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On Limited Fan-in Optimal Neural Networks

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Abstract

Because VLSI implementations do not cope well with highly interconnected nets—the area of a chip growing as the cube of the fan-in [25]—this paper analyses the influence of limited fan-in on the size and VLSI optimality of such nets.

Two different approaches will show that VLSI- and size-optimal discrete neural networks can be obtained for small (i.e. lower than linear) fan-in values. They have applications to hardware implementations of neural networks. The first approach is based on implementing a certain sub-class of Boolean functions, $\mathbb{F}_{n,m}$ functions [34]. We will show that this class of functions can be implemented in VLSI-optimal (i.e., minimising AT^2) neural networks of small constant fan-ins. The second approach is based on implementing Boolean functions for which the classical Shannon's decomposition can be used. Such a solution has already been used to prove bounds on neural networks with fan-ins limited to 2 [26]. We will generalise the result presented there to arbitrary fan-in, and prove that the size is minimised by small fan-in values, while relative minimum size solutions can be obtained for fan-ins strictly lower than linear.

Finally, a size-optimal neural network having small constant fan-ins will be suggested for $\mathbb{F}_{n,m}$ functions.

Keywords—neural networks, VLSI, fan-in, Boolean circuits, threshold circuits, $\mathbb{F}_{n,m}$ functions.

1. Introduction

In this paper we shall consider feedforward neural networks (NNs) made of linear threshold gates (TGs). A neuron (i.e., linear TG) will compute a Boolean function (BF) $f: \{0,1\}^n \rightarrow \{0,1\}$, where one of the k input vectors is $\mathbf{Z}_k = (z_{k,0}, \dots, z_{k,n-1}) \in \{0,1\}^n$ and $f(\mathbf{Z}_k) = \text{sgn}(\sum_{i=0}^{n-1} w_i z_{k,i} + \theta)$, with the synaptic weights $w_i \in \mathbb{R}$, thresholds $\theta \in \mathbb{R}$, and sgn the sign function. Two cost functions commonly associated to a NN are: (i) *depth*, which is the number of layers (or the number of edges—if we consider unit length for all the edges connecting the TGs) on the longest input to output path; and (ii) *size* (or node complexity), which is the number of neurons (TGs). Unfortunately, these measures are not the best criteria for ranking different solutions when going for silicon [21], as “comparing the number of nodes is inadequate for comparing the complexity of neural networks as the nodes themselves could implement quite complex functions” [41].

The fact that *size* is not a good estimate for *area* is explained as: (i) the *area* (of one neuron) can be related to its associated weights; and (ii) the *area* of the connections is—in most cases—neglected. Here are several alternatives of how the *area* could scale in relation to weights and thresholds:

- for purely digital implementation, the *area* scales at least with the cumulative size of the weights and thresholds (the bits for representing these weights and thresholds have to be stored);
- for certain types of analog implementations (e.g., using resistors or capacitors), the same type of scaling

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is valid (in particular cases, analog implementations can have binary encoding, thus the *area* would scale with the cumulative *log*-scale size of the parameters);

- there are some types of implementations (e.g., transconductance ones) which offer a constant size per element, thus scaling only with $\sum_{NN} fan-ins$.

All these 'cost functions' are linked to VLSI by the assumptions one makes on how the *area* of a chip scales with the *weights* and the *thresholds* [5, 9, 10]. That is why several other measures (i.e., 'cost functions')—beside *size*—have already been used:

- the total *number-of-connections*, or $\sum_{NN} fan-ins$, has been used by several authors [1, 25, 30, 33];
- the total *number-of-bits* needed to represent the *weights* and *thresholds* $\sum_{NN} (\sum_i \lceil \log |w_i| \rceil + \lceil \log |\theta| \rceil)^1$ has been used by others [22, 23, 41];
- the sum of all the absolute values of the *weights* and *thresholds* $\sum_{NN} (\sum_i |w_i| + |\theta|)$ has also been advocated [5, 10, 18, 19, 21], while another similar 'cost function' is $\sum_{NN} (\sum_i w_i^2 + \theta^2)$, which has been used in the context of genetic programming for reaching minimal NNs [44].

The sum of all the absolute values of the *weights* and *thresholds* has also been used as an optimum criterion for: (i) linear programming synthesis [32]; (ii) defining the minimum-integer TG realisation of a function [27]. Recently [3], the same measure (under the name of "total weight magnitude") has been used in the context of computational learning theory. It was proven that the generalisation error of NNs used for classification depends on the size of the *weights*—rather than the number of *weights*—by showing that the misclassification probability converges at a rate of $O\{(cA)^{1/l}/\sqrt{m}\}$. Here A is the sum of the magnitude of the *weights*, l is the *depth*, m is the number of examples, and c is a constant.

With respect to *delay*, two VLSI models have been commonly in use [38]:

- the simplest one assumes that *delay* is proportional to the input capacitance, hence a TG introduces a *delay* proportional to its *fan-in*;
- a more exact one considers the capacitance along any wire, hence the *delay* is proportional to the *length* of the connecting wires.

It is worth emphasising that it is anyhow desirable to limit the range of parameter values [42] for VLSI implementations—be they digital or analog—because both the maximum value of the *fan-in* [28, 40] and the maximal ratio between the largest and the smallest *weight* [23, 24, 29, 42] cannot grow over a certain (technological) limit.

¹ In this paper $\lfloor x \rfloor$ is the floor of x , i.e. the largest integer less than or equal to x , and $\lceil x \rceil$ is the ceiling of x , i.e. the smallest integer greater or equal to x . In this paper all the logarithms are to the base 2, except explicitly mentioned otherwise.

The focus of this paper will be on NNs having limited *fan-in* (the *fan-in* will be denoted by Δ). We will present both theoretical proofs and simulations in support of our claim that VLSI- and *size*-optimal NNs can be obtained for small *fan-ins*. For simplification, we shall consider only NNs having n binary inputs and μ binary outputs (if real inputs and outputs are needed, it is always possible to quantize them up to a certain number of bits such as to achieve a desired precision [4, 7, 11, 24]). In Section 2 we shall present a theoretical proof for the $\mathcal{F}_{n,m}$ class of functions, showing that their VLSI-optimal implementation is achieved with small constant *fan-ins*. Section 3 deals with BFs, and details the generalisation of a result from [26] to arbitrary *fan-ins*. Based on that generalisation we will show that the *size* can be minimised for small *fan-ins*. Finally, in Section 4 we will suggest how to implement $\mathcal{F}_{n,m}$ functions in *size*-optimal NNs having small constant *fan-ins*.

Conclusions, open questions and further directions for research complete the paper. Due to space limitations some of the lengthy mathematical proofs suggested in [6, 7] have been omitted, but can be found in [9, 10].

2. VLSI-optimal neural implementations of $\mathcal{F}_{n,m}$ functions

$\mathcal{F}_{n,m}$ is the class of BFs of n variables having m groups of ones in their truth table. Obviously, any BF can be represented by a suitable collection of its true values (ones), but for achieving that the number of groups of ones grows exponentially (i.e. $\mathcal{F}_{n,2^{n/2}}$ completely covers $\mathcal{B}_n$, the set of all n -ary BFs). This class of functions has been introduced and analysed by Red'kin [34]; he constructively proved that a *depth*-3 NN has *size*.

Proposition 1 (Theorem 1 from [34]) *The complexity realisation (i.e., number of threshold elements) of $\mathcal{F}_{n,m}$ (the class of Boolean functions $f(x_1, x_2, \dots, x_{n-1}, x_n)$ that have exactly m groups of ones) is at most $2\sqrt{2m} + 3$.*

The construction has: a first layer of $\lceil (2m)^{1/2} \rceil$ TGs (COMPARISONS) with *fan-in* = n and *weights* $\leq 2^{n-1}$; a second layer of $2\lceil (m/2)^{1/2} \rceil$ TGs of *fan-in* = $n + \lceil (2m)^{1/2} \rceil$ and *weights* $\leq 2^n$; one more TG of *fan-in* = $2\lceil (m/2)^{1/2} \rceil$ and *weights* $\in \{-1, +1\}$ in the third layer.

Red'kin also proved that if the implementation of BFs of this type is restricted to circuits having no more than three layers, than the upper bound—following his method of synthesis—is equal to the lower bound obtained from capacity considerations. Although this construction is *size*-optimal, it is not VLSI-optimal as the *weights* and *thresholds* are exponential.

Another solution was detailed in [20, 21] and later improved in [5, 15, 16]. It has a first layer of COMPARISONS

followed by a second layer of MAJORITY gates. This solution relies on the classical COMPARISON of two binary numbers $X = x_{n-1} x_{n-2} \dots x_1 x_0$ and $Y = y_{n-1} y_{n-2} \dots y_1 y_0$, which is a BF defined as:

$$C_n^{>(\geq)} = C_n^{>(\geq)}(X, Y) = \begin{cases} 1 & \text{if } X > Y \text{ (} X \geq Y \text{)} \\ 0 & \text{if } X \leq Y \text{ (} X < Y \text{)} \end{cases}$$

It is known from previous work that COMPARISON cannot be computed by a single TG with polynomially bounded integer weights, but can be computed by a depth-2 NN with $O(n^4)$ TGs and polynomially bounded weights [2]. This last result has been improved as follows.

Proposition 2 (Lemma 6 [37]) The COMPARISON function can be computed in a depth-3 neural network of size $3n$ with polynomially bounded integer weights.

This constructive solution (we shall call it **SRK**) has a first layer of n AND gates computing $x_i \wedge \bar{y}_i$, and n OR gates computing $x_i \vee \bar{y}_i$, followed by a layer of $n-1$ AND gates:

$$B_k = (x_k \wedge \bar{y}_k) \wedge \left\{ \bigwedge_{j=k+1}^{n-1} (x_j \vee \bar{y}_j) \right\},$$

and a third layer having one OR gate:

$$C_n^{\geq} = (x_{n-1} \wedge \bar{y}_{n-1}) \vee \left(\bigvee_{k=0}^{n-2} B_k \right).$$

This depth-3 NN has size $_{\text{SRK}} = 3n - 1$, thresholds $\leq n$, fan-in $\leq n$, and all the weights ± 1 .

Proposition 3 (Proposition 2 and 3 [7]) The COMPARISON of two n -bit numbers can be computed by a Δ -ary tree neural network having integer weights and thresholds which are: (i) polynomially bounded for all the values of the fan-in $3 \leq \Delta \leq O(\log n)$; (ii) super-polynomially bounded for all the values of the fan-in $O(\log n) < \Delta \leq O(\log^k n)$; and (iii) exponentially bounded for all the values of the fan-in $O(\log^k n) < \Delta \leq 2n$.

This constructive class of solutions (which we shall call

B_{Δ}), was proposed in [18, 19], and is based on decomposing COMPARISON in a tree structure. The NN has a first layer of 'partial' COMPARISONS $C_{\Delta}^{>}$ and $C_{\Delta}^{\geq}$ ($\lfloor \Delta/2 \rfloor$ bits from X and $\lfloor \Delta/2 \rfloor$ bits from Y) followed by a Δ -ary tree of TGs combining these partial results. The fact that the BFs implemented by the nodes are linear separable functions was proven in [17, 21]. The network has:

$$\text{depth}_{B_{\Delta}} = \lceil \log n / (\log \Delta - 1) \rceil, \text{ and}$$

$$\text{size}_{B_{\Delta}} = \lceil 4(n-1) / \Delta - 2 \rceil - \text{depth}_{B_{\Delta}}$$

with fan-in of Δ or $\Delta - 1$. The weights and the thresholds are lower than $2^{\Delta/2}$ (for $3 \leq \Delta \leq 2n$).

Proposition 4 (Theorem 3 [35]) The size complexity of COMPARISON implemented by generalised symmetric functions is $\Theta(n / \log n)$.

This constructive solution (we shall call it **ROS**) has a first layer of 'partial' COMPARISONS C_i (equivalent to $C_{2m}^{>}$) and $\tilde{C}_i$ (equivalent to $C_{2m}^{\geq}$) having m input bits from X and m input bits from Y . The first layer has $2 \lceil n/m \rceil - 1$ TGs of fan-in $= 2m$. The second layer has $\lceil n/m \rceil - 1$ AND gates with fan-in $= 2, 3, \dots, \lceil n/m \rceil$:

$$B_k = \tilde{C}_k \wedge \left(\bigwedge_{j=k+1}^{\lceil n/m \rceil} C_j \right).$$

The third layer has one OR gate $C_n^{\geq}(X, Y) = \bigvee_{k=1}^{\lceil n/m \rceil} B_k$, and by taking $m = \lceil \log n \rceil + 1$, size $_{\text{ROS}} = 3 \lceil n / (\lceil \log n \rceil + 1) \rceil - 1$. The NN has depth-3, weights and thresholds $\leq 2^{\lceil \log n \rceil}$ and fan-in $\leq \lceil n / (\lceil \log n \rceil + 1) \rceil$.

Proposition 5 (Corollary 2 [39]) The COMPARISON can be computed by a depth-2 linear threshold network of size $2 \lceil n / \lceil \sqrt{n} \rceil \rceil$, with weight values at most $2^{\lceil \sqrt{n} \rceil}$ and with an upper bound of $2^{\lceil \sqrt{n} \rceil} + 1$ for the maximum fan-in.

This constructive solution (we shall call it **VCB**) has a first layer of 'partial' COMPARISONS ($\lceil \sqrt{n} \rceil$ input bits from X and $\lceil \sqrt{n} \rceil$ input bits from Y), and the second layer computes the carry-out of the 2-1 binary ADDITION with carry.

Table 1 (from [7])
Size, weights and fan-ins of different solutions for COMPARISON (normal operand lengths).

Solution		VCB [39]	B ₂ $\lceil \sqrt{n} \rceil$ [6]	SRK [37]	ROS [35]	VCB [39]	B _Δ [18, 19]	B _Δ [18, 19]
Length		depth = 2		depth = 3			depth = 4	
32-bits	Size	13	12	95	19	18	19	45
	Max. weight	64	32	32	64	16	8	3
	Max. fan-in	13	12	32	12	9	8	5
64-bits	Size	17	16	191	31	26	39	63
	Max. weight	256	128	64	128	64	8	5
	Max. fan-in	17	16	64	14	13	8	7

Table 2
Different estimates for the AT^2 of COMPARISON for SRK, B₄, B_{log}, ROS and VCB.

Area \ Delay	Depth	Fan-in	Length
Size	$AT_{VCB}^2 = O(\sqrt{n})$	$AT_{B_4}^2 = O(n \log^2 n)$ (2)	$AT_{VCB}^2 = O(n^2 \sqrt{n})$
	$AT_{ROS}^2 = O(n / \log n)$	$AT_{B_{log}}^2 = O[n \log^3 n / \log^2(\log n)]$	$AT_{ROS}^2 \cong 3 \cdot n^3 / \log n$
	$AT_{SRK}^2 = O(n)$	$AT_{VCB}^2 = O(n \sqrt{n})$	$AT_{B_{log}}^2 \cong 4 \cdot n^3 / \log n$
	$AT_{B_{log}}^2 = O[n \log n / \log^2(\log n)]$	$AT_{ROS}^2 = O(n^3 / \log^3 n)$	$AT_{B_4}^2 \cong 4 \cdot n^3$
$\sum_{NN} fan-ins$	$AT_{B_4}^2 = O(n \log^2 n)$	$AT_{SRK}^2 = O(n^3)$	$AT_{SRK}^2 \cong 27 n^3 / 4$
	$AT_{VCB}^2 = O(n)$	$AT_{B_4}^2 = O(n \log^2 n)$ (3)	$AT_{B_{log}}^2 \cong 4 n^3$
	$AT_{B_{log}}^2 = O[n \log^2 n / \log^2(\log n)]$	$AT_{B_{log}}^2 = O[n \log^4 n / \log^2(\log n)]$	$AT_{VCB}^2 \cong 4 n^3$
	$AT_{B_4}^2 = O(n \log^2 n)$	$AT_{VCB}^2 = O(n^2)$	$AT_{B_4}^2 \cong 5 n^3$
$\sum_{NN} (\sum_i w_i + \theta)$	$AT_{ROS}^2 = O(n^2 / \log^2 n)$	$AT_{ROS}^2 = O(n^4 / \log^4 n)$	$AT_{ROS}^2 = O(n^4 / \log^2 n)$
	$AT_{SRK}^2 = O(n^2)$	$AT_{SRK}^2 = O(n^4)$	$AT_{SRK}^2 = O(n^4)$
	$AT_{B_4}^2 = O(n \log^2 n)$ (1)	$AT_{B_4}^2 = O(n \log^2 n)$ (4)	$AT_{B_4}^2 = O(n^3)$
	$AT_{B_{log}}^2 = O[n \sqrt{n} \log n / \log^2(\log n)]$	$AT_{B_{log}}^2 = O[n \sqrt{n} \log^3 n / \log^2(\log n)]$	$AT_{B_{log}}^2 = O(n^3 \sqrt{n} / \log n)$
	$AT_{ROS}^2 = O(n^2 / \log n)$	$AT_{ROS}^2 = O(n^4 / \log^3 n)$	$AT_{ROS}^2 = O(n^4 / \log n)$
	$AT_{SRK}^2 = O(n^2)$	$AT_{SRK}^2 = O(n^4)$	$AT_{SRK}^2 = O(n^4)$
	$AT_{VCB}^2 = O(n^{1/2} 2^{\sqrt{n}})$	$AT_{VCB}^2 = O(n^{3/2} 2^{\sqrt{n}})$	$AT_{VCB}^2 = O(n^{5/2} 2^{\sqrt{n}})$

Although size-optimal (like [34]), this solution is not very attractive from the VLSI complexity point of view as having exponentially growing *weights*. The complexity results are similar to those of a particular solution belonging to the B_Δ class [6, 7]: take $\Delta = 2 \lceil \sqrt{n} \rceil$ —which leads to B_{2 $\lceil \sqrt{n} \rceil$} —and the NN has *depth* = 2, *size* = $\lceil 2\sqrt{n} \rceil$, with *weights* and *thresholds* of at most $2^{\lceil \sqrt{n} \rceil}$.

For normal length COMPARISONS Vassiliadis *et al.* [39] claim improvements over ROS [35] and SRK [37]. We present in Table 1 the results reported in [7]. Both VCB and B_{2 $\lceil \sqrt{n} \rceil$} achieve better performances than SRK and ROS. For *depth* = 2, B_{2 $\lceil \sqrt{n} \rceil$} outperforms VCB (both for 32-bit and for 64-bit operand lengths), while for *depth* = 3, B_{2 $\lceil \sqrt{n} \rceil$} has lower *weights* and *fan-ins*, but slightly more gates. Still, B_Δ has two main advantages: (i) being a class of solutions it can be used for other *depths* (see the last column of Table I); (ii) because the *weights* and the *fan-ins* are lower, the *area* should also be lower.

It is known that a VLSI design is considered optimum when judged on a combined measure AT^2 [38], thus:

$$AT_{SRK}^2 = (3n - 1) \times 3^2 = O(n)$$

$$AT_{B_{\Delta}}^2 < \left\lceil \frac{4(n-1)}{\Delta-2} \right\rceil \times \left\lceil \frac{\log n}{\log \Delta - 1} \right\rceil^2 = O(n \log^2 n / \Delta \log \Delta)$$

$$AT_{ROS}^2 = 3 \lceil n / (\lceil \log n \rceil + 1) \rceil \times 3^2 = O(n / \log n).$$

The natural extension of the circuit complexity results to VLSI complexity ones is by using the closer estimates for the *area* and the *delay* (as discussed in Introduction).

Proposition 6 If the area of a neural network is estimated as $\sum_{NN} fan-ins$, there are neural networks computing the COMPARISON of two *n*-bit numbers which occupy between $O(n)$ and $O(n^2)$ area:

$$A_{SRK} = (n^2 + 11n - 6) / 2 = O(n^2)$$

$$A_{B_{\Delta}} < \Delta \times \left\lceil \frac{4(n-1)}{\Delta-2} \right\rceil + \left\lceil \frac{\log n}{\log \Delta - 1} \right\rceil + \left\lceil \frac{4n}{\Delta} \right\rceil = O(n)$$

$$A_{ROS} < 4n + \frac{1}{2} \cdot \left\lceil \frac{n}{\lceil \log n \rceil + 1} \right\rceil^2 + \frac{3}{2} \cdot \left\lceil \frac{n}{\lceil \log n \rceil + 1} \right\rceil = O(n^2 / \log^2 n).$$

Figure 1. The AT^2 values of COMPARISON—plotted as a 3D surface—versus the number of inputs n and the fan-in Δ for $4 \leq \Delta \leq 20$: (a) $n \leq 64$; (b) $n \leq 256$; (c) $n \leq 1024$; (d), (e), and (f) show the contour plots for the same cases. Clearly a 'valley' is formed, and that the 'deepest' points constantly lie somewhere between $\Delta_{\min} = 6$ and $\Delta_{\max} = 9$.

Proposition 7 If the area of a neural network is estimated as $\sum_{NN} (\sum_i |w_i| + |\theta|)$, there are neural networks computing the COMPARISON of two n -bit numbers which occupy between $O(n)$ and $O(n^2)$ area:

$$A_{SRK} = (2n^2 + 15n - 9)/2 = O(n^2)$$

$$A_{B\Delta} < \frac{2^{\Delta/2}}{\Delta} \cdot \frac{8n\Delta - 6n - 5\Delta}{\Delta - 2} = O(n \cdot 2^{\Delta/2}/\Delta)$$

$$A_{ROS} < \left(4n - \frac{5}{2}\right) \times \left\lceil \frac{n}{\lceil \log n \rceil + 1} \right\rceil + \left\lceil \frac{n}{\lceil \log n \rceil + 1} \right\rceil^2 = O(n^2 / \log n).$$

Proposition 8 If the delay of one neuron is proportional to its fan-in, there are neural networks computing the COMPARISON of two n -bit numbers which require between $O(\log n)$ and $O(n)$ time:

$$T_{SRK} = 2n + 2 = O(n)$$

$$T_{B\Delta} < (\Delta - 1) \times \left\lceil \frac{\log n}{\log \Delta - 1} \right\rceil + 1 = O(\Delta \log n / \log \Delta)$$

$$T_{ROS} < 2 \times \left(\left\lceil \frac{n}{\lceil \log n \rceil + 1} \right\rceil + \lceil \log n \rceil + 1 \right) = O(n / \log n).$$

Proposition 9 If the delay in a neural network is proportional to the length of the wires, there are neural networks computing the COMPARISON of two n -bit numbers which require $O(n)$ time:

$$T_{SRK} = (3n - 1)/2, \quad T_{B\Delta} < n, \quad T_{ROS} < n.$$

For all these different estimations of A and T , the AT^2 complexity values have been computed, ordered, and can be seen in Table 2.

Not wanting to complicate the proof, we shall determine the VLSI-optimal fan-in when implementing COMPARISON (in fact an $F_{n,1}$ function) for which several solutions were detailed in Propositions 2 to 9. The same result is valid for $F_{n,m}$ functions as can be intuitively expected as:

- the delay is determined by the first layer of COMPARISONS; while
- the area is mostly influenced by the same first layer of COMPARISONS (the area for the implementing the MAJORITY gate can be neglected [15, 21]).

From the alternatives presented in Table 2, we have chosen $\sum_{NN} (\sum_i |w_i| + |\theta|)$ for area and depth for delay, but other estimates lead to similar results (the optimal AT^2 being $O(n \log^2 n)$ in four out of nine cases—see Table 2). To get a better understanding, the AT^2 values have been computed for variable fan-ins and for different number of inputs n , and can be seen in Figure 1.

Proposition 11 (Theorem 3 [26]) Arbitrary Boolean logic functions $f: \{0, 1\}^n \rightarrow \{0, 1\}^m$ can be implemented in a neural network of perceptrons restricted to fan-in 2 with a node complexity of $\Theta\{\mu 2^n / (n + \log \mu)\}$. The resulting architecture requires $O(n)$ layers.

Sketch of proof The idea is to decompose each output BF into two subfunctions using Shannon's Decomposition [36]:

$$f(x_1 x_2 \dots x_{n-1} x_n) = \bar{x}_1 f_0(x_2 \dots x_{n-1} x_n) + x_1 f_1(x_2 \dots x_{n-1} x_n).$$

By doing this recursively for each subfunction, the output BFs will—in the end—be implemented by binary trees. Horne & Hush [26] use a trick for eliminating most of the lower level nodes by replacing them with a subnetwork that computes *all the possible BFs* needed by the higher level nodes. Each subcircuit eliminates one variable and has three nodes (one OR and two ANDs), thus the upper tree has:

$$size_{upper} = 3\mu \cdot \sum_{i=0}^{n-q-1} 2^i = 3\mu (2^{n-q} - 1) \quad (2)$$

nodes and $depth_{upper} = 2(n - q)$. The subfunctions now depend on only q variables, and a lower subnetwork that computes all the possible BFs of q variables is built. It has:

$$size_{lower} = 3 \cdot \sum_{i=1}^q 2^{2^i} < 4 \cdot 2^{2^q} \quad (3)$$

nodes and $depth_{lower} = 2q$ (see Figure 2 from [26]). That q which minimises the *size* of the two subnetworks is determined by solving $d(size)/dq = 0$, and gives:

$$q \approx \log[n + \log \mu - 2 \log(n + \log \mu)].$$

By substituting this value in (2) and (3), the minimum *size*:

$$size \approx 3\mu \cdot 2^{n-q} \approx 3\mu \cdot 2^n / (n + \log \mu)$$

is obtained. $\square$

We will use a similar approach for the case when the fan-in is limited by Δ .

Proposition 12 Arbitrary Boolean functions $f: \{0, 1\}^n \rightarrow \{0, 1\}^m$ can be implemented in a neural network of perceptrons restricted to fan-in Δ in $O(n / \log \Delta)$ layers.

Proof We use the approach of Horne & Hush [26] and limit the fan-in to Δ . Each output BF can be decomposed in $2^{\Delta-1}$ subfunctions (i.e., $2^{\Delta-1}$ AND gates). The OR gate would have $2^{\Delta-1}$ inputs. Thus, we have to decompose it in a Δ -ary tree of fan-in $= \Delta$ OR gates. This first decomposition step eliminates $\Delta - 1$ variables and generates a tree of:

$$depth = 1 + \lceil (\Delta - 1) / \log \Delta \rceil,$$

$$size = 2^{\Delta-1} + \lceil (2^{\Delta-1} - 1) / (\Delta - 1) \rceil.$$

Repeating this procedure recursively k times, we have:

$$depth_{upper} = k \cdot \{1 + \lceil (\Delta - 1) / \log \Delta \rceil\} \quad (4)$$

$$\begin{aligned} size_{upper} &= \{2^{\Delta-1} + \lceil (2^{\Delta-1} - 1) / (\Delta - 1) \rceil\} \cdot \sum_{i=0}^{k-1} 2^{i(\Delta-1)} \\ &= size \cdot \{2^{k(\Delta-1)} - 1\} / (2^{\Delta-1} - 1) \\ &\equiv 2^{k(\Delta-1)} (1 + 1/\Delta) \\ &\approx 2^{k\Delta - k}, \end{aligned} \quad (5)$$

where the subfunctions depend only on $q = n - k\Delta$ variables. We now generate all the possible subfunctions of q variables with a subnetwork of:

$$depth_{lower} = \lfloor (n - k\Delta) / \Delta \rfloor \cdot \{1 + \lceil (\Delta - 1) / \log \Delta \rceil\} \quad (6)$$

$$\begin{aligned} size_{lower} &= \{2^{\Delta-1} + \lceil (2^{\Delta-1} - 1) / (\Delta - 1) \rceil\} \cdot \sum_{i=1}^{\lfloor (n - k\Delta) / \Delta \rfloor} 2^{n - k\Delta - i\Delta} \\ &= size \cdot \{2^{2^0} + 2^{2^1} + \dots + 2^{2^{n - (k+1)\Delta}}\} \\ &< (size + 1) \cdot 2^{2^{n - (k+1)\Delta}} \end{aligned} \quad (7)$$

$$\approx 2^\Delta \cdot 2^{2^{n - k\Delta - \Delta}} \quad (8)$$

The inequality (7) can be proved by induction. Clearly, $size \cdot 2^{2^0} < (size + 1) \cdot 2^{2^0}$. Let us consider the statement true for α ; we prove it for $\alpha + 1$:

$$\begin{aligned} size \cdot \{2^{2^0} + \dots + 2^{2^{\alpha\Delta}}\} + size \cdot 2^{2^{(\alpha+1)\Delta}} \\ < size \cdot 2^{2^{(\alpha+1)\Delta}} + 2^{2^{(\alpha+1)\Delta}} \end{aligned}$$

$$size \cdot \{2^{2^0} + \dots + 2^{2^{\alpha\Delta}}\} < (size + 1) \cdot 2^{2^{\alpha\Delta}}$$

(due to hypothesis), thus:

$$(size + 1) \cdot 2^{2^{\alpha\Delta}} < 2^{2^{(\alpha+1)\Delta}}$$

and computing the logarithm of the left side:

$$\begin{aligned} 2^{\alpha\Delta} + \log(size + 1) \\ &= 2^{\alpha\Delta} + \log\{2^{\Delta-1} + \lceil (2^{\Delta-1} - 1) / (\Delta - 1) \rceil\} \\ &< 2^{\alpha\Delta} + \log\{2^{\Delta-1} + 2^{\Delta-1} / \Delta + 1\} \\ &< 2^{\alpha\Delta} + \Delta \\ &< 2^{(\alpha+1)\Delta}. \end{aligned}$$

From (4) and (6) we can estimate $depth_{BFs}$, and from (5) and (8) $size_{BFs}$ as:

$$depth_{BFs} = \{k + \lfloor (n - k\Delta) / \Delta \rfloor\} \cdot \{1 + \lceil (\Delta - 1) / \log \Delta \rceil\} \\ = (n / \Delta) \cdot (\Delta / \log \Delta + 1) \quad (9)$$

$$\approx n / \log \Delta = O(n / \log \Delta) \\ size_{BFs} = \mu \cdot size \cdot \{2^{k(\Delta-1)} - 1\} / (\Delta - 1) \\ + (size + 1) \cdot 2^{2^{n-(k+1)\Delta}} \\ \approx \mu \cdot 2^{k\Delta-k} + 2^\Delta \cdot 2^{2^{n-k\Delta-\Delta}} \quad (10)$$

concluding the proof. $\square$

Proposition 13 All the critical points of $size_{BFs}(\mu, n, k, \Delta)$ are relative minimum and are situated in the (close) vicinity of the parabola $k\Delta \approx n - \log(n + \log \mu)$.

Proof To determine the critical points, we equate the partial derivatives to zero. Starting from the approximation (10) of $size_{BFs}$ we compute $\partial size_{BFs} / \partial k = 0$:

$$\mu \cdot 2^{k\Delta-k} (\ln 2) (\Delta - 1) \\ + 2^\Delta \cdot 2^{2^{n-k\Delta-\Delta}} (\ln 2) \cdot 2^{n-k\Delta-\Delta} (\ln 2) \cdot (-\Delta) = 0 \\ \{\mu (\Delta - 1) / \Delta / (\ln 2)\} \cdot 2^{2k\Delta-k-n} = 2^{2^{n-k\Delta-\Delta}}$$

and using the notations $k\Delta = \gamma$, $\beta = \mu (\Delta - 1) / (\Delta \ln 2)$, and taking logarithms of both sides:

$$\log \beta + 2\gamma - k - n = 2^{n-\gamma-\Delta} \quad (11)$$

which has an approximate solution $\gamma \approx n - \log(n + \log \mu)$.

The same result can be obtained by computing with finite differences (instead of approximating the partial derivative):

$$size_{BFs}(\mu, n, k+1, \Delta) - size_{BFs}(\mu, n, k, \Delta) = 0$$

$$size \cdot \{\mu \cdot 2^{k\Delta-k} - 2^{2^{n-k\Delta-\Delta}}\} = 0 \\ \mu \cdot 2^{k\Delta-k} = 2^{2^{n-k\Delta-\Delta}}$$

and after taking twice the logarithm of both sides and using the same notations we have:

$$\log\{\log \mu + \gamma(1 - 1/\Delta)\} = n - \gamma - \Delta$$

$$\gamma = n - \{\Delta + \log(1 - 1/\Delta)\} - \log\{\gamma + \Delta / (\Delta - 1) \cdot \log \mu\} \\ \approx n - \Delta - \log(\gamma + \log \mu), \quad (12)$$

which has as the approximate solution:

$$\gamma = n - \log(n + \log \mu).$$

Starting again from (10), we compute $\partial size_{BFs} / \partial \Delta = 0$:

$$\mu 2^{k\Delta-k} (\ln 2) k + 2^\Delta (\ln 2) 2^{2^{n-k\Delta-\Delta}} \\ + 2^\Delta 2^{2^{n-k\Delta-\Delta}} (\ln 2) 2^{n-k\Delta-\Delta} (\ln 2) (-k) = 0 \\ \mu k \cdot 2^{\gamma-k} = k (\ln 2) \cdot 2^{n-\gamma} \cdot 2^{2^{n-\gamma-\Delta}} - 2^\Delta \cdot 2^{2^{n-\gamma-\Delta}} \\ \mu k \cdot 2^{\gamma-k} \cdot 2^{\gamma-n} \\ = k (\ln 2) \cdot 2^{2^{n-\gamma-\Delta}} - 2^\Delta \cdot 2^{\gamma-n} \cdot 2^{2^{n-\gamma-\Delta}} \\ \mu k \cdot 2^{2\gamma-k-n} = \{k (\ln 2) - 2^{\gamma+\Delta-n}\} \cdot 2^{2^{n-\gamma-\Delta}}$$

$$(\mu / \ln 2) \cdot 2^{2\gamma-k-n} = \{1 - 2^{\gamma+\Delta-n} / (k \ln 2)\} \cdot 2^{2^{n-\gamma-\Delta}}$$

which—by neglecting $2^{\gamma+\Delta} / \{k (\ln 2) \cdot 2^n\}$ —gives:

$$\log \beta + 2\gamma - k - n = 2^{n-\gamma-\Delta}$$

i.e., the same equation as (11).

These show that the critical points are situated in the (close) vicinity of the parabola $k\Delta \approx n - \log(n + \log \mu)$. $\square$

From *Proposition 12* and *13* it follows that *size-optimal* neural implementations of BFs are obtained for small *fan-ins* (i.e., from constant to at most $n - \log n < n$). The exact *size*:

$$size_{BFs} = size_{lower} + \mu \cdot size_{upper}$$

has been computed for many different values of n , μ , Δ and k . Some results of those extensive simulations are plotted in *Figure 3*. From *Figure 3(a)*, *(b)* and *(c)* it may seem that k and Δ used in the proof of *Proposition 12* have the same influence on $size_{BFs}$. The discrete parabola-like curves approximating $k\Delta \approx n - \log(n + \log \mu)$ can be seen in *Figure 3(d)*, *(e)* and *(f)*.

Proposition 14 The absolute minimum $size_{BFs}$ is obtained for *fan-in* $\Delta = 2$.

Sketch of proof We will analyse only the critical points by using the approximation $k\Delta \approx n - \log n$. Intuitively the claim can be understood if we replace this value in (10):

$$size_{BFs}^* \approx \mu \cdot 2^{n - \log n - k} + 2^\Delta \cdot 2^{2^{n-n+\log n-\Delta}} \\ < \mu \cdot 2^{n - \log n} + 2^\Delta \cdot 2^{2^{\log n}} \\ = \mu \cdot 2^n / n + 2^\Delta \cdot 2^n,$$

which clearly is minimised for $\Delta = 2$. $\square$

Figure 3. The size (in logarithmic scale) of NNs implementing arbitrary BFs for: (a) $n = 16$; (b) $n = 64$; (c) $n = 256$ (clipped at 2^{1000}), and the contour plots for the same cases (d), (e), (f).

The detailed proof computes $size_{BFs}(n, \mu, k, \Delta)$ for those $k \approx (n - \log n) / \Delta$ (i.e., $size_{BFs}^*(n, \mu, \Delta)$), and shows that:

$$size_{BFs}^*(n, \mu, \Delta + 1) - size_{BFs}^*(n, \mu, \Delta) > 0.$$

Hence, the function is monotonically increasing and the minimum is obtained for the smallest fan-in $\Delta = 2$. Because the proof has been obtained using successive approximations, several simulation results are presented in Table 3. It can be seen that while for relatively small n the size-optimal solutions are obtained even for $\Delta = 16$, starting from $n \geq 64$ all the size-optimal solutions are obtained for $\Delta = 2$. It is important that the other relative minima (on, or in the vicinity of the parabola $k\Delta \approx n - \log n$) are only slightly

larger than the absolute minimum. They might be of *practical interest* as leading to networks having fewer layers: $n / \log \Delta$ instead of n . Last, but not least, it is to be mentioned that all these relative minima are obtained for fan-ins strictly lower than linear, as $\Delta \leq n - \log n$.

4. Size-optimal neural implementations of $IF_{n,m}$ functions

A similar result can be obtained for $IF_{n,m}$, as the first layer is represented by COMPARISONS (i.e., $IF_{n,1}$) which can be decomposed to satisfy the limited fan-in condition [10, 16, 17, 21].

Table 3
Minimum $size_{BFs}$ for different values of n and $\mu = 1$.

n	$8 = 2^3$	$16 = 2^4$	$32 = 2^5$	$64 = 2^6$	$128 = 2^7$	$256 = 2^8$	$512 = 2^9$	$1024 = 2^{10}$	$2048 = 2^{11}$
size	110	1470	349,530	1.611×10^9	6.917×10^{18}	5.104×10^{38}	2.171×10^{76}	1.005×10^{154}	1.685×10^{307}
Δ	4	8	16	2	2	2	2	2	2
$k\Delta$	4	8	16	58	122	248	504	1014	2038

Figure 4. The reduced size (in logarithmic scale) of NNs implementing $IF_{n,m}$ functions for $m = 2^{\varepsilon n}$: (a) $\varepsilon = 0.1$; (b) $\varepsilon = 0.5$; (c) $\varepsilon = 0.9$; and the contour plots for the same cases (d), (e), (f). The lowest values are obtained for very small constant fan-in values.

Proposition 15 (Lemma 1 & Corollary 1 [21]) The COMPARISON of two n -bit numbers can be computed by a Δ -ary tree neural network having integer weights and thresholds bounded by $2^{\Delta/2}$ for any $3 \leq \Delta \leq n$.

The size complexity of the NN implementing one $IF_{n,m}$ function is [21]:

$$size_{B,\Delta} = 2nm \cdot \left\{ \frac{1}{\Delta/2} + \dots + \frac{1}{(\Delta/2)^{depth_{B,\Delta}}} \right\}, \quad (13)$$

where $depth_{B,\Delta} = \lceil \log n / (\log \Delta - 1) \rceil$, but a substantial enhancement is obtained if the fan-in is limited. Due to the limitation, the maximum number of **different BFs** which can be computed in each layer is:

$$(2n/\Delta) \cdot 2^\Delta, \quad \frac{2n/\Delta}{\Delta/2} \cdot 2^{\Delta(\Delta/2)}, \dots, \quad \frac{2n/\Delta}{(\Delta/2)^{depth_{B,\Delta}-1}} \cdot 2^{\Delta(\Delta/2)^{depth_{B,\Delta}-1}} \quad (14)$$

For large enough m (needed for achieving a certain precision [10, 23, 42]), and/or large enough n , the first terms

of the sum (13) will be larger than the equivalent ones from (14). This is equivalent to the trick from [26], as the lower levels will compute **all the possible functions** using only limited fan-in COMPARISONS. Hence, the optimum size becomes:

$$size_{B,\Delta}^{opt} = 2n \cdot \left\{ \sum_{i=1}^k \frac{2^{\Delta(\Delta/2)^{i-1}}}{\Delta(\Delta/2)^{i-1}} + \sum_{i=k+1}^{depth_{B,\Delta}} \frac{m}{(\Delta/2)^i} \right\}$$

explained as the same BFs are computed redundantly. In terms of fan-in, several exponentially decreasing terms will be replaced by double exponential increasing terms.

Following similar steps to the ones used in Proposition 13, it is possible to show that the minimum size is obtained for $\Delta = 3$. To get a better understanding we have done extensive simulations by considering that $m = 2^{\varepsilon n}$. Some of the results of these simulations can be seen in Figure 4. They show that it is always possible to obtain a significant reduction of the size by properly choosing a small constant fan-in. It is to be mentioned that the size reduction is by a huge factor which is of the form $2^{\varepsilon n - c}$ for very small fan-ins $\Delta_{opt} = 4 \dots 6$.

5. Conclusions and further work

The paper has focused on sparsely connected NNs, *i.e.* having either (small) constant *fan-ins*, or at most logarithmic in the number of inputs n . Using different cost functions—which are closer estimates than *size* and *depth* for the *area* and the *delay* of a VLSI chip—we have been able to prove that VLSI-optimal implementations of $IF_{n,m}$ functions are obtained by small constant *fan-ins*.

Concerning *size*-optimal solutions, we have shown that:

- arbitrary BFs require small—but not necessarily constant—*fan-ins* (at most $n - \log n$);
- $IF_{n,m}$ functions require small constant *fan-ins*.

Some of these results have already been applied to optimising the VLSI design of a neural constructive algorithm [5, 14, 15]. We are working on a mixed constructive algorithm which—after quantizing the input space as in [4, 11, 12, 13]—could synthesise $IF_{n,m}$ functions, arbitrary BFs, or a mixture of them such as to reduce the *area* of the resulting VLSI chip. This could also have applications to automatic synthesis of mixed analog/digital circuits [8, 16]. An alternative solution currently under investigation [14] is to use such a synthesis step after quantizing the input space as detailed in [24].

Future work should concentrate on:

- linking such results with the ‘Occam’s razor’ [44] and the ‘minimum description length’ [43];
- finding closer estimates (*i.e.*, other cost functions) for optimal mixed analogue/digital implementations.

The main conclusion is that VLSI-optimal solutions can be obtained for small constant *fan-ins*. We mention here that there are similar small constants relating to our capacity of processing information [31].

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