylindrical and 3-D x-y-z Geometries**

Quadratures for Eqs. (32), (33) and (29) for Integral of $F(\cos \theta, \sin \theta) \sin \theta$, $0 < \theta < \pi/2$
 $x = \Omega_z = \mu = \cos \theta$, $0 < x < 1$, $X(l) = \Omega_z(l) = \mu(l) = \cos \theta(l) = \text{Polar cone}$, $W(l) = \text{Weight}$

$X(l) = \Omega_z(l)$	$W(l)$	$X(l) = \Omega_z(l)$	$W(l)$
N = 11		N = 14	
9.992920885586E-01	2.363249379489E-03	9.996170854881E-01	1.275140251962E-03
9.922952654613E-01	1.356087006340E-02	9.958665467635E-01	7.246919569770E-03
9.678806036705E-01	3.743835918720E-02	9.828233055097E-01	2.004364812643E-02
9.136282961204E-01	7.249917577410E-02	9.534897570699E-01	3.963726169977E-02
8.213902876624E-01	1.119361737433E-01	9.019700831566E-01	6.391674922751E-02
6.915356004722E-01	1.460508077206E-01	8.252830440824E-01	8.929590299616E-02
5.341910490037E-01	1.655722503453E-01	7.244096885552E-01	1.115919285935E-01
3.673183880351E-01	1.644754130310E-01	6.044286758928E-01	1.269364987153E-01
2.126076545491E-01	1.414310116129E-01	4.738022216774E-01	1.325380596553E-01
9.075734500150E-02	9.958358775609E-02	3.430282977406E-01	1.271709942860E-01
1.767482481288E-02	4.508910138658E-02	2.229703514658E-01	1.113254048410E-01
		1.232133603037E-01	8.700579417403E-02
		5.079619961646E-02	5.717565864535E-02
		9.687886437649E-03	2.484003921791E-02
N = 12		N = 15	
9.993970302803E-01	2.004918473308E-03	9.997632309062E-01	7.920379444892E-04
9.935370452767E-01	1.127180916561E-02	9.973926436693E-01	4.634500596813E-03
9.734453021469E-01	3.062846365967E-02	9.888641877020E-01	1.332877149720E-02
9.292191586662E-01	5.904699117173E-02	9.688740084385E-01	2.755788867517E-02
8.538158901144E-01	9.200094292399E-02	9.321281905548E-01	4.660310036989E-02
7.459029638445E-01	1.228924149953E-01	8.747244787864E-01	6.844871370646E-02
6.109964087761E-01	1.449429063090E-01	7.952557673462E-01	9.020990454160E-02
4.607294608495E-01	1.529513576118E-01	6.953921827728E-01	1.087193004011E-01
3.106300371442E-01	1.444556260558E-01	5.798630725053E-01	1.211180483404E-01
1.771563731009E-01	1.200301577979E-01	4.558952186072E-01	1.253363869654E-01
7.487814130237E-02	8.272615110941E-02	3.322594845655E-01	1.203863062812E-01
1.450489918500E-02	3.704826072647E-02	2.181370404309E-01	1.064266159112E-01
		1.220337397713E-01	8.460730556392E-02
		5.078597925986E-02	5.675559476994E-02
		9.802485943013E-03	2.507552443509E-02
N = 13			
9.995274192866E-01	1.572493719984E-03		
9.949172584489E-01	8.887483788568E-03		
9.789978251220E-01	2.436812723985E-02		
9.435617728436E-01	4.761589249381E-02		
8.821668129905E-01	7.560289637280E-02		
7.923519365015E-01	1.035866084436E-01		
6.767400505904E-01	1.263574189966E-01		
5.428784724465E-01	1.394620415723E-01		
4.019874258074E-01	1.401249507999E-01		
2.670202255205E-01	1.276810275661E-01		
1.505701869180E-01	1.034499379946E-01		
6.315128776245E-02	7.014610183100E-02		
1.218167613961E-02	3.114501918089E-02		

Table III (Continued₂).

Quadratures for Polar Integration in 2-D r-z Cylindrical and 3-D x-y-z Geometries
 Quadratures for Eqs. (32), (33) and (29) for Integral of $F(\cos \theta, \sin \theta) \sin \theta$, $0 < \theta < \pi/2$
 $x = \Omega_z = \mu = \cos \theta$, $0 < x < 1$, $X(l) = \Omega_z(l) = \mu(l) = \cos \theta(l) = \text{Polar cone}$, $W(l) = \text{Weight}$

$X(l) = \Omega_z(l)$	$W(l)$	$X(l) = \Omega_z(l)$	$W(l)$
N = 16		N = 18	
9.997896472385E-01	7.023233644651E-04	9.998523225717E-01	4.934118937640E-04
9.977019229962E-01	4.063935407474E-03	9.983812828630E-01	2.869483686737E-03
9.902755734583E-01	1.155139827855E-02	9.931141477653E-01	8.222634865343E-03
9.730407399895E-01	2.367789882080E-02	9.807708234704E-01	1.704824897372E-02
9.415438316187E-01	3.990035550676E-02	9.579137463117E-01	2.916428667936E-02
8.923604533819E-01	5.873952495890E-02	9.216097073869E-01	4.376173529246E-02
8.238991514343E-01	7.806643068589E-02	8.699896686014E-01	5.955172938331E-02
7.368670786382E-01	9.547903622647E-02	8.026298174707E-01	7.496929897978E-02
6.343455751840E-01	1.086939116481E-01	7.207163541339E-01	8.839644943783E-02
5.214941228406E-01	1.158881310127E-01	6.269906514600E-01	9.837376875880E-02
4.049590018368E-01	1.159408872586E-01	5.254988845226E-01	1.037763783174E-01
2.921050727146E-01	1.085418257328E-01	4.211912361096E-01	1.039364062553E-01
1.902112850574E-01	9.415577547194E-02	3.194302332855E-01	9.870105623692E-02
1.057667095674E-01	7.386293112355E-02	2.254748926631E-01	8.842301765470E-02
4.396780923232E-02	4.912816931034E-02	1.440070754345E-01	7.388649161156E-02
8.442453300905E-03	2.160746519260E-02	7.876228729135E-02	5.617234520062E-02
N = 17		3.232701620474E-02	3.645921760242E-02
9.998257980038E-01	5.818153864441E-04	6.159451701344E-03	1.579403916998E-02
9.980938869938E-01	3.374668339833E-03		
9.919136853013E-01	9.630221407742E-03		
9.775013037540E-01	1.985379134631E-02		
9.509837451382E-01	3.371891114978E-02		
9.092025997722E-01	5.014814440052E-02		
8.503714503226E-01	6.751609980702E-02		
7.744917848873E-01	8.392037279589E-02		
6.834855427850E-01	9.746947435586E-02		
5.810494572649E-01	1.065448907082E-01		
4.722748580032E-01	1.100031630981E-01		
3.631061078406E-01	1.072933018603E-01		
2.597299615607E-01	9.847651125572E-02		
1.679938000049E-01	8.415063269939E-02		
9.293880391652E-02	6.530127756773E-02		
3.850138383127E-02	4.312325869183E-02		
7.378758334183E-03	1.889346512934E-02		

Table IV.

Quadratures for Polar Integration in 2-D r-z Cylindrical and 3-D x-y-z Geometries

Quadratures for Eqs. (32), (33) and (30) for Integral of $F(\cos \theta, \sin \theta) \sin \theta$, $0 < \theta < \pi/2$
 $x = \Omega_z = \mu = \cos \theta$, $0 < x < 1$, $X(l) = \Omega_z(l) = \mu(l) = \cos \theta(l) = \text{Polar cone}$, $W(l) = \text{Weight}$

$X(l) = \Omega_z(l)$	$W(l)$	$X(l) = \Omega_z(l)$	$W(l)$
N = 1		N = 8	
6.189908924467E-01	1.000000000000E+00	9.970063368207E-01	9.809563177794E-03
N = 2		9.699517300226E-01	4.993755387300E-02
8.634990514471E-01	3.867606081357E-01	8.879503955450E-01	1.167900832303E-01
2.707467653586E-01	6.132393918643E-01	7.361944193605E-01	1.840077166009E-01
N = 3		5.304474814371E-01	2.201579963795E-01
9.420141575125E-01	1.734817853566E-01	3.126552110074E-01	2.067710539408E-01
6.014839209434E-01	4.758504223783E-01	1.323869707831E-01	1.476215292233E-01
1.436154828008E-01	3.506677922651E-01	2.526885705460E-02	6.490450357423E-02
N = 4		N = 9	
9.692550874731E-01	9.483958290718E-02	9.985935590648E-01	4.687271581718E-03
7.603841957105E-01	3.239509561641E-01	9.848612009152E-01	2.638070556035E-02
3.781223125864E-01	3.911333839368E-01	9.384448738513E-01	6.974359023450E-02
7.288031758736E-02	1.900760769918E-01	8.409695379867E-01	1.259654703863E-01
N = 5		6.884671704442E-01	1.763082378214E-01
9.907228027705E-01	3.066865270574E-02	4.968907718281E-01	2.013205178931E-01
9.066754949760E-01	1.520873796776E-01	2.982703761263E-01	1.894159794100E-01
6.763407250056E-01	3.007989398347E-01	1.304306562034E-01	1.407412491186E-01
3.455719708920E-01	3.315500881013E-01	2.572642057725E-02	6.543697799409E-02
7.412304396110E-02	1.848949396807E-01	N = 10	
N = 6		9.988431905941E-01	3.831690452384E-03
9.934260252776E-01	2.136152921019E-02	9.878187060973E-01	2.096950939459E-02
9.365569210587E-01	1.021655746368E-01	9.512982489952E-01	5.461898814248E-02
7.781929016189E-01	2.139690494731E-01	8.748210152191E-01	9.932946005030E-02
5.233269543727E-01	2.824180551767E-01	7.528387780840E-01	1.433909126899E-01
2.469213715865E-01	2.523904145543E-01	5.927047772245E-01	1.735729930089E-01
5.065384196986E-02	1.276953769490E-01	4.137049559308E-01	1.799419832720E-01
N = 7		2.420294039117E-01	1.589187774698E-01
9.957315827450E-01	1.392973976943E-02	1.040133416870E-01	1.136148860623E-01
9.579357169756E-01	6.892898464294E-02	2.032527456069E-02	5.181079945732E-02
8.474708404724E-01	1.540022963383E-01		
6.540152494700E-01	2.268755020184E-01		
4.119940170234E-01	2.459222632522E-01		
1.854112547976E-01	1.963577927660E-01		
3.705166699011E-02	9.398342121276E-02		

Table IV (Continued₁).**Quadratures for Polar Integration in 2-D r-z Cylindrical and 3-D x-y-z Geometries**

Quadratures for Eqs. (32), (33) and (30) for Integral of $F(\cos \theta, \sin \theta) \sin \theta$, $0 < \theta < \pi/2$
 $x = \Omega_z = \mu = \cos \theta$, $0 < x < 1$, $X(l) = \Omega_z(l) = \mu(l) = \cos \theta(l) = \text{Polar cone}$, $W(l) = \text{Weight}$

$X(l) = \Omega_z(l)$	$W(l)$	$X(l) = \Omega_z(l)$	$W(l)$
N = 11		N = 14	
9.991302677320E-01	2.884161734127E-03	9.996558737820E-01	1.147057513783E-03
9.907904599945E-01	1.592166778168E-02	9.962693383644E-01	6.560585266642E-03
9.628342127515E-01	4.210429510321E-02	9.844006381495E-01	1.831225051549E-02
9.031675682025E-01	7.836749831056E-02	9.574371978087E-01	3.662153822498E-02
8.053034593427E-01	1.169567716752E-01	9.095151164539E-01	5.980800643841E-02
6.716836707816E-01	1.483035686839E-01	8.372368352245E-01	8.472257845895E-02
5.139106257595E-01	1.641342683895E-01	7.407906029852E-01	1.074785132224E-01
3.501888569958E-01	1.598016376376E-01	6.242752767306E-01	1.242764203624E-01
2.011107021879E-01	1.351922425517E-01	4.952311643392E-01	1.321396462151E-01
8.533860285514E-02	9.405053253703E-02	3.635415417531E-01	1.294179363033E-01
1.656084164276E-02	4.228335559564E-02	2.399856722056E-01	1.159714807527E-01
		1.347776768636E-01	9.301981285636E-02
		5.639551887688E-02	6.273486468456E-02
		1.086656296514E-02	2.778930918493E-02
N = 12		N = 15	
9.993182903074E-01	2.263464406185E-03	9.997217860289E-01	9.277955332396E-04
9.927399491150E-01	1.260438060930E-02	9.969770851516E-01	5.324881254380E-03
9.704415432003E-01	3.379182957262E-02	9.873133178911E-01	1.494948831251E-02
9.220724109667E-01	6.410234109849E-02	9.652017991798E-01	3.015294391042E-02
8.409912076571E-01	9.808702825277E-02	9.255042701452E-01	4.982592810351E-02
7.271112553886E-01	1.284451713994E-01	8.648133493431E-01	7.168598522528E-02
5.876609071491E-01	1.481987487236E-01	7.823872347417E-01	9.277828135504E-02
4.358871441498E-01	1.525460148381E-01	6.805340155142E-01	1.100543192029E-01
2.882445618572E-01	1.399867016900E-01	5.644276467831E-01	1.209126125418E-01
1.609145368599E-01	1.125198356935E-01	4.414412603907E-01	1.236103228459E-01
6.661300974079E-02	7.486149506603E-02	3.201611900935E-01	1.174821237352E-01
1.271017917531E-02	3.259298864993E-02	2.092924891631E-01	1.029369133400E-01
		1.166739455867E-01	8.124957900491E-02
		4.857478250334E-02	5.422386305719E-02
		9.333807675231E-03	2.388496257768E-02
N = 13			
9.996056789222E-01	1.317932282413E-03		
9.956783395241E-01	7.651720102310E-03		
9.817170248781E-01	2.166000129594E-02		
9.496614980081E-01	4.366484358444E-02		
8.924880178733E-01	7.129522897898E-02		
8.066758436297E-01	1.000807915314E-01		
6.937364298650E-01	1.246517659564E-01		
5.604409274254E-01	1.400696627407E-01		
4.178107622661E-01	1.429204365193E-01		
2.792375210342E-01	1.319338252731E-01		
1.582700765155E-01	1.080243650166E-01		
6.654296956713E-02	7.381055437147E-02		
1.288229858096E-02	3.291887234697E-02		

Table IV (Continued₂).**Quadratures for Polar Integration in 2-D r-z Cylindrical and 3-D x-y-z Geometries**

Quadratures for Eqs. (32), (33) and (30) for Integral of $F(\cos \theta, \sin \theta) \sin \theta$, $0 < \theta < \pi/2$
 $x = \Omega_z = \mu = \cos \theta$, $0 < x < 1$, $X(l) = \Omega_z(l) = \mu(l) = \cos \theta(l) = \text{Polar cone}$, $W(l) = \text{Weight}$

$X(l) = \Omega_z(l)$	$W(l)$	$X(l) = \Omega_z(l)$	$W(l)$
N = 16		N = 18	
9.997688033812E-01	7.714118602424E-04	9.998643186575E-01	4.535444460650E-04
9.974817188842E-01	4.443946511975E-03	9.985095657004E-01	2.646122049826E-03
9.893904569635E-01	1.254977819319E-02	9.936394403828E-01	7.618786987172E-03
9.707482915922E-01	2.551711882156E-02	9.821650338648E-01	1.589159962766E-02
9.369723698241E-01	4.260265070016E-02	9.607790068479E-01	2.737618872149E-02
8.847379407654E-01	6.208231592071E-02	9.265617655799E-01	4.139739026261E-02
8.127896715603E-01	8.160983285973E-02	8.775185305293E-01	5.680552361525E-02
7.223510099119E-01	9.864709537668E-02	8.129709750266E-01	7.214957225791E-02
6.171016248194E-01	1.108857393961E-01	7.337588102852E-01	8.587880206166E-02
5.027643330792E-01	1.165977355328E-01	6.422382246552E-01	9.654276634823E-02
3.863961603479E-01	1.148706343615E-01	5.420919860080E-01	1.029641396646E-01
2.755133143512E-01	1.057019241225E-01	4.379889494132E-01	1.043642048301E-01
1.771953459552E-01	8.994684678464E-02	3.351477918895E-01	1.004272305469E-01
9.731207205116E-02	6.912603219075E-02	2.388696773826E-01	9.129808005242E-02
4.001376430336E-02	4.508897765036E-02	1.541052327805E-01	7.751774985588E-02
7.627772606466E-03	1.955795971710E-02	8.511044202699E-02	5.991393459237E-02
		3.522248344992E-02	3.947666553464E-02
		6.746971030014E-03	1.727769854532E-02
N = 17			
9.998496261600E-01	5.033980879584E-04		
9.983377240080E-01	2.963411954252E-03		
9.928507662419E-01	8.619903084610E-03		
9.798024424910E-01	1.813659400784E-02		
9.553151902739E-01	3.141368012838E-02		
9.160087673050E-01	4.756815316756E-02		
8.597217960590E-01	6.508105325269E-02		
7.860255025427E-01	8.205441506288E-02		
6.964538247577E-01	9.651113160550E-02		
5.944358823670E-01	1.066822953425E-01		
4.849651757731E-01	1.112404525269E-01		
3.740752050608E-01	1.094493588037E-01		
2.682147415396E-01	1.012120443687E-01		
1.736297383459E-01	8.700201391745E-02		
9.591061666473E-02	6.748967807090E-02		
3.889663020451E-02	4.562281589888E-02		
7.709154866395E-03	1.844960071929E-02		

Table V.

Quadratures for Polar Integration in 2-D x-y Rectangular and Non-Rectangular Geometries

Quadratures for Integral of $F(\sin \theta) \sin \theta$, $0 < \theta < \pi/2$, or

$F(x) \omega(x)$, $\omega(x) = x / (1-x^2)^{1/2}$, $0 < x < 1$, $x = \sin \theta$

$X(l)$, $W(l)$, $X(l) = \sin \theta(l)$

$X(l)$	$W(l)$	$X(l)$	$W(l)$
N = 1		N = 8	
7.853981633974E-01	1.000000000000E+00	4.619235546733E-02	3.535298053836E-03
N = 2		1.495785939930E-01	1.939781153201E-02
3.993735638824E-01	2.505459190576E-01	2.974141685414E-01	5.120081207156E-02
9.144479484660E-01	7.494540809424E-01	4.714033535760E-01	9.615210566300E-02
N = 3		6.494576043473E-01	1.471910521216E-01
2.311522961279E-01	8.530082397263E-02	8.085613275367E-01	1.956194174053E-01
6.399711962341E-01	3.414562579328E-01	9.279142834478E-01	2.332109926667E-01
9.544967673433E-01	5.732429180946E-01	9.918094732391E-01	2.536925104860E-01
N = 4		N = 9	
1.492491465731E-01	3.605010432091E-02	3.738717543137E-02	2.321532538025E-03
4.452174134056E-01	1.651009783738E-01	1.218673184482E-01	1.295685275068E-02
7.657198751837E-01	3.388394165669E-01	2.449791366336E-01	3.503122447209E-02
9.718403838896E-01	4.600095007383E-01	3.944837657444E-01	6.774755109636E-02
N = 5		5.551788717533E-01	1.073193796372E-01
1.039469054750E-01	1.765116967436E-02	7.104369578734E-01	1.483922364818E-01
3.218997600306E-01	8.726300032090E-02	8.440015993099E-01	1.853051377127E-01
5.925641316724E-01	1.993860321943E-01	9.417816602326E-01	2.130391465706E-01
8.367574981863E-01	3.127359798582E-01	9.934143180700E-01	2.278869387406E-01
9.808766137327E-01	3.829638179523E-01	N = 10	
N = 6		3.087344674321E-02	1.585962741949E-03
7.642225770190E-02	9.602183026857E-03	1.011256691908E-01	8.966281037954E-03
2.418380224706E-01	4.982286016750E-02	2.049120334874E-01	2.469133544735E-02
4.624149310771E-01	1.216972042400E-01	3.337559301018E-01	4.884013052964E-02
6.917637890192E-01	2.077474677705E-01	4.769246984678E-01	7.941386493165E-02
8.801887364310E-01	2.835447130159E-01	6.222949084152E-01	1.131174844298E-01
9.861717557162E-01	3.275855717792E-01	7.574090067647E-01	1.461301662904E-01
N = 7		8.705939190886E-01	1.747226071284E-01
5.849788427341E-02	5.651443322490E-03	9.520192454053E-01	1.957168918510E-01
1.876744099204E-01	3.029834441270E-02	9.945898978712E-01	2.068152756119E-01
3.673808738342E-01	7.742885116751E-02		
5.691413298688E-01	1.397678418675E-01		
7.600739892315E-01	2.041795195416E-01		
9.084990189032E-01	2.566694985682E-01		
9.895378462167E-01	2.860045011201E-01		

Table VI.
Quadratures for Azimuthal Integration in 2-D r-z Cylindrical Geometries

Quadratures for Integral of $F(\cos \phi)$, $0 < \phi < \pi/2$, or
 $F(x) \omega(x)$, $\omega(x) = 1 / (1-x^2)^{1/2}$, $0 < x < 1$, $x = \cos \phi$
 $X(l)$, $W(l)$, $X(l) = \cos \phi(l)$

$X(l)$	$W(l)$	$X(l)$	$W(l)$
N = 1		N = 8	
6.366197723676E-01	1.570796326795E+00	2.058150321726E-02	5.249421382819E-02
N = 2		1.055080677429E-01	1.161797635343E-01
2.429927416501E-01	5.959299768839E-01	2.464757489764E-01	1.683422031465E-01
8.772416251352E-01	9.748663499110E-01	4.241955600149E-01	2.076474812112E-01
N = 3		6.135274802892E-01	2.355229189552E-01
1.238764288792E-01	3.087609757690E-01	7.870729700953E-01	2.541768525953E-01
5.500785229008E-01	5.768331132780E-01	9.193594392658E-01	2.655355245488E-01
9.405227398999E-01	6.852022377479E-01	9.908112964227E-01	2.708973689754E-01
N = 4		N = 9	
7.455877012217E-02	1.876504950502E-01	1.643732212932E-02	4.197350818481E-02
3.556486768115E-01	3.759446152299E-01	8.472238061017E-02	9.376560322843E-02
7.162558109838E-01	4.813994067786E-01	1.999677384679E-01	1.377358446966E-01
9.651919658866E-01	5.258018097362E-01	3.496474051891E-01	1.724679735452E-01
N = 5		5.170080886249E-01	1.984983187666E-01
4.966953454771E-02	1.257279221582E-01	6.829107634870E-01	2.171740422782E-01
2.450638068085E-01	2.626395733979E-01	8.281045708332E-01	2.298930197517E-01
5.304231821113E-01	3.528347422000E-01	9.355858649411E-01	2.377652951680E-01
8.076170369864E-01	4.036134763392E-01	9.926986564284E-01	2.415227211753E-01
9.772154881587E-01	4.259806126995E-01	N = 10	
N = 6		1.342824838436E-02	3.431954126375E-02
3.541634230105E-02	8.997072729173E-02	6.948712385061E-02	7.720513474657E-02
1.780640541547E-01	1.930565561353E-01	1.652292828984E-01	1.146175823175E-01
4.004819749054E-01	2.681726916929E-01	2.921823096087E-01	1.452645689176E-01
6.491670135146E-01	3.167912172320E-01	4.388173096638E-01	1.692218673749E-01
8.618143747521E-01	3.449903291201E-01	5.915513208592E-01	1.872741688069E-01
9.839469056393E-01	3.578148053229E-01	7.360308895521E-01	2.003961201296E-01
N = 7		8.585450000021E-01	2.094649247126E-01
2.651100154605E-02	6.750727300065E-02	9.473937073276E-01	2.151486323949E-01
1.348546683446E-01	1.475078835334E-01	9.940594766523E-01	2.178837861306E-01
3.103573950750E-01	2.099229174374E-01		
5.218230715245E-01	2.540781440086E-01		
7.302364923763E-01	2.830648798432E-01		
8.962429964403E-01	3.003419147495E-01		
9.880867243590E-01	3.083733142221E-01		

Table VII.

Angular Quadratures for Polar Integration Useful for Gamma Penetrations

Abbreviation	Quadrature Identification ^a
MG 26	MG 26 (4, 3.0; 4, 2.0; 9; 0, 0.0; 0.0, 0.0, .95, 1.0)
MP 20-1	MP 20-1 (16, 1.5; 4, .67; 0, 0.0)
MP 20-2	MP 20-2 (10, 2.0; 6, 1.5; 4, .67)
MP 20-3	MP 20-3 (20, 1.5; 0, 0.0; 0, 0.0)
MG 20-1	MG 20-1 (8, 2.0; 8, 1.5; 2; 0, 0.0; 0.0, 0.0, .5, 1.0)
MG 20-2	MG 20-2 (10, 2.0; 6, 1.5; 2; 0, 0.0; 0.0, 0.0, .5, 1.0)
MMG 20-1	MMG 20-1 (4, 3.0; 6, 2.0; 6, 1.5; 2; 0.0, .5, 1.0)
MMG 20-2	MMG 20-2 (4, 3.0; 6, 2.0; 6, 1.5; 2; 0.0, .58, 1.0)
MP 16-1	MP 16-1 (6, 2.0; 6, 1.5; 4, .67)
MP 16-2	MP 16-2 (6, 2.0; 6, 1.5; 4, .5)
MP 16-3	MP 16-3 (4, 2.0; 8, 1.5; 4, .67)
MP 16-4	MP 16-4 (4, 2.0; 8, 1.5; 4, .5)
MP 16-5	MP 16-5 (4, 3.0; 8, 1.5; 4, .67)
MG 16-1	MG 16-1 (6, 2.0; 6, 1.5; 2; 0, 0.0; 0.0, 0.0, .58, 1.0)
MG 16-2	MG 16-2 (6, 2.0; 4, 1.5; 3; 0, 0.0; 0.0, 0.0, .77, 1.0)
MMG 16	MMG 16 (3, 3.0; 3, 2.0; 6, 1.5; 2; 0.0, .58, 1.0)
MG 14-1	MG 14-1 (4, 2.0; 6, 1.5; 2; 0, 0.0; 0.0, 0.0, .58, 1.0)
MG 14-2	MG 14-2 (5, 2.0; 5, 1.5; 2; 0, 0.0; 0.0, 0.0, .58, 1.0)
MG 14-3	MG 14-3 (6, 2.0; 4, 1.5; 2; 0, 0.0; 0.0, 0.0, .58, 1.0)
MG 14-4	MG 14-4 (5, 2.0; 3, 1.5; 3; 0, 0.0; 0.0, 0.0, .77, 1.0)
MG 14-5	MG 14-5 (4, 2.0; 4, 1.5; 3; 0, 0.0; 0.0, 0.0, .77, 1.0)
MG 14-6	MG 14-6 (8, 1.6; 0, 0.0; 3; 0, 0.0; 0.0, 0.0, .77, 1.0)
MG 14-7	MG 14-7 (10, 1.6; 0, 0.0; 2; 0, 0.0; 0.0, 0.0, .58, 1.0)
MG 12-1	MG 12-1 (4, 2.0; 6, 1.5; 0; 2, .3; 0.0, 0.3, 0.3, 1.0)
MG 12-2	MG 12-2 (5, 2.0; 5, 1.5; 0; 2, .3; 0.0, 0.3, 0.3, 1.0)
MG 12-3	MG 12-3 (3, 2.0; 5, 1.5; 2; 0, 0.0; 0.0, 0.0, .58, 1.0)
MG 12-4	MG 12-4 (4, 2.0; 4, 1.5; 2; 0, 0.0; 0.0, 0.0, .58, 1.0)
MHG 12-1	MHG 12-1 (5, 2.0; 5, 1.5; 2, .29)
MHG 12-2	MHG 12-2 (4, 2.0; 5, 1.5; 3, .36)
MHG 12-3	MHG 12-3 (5, 2.0; 4, 1.5; 3, .36)

^a See text description in Section 5 for more detail.

Table VIII.

Mid-Point Rule and Composite Type Quadratures for Polar Integration

Quadratures for Integral of $F(\mu) = F(x)$, $x = \Omega_z = \mu = \cos \theta$, $0 < \theta < \pi/2$, $0 < x = \mu < 1$
 $X(l) = \Omega_z(l) = \mu(l) = \cos \theta(l) = \text{Polar cone}$, $W(l) = \text{Weight}$, $L(l) = \text{angles per octant on cone}$

$X(l) = \Omega_z(l)$	$W(l)$	$L(l)$	$X(l) = \Omega_z(l)$	$W(l)$	$L(l)$
MG 12/3-1 MG 12-1 (4,2.0;6,1.5;0;2,3;0,3,3,1.)			MG 16/3-1 MG 16-1 (6,2.0;6,1.5;2;0,0;0,0,58,1.)		
9.986761229314E-01	2.647754137116E-03	3	9.998019801980E-01	3.960396039604E-04	3
9.947044917258E-01	5.295508274232E-03	3	9.992079207921E-01	7.920792079208E-04	3
9.867612293144E-01	1.059101654846E-02	3	9.980198019802E-01	1.584158415842E-03	3
9.708747044917E-01	2.118203309693E-02	3	9.956435643564E-01	3.168316831683E-03	3
9.443971631206E-01	3.177304964539E-02	3	9.908910891089E-01	6.336633663366E-03	3
9.046808510638E-01	4.765957446808E-02	3	9.813861386139E-01	1.267326732673E-02	3
8.451063829787E-01	7.148936170213E-02	3	9.655445544554E-01	1.900990099010E-02	3
7.557446808510E-01	1.072340425532E-01	3	9.417821782178E-01	2.851485148515E-02	3
6.217021276596E-01	1.608510638298E-01	3	9.061386138614E-01	4.277227722772E-02	3
4.206382978723E-01	2.412765957447E-01	3	8.526732673267E-01	6.415841584158E-02	3
1.846153846154E-01	2.307692307692E-01	3	7.724752475247E-01	9.623762376238E-02	3
3.461538461538E-02	6.923076923077E-02	3	6.521782178218E-01	1.443564356436E-01	3
MG 12/3-2 MG 12-2 (5,2.0;5,1.5;0;2,3;0,3,3,1.)			5.397295303623E-01	1.008779050899E-01	3
9.989928057554E-01	2.014388489209E-03	3	3.885945026396E-01	1.891220949101E-01	3
9.959712230216E-01	4.028776978417E-03	3	1.914054973604E-01	1.891220949101E-01	3
9.899280575539E-01	8.057553956834E-03	3	4.027046963773E-02	1.008779050899E-01	3
9.778417266187E-01	1.611510791367E-02	3	MP 16/3-1 MP16-1 (6,2.0;6,1.5;4,67)		
9.536690647482E-01	3.223021582734E-02	3	9.996961409907E-01	6.077180186215E-04	3
9.133812949640E-01	4.834532374101E-02	3	9.987845639628E-01	1.215436037243E-03	3
8.529496402878E-01	7.251798561151E-02	3	9.969614099069E-01	2.430872074486E-03	3
7.623021582734E-01	1.087769784173E-01	3	9.933151017952E-01	4.861744148972E-03	3
6.263309352518E-01	1.631654676259E-01	3	9.860224855717E-01	9.723488297945E-03	3
4.223741007194E-01	2.447482014388E-01	3	9.714372531248E-01	1.944697659589E-02	3
1.846153846154E-01	2.307692307692E-01	3	9.471285323799E-01	2.917046489383E-02	3
3.461538461538E-02	6.923076923077E-02	3	9.106654512626E-01	4.375569734075E-02	3
MG 14/3-1 MG 14-1 (4,2.0; 6,1.5;2;0,0;0,0,58,1.)			8.559708295867E-01	6.563354601113E-02	3
9.992056737589E-01	1.588652482270E-03	3	7.739288970728E-01	9.845031901669E-02	3
9.968226950355E-01	3.177304964539E-03	3	6.508659983019E-01	1.476754785250E-01	3
9.920567375886E-01	6.354609929078E-03	3	4.662716501456E-01	2.215132177875E-01	3
9.825248226950E-01	1.270921985816E-02	3	2.816773019524E-01	1.476754785989E-01	3
9.666382978723E-01	1.906382978723E-02	3	1.586144030954E-01	9.845031911514E-02	3
9.428085106383E-01	2.859574468085E-02	3	7.657247048305E-02	6.563354610958E-02	3
9.070638297872E-01	4.289361702128E-02	3	2.187784871413E-02	4.375569742826E-02	3
8.534468085106E-01	6.434042553191E-02	3			
7.730212765957E-01	9.651063829787E-02	3			
6.523829787234E-01	1.447659574468E-01	3			
5.397295303623E-01	1.008779050899E-01	3			
3.885945026396E-01	1.891220949101E-01	3			
1.914054973604E-01	1.891220949101E-01	3			
4.027046963773E-02	1.008779050899E-01	3			

Table VIII (Continued₁).**Mid-Point Rule and Composite Type Quadratures for Polar Integration**

Quadratures for Integral of $F(\mu) = F(x)$, $x = \Omega_z = \mu = \cos \theta$, $0 < \theta < \pi/2$, $0 < x = \mu < 1$
 $X(l) = \Omega_z(l) = \mu(l) = \cos \theta(l) = \text{Polar cone}$, $W(l) = \text{Weight}$, $L(l) = \text{angles per octant on cone}$

$X(l) = \Omega_z(l)$	$W(l)$	$L(l)$	$X(l) = \Omega_z(l)$	$W(l)$	$L(l)$
MG 20/3-1 MG 20-1 (8,2.0; 8,1.5;2;0,0;0,0,.,5,1.)			MG 26/v MG 26 (4, 3.0;4, 2.;9;0, 0.;0,0,.,95,1.)		
9.999742599743E-01	5.148005148005E-05	3	9.999705882353E-01	5.882352941177E-05	6
9.998970398970E-01	1.029601029601E-04	3	9.998529411765E-01	1.764705882353E-04	6
9.997425997426E-01	2.059202059202E-04	3	9.995000000000E-01	5.294117647059E-04	6
9.994337194337E-01	4.118404118404E-04	3	9.984411764706E-01	1.588235294118E-03	6
9.988159588159E-01	8.236808236808E-04	3	9.960588235294E-01	3.176470588236E-03	6
9.975804375804E-01	1.647361647362E-03	3	9.912941176471E-01	6.352941176471E-03	6
9.951093951094E-01	3.294723294723E-03	3	9.817647058824E-01	1.270588235294E-02	6
9.901673101673E-01	6.589446589447E-03	3	9.627058823530E-01	2.541176470588E-02	6
9.819305019305E-01	9.884169884170E-03	3	9.459934549999E-01	1.026760642508E-02	6
9.695752895753E-01	1.482625482625E-02	3	9.290163760464E-01	2.361441072511E-02	6
9.510424710425E-01	2.223938223938E-02	3	8.989861715863E-01	3.630222187107E-02	6
9.232432432432E-01	3.335907335907E-02	3	8.567598555119E-01	4.794747095049E-02	6
8.815444015444E-01	5.003861003861E-02	3	8.035513454537E-01	5.821372318795E-02	6
8.189961389961E-01	7.505791505791E-02	3	7.408911447601E-01	6.680538446856E-02	6
7.251737451737E-01	1.125868725869E-01	3	6.705818016948E-01	7.347522068498E-02	6
5.844401544401E-01	1.688803088803E-01	3	5.946459572035E-01	7.803132977927E-02	6
4.652840778985E-01	8.696371128436E-02	3	5.152681311948E-01	8.034263190749E-02	7
3.349952608962E-01	1.630362887156E-01	3	4.347318688052E-01	8.034263190749E-02	7
1.650047391038E-01	1.630362887156E-01	3	3.553540427965E-01	7.803132977927E-02	7
3.471592210149E-02	8.696371128436E-02	3	2.794181983052E-01	7.347522068498E-02	7
MMG 20/3-1 MMG20-1(4, 3.; 6, 2.; 6, 1.5; 2; 0,.,5,1.)			2.091088552399E-01	6.680538446856E-02	7
9.999956375312E-01	8.724937616696E-06	3	1.464486545463E-01	5.821372318795E-02	7
9.999781876560E-01	2.617481285009E-05	3	9.324014448805E-02	4.794747095049E-02	7
9.999258380303E-01	7.852443855026E-05	3	5.101382841366E-02	3.630222187107E-02	7
9.997687891531E-01	2.355733156508E-04	3	2.098362395359E-02	2.361441072511E-02	7
9.994154291797E-01	4.711466313016E-04	3	4.006545000060E-03	1.026760642508E-02	7
9.987087092327E-01	9.422932626032E-04	3			
9.972952693388E-01	1.884586525206E-03	3			
9.944683895510E-01	3.769173050413E-03	3			
9.888146299754E-01	7.538346100825E-03	3			
9.775071108241E-01	1.507669220165E-02	3			
9.586612455721E-01	2.261503830248E-02	3			
9.303924476940E-01	3.392255745371E-02	3			
8.879892508768E-01	5.088383618057E-02	3			
8.243844556511E-01	7.632575427086E-02	3			
7.289772628126E-01	1.144886314063E-01	3			
5.858664735547E-01	1.717329471094E-01	3			
4.652840778985E-01	8.696371128436E-02	3			
3.349952608962E-01	1.630362887156E-01	3			
1.650047391038E-01	1.630362887156E-01	3			
3.471592210149E-02	8.696371128436E-02	3			

Table IX.
Notation for Quadratures for General X-Y Problems

Quadrature	Description
s16 = S_{16}	Standard axially invariant symmetric S_{16} quadrature with 36 angles/octant.
sg16 dg16	S_{16} -like quadrature with 36 angles/octant based respectively on 8 SG and 8 DG cones, one angle on the top cone closest to the z-axis, two on the second cone, ..., and eight on the bottom cone closest to the x-y plane. The azimuthal angles on each cone are uniformly distributed.
sg10x20 dg10x20	Respectively 10 SG and DG polar cone quadratures each with 20 uniformly distributed azimuthal angles/cone/octant. Total 200 angles/octant.
s10x20, d10x20 q10x20	Benchmarks with respectively 10 SG, DG and QR polar cone quadratures each with 20 QR azimuthal angles/cone/octant. Total 200 angles/octant.
sg4x20 dg4x20	Respectively 4 SG and DG polar cone quadratures each with 20 uniformly distributed azimuthal angles/cone/octant. Total 80 angles/octant.
s4x20, d4x20 q4x20	Respectively 4 SG, DG and QR polar cone quadratures each with 20 QR azimuthal angles/cone/octant. Total 80 angles/octant.
sgijkl dgijkl	Respectively 4 SG and DG polar cone quadratures each with i, j, k and l uniformly distributed azimuthal angles/octant on the various cones. Total i+j+k+l angles/octant.
sijkl dijkl qijkl	Respectively 4 SG DG and QR polar cone quadratures each with i, j, k and l QR azimuthal angles/octant on the various cones. Total i+j+k+l angles/octant.
dg10111213 = dg13121110	Four DG polar cones and respectively 10, 11, 12, and 13 uniformly distributed azimuthal angles/octant on top, second, third, and bottom cones. Total 46 angles/octant (Developed by Gelbard and Shure [6])

Table X

Deviation of Edge Fluxes for the Ray Effect Problem of Figure 2 Corresponding to Various Quadratures from Edge Fluxes Corresponding to Benchmark Quadratures

Quadrature	Range of Edge Flux Deviations	Quadrature	Range of Edge Flux Deviations
dg4444	-8% to 13%	dg2446	-8.0% to 11.2%
sg4444	-8% to 13%	sg2446	-7.1% to 9.6%
d4444	-11% to 10%	d2446	-5.8% to 4.3%
s4444	-11% to 10%	<u>s2446^a</u>	<u>-5.2% to 5.5%</u>
		q2446	-5.3% to 5.7%
dg4466	-6.5% to 8.2%	dg2468	-6.5% to 7.5%
sg4466	-6.0% to 6.7%	sg2468	-5.1% to 5.6%
d4466	-5.9% to 5.0%	d2468	-7.2% to 3.5%
s4466	-6.5% to 3.7%	s2468	-4.8% to 2.3%
		<u>q2468</u>	<u>-4.3% to 2.1%</u>
dg881010	-3.3% to 3.3%	dg681012	-3.2% to 3.4%
sg881010	-3.5% to 2.9%	sg681012	-2.9% to 2.9%
d881010	-3.8% to 1.7%	d681012	-3.4% to 1.2%
s881010	-3.5% to 2.1%	s681012	-2.9% to 0.9%
		q681012	-2.7% to 1.0%
dg481014	-3.2% to 3.3%	dg461214	-3.6% to 3.5%
sg481014	-2.6% to 2.4%	sg461214	-2.8% to 2.5%
d481014	-2.6% to 1.3%	d461214	-3.1% to 1.4%
s481014	-1.1% to 1.1%	s461214	-1.4% to 1.0%
q481014	-0.8% to 1.6%	<u>q461214</u>	<u>-0.6% to 1.0%</u>
d241416	-4.8% to 3.0%	d241218	-5.0% to 2.9%
s241416	-2.2% to 1.3%	s241218	-2.2% to 1.3%
dg16	-7.5% to 8.0%	dg13121110	-2.3% to 2.3%
<u>sg16</u>	<u>-5.4% to 6.2%</u>	<u>sg13121110</u>	<u>-2.0% to 2.0%</u>
S ₁₆	-7.0% to 7.7%	u13121110	-0.4% to 4.3%
q461820	-.50% to .55%	<u>q681420</u>	<u>-.37% to .53%</u>
<u>q481620</u>	<u>-.56% to .32%</u>	q6101220	-1.2% to .44%
q4101420	-.26% to .78%	q681618	-.61% to .68%
		q6101418	-.55% to .93%
dg4x20	-1.3% to 0.9%	d4x20	-.34% to .01%
sg4x20	-1.3% to 0.8%	s4x20	-.14% to -.09%
		<u>q4x20</u>	<u>+.01% to .05%</u>

^a The best quadrature within each family is highlighted in bold face and is underlined.

Table XI.

Q36 = q461214 Octant-Range or Quadruple-Range Quadrature for Rectangular x-y Geometry

m	Ω_x^m	Ω_y^m	W_m
1	9.717784813336E-01	1.096881837272E-02	8.454511187252E-03
2	7.656319455497E-01	1.160393058611E-02	8.352354145856E-03
3	4.445439440056E-01	2.447911451942E-02	1.460888798152E-02
4	9.701698603928E-01	5.695764868253E-02	1.913728513580E-02
5	7.633693960835E-01	5.995074957044E-02	1.873220073879E-02
6	1.483114568272E-01	1.670387759191E-02	6.404244616724E-03
7	9.622473153642E-01	1.362124657777E-01	2.863542971348E-02
8	7.524467626583E-01	1.419535016004E-01	2.759429759588E-02
9	9.410672772109E-01	2.426233944222E-01	3.648716160597E-02
10	4.288508824476E-01	1.196054590036E-01	2.995376809966E-02
11	7.241384940891E-01	2.488983098158E-01	3.442681426024E-02
12	8.997294996538E-01	3.673697853806E-01	4.244873302980E-02
13	6.711819639118E-01	3.685670882907E-01	3.901232700510E-02
14	1.293388490485E-01	7.447663982495E-02	1.162080754372E-02
15	8.335743322378E-01	4.996274255819E-01	4.642823955812E-02
16	3.670788892962E-01	2.519357740235E-01	3.798783310581E-02
17	5.909368760506E-01	4.869502395267E-01	4.130171453748E-02
18	7.417637460141E-01	6.279014865859E-01	4.841339013884E-02
19	6.279014865859E-01	7.417637460141E-01	4.841339013884E-02
20	4.869502395267E-01	5.909368760506E-01	4.130171453748E-02
21	2.519357740235E-01	3.670788892962E-01	3.798783310581E-02
22	4.996274255819E-01	8.335743322378E-01	4.642823955812E-02
23	7.447663982495E-02	1.293388490485E-01	1.162080754372E-02
24	3.685670882907E-01	6.711819639118E-01	3.901232700510E-02
25	3.673697853806E-01	8.997294996538E-01	4.244873302980E-02
26	2.488983098158E-01	7.241384940891E-01	3.442681426024E-02
27	1.196054590036E-01	4.288508824476E-01	2.995376809966E-02
28	2.426233944222E-01	9.410672772109E-01	3.648716160597E-02
29	1.419535016004E-01	7.524467626583E-01	2.759429759588E-02
30	1.362124657777E-01	9.622473153642E-01	2.863542971348E-02
31	1.670387759191E-02	1.483114568272E-01	6.404244616724E-03
32	5.995074957044E-02	7.633693960835E-01	1.873220073879E-02
33	5.695764868253E-02	9.701698603928E-01	1.913728513580E-02
34	2.447911451942E-02	4.445439440056E-01	1.460888798152E-02
35	1.160393058611E-02	7.656319455497E-01	8.352354145856E-03
36	1.096881837272E-02	9.717784813336E-01	8.454511187252E-03

Table XII.

Q20 = q2468 Octant-Range or Quadruple-Range Quadrature for Rectangular x-y Geometry

m	Ω_x^m	Ω_y^m	w_m
1	9.713274064903E-01	3.157215799340E-02	2.419260514149E-02
2	7.645615896150E-01	4.210110375297E-02	2.998205782366E-02
3	4.424202396002E-01	4.982847370367E-02	2.932993043666E-02
4	9.586898685237E-01	1.593344524838E-01	5.213067212540E-02
5	7.375714298063E-01	2.057068622698E-01	6.147460425028E-02
6	1.409476441875E-01	4.908227124734E-02	1.802505216045E-02
7	9.028558915298E-01	3.596178122512E-01	7.185542471164E-02
8	3.858240341629E-01	2.221674140412E-01	5.322055875020E-02
9	6.313311043797E-01	4.332989313333E-01	7.796304620960E-02
10	7.770210099715E-01	5.837054752370E-01	8.182604839076E-02
11	5.837054752370E-01	7.770210099715E-01	8.182604839076E-02
12	4.332989313333E-01	6.313311043797E-01	7.796304620960E-02
13	2.221674140412E-01	3.858240341629E-01	5.322055875020E-02
14	3.596178122512E-01	9.028558915298E-01	7.185542471164E-02
15	4.908227124734E-02	1.409476441875E-01	1.802505216045E-02
16	2.057068622698E-01	7.375714298063E-01	6.147460425028E-02
17	1.593344524838E-01	9.586898685237E-01	5.213067212540E-02
18	4.982847370367E-02	4.424202396002E-01	2.932993043666E-02
19	4.210110375297E-02	7.645615896150E-01	2.998205782366E-02
20	3.157215799340E-02	9.713274064903E-01	2.419260514149E-02