EFFECT OF MATERIAL HETEROGENEITY ON THE PERFORMANCE OF DSA FOR EVEN-PARITY $S_n$ METHODS

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*Research sponsored by the Oak Ridge National Laboratory managed by Lockheed Martin Energy Research Corporation for the U.S. Department of Energy under contract No. DE-AC05-96OR22464.
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Effect of Material Heterogeneity on the Performance of DSA for Even-Parity Sn Methods

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Abstract

A spectral analysis is conducted for the Source Iteration (SI), and Diffusion Synthetic Acceleration (DSA) operators previously formulated for solving the Even-Parity Method (EPM) equations. In order to accommodate material heterogeneity, the analysis is performed for the Periodic Horizontal Interface (PHI) configuration. The dependence of the spectral radius on the optical thickness of the two PHI layers illustrates the deterioration in the rate of convergence with increasing material discontinuity, especially when one of the layers approaches a void. The rate at which this deterioration occurs is determined for a specific material discontinuity in order to demonstrate the conditional robustness of the EPM-DSA iterations. The results of the analysis are put in perspective via numerical tests with the DANTE code (McGhee, et. al., 1997) which exhibits a deterioration in the spectral radius consistent with the theory.

1 Introduction

Diffusion Synthetic Acceleration (DSA) schemes have been developed for a wide variety of methods for solving neutral particle transport problems (Alcouffe, 1977; Larsen, 1982). Typically, DSA schemes are developed and analyzed via a Fourier mode decomposition on model problem configurations, i.e. uniform mesh and homogeneous material composition. However, recent research on Adjacent-cell Preconditioners (AP) acceleration techniques that possess the same cell-centered coupling stencil as discrete variable diffusion operators revealed the adverse effect of material discontinuity on the spectral radius of the accelerated scheme (Azmy, 1997). Furthermore, it was shown that the deterioration in spectral properties of the accelerated operator in the case of Weighted Difference (WD) formulations of the first order form of the Sn equation is not due to a poor choice of the diffusion parameters. Rather, as the material heterogeneity across a cell interface becomes sharper, with the optical thicknesses of the neighboring cells asymptotically approaching zero and infinity, the condition for two specific eigenvalues being bounded away from unity conflict (Azmy, 1998). This implies that for every acceleration scheme of first order Sn methods that has a cell-centered diffusion-like coupling stencil, there exists a problem configuration with a material discontinuity so sharp that the spectral radius can be made arbitrarily close to unity, thus the iterative convergence arbitrarily slow.

In this paper we illustrate by way of example that the above stated conclusion might apply more
generally to other formulations of $S_n$ methods, namely Even Parity Methods (EPM), accelerated via DSA (Morel and McGhee, 1995). For the sake of simplifying the analysis we restrict this method’s three-dimensional equations to their two-dimensional analogues, and assume isotropic scattering. We conjecture the results will not change much in the case of three-dimensional geometry, and the spectral radius will increase with scattering anisotropy.

The remainder of this paper is organized as follows. As in previous studies, in Sec. 2 we apply this method’s equations to the Periodic Horizontal Interface (PHI) configuration and perform a Fourier mode decomposition in the presence of a periodic material discontinuity. Then in Sec. 3 we demonstrate the approach to unity of the spectral radius of the accelerated iterative procedure as the material discontinuity becomes sharper. In addition to this analytic result, in Sec. 4 we also conduct some numerical experiments on finite size configurations and show the same poor behavior of the spectral radius in the same limit of the material discontinuity.

2 Fourier Analysis of EPM-DSA for the PHI Configuration

It is possible to show in two dimensional geometry that the EPM-DSA iterations are indeed unconditionally stable and robust for model problem configurations characterized by uniform grids and homogeneous material composition. In order to investigate the effect of material and mesh discontinuity on the efficiency, i.e. stability and robustness, of the DSA for accelerating the inner iterations for the EPM equations we introduce the Periodic Horizontal Interface (PHI) configuration. PHI is comprised of slabs infinite in the $x$ extent of two distinct materials interchangeably stacked on top of one another ad infinitum along the $y$-dimension. In the Cartesian grids considered here the cell width, i.e. cell size in the $x$-dimension, is the same in all layers and is denoted $a$. The cell height, i.e. cell size in the $y$-dimension, is not necessarily the same in the two distinct layers and is denoted $b_j$, with the corresponding total cross section, scattering ratio, and diffusion coefficient denoted $\sigma_j$, $c_j$, and $D_j$, $j = 0, 1$, respectively.

We now derive the spectrum of the Source Iterations (SI) and DSA schemes for this configuration. We strictly abide with the notation in (Morel and McGhee, 1995) except that we drop all $z$ quantities in order to restrict the equations to two dimensions. The EPM discrete variable equations are comprised of:

- a balance equation for the even-parity flux, $\psi_{i,j}^+$, over each median-mesh cell centered about vertex $i$, $j$;
- an expression for the median-mesh odd-parity flux, $\psi_{i+\frac{1}{2},j}^-$, at edge-center $i + \frac{1}{2}, j$, in terms of the corner odd-parity fluxes, $\psi_{i,j}^{-i+\frac{1}{2},j+\frac{1}{2}}$, associated with vertex $i$, $j$ and cell centers $i\pm\frac{1}{2}$, $j\pm\frac{1}{2}$; and
- an expression for the corner odd-parity flux, $\psi_{i,j}^{-i+\frac{1}{2},j+\frac{1}{2}}$, associated with vertex $i$, $j$ and cell $i + \frac{1}{2}, j + \frac{1}{2}$ in terms of the even-parity fluxes on the vertices of this cell.

These equations can be manipulated to eliminate the odd-parity flux discrete variables retaining only the even-parity and scalar flux variables. Evaluating the balance equation for the $j \neq j' = 0$, and 1
layers of PHI separately we obtain,
\[
\frac{\zeta_\eta}{2ab} \left( \frac{\psi_{i-1,j+1}^+ - \psi_{i+1,j+1}^+}{\sigma_j} + \frac{\psi_{i+1,j-1}^+ - \psi_{i-1,j-1}^+}{\sigma_j} \right) - \tilde{\sigma} \left( \frac{\zeta}{a} \right)^2 (\psi_{i+1,j}^+ + \psi_{i-1,j}^+) \\
- \frac{\eta^2}{b} \left( \frac{\psi_{i,j}^+}{b_j \sigma_j'} + \frac{\psi_{i,j-1}^+}{b_j \sigma_j} \right) + \left[ \tilde{\sigma} + 2\tilde{\sigma} \left( \frac{\xi}{a} \right)^2 + \frac{\eta^2}{b} \left( \frac{1}{b_j \sigma_j'} + \frac{1}{b_j \sigma_j} \right) \right] \psi_{i,j}^+ = \sigma_\phi \tilde{\phi}_{i,j} \tag{1}
\]
where \(\tilde{\phi}_{i,j}\) is the previous iterate of the scalar flux, and we have defined,
\[
\bar{b} \equiv \frac{b_0 + b_1}{2}; \quad \tilde{\sigma} \equiv \frac{b_0 / \sigma_0 + b_1 / \sigma_1}{b_0 + b_1} \\
\bar{\sigma} \equiv \frac{b_0 \sigma_0 + b_1 \sigma_1}{b_0 + b_1}; \quad \sigma_\phi \equiv \frac{b_0 c_0 \sigma_0 + b_1 c_1 \sigma_1}{b_0 + b_1}. \tag{2}
\]
The discrete-variable DSA equation provided in (Morel and McGhee, 1995) is manipulated to eliminate the change in the net current in favor of the change in the scalar flux, \(\delta \phi_{i,j}\),
\[
- \frac{\tilde{\sigma}}{3a^2} (\delta \phi_{i+1,j} - 2 \delta \phi_{i,j} + \delta \phi_{i-1,j}) - \frac{1}{3b} \left( \frac{\delta \phi_{i,j+1} - \delta \phi_{i,j}}{b_0 \sigma_0} - \frac{\delta \phi_{i,j} - \delta \phi_{i,j-1}}{b_1 \sigma_1} \right) \\
+ (1 - \bar{\epsilon}) \tilde{\sigma} \delta \phi_{i,j} = \bar{\epsilon} \tilde{\sigma} (\phi_{i,j} - \tilde{\phi}_{i,j}), \tag{3}
\]
where \(\bar{\epsilon} \equiv \sigma_\phi / \tilde{\sigma}\).

Now we conduct a Fourier analysis of the SI and DSA iterative procedures in the context of the PHI configuration, then explore their respective stability and robustness via their spectral radii for a variety of parameter choices.

2.1 SI Spectrum

Decomposing the dependent variables in Eq. (1) into their Fourier modes separately in each layer, and manipulating the resulting expressions yields,
\[
\begin{bmatrix}
A & B_{1,2} \\
B_{2,1} & A
\end{bmatrix}
\begin{bmatrix}
\Psi_0 \\
\Psi_1
\end{bmatrix} = \bar{\epsilon}
\begin{bmatrix}
\hat{\phi}_0 \\
\hat{\phi}_1
\end{bmatrix}, \tag{4}
\]
where \(\Psi_j\), and \(\hat{\phi}_j\) are the Fourier modes of the even-parity flux, and the previous iterate of the scalar flux, respectively, in PHI layer \(j=0,1\), and the matrix elements are given by,
\[
A \equiv 1 + 4 \gamma \left( \frac{\zeta}{\alpha} \right)^2 \sin^2 (s_x / 2) + \frac{\eta^2}{\bar{\sigma} b} \left( \frac{1}{b_0 \sigma_0} + \frac{1}{b_1 \sigma_1} \right), \\
B_{1,2} \equiv -i \frac{\zeta \eta}{\alpha b} \sin (s_z) \left( \frac{1}{\sigma_0} - \frac{e^{-i s_y}}{\sigma_1} \right) - \frac{\eta^2}{\bar{\sigma} b} \left( \frac{1}{b_0 \sigma_0} + \frac{e^{-i s_y}}{b_1 \sigma_1} \right), \\
B_{2,1} \equiv -i \frac{\zeta \eta}{\alpha b} \sin (s_z) \left( \frac{e^{i s_y}}{\sigma_1} - \frac{1}{\sigma_0} \right) - \frac{\eta^2}{\bar{\sigma} b} \left( \frac{1}{b_0 \sigma_0} + \frac{e^{i s_y}}{b_1 \sigma_1} \right). \tag{5}
\]
In Eqs. (5) we have defined the dimensionless Fourier variables, \(s_x, s_y\), along the \(x\)- and \(y\)-axes;
\(i \equiv \sqrt{-1}; \alpha \equiv \tilde{\sigma} \alpha; \) and \(\gamma \equiv 1 + \frac{b_0 b_1}{\sigma_0 \sigma_1} \left( \frac{\sigma_0 - \sigma_1}{b_0 + b_1} \right)^2\).
Solving the set of decomposed balance equations for the modes of the even-parity flux, then summing over discrete ordinates, we obtain an expression for the mapping of the eigenmodes of the old iterate of the scalar flux into the eigenmodes of the new iterate of the scalar flux. The eigenvalues of the matrix operator that effects this mapping provide the amplification factors of the corresponding eigenmodes. For the purposes of the stability analysis conducted here only the largest magnitude amplification factor is of interest, hence for each mode we compute such factor, and loosely refer to it, over the Fourier space, as the spectrum,

$$\nu_{SI}(s_x, s_y) = \sup |\mathcal{E}(M(s_x, s_y))| = \sup |\mathcal{E}\left(\bar{c} \sum_n w_n \begin{bmatrix} A & B_{1,2} \\ B_{2,1} & A \end{bmatrix}^{-1}\right)|,$$

where $n$ is the index of the discrete ordinates quadrature set and $w_n$ is the associated weight; $\mathcal{E}$ computes the eigenvalues of its matrix argument.

Since the SI spectrum in Eq. (6) is proportional to the effective scattering ratio, $\bar{c}$, the worst case corresponds to the case of perfect scattering in both layers, $\bar{c} = c_0 = c_1 = 1$. It is easy to show that the spectral radius of SI for the PHI configuration in this case is at least one. This is accomplished by evaluating $\nu_{SI}$ at the origin in Fourier space, $s_x = 0 = s_y$, where the mapping matrix becomes,

$$M_0 = \begin{bmatrix} 1 - \bar{\chi} & \bar{\chi} \\ \bar{\chi} & 1 - \bar{\chi} \end{bmatrix},$$

where $\bar{\chi} = \sum n w_n \frac{x_n}{1 + 2 x_n}, x_n \equiv \frac{x_n^2}{\bar{\sigma} \bar{\sigma}} \left( \frac{1}{b_0 \sigma_0} + \frac{1}{b_1 \sigma_1} \right)$. The eigenvalues of $M_0$ are 1 and $1 - 2 \bar{\chi}$, thus the maximum eigenvalue for this matrix is one. Since we cannot rigorously check all eigenvalues for values of $s_x$ and $s_y$, it is conceivable that an eigenvalue greater than one exists at one or more points in Fourier space. However, we have not been able to find such an eigenvalue. Therefore we conclude that the spectral radius for the source iteration process is indeed unity and occurs at the origin in Fourier space.

### 2.2 DSA Spectrum

Analogously, decomposing the dependent variables in Eq. (3) separately for each PHI layer into their Fourier modes yields,

$$\begin{bmatrix} G & E_{1,2} \\ E_{2,1} & G \end{bmatrix} \begin{bmatrix} \Delta_0 \\ \Delta_1 \end{bmatrix} = \bar{c} \begin{bmatrix} \hat{\Phi}_0 \\ \hat{\Phi}_1 \end{bmatrix} - \bar{c} \begin{bmatrix} \hat{\Phi}_0 \\ \hat{\Phi}_1 \end{bmatrix},$$

where we have defined the matrix elements,

$$G \equiv 1 - \bar{c} + \frac{4\gamma}{3\alpha^2} \sin^2(s_x/2) + \frac{1}{3\bar{\sigma} b} \left( \frac{1}{b_0 \sigma_0} + \frac{1}{b_1 \sigma_1} \right),$$

$$E_{1,2} \equiv - \frac{1}{3\bar{\sigma} b} \left( \frac{1}{b_0 \sigma_0} + \frac{e^{-\bar{\sigma} s_y}}{b_1 \sigma_1} \right),$$

$$E_{2,1} \equiv - \frac{1}{3\bar{\sigma} b} \left( \frac{1}{b_0 \sigma_0} + \frac{e^{\bar{\sigma} s_y}}{b_1 \sigma_1} \right),$$

and $\Delta_j$ and $\hat{\Phi}_j$ are the Fourier modes of $\delta \phi$ and $\phi$, respectively, in PHI layer $j = 0, 1$. 
Now we solve this set of decomposed DSA equations for the modes of the scalar flux iteration residual, $\Delta_j$, in terms of the mesh sweep residual eigenmodes, $\Phi_j$. Then, we eliminate the latter using Eq. (4) to obtain an expression for the mapping of the modes of the old iterate of the scalar flux into the modes of the new, accelerated iterate of the scalar flux. The eigenvalues of the matrix operator that effects this mapping provide the amplification factors of the corresponding eigenmodes in the iteration residual by the DSA scheme, i.e. its spectrum. As before, here we refer to the largest magnitude eigenvalue for each Fourier mode as the DSA spectrum,

$$\nu_{DSA}(s_x, s_y) = \sup | \mathcal{E} \{ I + \bar{c} M_{DSA}(s_x, s_y) \} M(s_x, s_y) - \bar{c} M_{DSA}(s_x, s_y) |,$$

where $I$ is the identity matrix of order 2, and $M_{DSA}$ is the inverse of the matrix on the left hand side of Eq. (8).

We note that in the case of a symmetric angular quadrature the DSA eigenvalue is an even function of its arguments, i.e. $\nu_{DSA}(\pm s_x, \pm s_y) = \nu_{DSA}(s_x, s_y)$, so that the period of the eigenvalue is reduced to the square $[0,\pi] \times [0,\pi]$ in the Fourier plane $(s_x, s_y)$. It is possible, but very elaborate, to show that the spectrum of the accelerated scheme has a finite limit near the origin in Fourier space along an arbitrary trajectory in the $(s_x, s_y)$ plane. Numerical experiments give us confidence that this is indeed the case, and in the next section we derive an expression for this limit along the $s_x$-axis as $s_x \to 0$. To illustrate the spectrum of the DSA scheme for the PHI configuration we plot $\nu_{DSA}$ for various material discontinuity combinations, i.e. pairs of $\sigma_0, \sigma_1$ in Fig. 1.

It is evident from Fig. 1, especially the cases $\sigma_0 = .01$ and $\sigma_1 = .1, 1, 10$, that the spectral properties of the accelerated iterations deteriorate seriously with increasing sharpness of material discontinuity, particularly when one of the layers is optically very thin. This raises the potential for conditional robustness as demonstrated in the following section for a specific limit of material properties in the PHI configuration, and later verified for finite problem domains in Sec. 4.

3 Conditional Robustness of EPM-DSA

In this section we establish the conditional robustness of DSA for the PHI configuration by demonstrating the existence of an eigenvalue that approaches one under certain conditions. Expanding the elements of the mapping matrix for the even-parity flux eigenmodes near the origin in Fourier space, and in particular along the trajectory $s_y=0$, $s_x \to 0$, then summing over the angular quadrature we obtain the mapping matrix for the SI scalar flux,

$$M \to \bar{c} \begin{bmatrix} d_0 + s_x^2 d_2 & 1 - d_0 + s_x^2 o_2 \\ 1 - d_0 + s_x^2 o_2 & d_0 + s_x^2 d_2 \end{bmatrix},$$

where we have assumed full symmetry of the angular quadrature, and defined,

$$d_0 \equiv 1 - \sum_n w_n \frac{\Gamma \eta^2}{1 + 2 \Gamma \eta^2}; \quad d_2 \equiv - \sum_n w_n \frac{\zeta^2 \gamma + 2 \Gamma \eta^2 (1 + \Gamma \eta^2)}{(1 + 2 \Gamma \eta^2)^2};$$
$$o_2 \equiv - \sum_n w_n \frac{2 \eta^2 \zeta^2 \Gamma}{\alpha^2} \frac{\gamma + \Gamma \eta^2}{(1 + 2 \Gamma \eta^2)^2}; \quad \Gamma \equiv \frac{2}{b_0 b_1 \sigma_0 \sigma_1}. $$

Similarly, expanding the DSA mapping matrix, the inverse of the matrix appearing on the left hand
Figure 1: EPM-DSA spectrum for PHI with $S_4$ quadrature, $\bar{c} = 1$, $\alpha = b_0 = b_1 = 1$. 

- $\sigma_0 = 0.01, \sigma_1 = 0.01$
- $\sigma_0 = 0.01, \sigma_1 = 0.1$
- $\sigma_0 = 0.01, \sigma_1 = 1$
- $\sigma_0 = 0.1, \sigma_1 = 0.1$
- $\sigma_0 = 0.1, \sigma_1 = 1$
- $\sigma_0 = 1, \sigma_1 = 10$
- $\sigma_0 = 1, \sigma_1 = 1$
- $\sigma_0 = 10, \sigma_1 = 10$

<table>
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<th>$s_x$</th>
<th>$\sigma_0$</th>
<th>$\sigma_1$</th>
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<td>$0.01$</td>
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<td>$0.1$</td>
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<td>0</td>
<td>$10$</td>
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</tbody>
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DSA Spectrum: 2D, $S_4$, $\bar{c} = 1$. 

- $s_x = 0$, $s_x = \pi/8$, $s_x = \pi/4$
- $s_x = 3\pi/8$, $s_x = \pi/2$, $s_x = 5\pi/8$
- $s_x = 3\pi/4$, $s_x = 7\pi/8$, $s_x = \pi$
side of Eq. (8), in the same asymptotic limit we obtain,

\[
M_{DSA} \rightarrow \begin{bmatrix}
    s_{y}^{-2}h_{2} + h_{0} + h_{2}/12 & s_{y}^{-2}h_{2} - h_{0} + h_{2}/12 \\
    s_{y}^{-2}h_{2} - h_{0} + h_{2}/12 & s_{y}^{-2}h_{2} + h_{0} + h_{2}/12
\end{bmatrix},
\]  

(13)

where \(h_{0} \equiv 3/(4 \Gamma), h_{2} \equiv 3 \alpha^{2}/(2 \gamma)\).

Substituting Eqs. (11) and (13) into Eq. (10) yields the \(O(s_{x}^{0})\) eigenvalues for the mapping of the accelerated iterations operator,

\[
\nu_{DSA}(s_{x} \to 0, 0) = \bar{c} \sup \left| \left[ 1 + 4h_{0} - 2d_{0}(1 + 2h_{0}), 1 + 2h_{2}(d_{2} + o_{2}) \right] \right| + O(s_{x}).
\]  

(14)

Evaluation of Eq. (14) yields a maximum eigenvalue of one when \(\bar{c} = 1\). Thus, the spectral radius of the DSA scheme is at least one. Since we have not been able to find any larger eigenvalues, we conclude that the spectral radius is indeed one.

In order to illustrate the loss of robustness of the DSA scheme, and the rate at which it deteriorates with material discontinuity we consider the specific material discontinuity: \(\sigma_{0} = \sigma, \sigma_{1} = 1/\sigma\). In the limit \(\sigma \to 0\),

\[
d_{0} \to \sigma \frac{b_{0}b_{1}}{2} \sum_{n} \frac{w_{n}}{\eta^{2}}; \quad d_{2} \to -\sigma^{-1} \frac{(b_{0}b_{1})^{3}}{64 \alpha^{2}b^{2}} \sum_{n} \frac{w_{n}c^{2}}{\eta^{2}}; \quad \\
o_{2} \to -\sigma^{-2} \frac{(b_{0}b_{1})^{2}}{16 \alpha^{2}} \sum_{n} \frac{w_{n}c^{2}}{\eta^{2}}; \quad h_{0} \to \sigma \frac{b_{0}b_{1}}{8}; \quad h_{2} \to 6\sigma^{3} \left( \frac{\alpha b}{b_{0}b_{1}} \right)^{2},
\]  

(15)

and Eq. (14) approaches,

\[
\nu_{DSA}(s_{x} \to 0, 0) \to \\
\bar{c} \sup \left| \left[ 1 - \sigma b_{0}b_{1} \left( \sum_{n} \frac{w_{n}}{\eta^{2}} - \frac{1}{2} \right), 1 - \sigma \frac{3}{16} (b_{0} + b_{1})^{2} \sum_{n} \frac{w_{n}c^{2}}{\eta^{2}} \right] \right| + O(s_{x}).
\]  

(16)

Evidently both arguments of the \(\sup\) function approach 1 as \(\sigma \to 0\). This means that the EPM-DSA can be made arbitrarily slow by sharpening the material discontinuity to bring the lower bound on the spectral radius, Eq. (16), arbitrarily close to unity. Hence the conditional robustness of the acceleration scheme.

4 Numerical Results

The first test problem is a square within a square. The outer square has a side length of 1.0 cm. The inner square has a side length of 0.4 cm and is placed at the center of the outer square. The inner square contains material 1 and the outer square contains material 2, both of which are purely scattering materials. The total cross section in each material is varied between 0.01 cm\(^{-1}\) and 1000.0 cm\(^{-1}\). All boundaries are vacuum. The problem is meshed into a 10 x 10 array of cells, and each cell has an edge width of 0.1 cm.

The spectral radii for the first test problem are given in Table (1).
Table 1: Spectral radii for the first test problem

<table>
<thead>
<tr>
<th>$\sigma_2$</th>
<th>0.01</th>
<th>0.1</th>
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<tr>
<td>100.0</td>
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<td>0.59</td>
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<td>0.03</td>
<td>0.02</td>
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<tr>
<td>1000.0</td>
<td>0.59</td>
<td>0.59</td>
<td>0.48</td>
<td>0.22</td>
<td>0.02</td>
<td>0.004</td>
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</tbody>
</table>

The second test problem is a square within a square within a square. The outer square has a side length of 1.0 cm. The middle square has a side length of 0.6 cm and the inner square has a side length of 0.2 cm and is placed at the center of the outer square. The inner and outer squares contain material 1 and the middle square contains material 2, both of which are purely scattering materials. The total cross section in each material is varied between 0.01 cm$^{-1}$ and 1000.0 cm$^{-1}$. All boundaries are vacuum. The problem is meshed into a 10 x 10 array of cells, and each cell has an edge width of 0.1 cm.

The spectral radii for the second test problem are given in Table (2).

Table 2: Spectral radii for the second test problem

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<tr>
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<td>0.13</td>
<td>0.13</td>
<td>0.14</td>
<td>0.31</td>
<td>0.52</td>
<td>0.63</td>
</tr>
<tr>
<td>10.0</td>
<td>0.40</td>
<td>0.41</td>
<td>0.32</td>
<td>0.18</td>
<td>0.23</td>
<td>0.25</td>
</tr>
<tr>
<td>100.0</td>
<td>0.52</td>
<td>0.56</td>
<td>0.43</td>
<td>0.20</td>
<td>0.03</td>
<td>0.04</td>
</tr>
<tr>
<td>1000.0</td>
<td>0.57</td>
<td>0.46</td>
<td>0.39</td>
<td>0.21</td>
<td>0.02</td>
<td>0.004</td>
</tr>
</tbody>
</table>

It can be seen from Tables (1) and (2) that the computational spectral radius increases as the magnitude of the material discontinuity increases. Thus the trend suggested by our Fourier analysis is observed in the test calculations.

5 Conclusions

A Diffusion Synthetic Acceleration (DSA) scheme that was derived, implemented and tested earlier for the Even-Parity Method (EPM) possesses excellent spectral properties in model problem configurations. We investigated the effect of material heterogeneity and mesh non-uniformity on the convergence rate of the DSA iterations for the EPM by performing a spectral analysis of the Periodic Horizontal Interface (PHI) configuration. In this case the spectral radius appears to approach unity, thereby indicating loss of robustness by the DSA, as one of the layers in the PHI configuration approaches a void, $\sigma_0 = \sigma^2$, while the other layer becomes thick like $\sigma_1 = 1/\sigma$, $\sigma \to 0$. This leads
us to conclude that the DSA scheme analyzed in this work is not unconditionally stable for problems with sharp material discontinuities. This result was verified via numerical experiments involving finite problem domains and realistic, albeit simplified, geometric configurations.

References


