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I. INTRODUCTION

In an asymptotic development of the equations governing the equilibria and linear stability of rapidly rotating polytropes (papers I¹ and II²) we employed the slender aspect of these objects to reduce the three-dimensional partial differential equations to a somewhat simpler, ordinary integro-differential form. Specifically, the polytrope is characterized by an equatorial radius a and a characteristic thickness εa . The dimensionless parameter $\varepsilon = k_J a$, where k_J is the Jeans wavenumber³ based on conditions at the center of the disk. The equations are then asymptotically solved in the limit $\varepsilon \rightarrow 0$ and, once this solution is in hand, ε may be expressed in terms of known global parameters of the disk, such as density, entropy, mass and angular momentum.

The earlier calculations dealt with isolated objects that were in centrifugal balance, that is, the centrifugal acceleration of the configuration was balanced largely by self gravity with small contributions from the pressure gradient. Another interesting situation is that in which the polytrope rotates subject to externally imposed gravitational fields. In astrophysics, this

^{a)}To be published in the *Proceedings of the 10th Florida Workshop in Nonlinear Astronomy: Waves in Astrophysics*, eds. J. Hunter and R. Wilson, New York Academy of Sciences (1994).

is common in the theory of galactic dynamics because disks are unlikely to be isolated objects.⁴

The dark halos associated with disks also provide one possible explanation of the apparent warping of many galaxies. If the axis of the highly flattened disk is not aligned with that of the much less flattened halo, then the resultant torque of the halo gravity on the disk might provide a nonaxisymmetric distortion or disk warp.⁴

Motivated by these possibilities we shall here build models of polytropic disks of small but finite thickness which are subjected to prescribed, external gravitational fields. First we estimate how a symmetrical potential distorts the structure of the disk (Sec. II), then we examine its sinuous oscillations to confirm that they freely decay, hence suggesting that a warp must be externally forced (Sec. III). Finally (in Sec. IV), we consider steady warps of the disk plane when the axis of the disk does not coincide with that of the halo.

II. POLYTROPES IN SYMMETRICAL POTENTIALS

We first consider polytropes whose structures are distorted under the influence of a massive ambient halo. Our purpose, in part, is to determine the leading-order effect of a halo, which we assume to be symmetrical about the disk plane. We also use this opportunity to reintroduce the notation and method developed in paper I.

A. Equations

Our study begins with the equations of cosmical gas dynamics, supplemented by the polytropic equation of state (with index n). These are given explicitly in papers I and II, and are in each case nondimensionalized to reveal the leading-order balances when ε is small. Here we shall write down and deal with only the dimensionless form of the equations with, in particular, the radius a as unit of length. Therefore, we also stretch the axial coordinate, z , so that $\zeta = \varepsilon^{-1}z$ is of order unity within the disk. Axial derivatives are then of order ε^{-1}

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and evidently large, as they must be in a slender object.

The conservation equations of momentum and mass are, in cylindrical polar coordinates $(r, \theta, \varepsilon\zeta)$,

$$\partial_t u + u \partial_r u + \Omega \partial_\theta u + \varepsilon^{-1} w \partial_\zeta u - r \Omega^2 = -\partial_r(\Phi + \Phi_h + H), \quad (1)$$

$$r \partial_t \Omega + r u \partial_r \Omega + r \Omega \partial_\theta \Omega + \varepsilon^{-1} r w \partial_\zeta \Omega + 2 \Omega u = -\frac{1}{r} \partial_\theta(\Phi + \Phi_h + H), \quad (2)$$

$$\partial_t w + u \partial_r w + \Omega \partial_\theta w + \varepsilon^{-1} w \partial_\zeta w = -\varepsilon^{-1} \partial_\zeta(\Phi + \Phi_h + H) \quad (3)$$

and

$$\partial_t \rho + \frac{1}{r} \partial_r(r \rho u) + \partial_\theta(\rho \Omega) + \varepsilon^{-1} \partial_\zeta(\rho w) = 0, \quad (4)$$

where (u, v, w) is the velocity vector, $\Omega = v/r$ is the angular velocity, Φ_h is the gravitational potential of the halo, Φ is that of the disk, and the specific enthalpy is given by

$$H = n \rho^{1/n}, \quad (5)$$

where ρ is the density.

Poisson's equation is

$$\partial_\zeta^2 \Phi - \rho = -\varepsilon^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} \right]. \quad (6)$$

B. Halo models

As a simple model of a halo, we consider a spheroidal distribution of inert material which interacts with the matter within the disk only through gravity. We presume the halo to be approximately a Maclaurin spheroid with gravitational potential,

$$\Phi_h = \frac{1}{2} C_h \Omega_h^2 [(1-e)z^2 + r^2], \quad (7)$$

where Ω_h is the (constant) rotation rate of the halo, e is its ellipticity, and C_h is a scaling factor measuring the halo mass. In scaled variables, we write this as

$$\Phi_h = \frac{1}{2} \beta \varepsilon^{-1} r^2 + \frac{1}{2} \alpha \zeta^2, \quad (8)$$

where α and β are parameters. Using the external potential (8), we model the influence of externally imposed gravity on the disk's structure. More sophisticated halo structures can also be modeled with (8), since it is the general form of the Taylor expansion of Φ_h about the disk midplane.

C. Steady, symmetrical solutions

When the basic state is axisymmetric, time-independent and symmetric about the midplane, the derivatives with respect to θ and t and the velocity components u and w all vanish. As in paper I, the centrifugal potential is then an integral of the equations of motion, depending only on radius:

$$F(r) = \Phi_h + \Phi + H = \int r \Omega^2 dr. \quad (9)$$

To solve the equations we seek the following asymptotic series:

$$F \sim \varepsilon^{-1} F_0 + F_1 + \varepsilon F_2, \quad \Phi \sim \varepsilon^{-1} \Phi_0 + \Phi_1 + \varepsilon \Phi_2, \quad (10a)$$

$$H \sim H_1 + \varepsilon H_2 \quad \text{and} \quad \rho \sim \rho_1 + \varepsilon \rho_2. \quad (10b)$$

The leading-order terms of Eq. (9) are

$$F_0 = \frac{1}{2} \beta r^2 + \Phi_0(r), \quad (11)$$

which also satisfies the leading-order expression resulting from Poisson's Eq. (6). Thus the halo supplements the gravitational field of the disk in balancing the internal rotation, as is

commonly assumed in fitting the rotation curves of galaxies to their observed visible mass distributions.

The equation determining the vertical structure is given at $O(1)$ by combining Eqs. (6) and (9), and is

$$\partial_\zeta^2 H_1 + \frac{H_1^n}{n^n} = -\alpha. \quad (12)$$

Thus the density structure is

$$\rho_1(r, \zeta) = \rho_{10} \mathcal{S}_n [\zeta/\zeta_0; \alpha(n+1)/\rho_{10}], \quad (13)$$

where $\rho_{10}(r)$ is the density in the midplane, $\mathcal{S}_n(\xi; \varpi)$ is defined by the integral relation

$$\xi = \int_{\mathcal{S}_n}^1 \frac{ds}{s^{(n-1)/n} [1 - s^{(n+1)/n} + \varpi(1 - s^{1/n})]^{1/2}} \quad (14)$$

and

$$\zeta_0 = \left[\frac{2n}{(n+1)} \rho_{10}^{(n-1)/n} \right]^{-1/2}. \quad (15)$$

The disk's surface is then located at $\zeta = \Theta_1$ in leading order, where

$$\Theta_1 = \zeta_0 \int_0^1 \frac{ds}{s^{(n-1)/n} [1 - s^{(n+1)/n} + \varpi(1 - s^{1/n})]^{1/2}}. \quad (16)$$

Finally one must match the gravitational potential to the vacuum field present outside the disk, on the surface of the disk (see paper I, and later sections). This implies that the function ρ_{10} is related to the function F_0 through the relations

$$F_0 = -\frac{1}{2} \int_0^1 \mathcal{K}(r, s) \Sigma_1(s) s ds, \quad (17)$$

and

$$\frac{1}{2} \Sigma_1 = \sqrt{\frac{2n}{(n+1)} \rho_{10}^{(n+1)/n} \left[1 + \frac{\alpha(n+1)}{\rho_{10}} \right]}, \quad (18)$$

where Σ_1 is the surface density and

$$\mathcal{K}(r, s) = \frac{\pi}{2} \sum_{k=0}^{\infty} (4k+1) \left[\frac{(2k)!}{(2^k k!)^2} \right]^2 P_k(\sqrt{1-r^2}) P_k(\sqrt{1-s^2}), \quad (19)$$

with $P_k(x)$ a Legendre polynomial.

The halo suppresses the natural tendency of the disk to flare toward its edge and compresses the structure as a whole. Figure 1 illustrates this distortion graphically, giving polytropic shapes for various values of α . These disks are characterized by the surface density, $\Sigma_1 = 2(1 - r^2)^{3.5}$, and polytropic index, $n = 1.5$. Provided the halo is sufficiently massive, polytropes over wide ranges in n become lenticularly shaped, and there is no need to apply boundary layer methods to correct the solution near the rim of the disk, $r = 1$, as there is for a disk with only self gravity.¹

III. SINUOUS NORMAL MODES

The sinuous modes, which are antisymmetric about the midplane, can be treated in a very similar manner to the varicose normal modes.² We first split the azimuthal velocity into a mean and a perturbation, $v = r\Omega + v'$, and then, since v no longer appears, we drop the prime on v' . The linear equations of motion and continuity are then

$$(\partial_t + \Omega\partial_\theta)u - 2\Omega w = -\partial_r f, \quad (22)$$

$$(\partial_t + \Omega\partial_\theta)v + (2\Omega + r\partial_r\Omega)u = -\frac{1}{r}\partial_\theta f, \quad (23)$$

$$(\partial_t + \Omega\partial_\theta)w = -\varepsilon^{-1}\partial_z f \quad (24)$$

and

$$(\partial_t + \Omega\partial_\theta)\rho' + \frac{1}{r}\partial_r(r\rho u) + \rho\partial_\theta v + \varepsilon^{-1}\partial_z(\rho w) = 0. \quad (25)$$

Here f represents the linear, time-dependent perturbation to F , given by

$$f = \phi + h, \quad (26)$$

and is no longer solely a function of r . Moreover, the perturbations to the thermodynamic state are described by the enthalpy change, h , and density change, ρ' . The linear version of

Poisson's equation is

$$\partial_\zeta^2 \phi = \rho' - \varepsilon^2 \left[\frac{1}{r} \partial_r (r \partial_r \phi) + \frac{1}{r^2} \partial_\theta^2 \phi \right], \quad (27)$$

where ϕ is the potential perturbation.

Equations (22)–(24) admit normal-mode solutions whose temporal and azimuthal dependence have the form $\exp(\eta t + im\theta)$, where η is the (complex) growth rate and m is the azimuthal quantum number. Hence these equations can be rearranged to give the velocity components in terms of the other variables:

$$u = -\frac{1}{\kappa^2} \left(\nu \partial_r f + \frac{2im\Omega}{r} f \right), \quad (28)$$

$$v = \frac{1}{\kappa^2} \left(2B \partial_r f - \frac{im\nu}{r} f \right) \quad (29)$$

and

$$w = -\frac{1}{\varepsilon\nu} \partial_\zeta f, \quad (30)$$

where

$$B = 2\Omega + r \partial_r \Omega, \quad (31)$$

$$\nu = \eta + im\Omega \quad (32)$$

and

$$\kappa^2 = \nu^2 + 4\varepsilon\Omega B =: \nu^2 + \Delta. \quad (33)$$

If we insert these expressions into the continuity equation, we obtain the dynamical perturbation equation,

$$\partial_\zeta (\rho \partial_\zeta f) = \varepsilon^2 \nu^2 \rho' - \varepsilon^2 \nu^2 \left\{ \frac{1}{r} \partial_r \left(\frac{r\rho}{\kappa^2} \partial_r f \right) + \left[\frac{2im}{r\nu} \partial_r \left(\frac{\rho\Omega}{\kappa^2} \right) - \frac{\rho m^2}{r^2 \kappa^2} \right] f \right\}. \quad (34)$$

A. Asymptotic solution

We now develop the equations asymptotically:

$$f \sim \varepsilon^{-1} f_0 + f_1 + \varepsilon f_2, \quad \phi \sim \varepsilon^{-1} \phi_0 + \phi_1 + \varepsilon \phi_2, \quad (35a)$$

$$h \sim h_1 + \varepsilon h_2 \quad \text{and} \quad \rho' \sim \rho_1 + \varepsilon \rho_2, \quad (35b)$$

with

$$\eta \sim \varepsilon^{-1/2}(\eta_0 + \varepsilon \eta_1), \quad \Omega \sim \varepsilon^{-1/2}(\Omega_0 + \varepsilon \Omega_1) \quad \text{and} \quad \nu \sim \varepsilon^{-1/2}(\nu_0 + \varepsilon \nu_1). \quad (35c)$$

In addition, since we consider perturbations that are *antisymmetric* about the midplane of the disk, we impose the boundary conditions:

$$\phi = 0 \quad \text{and} \quad \frac{1}{\rho_{10}} \partial_\zeta \phi = -A(r) \quad \text{on} \quad \zeta = 0, \quad (36)$$

where the function $A(r)$ is as yet unknown, and must be specified by matching the solution for the perturbation to the external potential. This yields an eigenvalue problem with $A(r)$ as the eigenfunction.

The leading-order terms of Eqs. (27) and (34) are

$$\partial_\zeta(\rho_1 \partial_\zeta f_0) = 0 \quad \text{and} \quad \partial_\zeta^2 \phi_0 = 0. \quad (37)$$

The $O(\varepsilon^{-1})$ solution is therefore trivial: $\phi_0 = f_0 = 0$. At next order, we have

$$\partial_\zeta(\rho_1 \partial_\zeta f_1) = 0 \quad (38)$$

and

$$\partial_\zeta^2 \phi_1 = \rho'_1. \quad (39)$$

To avoid a logarithmic singularity in f_1 at the surface of the disk, we require $f_1 = 0$.

Equation (39) can then be solved to give

$$\phi_1(r, \zeta) = -A_1 \int_0^\zeta \rho_1(r, \zeta') d\zeta' = -\frac{1}{2} A_1 \mu_1(r, \zeta), \quad (40)$$

where $\mu_1(r, \zeta)$ is the unperturbed surface density between ζ and $-\zeta$. The convenient relation,

$$\partial_\zeta \phi_1 = -A_1 \rho_1, \quad (41)$$

then follows.

The $O(\varepsilon)$ terms of Eqs. (27) and (34) now yield

$$\partial_\zeta(\rho_1 \partial_\zeta f_2) = \nu_0^2 \rho_1' \quad (42)$$

and

$$\partial_\zeta^2 \phi_2 = \rho_2'. \quad (43)$$

Equation (42) can be integrated once, with the help of (39), to

$$\rho_1 \partial_\zeta f_2 = \nu_0^2 \partial_\zeta \phi_1. \quad (44)$$

Introduction of Eq. (41) into the right-hand side of this equation and integration yield

$$f_2(r, \zeta) = \nu_0^2 A_1 \zeta. \quad (45)$$

The amplitude of the axial velocity, $w = -\nu^{-1} \varepsilon^{-1/2} \partial_\zeta f$, is therefore constant with distance from the midplane in leading order. This means that the disturbance does not induce a true compression or rarefaction anywhere within the disk; density changes are caused only by the advection of stratified fluid.

The $O(\varepsilon)$ correction to the gravitational perturbation can be found by integrating Eq. (43). This gives

$$\phi_2(r, \zeta) = -\frac{1}{2} \left[2\nu_0^2 A_1(r) \zeta - 2G_0(r) A_1(r) \zeta + A_1(r) \mu_2(r, \zeta) + A_2(r) \mu_1(r, \zeta) \right], \quad (46)$$

where

$$G_0(r) = \frac{1}{r} \partial_r(r \partial_r \Phi_0) = \frac{1}{r} \partial_r(r \partial_r F_0) = 2\Omega_0^2 + r \frac{d\Omega_0^2}{dr}, \quad (47)$$

μ_2 is the $O(\varepsilon)$ correction to μ_1 , namely,

$$\mu_2(r, \zeta) = \int_{-\zeta}^{+\zeta} \rho_2 d\zeta, \quad (48)$$

and $A_2(r)$ is another undetermined function. Accordingly,

$$[\partial_\zeta \phi_2 + \Theta_2 \rho_1']_{\zeta=\Theta_1} = [G_0(r) - \nu_0^2] A_1, \quad (49)$$

where Θ_2 is the $O(\varepsilon)$ correction to the position of the surface of the disk.¹

B. The eigenvalue problem

As before, we match the internal solution

$$\phi(r, \Theta_1) = \phi_1(r, \Theta_1) + \dots, \quad \phi_z(r, \Theta_1) = \phi_{2\zeta}(r, \Theta_1) + \Theta_2 \phi_{1\zeta\zeta}(r_1, \Theta_1) + \dots \quad (50)$$

(where we have used $\phi_{1\zeta}(r, \Theta_1) = 0$, following from (41)) to the external solution

$$\phi^{\text{ext}}(r, \varepsilon \Theta_1) = \phi_1^{\text{ext}}(r, 0) + \dots, \quad \phi_z^{\text{ext}}(r, \varepsilon \Theta_1) = \phi_{1z}^{\text{ext}}(r, 0) + \dots \quad (51)$$

(It is not necessary to consider the time-dependent perturbation of the surface of the disk, since the continuity of the undisturbed potential and its derivative across the surface ensure that such terms always cancel. This is explicitly seen in the fuller derivation presented in paper II.) The resulting equation can be manipulated into the eigenvalue problem,

$$[\nu_0^2 - G_0(x)]A_1(x) = -\frac{1}{2} \int_0^1 Q_m(x, y) A_1(y) \Sigma_1(y) y dy, \quad (52)$$

where we have introduced the new coordinates, $x^2 = 1 - r^2$ and $y^2 = 1 - s^2$. On account of the antisymmetry about the midplane, the kernel Q may be put in the convenient form,⁵

$$Q_m(x, y) = \frac{1}{xy} \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k-1+2m)!} (4k+2m-1) \Lambda_k^m P_{2k+m-1}^m(x) P_{2k+m-1}^m(y), \quad (53)$$

where $P_j^m(x)$ is a Legendre function, and

$$\Lambda_k^m = \frac{(2k-1)!(2k+m-1)!}{[2^{2k+m-1}(k-1)!(k+m-1)!]^2} \frac{\pi}{2}. \quad (54)$$

If we further set

$$A_1 = \frac{1}{\Sigma_1} d_1(x) \quad (55)$$

and

$$\varphi_k^m(x) = P_{2k+m-1}^m(x) \sqrt{(4k+2m-1) \frac{(2k-1)!}{(2k-1+2m)!}}, \quad (56)$$

then

$$[\nu_0^2 - G_0(x)] \frac{2x d_1(x)}{\Sigma_1} = - \sum_{k=1}^{\infty} \Lambda_k^m \varphi_k^m(x) \int_0^1 \varphi_k^m(y) d_1(y) dy. \quad (57)$$

Equation (57) is the eigenvalue problem for the sinuous modes. It is similar to that derived by Hunter and Toomre⁵ for the sinuous oscillations of infinitely thin, pressureless disks. In the rigidly rotating case, the problem reduces to a much simpler form: $\Sigma_1(x) = 2x$ and the k^{th} eigenvalue is

$$\nu_{0,k}^2 = (\eta_{0,k} + im\Omega_0)^2 = -\left(\Lambda_k^m - \frac{\pi}{2}\right). \quad (58)$$

This indicates that all of the sinuous modes are neutral in leading order for the rigid rotator.

For axisymmetrical oscillations of general disk models, $m = 0$, and we may cast the eigenvalue equation into a convenient, variational form by multiplying by $d_1(x)$ and integrating over x :

$$\eta_0^2 \int_0^1 \frac{2d_1^2}{\Sigma_1} x dx = \int_0^1 G_0(x) \frac{2d_1^2}{\Sigma_1} x dx - \sum_{k=1}^{\infty} \Lambda_k^0 \left[\int_0^1 \varphi_k^0 d_1 dy \right]^2. \quad (59)$$

The form of $G_0(x)$ depends on the model considered. For highly centrally condensed models, G_0 is negative throughout substantial regions, whereas for models that are very nearly in uniform rotation, G_0 is close to the value $\pi/2$ throughout the disk. In either circumstance, the second term on the right-hand side of Eq. (59) is a strongly increasing function of radial order and dominates the first term. Axisymmetrical modes are therefore always neutral.

As illustrated by these two special examples, sinuous modes appear to be neutral at leading order. Hunter and Toomre give further examples and complementary arguments to suggest this is always the case (and that the spectrum may also be continuous). The question then arises here as to whether the effects of compressibility introduced through finite thickness may change this trend. Such effects appear at second order in ϵ . If, in that order, we evaluate ϕ_3 and match the result to the analogous term in the perturbed potential outside the polytrope, we get an integral equation for the function $A_2(r)$ and the eigenvalue correction, ν_1 , as for varicose modes.² Then we find that the effect of compressibility leads to further stabilization of the higher-order modes. Therefore, it is not necessary to go to second order to characterize the stability of polytropic disks of finite thickness to sinuous modes.

This suggests that *warps must be externally forced rather than self excited*. We turn next to the study of externally driven warps.

IV. STEADY WARPS

A common feature of observed galaxies is a warping of the disk plane and, currently, one of the most plausible explanations for this phenomenon is that the symmetry axis of the halo which shrouds the disk is not aligned with that of the disk itself.⁴ Then, a weakly dipolar force acts upon the disk in addition to the leading-order, symmetrical compression discussed in Sec. II. For this case, we assume a form for the external potential given by

$$\Phi_h = \frac{1}{2}\varepsilon^{-1}\beta r^2 + \varepsilon\gamma(\theta)r\zeta, \quad (60)$$

where $\gamma(\theta) = \tilde{\gamma}e^{i\theta}$.

Although the disk structure is steady, under the influence of the external potential, it is not axisymmetric. Furthermore, the warping of the midplane introduces noncircular motions. In particular, if the axial displacement of the midplane is $O(\varepsilon\lambda)$, we expect vertical velocities of the order of

$$w \sim \frac{2}{\pi}\varepsilon^{1/2}\Omega_0\lambda \quad (61)$$

and radial motions of similar magnitude. Such considerations provide scalings for the velocity components of a disk with a steady warp.

We also must retain the time derivatives because the external potential can provide a torque on the warped disk that forces its axis to precess.⁵ In that circumstance, the disk properties can depend upon azimuthal angle and time through a term of the form $-ir\Omega_p te^{i\theta}$, where Ω_p is some constant angular precession speed. This term represents precession. There are also other nonaxisymmetrical terms that are independent of time; these represent the permanent warp.

In addition to the conventional asymptotic scalings (10), we pose

$$w = \varepsilon^{1/2} w_1 + O(\varepsilon^{3/2}), \quad u = \varepsilon^{3/2} u_1 + O(\varepsilon^{5/2}), \quad (62)$$

guided by the estimate (61). Moreover, we set

$$F = \varepsilon^{-1} F_0 + F_1 + \dots \quad \text{and} \quad \Omega = \varepsilon^{-1/2} (\Omega_0 + \varepsilon \Omega_1 + \dots), \quad (63)$$

where both the centrifugal potential and angular momentum may now depend on time, θ and ζ , in addition to radius.

A. The expansion

The leading-order asymptotic equations are

$$r\Omega_0^2 = \partial_r F_0, \quad (64)$$

$$r^2(\partial_\theta \Omega_0 + r w_1 \partial_\zeta \Omega_0) = -\partial_\theta F_0, \quad (65)$$

$$0 = \partial_\zeta F_0 = \partial_\zeta \Phi_0 \quad (66)$$

and

$$\partial_t \rho_1 + \partial_\theta (\rho_1 \Omega_0) + \partial_\zeta (\rho_1 w_1) = 0, \quad (67)$$

where the operator,

$$\partial_\theta = \partial_t + \Omega_0 \partial_\theta. \quad (68)$$

Equations (64)–(66) imply that $F_0 = F_0(r) = \Phi_0$ and $\Omega_0 = \Omega_0(r)$. This provides the leading-order solution which is, as expected, uninfluenced by the $O(1)$ dipolar force.

The second-order equations of motion are

$$2r\Omega_0\Omega_1 = \partial_r F_1, \quad (69)$$

$$r^2(\partial_\theta \Omega_1 + w_1 \partial_\zeta \Omega_1) = -\partial_\theta F_1 \quad (70)$$

and

$$0 = \partial_{\zeta} F_1 = \partial_{\zeta} (\Phi_1 + H_1). \quad (71)$$

Thus F_1 and Ω_1 are again independent of ζ , θ and t .

This information alone permits us to construct the vertical stratification. The Poisson equation can be developed to leading and second order into the relations,

$$\partial_{\zeta}^2 \Phi_0 = 0 \quad (72)$$

and

$$\partial_{\zeta}^2 \Phi_1 = -\partial_{\zeta}^2 H_1 = \frac{H_1^n}{n^n}. \quad (73)$$

Consequently,

$$\frac{1}{2}(\partial_{\zeta} H_1)^2 = \frac{(H_{1m}^{n+1} - H_1^{n+1})}{(n+1)n^n} \quad (74)$$

and so

$$\rho_1(r, \theta, \zeta) = \rho_{1m}(r, \theta) C_n[(\zeta - \zeta_1)/\zeta_0], \quad (75)$$

where $\partial_{\zeta} H_1 = 0$, $H_1 = H_{1m}$ and $\rho_1 = \rho_{1m}$ at $\zeta = \zeta_1$. Also, the integral function $C_n(\xi) = S_n(\xi; 0)$, and

$$\zeta_0 = \sqrt{\frac{n^{n-1}(n+1)}{2H_{1m}^{n-1}}}. \quad (76)$$

The structure is therefore vertically symmetric about $\zeta = \zeta_1$ which denotes the location of the displaced midplane. The polytrope's surfaces are located at

$$\zeta = \zeta_1 \pm \Theta_1, \quad (77)$$

where Θ_1 , as given by Eq. (16) with $\varpi = 0$, is the equilibrium half-thickness in the absence of any dipolar distortion.

B. The potential of the warped disk

The internal and external potential fields must now be matched at the surface of the polytrope. Proceeding as in paper I, we derive

$$F_0(r) = \Phi_0^{\text{ext}}(r, \theta, t, 0) \quad \text{and} \quad [\Phi_{1\zeta}]_{\zeta=\Theta_1+\zeta_1} = \Phi_{0z}^{\text{ext}}(r, \theta, t, 0), \quad (78)$$

at the upper surface; conditions at the lower surface follow from simply changing the sign of Θ_1 . These relations imply that H_{1m} is independent of θ and t , and that

$$F_0 = \frac{1}{2}\beta r^2 - \frac{1}{2} \int \mathcal{K}(r, s) \Sigma_1(s) s ds, \quad (79)$$

where

$$\frac{1}{2}\Sigma_1 = \sqrt{\frac{2H_{1m}^{n+1}}{(n+1)n^n}}. \quad (80)$$

At following order, the matching conditions are

$$F_1(r) = \Phi_1^{\text{ext}}(r, \theta, t, 0) + \frac{1}{2}(\zeta_1 + \Theta_1)\Sigma_1 \quad (81)$$

and

$$[\Phi_{2\zeta}]_{\zeta=\Theta_1+\zeta_1} = \Phi_{1z}^{\text{ext}}(r, \theta, t, 0) + (\zeta_1 + \Theta_1)\Phi_{0zz}^{\text{ext}}(r, 0). \quad (82)$$

The $O(1)$ external potential field may now be split into dipolar and axisymmetric parts:

$$\Phi_1^{\text{ext}} = \Phi_1^{\text{ext},d} + \Phi_1^{\text{ext},s}. \quad (83)$$

Equation (81) can then be divided into the components,

$$F_1(r) = \Phi_1^{\text{ext},s}(r, 0) + \frac{1}{2}\Theta_1(r)\Sigma_1(r) \quad (84)$$

and

$$0 = \Phi_1^{\text{ext},d}(r, \theta, t, 0) + \frac{1}{2}\zeta_1(r, \theta, t) \Sigma_1(r). \quad (85)$$

Similarly, if we divide the left-hand side of (82) into pieces with different symmetry,

$$[\Phi_{2\zeta}]_{\zeta=\Theta_1+\zeta_1} = X_d(r, \theta, t) + X_s(r), \quad (86)$$

then that equation can be divided into the two expressions,

$$X_s(r) = \Phi_{1z}^{\text{ext},s}(r, 0) + \Theta_1(r)\Phi_{0zz}^{\text{ext}}(r, 0) \quad (87)$$

and

$$X_d(r, \theta, t) = \Phi_{1z}^{\text{ext},d}(r, \theta, t, 0) + \zeta_1(r, \theta, t)\Phi_{0zz}^{\text{ext}}(r, 0). \quad (88)$$

The axisymmetric relations (84) and (87) can now be combined into

$$F_1 = \frac{1}{2}\Theta_1\Sigma_1 - \frac{1}{2}\int \mathcal{K}(s, r)\Sigma_2(s)sds, \quad (89)$$

where

$$\frac{1}{2}\Sigma_2 \equiv X_s - \Theta_1 G_0, \quad (90)$$

and G_0 is defined by Eq. (47).

The asymmetric equations (85) and (88) can be combined into

$$\frac{1}{2}\int_0^1 \zeta_1(s, \theta, t)\Sigma_1(s)\mathcal{Q}_1(r, s)sds = X_d(r, \theta, t) - \zeta_1(r, \theta, t)G_0(r), \quad (91)$$

where $\mathcal{Q}_m$ is given in (53).

C. Warped galactic models

The leading-order density profile (75) depends on θ and t only through the midplane displacement $\zeta_1(r, \theta, t)$. Hence,

$$\partial_\theta \rho_1 = -\partial_\theta \zeta_1 \partial_\zeta \rho_1. \quad (92)$$

If we insert this expression into the leading-order continuity equation, we find

$$w_1 = \partial_\theta \zeta_1, \quad (93)$$

which indicates that the vertical velocity is independent of ζ .

The third-order vertical equation of motion is

$$\partial_\theta w_1 = -\partial_\zeta F_2, \quad (94)$$

which, when combined with Eq. (93), provides the relation,

$$F_2 = F_{20} - (\zeta - \zeta_1)\partial_\theta^2\zeta_1, \quad (95)$$

where F_{20} is the value of F_2 on the displaced midplane. We also have the relation,

$$\Phi_2 = F_2 - H_2 - \gamma r\zeta, \quad (96)$$

which yields the quantity

$$-[\Phi_{2\zeta}]_{\zeta=\zeta_1+\Theta_1} = \partial_\theta^2\zeta_1 + [H_{2\zeta}]_{\zeta=\zeta_1+\Theta_1} + \gamma r. \quad (97)$$

An important feature of the second-order correction H_2 is that $[H_{2\zeta}]_{\zeta=\zeta_1+\Theta_1}$ is independent of both θ and t if the value of H_2 on $\zeta = \zeta_1$ is also axisymmetric. We currently have sufficient freedom to choose $H_2(r, \theta, t, \zeta_1)$ to be axisymmetric in the scheme of asymptotic expansion. This we shall do, and so the dipolar piece of the surface gravity is given by

$$X_a = -\partial_\theta^2\zeta_1 - \gamma r. \quad (98)$$

We may therefore rewrite (91) as an integral equation that determines the displacement of the midplane:

$$\frac{1}{2} \int_0^1 \zeta_1 \Sigma_1 \mathcal{Q}_1(r, s) s ds = -(G_0 + \partial_\theta^2)\zeta_1 - \gamma r. \quad (99)$$

The integral expression (99) is very similar to the equation describing infinitesimal, steady warps of zero-pressure disks proposed by Hunter and Toomre.⁵ It is here derived under the assumption that the warp of the galactic disk is as large as its characteristic thickness. Equation (99) is linear in the midplane displacement, ζ_1 , because the external potential (60) is linear in ζ . This makes Eq. (99) relatively straightforward to solve. We call such a solution, a *linear warp*, and we present an example below. We could allow the external potential to have an arbitrary dependence on the axial coordinate. We would then replace the final term of Eq. (99), $-\gamma r$, with one having general nonlinear dependence on ζ_1 . This would describe

a *nonlinear warp*, but then the integral equation is harder to solve and we do not pursue these ideas here.

If we change variables to x and y , the warp equation becomes

$$\frac{1}{2x} \sum_{k=0}^{\infty} \Lambda_k^1 \varphi_k^1 \int_0^1 \tilde{\zeta}_1 \Sigma_1 \varphi_k^1 dy = (\Omega_0^2 + D_0) \tilde{\zeta}_1 - 2i\Omega_0 \partial_t \tilde{\zeta}_1 - \tilde{\gamma} \sqrt{1-x^2}, \quad (100)$$

where $\zeta_1 = \tilde{\zeta}_1(r) \exp i\theta$. For models with particular surface densities, expression (100) can be written as a finite-dimensional matrix equation by separating ζ_1 into terms representing precession (which are linear in t) and the permanent warp (independent of t), then expanding in a truncated series of the functions φ_k^1 . For a rigidly rotating disk, $\Sigma_1 = 2x$ and the solution is $\tilde{\zeta}_1 \propto r$. The warp of a more centrally condensed, differentially rotating model is shown in Fig. 2. The surface density law is given by $\Sigma_1 = 2x^7$, the polytropic index is 1.5 and the parameter $\tilde{\gamma}$ is set to unity (an edge correction has also been applied based on the method outlined in paper I and the value for ε of 0.01).

V. REMARKS

In line with earlier work, we have found that in the purely gravitational case, disk warps are not self-excited. We then went on to extend previous work⁵ on warped disks to develop the theory for disks with finite thickness. Our results may be of use for some further studies and we close by mentioning a few.

Although we have been concerned with disk galaxies, the asymptotic theory may also be applied to other astrophysical objects like accretion disks. These disks are normally taken to be turbulent and this raises the issue of the origin of the turbulence. Nonlinear instability is a natural process⁶ that may be expected to occur in disks as it does in laboratory shears, even if over-reflective instabilities^{7,8} prove too mild. It is commonly believed that nonlinear instability leads to complicated two-dimensional motions which give way to fully three-dimensional turbulence through secondary instability.⁹ However, in a warped disk, we face

a three-dimensional flow at the outset and this may act as a powerful catalyst favoring a primary three-dimensional instability. Hence the warp may provide a direct route to turbulence.

Moreover, magnetic fields can certainly catalyze disk turbulence.¹⁰ On the other hand, to produce an appreciable magnetic field, we need a sufficiently complicated flow to begin with. Perhaps disks can bootstrap themselves into a situation with both magnetic fields and turbulence in mutual support. However, the solutions found here provide basic states containing nonzero helicity densities, hence describe equilibria that may allow a dynamo of the $\alpha - \omega$ variety. It might be worthwhile to examine the dynamo action of disk warps, at least in the kinematic case. To do this, we could use the periodic Lagrangian orbits of the warped disk flows to get the dynamo growth rate.¹¹ In short, the complications considered here, may simplify some other problems in surprising ways.

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FIGURE CAPTIONS

FIG. 1. The shape of polytropic disks subjected to external gravitational fields. The surface density is given by $\Sigma_1 = 2x^7$. The polytropes with $n = 0.5, 1.0$ and 1.5 are shown. The curves drawn for each polytrope correspond to different values for the parameter α . The continuous, dotted, dashed and dot-dashed curves indicate the particular values of α of $0, 0.01, 0.1$ and 0.5 . As α increases the characteristic density of the surrounding halo also increases. For the polytrope with $n = 1.5$ the curve representing the shape for $\alpha = 10^{-3}$ is included.

FIG. 2. A model of a warped galaxy. A massive halo whose axis is not aligned with that of the disk is assumed to be responsible for the warp, and creates an external gravitational field given by equation (60). The resulting midplane distortion is computed using equation (100), and the axial stratification from equation (13) with $\varpi = 0$. In addition, the leading-order axisymmetric surface density has the form $2x^7$, and the polytropic index $n = 1.5$. Irregular behaviour at the edge has been corrected by the method outlined in paper I. The particular contour levels shown are those with values linearly spaced in increments of 0.1 between 0.1 and 0.9 (continuous lines), and logarithmically spaced between 10^{-2} and 10^{-5} (dashed lines).

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