UW-CPTC --98-10

UNIVERSITY OF WISCONSIN

CENTER FOR PLASMA THEORY AND COMPUTATION

REPORT

RECEIVED

SEP 22 1998

OSTI

Finite Pressure Effects on the Tokamak Sawtooth Crash

Yasutaro Nishimura

Department of Nuclear Engineering And Engineering Physics
University of Wisconsin
Madison, WI 53706-1687

July 1998

UW-CPTC 98-10

MASTER
JNT

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

MADISON, WISCONSIN 53706-1687

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, make any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

Finite Pressure Effects on the Tokamak Sawtooth Crash

by

Yasutaro Nishimura

A dissertation submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

(Nuclear Engineering And Engineering Physics)

at the

The University of Wisconsin-Madison

1998

Contents

List of Tables

List of Figures

Abstract

Chapter 1 Introduction

Chapter 2 Experimental Observations Of Sawtooth Oscillations

- 2.1 Early Tokamak Experiments 6
- 2.2 Direct measurement of magnetic field 7
- 2.3 ECE and soft X-ray tomography in TFTR 9
- 2.4 Partial Temperature Crash in TFTR 11
- 2.5 Theoretical interpretation: Finite pressure effects and the role of stochasticity 12

Chapter 3 The Model Equations and Numerical Schemes

- 3.1 Basic MHD equations 17
- 3.2 FAR code formulation 22
 - 3.2.1 Equilibrium 22

i

ii

- 3.2.2 Dynamics 24
- 3.2.3 Boundary conditions 26
- 3.3 Energy Conservation 27

Chapter 4 Linear And Nonlinear Resistive Kink Modes

- 4.1 Linear growth and nonlinear saturation of the ideal $m = 1$ kink mode 29
- 4.2 Linear layer reconnection theory 37
- 4.3 Kadomtsev helical flux reconnection model 39
- 4.4 Mode coupling in toroidal geometry 41
- 4.5 Low β simulation results 44
 - 4.5.1 Rounded q profile ($\lambda_{rs} = 2$) case 45
 - 4.5.2 Peaked q profile ($\lambda_{rs} = 1$) case 49

Chapter 5 Nonlinear Excitation Of Pressure-Driven Instabilities

- 5.1 Suydam and Mercier criteria 66
- 5.2 Finite pressure effects on internal kink modes 70
- 5.3 Eikonal representation and ballooning transformation 71
 - 5.3.1 Ballooning equation 73
 - 5.3.2 Ballooning $s - \alpha$ diagram 76
- 5.4 High β simulation results 76
 - 5.4.1 Magnetic shear reduction near the current sheet 82
 - 5.5 Effect of heat transport along stochastic field lines 83

Chapter 6 Summary and Discussion

Appendix A Magnetic Field Line Behavior

- A.1 Magnetic field line equation in a straight field line coordinate system 105

A.2 Magnetic field line equation in a Cartesian coordinate system	109
---	-----

Appendix B Statistical Analysis Of The Magnetic Field Line Stochasticity

	112
B.1 Lyapunov exponents	113
B.2 Spatial correlations and magnetic field line diffusion	117
B.3 Thermal transport term along stochastic field lines	121

Bibliography	123
------------------------	-----

List of Figures

2.1 (a) Experimental arrangement of X-ray detectors in the ST tokamak;	
(b) The soft X-ray signals obtained exhibit sawteeth or internal disruptions. The signal above is at $r = 3.9$ cm. The signal below is at $r = 0$ cm. (From S. von Goeler et al., Ref. [1].)	13
2.2 Series of ECE images during the sawtooth crash in TFTR. Red represents the highest electron temperature, 6000 eV; the separation between isotherms is 500 eV. The relaxation shown takes about 3 ms; the fourth and the sixth frames are separated by 130 μ s. All the figures are on the bad curvature side. (From J.D. Callen et al., Ref. [6], courtesy of Y. Nagayama.)	14
2.3 Heat flow contours superimposed on $T_e(r, \theta)$ contours (solid lines) for (a) $t = 216$ μ s, (b) $t = 314$ μ s, (c) $t = 368$ μ s, and (d) $t = 422$ μ s. Blue (red) represents a heat loss (gain) of about 1×10^4 keV/s. The dashed lines depict the inversion radius, which is close to the $q = 1$ radius measured by MSE. (From M. Yamada et al., Ref. [10].)	15
2.4 Time evolution of the central q -value and the ECE based T_e signal. (From M. Yamada et al., Ref. [9].)	16
4.1 The behavior of the analytical solution for the internal column displacement ξ .	32
4.2 The relation between x and the function $f(\psi)$. Curve B added on top of the analysis by Waelbroeck [38].	36
4.3 The behavior of eigenfunctions ξ (thicker solid line) and ψ (thinner solid line). The dashed lines indicate the ideal kink mode behavior. The solid lines indicate the resistive kink mode behavior.	39
4.4 Schematic diagrams showing a rigid shift of the core and the reconnection of an $m = 1, n = 1$ magnetic island, i.e., the Kadomtsev model. Solid lines show the poloidal magnetic field in the plasma. (a) Initial state. (b) Nonlinear saturation after the establishment of an equilibrium with infinite electrical conductivity. (c) During reconnection of the magnetic field lines due to the finite conductivity. (d) After reconnection. (From B.B. Kadomtsev, [2].)	52
4.5 A schematic diagram showing the behavior in the thin layer where the magnetic reconnection takes place. (a) Initial state, (b) After the layer was squeezed by the shift of the center plasma column. Here r is the initial length of the layer, δ is the width of the layer, v_e is the velocity of the column shift, and v_e is the velocity of the ejection.	53
4.6 Linear growth rate of an $m/n = 1/1$ resistive kink mode versus Lundquist number. The dashed line is from numerical calculations.	54
4.7 Eigenmodes of an $m/n = 1/1$ resistive kink mode with a rounded q -profile and $\beta = 0.3\%$ at $t = 900\tau_s$. The dashed line at $\rho = 0.56$ corresponds to the $q = 1$ rational surface: (a) $\psi_{1/1}$, (b) $\phi_{1/1}$, (c) $p_{1/1}$, and (d) $U_{1/1}$ profiles.	55

4.8 Eigenmodes of an $m/n = 1/1$ resistive kink mode at $t = 1058\tau_a$. The dashed line at $\rho = 0.56$ corresponds to the $q = 1$ rational surface: (a) $\psi_{1/1}$, (b) $\phi_{1/1}$, (c) $p_{1/1}$, and (d) $U_{1/1}$ profiles.	56
4.9 (a), (c), and (e) Poincaré plots of magnetic field line trajectories at a toroidal angle $\zeta = 0$, and (b), (d), and (f) at a toroidal angle $\zeta = \pi$, for nonlinear evolution of an internal kink mode in a $\beta = 0.3\%$ and $S = 10^5$ plasma.	57
4.10 Poincaré plots for further time evolution of plots in Figs. 4.9.	58
4.11 The values of the Lyapunov exponents after 200 revolutions, versus the initial position at ($\zeta = 0$) of the field line trajectories. Corresponds to Figs. 4.9 and Figs. 4.10: (a) $t=1058 \tau_a$, (b) 1093, (c) 1113, (d) 1133, (e) 1300, and (f) $t=1800 \tau_a$	59
4.12 Eigenmodes of an $m/n = 1/1$ resistive kink mode with a peaked q -profile and $\beta = 0.3\%$, at $t = 3000\tau_a$. The dashed line at $\rho = 0.31$ corresponds to the $q = 1$ rational surface: (a) $\psi_{1/1}$, (b) $\phi_{1/1}$, (c) $p_{1/1}$, and (d) $U_{1/1}$ profiles.	60
4.13 Eigenmodes of an $m/n = 1/1$ resistive kink mode with a peaked q -profile and $\beta = 0.3\%$, at $t = 3100\tau_a$ (right after complete reconnection). The dashed line at $\rho = 0.31$ corresponds to the $q = 1$ rational surface: (a) $\psi_{1/1}$, (b) $\phi_{1/1}$, (c) $p_{1/1}$, and (d) $U_{1/1}$ profiles.	61
4.14 (a), (c), and (e) Velocity fields at a toroidal angle $\zeta = \pi$, and (b), (d), and (f) Poincaré plots of magnetic field line trajectories at a toroidal angle $\zeta = \pi$, for nonlinear evolution of an internal kink mode in a $\beta = 0.5\%$ and $S = 10^5$ plasma.	62

4.15 Poincaré plots and the velocity field for further time evolution of the plots in Figs. 4.14.	63
4.16 (a), (c), and (e) Poincaré plots of magnetic field line trajectories at a toroidal angle $\zeta = 0$, and (b), (d), and (f) at a toroidal angle $\zeta = \pi$, for nonlinear evolution of an internal kink mode in a $\beta = 5\%$ and $S = 10^5$ plasma.	64
5.1 The equilibrium flux surfaces and θ contour lines in a toroidal geometry with a very large Shafranov shift $\beta = 20\%$. Magnetic field lines spend a longer "time" on the good curvature side than on the bad curvature side.	69
5.2 The eigenfunction of the ballooning equation Eq. (5.14).	75
5.3 The $s - \alpha$ diagram from Eq. (5.19).	77
5.4 Mode spectrum obtained from the high β simulation. The abscissa and ordinate corresponds to poloidal and toroidal mode numbers, respectively. A total of 151 modes are used in the simulations.	78
5.5 Pressure contour for the $\beta = 5\%$ simulation. The symmetry axis of the tokamak equilibrium is a vertical line through the middle of the page. Thus, the inboard side of the torus is located between the two pictures on each row. The outboard side of the torus is toward the edges of the page.	85
5.6 Pressure contour for the $\beta = 5\%$ simulation. The axis of symmetry of the tokamak plasma is a vertical line down the middle of the page.	86
5.7 The velocity field and the corresponding pressure contour for the $\beta = 5\%$ simulation. Both the figures are located on the bad curvature (outboard) side of the torus.	87

5.8 Fourier spectrum of the stream functions ϕ for the $\beta = 5\%$ simulation, at $t = 0\ t_e$. Same helicity modes, $m/n = 1/1$ to $m/n = 10/10$	88
5.9 Fourier spectrum of stream function ϕ for the $\beta = 5\%$ simulation, at $t = 0\ t_e$. Modes with same toroidal mode number $n = 10$	89
5.10 Fourier spectrum of stream function ϕ for the $\beta = 5\%$ simulation, at $t = 70\ t_e$. Same helicity modes, $m/n = 1/1$ to $m/n = 10/10$	90
5.11 Fourier spectrum of stream function ϕ for the $\beta = 5\%$ simulation, at $t = 70\ t_e$. All modes shown have the same toroidal mode number $n = 10$, but they have incommensurate helicities.	91
5.12 Poincaré plot for the $\beta = 5\%$ simulation. To the right is the bad curvature side. These field lines are for the case as that for the pressure contours in Fig. 5.6(b).	92
5.13 The values of the Kolmogorov entropy after 200 revolutions, versus the initial position ($\zeta = \pi$) of the field line trajectories. Case corresponds to the Poincaré plot of Fig. 5.12.	92
5.14 Pressure contours for the $\beta = 4\%$ simulation. The axisymmetry center line is located at the middle of the page.	93
5.15 Pressure contours for the $\beta = 3\%$ simulation. The axisymmetry center line is located at the middle of the page.	94
5.16 Pressure contours for the $\beta = 2\%$ simulation. The axisymmetry center line is located at the middle of the page.	95
5.17 Poincaré plots of magnetic field line trajectories in the vicinity of the X-point ($\beta = 5\%$ and $S = 10^5$ plasma).	96

5.18 Pressure contours for the $\beta = 5\%$ simulation in the presence of finite parallel diffusion. The axisymmetry center line is located at the middle of the page.	97
A.1 Equilibrium q profile (solid line) and q profile in the presence of a magnetic island (solid line).	109
B.1 Toroidal revolution history of the Lyapunov exponents of field line trajectories up to 200 revolutions around the torus. Stable trajectories occur with initial positions $x_0 = -0.96, 0.86, Z = 0$ (green and red curves); unstable trajectories occur with initial positions $x_0 = -0.36, 0.26, Z = 0$ (black and blue curves). Poincaré plots for these cases are shown in Fig. B.2.	115
B.2 Poincaré mappings of magnetic field line trajectories. The colors correspond to those of Fig. B.1.	116
B.3 Auto correlations.	118
B.4 Short range behavior of cross correlation.	119
B.5 Long range behavior of cross correlation.	120

Abstract

FINITE PRESSURE EFFECTS ON THE TOKAMAK SAWTOOTH CRASH

Yasutaro Nishimura

Under the supervision of Professor James D. Callen
at the University of Wisconsin-Madison

The sawtooth crash [S. von Goeler et al., Phys. Rev. Lett. **33**, 1201 (1974)] is a hazardous, disruptive phenomenon that is observed in tokamaks whenever the safety factor at the magnetic axis is below unity. Recently, Tokamak Test Fusion Reactor (TFTR) experimental data [Y. Nagayama et al., Phys. Plasmas **3**, 1647 (1996)] has revealed interesting features of the dynamical pressure evolution during the crash phase.

Motivated by the experimental results, this dissertation focuses on theoretical modeling of the finite pressure effects on the nonlinear stage of the sawtooth crash. The crash phase has been studied numerically employing a toroidal magnetohydrodynamic (MHD) initial value code deduced from the FAR code [L. A. Charlton et al., J. Comput. Phys. **86**, 270 (1990)]. For the first time, by starting from a concentric equilibrium, it has been shown that the evolution through an $m/n = 1/1$ magnetic island induces secondary high- n ballooning instabilities. The magnetic island evolution gives rise to convection of the pressure inside the inversion radius and builds up

a steep pressure gradient across the island separatrix, or current sheet, and thereby triggers ballooning instabilities below the threshold for the axisymmetric equilibrium.

A magnetic field line solver with high spatial resolution has been developed and incorporated to explore the resultant magnetic field line stochasticity. The microscopic combined structure of the regular and destroyed magnetic flux surfaces is investigated quantitatively. The "radial" transport due to stochastic magnetic field lines and the resultant electron temperature response are briefly examined.

Due to the onset of secondary ballooning modes, concomitant fine scale vortices and magnetic stochasticity are generated. These effects produce strong flows across the current sheet, and thereby significantly modify the $m = 1$ driven magnetic reconnection process. The resultant interaction of the high- n ballooning modes with the magnetic reconnection process is discussed.

Acknowledgment

First of all, I would like to thank my thesis advisor, Professor James Callen. I would like to thank him for providing me with an opportunity to work on the sawteeth analysis and his sharing enthusiasm. I am very thankful to Professor Callen for thoroughly correcting my manuscripts.

Second of all, I would like to thank my defense committee Professors Gilbert Emmert, Raymond Fonck, Carry Forest, Stewart Prager, and Paul Terry.

I would like to thank Drs. B. Carreras, K. Corner, B. Coppi, R. Dagazian, E. Fernandez, J. Gilland, B. Griffith, C. Hegna, E. Held, N. Hershkovitz, H. Ji, S. Jones, K. Kelly, J. Kim, M. Kissick, S. Kruger, J.-N. Leboeuf, V. Lynch, L. Louis, Y. Mouzouris, Y. Nagayama, D. Newman, W. Park, C. Ren, J. Santarius, L. Schilling, K. Shaing, D. Smitherman, S. Spear, L. Sugiyama, M. Swandby, J. Tataronis, R. Torasso, F. Waelbroeck, J. Wheaton, S. Wukitch, E. Uchimoto, S. von Goeler, M. Yamada, and L. Zakharov. In particular, I appreciate discussions with C. Hegna and C. Ren.

Finally, I would like to acknowledge an uncountable credit from L. Charlton, H. Hicks, J. Holmes, J.-N. Leboeuf, D. Lee, V. Lynch, D. Spong, and B. Waddell.

This research was supported by United States Department of Energy Grant No. DE-FG02-86ER53218.

Chapter 1

Introduction

The basic concept of magnetic plasma confinement relies on the existence of magnetic flux surfaces. A flux surface is a surface ergodized by one magnetic field line. Since the center of charged particle gyro-motion follows magnetic field lines, topologically bounded flux surfaces, (or invariant tori) provide an ideal plasma confinement system. Tokamaks consist of axisymmetric nested magnetic flux surfaces in equilibrium. The magnetic field lines in tokamaks consist of *toroidal* fields generated by external coils, and a *toroidal* current-induced *poloidal* magnetic field.

As long as magnetic surfaces have a nested structure, the concept of toroidal confinement is valid. However, the response to a magnetic perturbation is resonant on a *rational surface*, where magnetic field lines have a pitch that is a ratio of integers so that field lines close on themselves after a rational number of toroidal (or poloidal) transits. This resonant response in the presence of dissipation can reconnect magnetic field lines and give rise to bifurcation of magnetic flux surface and produces so-called magnetic islands.

Major disruptions which are accompanied by drastic confinement deteriora-

tion can be caused by a magnetic surface bifurcation. A major class of disruptions is thought to be caused by overlapping of magnetic islands of incommensurate helicity. Another important disruptive phenomena are known as sawtooth crashes [1], which are sometimes referred to as internal disruptions. Understanding and control of the detailed mechanisms of both these disruptive phenomena is crucial for obtaining stable, long pulsed tokamak operations.

Sawtooth crashes [1] are generally observed in tokamak plasmas whenever the safety factor q at the magnetic axis drops below unity. The temporally discontinuous soft X-ray signals that are observed experimentally suggest sudden electron temperature drops or crashes in the core of the plasma column. After the crash, the temperature profile is flattened inside the core plasma region where $q \leq 1$.

There is a large gap between theoretical understanding of sawteeth and experimental results; the mechanism responsible for the nonlinear regime of the crash phase is far from being understood. The following principal problems remain unsolved for sawtooth studies: (1) The central q -value remains below unity during the crash in most experiments whereas theory predicts [2] a full magnetic field reconnection and that the q rises to above unity in the core of the plasma. (2) The time scale of the final crash phase is much shorter than the one predicted by the Kadomtsev reconnection [2] or Sweet-Parker layer model [3, 4]. In today's high temperature tokamak plasmas, the time crash scale can be on the order of tens of microseconds. (Some crashes are still as slow as a few milliseconds [5].) (3) Although the precursor signals of the X-ray and ECE experimental data indicates the excitation and slow growth of an $m/n = 1/1$ MHD-like mode, the mechanism for a sudden transition from a marginally stable state to a disruptively unstable state is not understood.

Recent experimental results, for example those in Refs. [6] and [7], indicate

a dynamical temperature evolution during the crash phase of a sawtooth oscillation. Furthermore, a couple of experimental papers [8, 9] have suggested that magnetic field line stochasticity in the vicinity of the $q = 1$ inversion radius plays an important role in thermal transport and the resultant rapid changes in the magnetic field structure. Experimental measurements in the Tokamak Fusion Test Reactor (TFTR) [10] suggest that the pressure contours and the magnetic flux surfaces do not coincide during the nonlinearly developed stages of the crash [5].

In the presence of magnetic field line stochasticity, the effective radial heat diffusivity in the collisional limit [11] is given by $\kappa_{eff} \sim \kappa_{\perp} + (B_p/B)^2 \kappa_{\parallel}$ [12], where $\kappa_{\perp}$ and $\kappa_{\parallel}$ are the perpendicular and the parallel thermal diffusivity, B_p is the stochastic component of the radial component of the magnetic perturbation, and B is the strength of the external magnetic field. Thus, parallel transport contribution effects are significant if $B_p/B \gtrsim (\kappa_{\perp}/\kappa_{\parallel})^{1/2}$, which is often the case when the magnetic field lines become stochastic through nonlinear effects.

Also indicated by ECE tomography experimental data is the "inside-outside" asymmetry of the signals. This asymmetry indicates the localization of the mode structure in toroidal angle, typically on the bad curvature side [5, 13]. The localization of the mode on the bad curvature side could be due to ballooning modes. Ballooning modes are analogous to Rayleigh-Taylor instabilities (instability of heavy fluid over a light fluid); in tokamak plasmas, a curvature-induced centrifugal force is equivalent in its effect to the earth's gravity for Rayleigh-Taylor instabilities. Furthermore, the toroidal localization of the ballooning mode can break the toroidal and near helical symmetry and thereby induce magnetic stochasticity.

The goal of our research is to investigate pressure evolution feedback effects on the magnetic reconnection process during a sawtooth crash. First, we investigate

the magnetic field line structure in low β plasmas. Second, we investigate the onset and nonlinear evolution of the pressure-driven modes, that is the ballooning mode, in the final stage of sawtooth crash. The ballooning mode is nonlinearly triggered by the evolution of the $m/n = 1/1$ magnetic island. The mode spectrum, obtained from numerical simulation, will be investigated in detail. Finally, we discuss our initial results on the combined effects of the finite thermal conduction along the stochastic magnetic field lines and the ballooning mode. We suggest a possible scenario for incomplete magnetic reconnection during the crash phase.

Overall, the numerical studies of the nonlinear evolution of $m/n = 1/1$ internal kink mode require numerous considerations to avoid numerical instabilities. We will emphasize the numerical treatment on how to generate smooth and physically valid solutions of the MHD equations.

This thesis is organized as follows. In Chapter 2 experimental results on sawtooth oscillations in tokamak discharges will be reviewed. We will put an emphasis on the observation of toroidal localization of the pressure-driven modes and magnetic field line stochasticity. In Chapter 3 a formulation of the resistive, nonlinear incompressible MHD equations will be presented. In Chapter 4 a theoretical basis for ideal and the resistive internal kink modes, and MHD simulation results for low β cases is presented. Magnetic field line stochasticity will be evaluated quantitatively as well. In Chapter 5 results for high β cases will be presented. Finally, we summarize this dissertation in Chapter 6. Appendices A and B discuss the magnetic field used in the calculations and how their stochastic character is quantified, respectively.

Chapter 2

Experimental Observations Of Sawtooth Oscillations

In this chapter, the detailed information on sawteeth available from experiments will be recapitulated. Subsequently, theoretical interpretations will be discussed. The sawtooth crash is a complex phenomenon, caused by multiple factors, with free energy sources from the plasma current and pressure, and possibly including effects due to turbulence. It is a highly nonlinear phenomenon in that multiple Fourier mode components come into play. Before going into the theoretical modeling and computation, a chapter is spent clarifying what is understood in the plasma physics community and what remain as discrepancies with the conventional resistive MHD theory (theoretical model of $m=1$ reconnection).

The study of sawteeth is rather empirical, most of the time motivated and driven by experimental findings. A careful observation of experimental results is indispensable to extract the physical essence, and hence to serve as the basis for appropriate theoretical/numerical modeling. The approach to sawtooth analysis is

thus to include only a few physics effects at a time. It is worth pointing out that, up to this point, there is no literature that coherently compares sawtooth experiments in different machines.

Experimental results are chronologically described. The sawtooth oscillation was named after the sawtooth-like behavior observed in the core soft X-ray emission in the Princeton ST-tokamak [1]. In the 1980's the Tokapole-II [15], TEXT [16], and TEXTOR [17, 18] experiments explored the detailed magnetic structure. The latter experiments found q_0 to be below unity at the plasma center. Finite pressure effects were exhibited by TFTR through simultaneous measurements by Electron Cyclotron Emission (ECE), soft X-ray tomography and the Motional Stark Effect (MSE) [10] of sawtooth crashes in high pressure plasma.

2.1 Early Tokamak Experiments

Figure (2.1) shows the soft X-ray signals which exhibit sawtooth-like oscillations that were first observed in the ST-tokamak [1]. The classic ST-tokamak experiments exhibit all the important features of large aspect ratio, resistive (or low temperature) tokamak experiments. The X-ray traces in the figure indicate a sudden drop of the temperature which accompanies what are called internal disruptions of the plasma column. An X-ray image of the plasma column was formed by a slot aperture with X-ray emission coming from different regions, as shown in Fig. (4.2)(a). The radiation intensity is a function of the electron density n_e , electron temperature T_e , and average impurity charge. The oscillograms of the X-ray emissions shown in the figure are typical for high density discharges in tokamaks. The lower trace is at $r = 0$ cm and shows a linear rise followed by a rapid drop (i.e., a sawtooth), which suggests that

the energy is transported radially outward at the drop. The upper trace in Fig. 2.1(b) is for $r = 3.9$ cm and shows a rapid rise (due to the outward energy transport) and a slow decay during each sawtooth. The rapid radial transport during the crash is why the signals in the two traces are inverted. The safety factor q at the center of the plasma has been calculated to be roughly 0.8 and the $q = 1$ inversion radius is found to be $r \approx 2$ cm. The minor radius of the ST-tokamak plasma was $r = 14$ cm. Simultaneous measurement of the sawteeth at different radii and at different toroidal angles show that they occur at the same time everywhere.

Each sawtooth crash was preceded by and caused by a growing sinusoidal mode oscillation. The amplitude of the oscillation is maximum close to the $q = 1$ surface and vanishes abruptly outside the $q = 1$ surface, which is a unique property of an internal mode. From simultaneous measurements at different poloidal and toroidal angles, it was inferred that the oscillation has mode numbers of $m = 1$ and $n = 1$. Here, m and n stand for the poloidal and toroidal wave numbers of the magnetic perturbation, respectively.

The time scale calculated for an ideal kink mode is far too rapid to explain the phenomena. This suggests the existence of slower time scale effects which arise from the finite plasma resistivity that accompanies magnetic reconnection and energy dissipation. This led to the Kadomtsev model [2] which combines an internal kink mode with magnetic reconnection and the evolves into an $m = 1$ magnetic island.

2.2 Direct measurement of magnetic field

In the context of direct magnetic field measurement, Osborne et al. [15] first showed that the central q -value remains below unity throughout the sawtooth oscillation

phase in the noncircular poloidal diverter tokamak Tokapole II. As a reminder, Tokapole II experiments were conducted in a relatively low β and low Lundquist number ($S \sim 10^4$) regime. This fact implies that the fast reconnection (sawtooth crash) problem and the q_0 problem may be independent.

In TEXTOR experiments [17, 18, 8], it was also shown that in a large, hotter tokamak plasma the central q -value remains below unity after the sawtooth crash. These results suggest a *partial reconnection* of the $m/n = 1/1$ magnetic island. They are thus obviously in contradiction with the theoretical predictions. In the first place, the $m = 1$ mode in a tokamak can be shown to be marginally stable to ideal MHD kink instabilities. This kink mode gives rise to current sheet formation and thus with resistivity the $m = 1$ reconnection. According to the Kadomtsev model, a full resistive reconnection should take place; the flux surface at the magnetic axis should reconnect with a flux surface outside the $q = 1$ inversion surface in the final state. If such a full reconnection takes place, the central q -value should be slightly above unity after each sawtooth event. The TEXTOR device is a medium size tokamak with a circular plasma cross section, a major radius of 1.75 m, a limiter radius of 0.5 m, a toroidal field of 2 T, and a toroidal current of 2 MA; these results are representative of tokamaks in general.

The noise level in Faraday rotation signals has been reduced by forming a coherent average over many sawtooth periods. In the TEXTOR experiments [8], direct measurements of the internal magnetic field structure or the related current density profile with high spatial and temporal resolutions have been undertaken. The refined data in TEXTOR have revealed some new features of the sawtooth crash. A new observation was the periodic appearance of a poloidal field perturbation correlated with the sawtooth collapse which is localized on the weak field side of the $q = 1$ surface.

The perturbation is axisymmetric with mode numbers $m = 1, n = 0$. In the presence of an $m = n = 1$ island, the interaction between the modes may give rise to magnetic field line stochasticity and thus may play an important role in thermal transport during the crash. Another new finding was the existence of an $m/n = 1/0$ mode on top of the $m/n = 1/1$, which gave rise to a toroidally symmetric current sheet. Based on this observation, in Ref. [8] Soltwisch et al. suggested that the interaction of a helical $1/1$ mode and the symmetric $1/0$ mode cause formation of stochastic regions, and thus a loss of particles and energy from the plasma core. The experiment was conducted in a relatively low β regime ($\beta_p \sim 0.3$).

2.3 ECE and soft X-ray tomography in TFTR

The electron temperature distribution in TFTR has been studied by Nagayama et al. [6, 7] using both the ECE and soft X-ray signals. The experimental results revealed that the oval-shaped hot core on the bad curvature side has a larger bulge compared to the one on the good curvature side (see Fig. 2.2), which can be explained by finite pressure effects. The ECE data have exhibited complete pressure flattening [6], though this does not imply a complete magnetic reconnection.

The ECE measurements [5, 13] have elucidated the local electron heat flow during a sawtooth oscillation. During the crash phase, the ECE diagnostic revealed a fast flattening of the electron temperature suggesting that the magnetic surfaces and the pressure contours do not coincide. (This implies a possible stochasticization of the magnetic field lines in the vicinity of the $q = 1$ surface).

Figure 2.3 shows the electron heat flow pattern which is represented by a quantity $dT_e/dt \sim [T_e(r, \theta, t + \tau) - T_e(r, \theta, t)]/\tau$ where $\tau \sim 50 \mu s$ is the plasma

toroidal rotation period. The figures show a typical flow pattern by overlaying a graded color representation for contour lines for a sawtooth discharge. A blue (red) shading represents a heat loss (gain) of 10^4 keV/s. During the period of development of the crescent-shaped flat temperature region, heat flow is confined within the $q = 1$ surface. Heat flows from the hot core toward the $q = 1$ surface and escapes through the X-point region.

As a complementary work, the magnetic field structure has been measured in TFTR [10]. The MSE (Motional Stark Effect) diagnostic is employed to measure the pitch angle of the magnetic field lines and hence the q profile. The MSE measurements in TFTR indicated that central q values do not relax up to unity after the crash, but increase only by 5 – 10% — typically from 0.7 to 0.8 or less. Figure 2.4 presents the time evolution of the central q value obtained from MSE measurements. This observation is in agreement with the TEXTOR experiments referred to in the previous section [8, 17, 18].

Further detailed information on the ECE measurements have been presented quite recently by Nagayama et al. [7]. The results give supporting evidence for incomplete magnetic reconnection, with an emphasis on the localization of the mode on the bad curvature side. One new finding is activity of an $m = 1$ mode in the middle of the sawtooth period which is called the *mid-cursor*. Following the post-cursor, a mid-cursor occurs in the $q = 1$ ring region. During the mid-cursor, the hot region shrinks and the cold region expands, suggesting that heat is transported from the ring region to the peripheral region during this period.

By comparing the phase of the precursor and post-cursor signals, it has been shown [7] that the structure between the hot spot and the island remains after the crash phase. Both the post-cursor and mid-cursor signals indicate that the $m = 1$

mode and $q = 1$ flux surface remain after the crash rather than going through a secondary reconnection phase. In addition, there was no evidence for heat being transported to the other side of the X-point region as in the Biskamp-Drake secondary reconnection model [19].

2.4 Partial Temperature Crash in TFTR

A partial crash occurs not only for the magnetic fields. Some of the ECE data (which indicates the electron temperature profile) in TFTR exhibits a partial crash. The phenomena are also known as *double sawteeth* in Alcator C-Mod [20]. A double sawteeth experiences sequential temperature drops, one minor drop followed by a major sawtooth drop; the $m/n=1/1$ island grows, but the crash process stops before pushing out the entire hot core. After the crash process stops the hot spot remains, and the large amplitude of the $m/n = 1/1$ mode persists until the full crash. It has been shown [13] that a heat ring (stochastic ring region) is created after the partial crash. Using ECE data with fine time resolution, it is observed that the heat ring gradually diffuses, and becomes small before the full crash starts.

From mode analyses these partial crashes have been identified as transition processes from the (2,2) or (3,3) modes to the (1,1) mode [6]; during this process the heat is quickly transported from the core region to the peripheral region of the $q = 1$ surface. Localized magnetic reconnection on the bad curvature side is suggested as well.

2.5 Theoretical interpretation: Finite pressure effects and the role of stochasticity

As we have seen in the experimental results, the magnetic surfaces are no longer conserved during the sawtooth crashes. Once the $m = 1$ magnetic island evolves to a finite width, magnetic field line stochasticity comes into play. Experimental results from TFTR and TEXTOR both agree that the magnetic flux and the pressure contour do not coincide during the crash phase.

Furthermore, it is likely that the toroidally localized finite pressure effects, which are possibly ballooning modes, are helping to destroy magnetic flux surfaces and hence flattening the pressure profile. The experimental results also suggest that the $m/n = 1/1$ single helicity mode is not the only component controlling the phenomena.

We would like to emphasize the finite pressure effects and the role of magnetic field line stochasticity in this dissertation. To investigate both effects from a theoretical point of view, we need to include a wide spectrum of Fourier modes in the simulation model. In particular, to describe the ballooning modes correctly, one needs to retain as many Fourier modes as possible.

In this dissertation, we only consider a circular cross section, since the onset of sawtooth depends on the existence of $q = 1$ surface, and the geometrical effects do not alter the basic property of the magnetic reconnection processes. The model MHD equation and the numerical formulation of the MHD code are described in the next Chapter.

Figure 2.1: (a) Experimental arrangement of X-ray detectors in the ST tokamak; (b) The soft X-ray signals obtained exhibit sawtooth or internal disruptions. The signal above is at $r = 3.9$ cm. The signal below is at $r = 0$ cm. (From S. von Goeler et al., Ref. [1].)

Figure 2.2: Series of ECE images during the sawtooth crash in TFTR. Red represents the highest electron temperature, 6000 eV; the separation between isotherms is 500 eV. The relaxation shown takes about 3 ms; the fourth and the sixth frames are separated by 130 μ s. All the figures are on the bad curvature side. (From J.D.Callen et al., Ref. [6], courtesy of Y.Nagayama.)

Figure 2.3: Heat flow contours superimposed on $T_e(r, \theta)$ contours (solid lines) for (a) $t = 216 \mu$ s, (b) $t = 314 \mu$ s, (c) $t = 368 \mu$ s, and (d) $t = 422 \mu$ s. Blue (red) represents a heat loss (gain) of about 1×10^4 keV/s. The dashed lines depict the inversion radius, which is close to the $q = 1$ radius measured by MSE. (From M.Yamada et al., Ref. [10].)

Chapter 3

The Model Equations and Numerical Schemes

In this chapter, the formulation of the MHD equations to be used in our code will be presented. In the first section, basic incompressible MHD equations will be derived. In the second section, the numerical formulation used to time advance reduced MHD equations in the FAR code will be shown. In the last section, energy conservation will be discussed.

3.1 Basic MHD equations

We begin with the standard one fluid resistive MHD equations which are a combination of Maxwell's equations for the electromagnetic fields, and mass, momentum, and energy balance equations for the plasma. Here, the normalized MKS unit system ($\mu_0 = 1$) is employed. They are: Ampere's law,

$$\mathbf{J} = \nabla \times \mathbf{B}, \quad (3.1)$$

Figure 2.4: Time evolution of the central q -value and the ECE based T_e signal. (From M.Yamada et al., Ref. [9].)

Gauss's law,

$$\nabla \cdot \mathbf{B} = 0, \quad (3.2)$$

the quasi-neutrality condition,

$$\nabla \cdot \mathbf{J} = 0, \quad (3.3)$$

Faraday's law,

$$-\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}, \quad (3.4)$$

and Ohm's law,

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}. \quad (3.5)$$

The plasma equations used are the mass balance equation,

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot \rho_m \mathbf{v} = 0, \quad (3.6)$$

the momentum balance equation,

$$\rho_m \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mathbf{J} \times \mathbf{B}, \quad (3.7)$$

and the isentropic (or dissipationless) energy balance equation

$$\frac{\partial p}{\partial t} = -\mathbf{v} \cdot \nabla p - p \nabla \cdot \mathbf{v}. \quad (3.8)$$

Here, $\mathbf{v}$, $\mathbf{B}$, $\mathbf{E}$, $\mathbf{J}$, ρ_m , p , η , and Γ stand for flow velocity, magnetic field, electric field, current, mass density, pressure, resistivity, and specific heat, respectively. The magnetic field is written in terms of the vector potential $\mathbf{A}$ as

$$\mathbf{B} \equiv \nabla \times \mathbf{A}. \quad (3.9)$$

From Eqs. (3.4) and (3.5), we find that

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \times \mathbf{B} - \eta \mathbf{J} + \nabla \alpha \quad (3.10)$$

in which α is the electrostatic potential.

Next, we simplify these equations to the reduced MHD equations [21] for only three scalar function evolution equations. The condition of compressibility is addressed first. The following constraint [25] will be added on top of fluid compressibility $\nabla \cdot \mathbf{v} = 0$:

$$\nabla \cdot (\mathbf{B}^2 \mathbf{v}) = 0. \quad (3.11)$$

Since $B \sim 1/R$, Eq. (3.11) can be rewritten as

$$\nabla \cdot (R^{-2} \mathbf{v}) = 0.$$

The velocity is described by a velocity potential $\Omega \equiv -\phi \nabla \zeta - \lambda \nabla \theta$,

$$\begin{aligned} \mathbf{v} &= R^2 \nabla \times \Omega \\ &= R^2 (\nabla \zeta \times \nabla \phi + \nabla \theta \times \nabla \lambda). \end{aligned}$$

Nominally, vorticity in an incompressible fluid is defined by $\mathbf{U} = \nabla \times \mathbf{v}$. To be consistent with our compressibility condition, we take

$$\mathbf{U} = R^2 \nabla \times R^2 \rho_m \mathbf{v}. \quad (3.12)$$

Multiplying by R^2 and taking the curl of the momentum balance equation, one obtains

$$\begin{aligned} \frac{\partial \mathbf{U}}{\partial t} &= (\mathbf{U} \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{U} \\ &\quad - R^2 \nabla \times \{ [\mathbf{v} \cdot \nabla (R^2 \rho_m \mathbf{v})] \} - R^2 \nabla (R^2 \rho_m) \times \nabla \left(\frac{v^2}{2} \right) \\ &\quad + R^2 [\mathbf{B} \cdot \nabla (R^2 \mathbf{J}) - \mathbf{J} \cdot \nabla (R^2 \mathbf{B}) - (\nabla R^2 \times \nabla p)]. \end{aligned}$$

Applying the $\nabla \cdot (R^{-2}\mathbf{v}) = 0$ condition, the mass and the energy balance equation can be rewritten as

$$\frac{d}{dt}(R^2\rho_m) = 0, \quad (3.13)$$

$$\frac{d}{dt}(R^{2\epsilon}p) = 0. \quad (3.14)$$

Letting $R^2\rho_m = \text{const}$, we obtain a simplified normalized form

$$\begin{aligned} \frac{\partial \mathbf{U}}{\partial t} &= (\mathbf{U} \cdot \nabla) \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{U} \\ &+ S^2 R^2 \left[\mathbf{B} \cdot \nabla (R^2 \mathbf{J}) - \mathbf{J} \cdot \nabla (R^2 \mathbf{B}) - \frac{\beta_0}{2\epsilon^2} (\nabla R^2 \times \nabla p) \right]. \end{aligned}$$

Toroidal curvature effects are included in the last term. Here, S is the Lundquist number, and $\beta_0 \equiv 2p_0/B^2$ is the maximum beta value at the center of the plasma column. Hereafter, the time has been normalized by the resistive time η/a^2 , and the scale length by the minor radius. The equation can be calculated in the ideal limit (infinite electrical conductivity) by normalizing time variables by the poloidal Alfvén transit time τ_a , and setting $S = 1$, $\eta = 0$ in the equation above. (Note that the resistivity only comes in through the Ohm's law). We take the specific heat $\Gamma = 0$ on the right-hand side of Eq. (3.8) since we consider an incompressible fluid.

In this study the magnetic field is represented by $\mathbf{B} = \nabla\theta \times \nabla(\rho\chi) + \nabla\zeta \times \nabla\psi$. (Here, χ and ψ are the two components of the vector potential. See Appendix A for details). For Ohm's law, we have

$$\begin{aligned} \frac{\partial}{\partial t} [\chi \nabla\theta + \psi \nabla\zeta] &= -[\mathbf{v} \cdot \nabla(\rho\chi)] \nabla\theta - (\mathbf{v} \cdot \nabla\psi) \nabla\zeta \\ &+ (\mathbf{v} \cdot \nabla\theta) \nabla(\rho\chi) + (\mathbf{v} \cdot \nabla\zeta) \nabla\psi \\ &+ \nabla\alpha - \eta \mathbf{J}. \end{aligned}$$

In this set of equation, the poloidal component scalars λ and χ are on the order of ϵ^2 smaller than the toroidal component functions ϕ and ψ . (Here, ϵ is the inverse aspect ratio). Neglecting these ϵ^2 order terms and taking the scalar product with $\nabla\zeta$, the set of equation reduces to [21]

$$\frac{\partial\psi}{\partial t} = -R^2 [(\mathbf{v} \times \mathbf{B}) \cdot \nabla\zeta - \eta J^{\zeta}], \quad (3.15)$$

$$\frac{dU^{\zeta}}{dt} = S^2 R^2 \left[\mathbf{B} \cdot \nabla(J_{\zeta}) - \frac{\beta_0}{2\epsilon^2} (\nabla R^2 \times \nabla p) \cdot \nabla\zeta \right] + \nu \nabla^2 U^{\zeta}, \quad (3.16)$$

$$\frac{dp}{dt} = \kappa_{\perp} \nabla_{\perp}^2 p + \kappa_{\parallel} \nabla_{\parallel}^2 p. \quad (3.17)$$

Note that additional dissipation terms are added on the right hand side of Eq. (3.16) and Eq. (3.17). Here, ν , $\kappa_{\perp}$, and $\kappa_{\parallel}$ stand for the viscosity, perpendicular and parallel thermal conduction coefficients, respectively. A caveat on the use of these reduced equations is that the onset conditions for the $m = 1$ internal kink mode is not described correctly in the reduced MHD formulation since its effect enters through order ϵ^4 in the variational principle [26, 27]. The effect of this approximation will be discussed below.

By taking the radial component of the Ohm's law, the electrostatic potential α is related to the velocity potential by

$$\alpha = \frac{F}{R^2} \phi. \quad (3.18)$$

With our definition of the coordinate system, the gauge can now be specified and was chosen to be

$$A_p = 0, \quad (3.19)$$

with a similar gauge-like constraint on the velocity potential

$$\Omega_p = 0. \quad (3.20)$$

By the assumption of a large aspect ratio ordering [21], A_θ , Ω_θ were also taken to be zero.

3.2 FAR code formulation

In this section, the numerical formulation in the FAR code will be presented. The code consists of an equilibrium part RSTEQ [22] and a perturbation part FAR [23, 24]. The latter, the *initial-value code*, time advances the perturbation variables.

3.2.1 Equilibrium

First, in the equilibrium part, the Grad-Shafranov equation is solved treating the fluid as ideal, in an axisymmetric geometry. An equilibrium flux surface is labeled by the straight magnetic field line coordinates: ρ , a radial like variable; θ , a poloidal angle like variable; and ζ the geometric toroidal angle variable. The equilibrium magnetic field is given by

$$\mathbf{B}_{eq} = -F\nabla\zeta + \nabla\zeta \times \nabla\psi_{eq}, \quad (3.21)$$

where $\psi_{eq} = \psi_{eq}(\rho)$ is the poloidal magnetic flux function, and $F(\rho)$ is the poloidal current.

The Grad-Shafranov equation is given by

$$\Delta^*\psi = -R^2 \frac{dp}{d\psi} - F(\rho) \frac{dF(\rho)}{d\psi}, \quad (3.22)$$

where

$$\Delta^*\psi = R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \psi}{\partial R} \right) + \frac{\partial^2 \psi}{\partial \zeta^2}. \quad (3.23)$$

It is solved to obtain the equilibrium flux function ψ_{eq} . Here, $R = R(\rho, \theta)$ stands for the major radius, $Z = Z(\rho, \theta)$ stands for a vertical coordinate, and $p = p(\psi)$ is the pressure. To solve the Grad-Shafranov equation, as we can see from Eq. (3.22), both the pressure profile and the poloidal current profile (or q profile) must be given. In the following calculations, both the profiles were given in the form of polynomial functions representing the case of circular tokamaks such as TFTR and TEXTOR. The equilibrium q value at the magnetic axis is always taken to be below unity, typically 0.81 in our calculations. Since the onset of a sawtooth and the reconnection process depends on the existence of a $q = 1$ surface, we regard noncircularity to be insignificant in changing the stability properties.

The shape of the equilibrium magnetic flux surface will be represented by the contravariant metric elements $g^{\rho\rho} \equiv \nabla\rho \cdot \nabla\rho$, $g^{\theta\theta} \equiv \nabla\theta \cdot \nabla\theta$, $g^{\rho\theta} \equiv \nabla\rho \cdot \nabla\theta$ and $g^{\zeta\zeta} = 1/R^2$. These are related to and obtained from the covariant metric elements,

$$g_{\rho\rho} = \frac{J^2}{\rho^2 R^2} g^{\theta\theta},$$

$$g_{\theta\theta} = \frac{J^2}{R^2} g^{\rho\rho},$$

$$g_{\rho\theta} = -\frac{J^2}{\rho R^2} g^{\rho\theta},$$

where $J \equiv 1/(\nabla\rho \cdot \nabla\theta \times \nabla\zeta)$ is the Jacobian. The covariant metric elements,

$$g_{\rho\rho} = \left(\frac{\partial R}{\partial \rho} \right)^2 + \left(\frac{\partial Z}{\partial \rho} \right)^2, \quad (3.24)$$

$$g_{\theta\theta} = \left(\frac{\partial R}{\partial \theta} \right)^2 + \left(\frac{\partial Z}{\partial \theta} \right)^2, \quad (3.25)$$

$$g_{\rho\theta} = \left(\frac{\partial R}{\partial \rho} \right) \left(\frac{\partial R}{\partial \theta} \right) + \left(\frac{\partial Z}{\partial \rho} \right) \left(\frac{\partial Z}{\partial \theta} \right), \quad (3.26)$$

are obtained from the solution of the Grad-Shafranov equation, Eq. (3.22). Here, $g_{\zeta\zeta} = R^2$.

3.2.2 Dynamics

Based on the equilibrium values, the reduced MHD equations are time advanced. The toroidal geometry is introduced from RSTEQ into the initial-value code FAR via the above metric elements, as well as R . To time advance the reduced MHD equations, the radial component was evolved by a finite difference scheme, while the poloidal and the toroidal components are expanded in Fourier series. The unknown quantities are separated into equilibrium (X_{eq}) and perturbation ($\tilde{X}$) parts:

$$X(\rho, \theta, \zeta) = X_{eq}(\rho, \theta) + \tilde{X}(\rho, \theta, \zeta). \quad (3.27)$$

The purely equilibrium terms are discarded in FAR, leaving only terms that are linear or quadratic in the perturbed quantities. Linear calculations time-advance only the $X_{eq} \times \tilde{X}$ terms (implicitly), while nonlinear calculations integrate quadratic terms $\tilde{X} \times \tilde{X}$ explicitly.

The perturbed quantities are expanded in Fourier series. Assuming up-down symmetry of the equilibrium, the perturbed quantities ψ and p can be expanded in cosine functions

$$f(\rho, \theta, \zeta) = \sum_{m/n} f_{m/n}(\rho) Re[\exp(im\theta + in\zeta)] \quad (3.28)$$

and ϕ and α in sine functions

$$f(\rho, \theta, \zeta) = \sum_{m/n} f_{m/n}(\rho) Im[\exp(im\theta + in\zeta)] \quad (3.29)$$

where $i \equiv \sqrt{-1}$ is the imaginary number. The FAR code is a full spectral code [28]. This is quite unique to MHD simulations and differs from neutral fluids code where

a pseudospectral method is typically employed; plasma dynamics is based on the resonance of the Alfvén wave on the mode rational surface where $q = m/n$. In the full spectral representation the $B \cdot \nabla$ operator is algebraic ($\sim m - nq$) to the lowest order in the perturbation amplitude, and hence typically more accurate than the discrete differential operator used in the pseudospectral or finite difference scheme.

Defining three field vectors $X_{eq} = (\psi_{eq}, 0, p_{eq})$ and $\tilde{X} = (\psi, \phi, p)$, the reduced equations can be written symbolically as

$$L \frac{\partial X}{\partial t} = MX + X^T NX \quad (3.30)$$

where L and R are two dimensional differential operators and the three dimensional matrix N stands for the nonlinear differential operator. Here, the superscript T represents the transpose. Substituting Eq. (3.31) into Eq. (3.30), we obtain

$$L \frac{\partial \tilde{X}}{\partial t} = R\tilde{X} + \tilde{X}^T N \tilde{X} \quad (3.31)$$

where $R\tilde{X} = M\tilde{X} + X_{eq}^T N \tilde{X} + \tilde{X}^T N X_{eq}$.

The linear part of the equation can be written in the implicit form [23]

$$\left(\hat{L} - \frac{\Delta t}{2} \hat{R} \right) X^{i+1} = \left(\hat{L} - \frac{\Delta t}{2} \hat{R} \right) X^i. \quad (3.32)$$

Here, $\hat{L}$ and $\hat{R}$ are defined as the matrices resulting from expressing the operators L and R in centered finite difference form in ρ . Equation (3.32) is solved iteratively for each toroidal mode number. (In toroidal geometry the linear solver decouples into separate toroidal mode numbers n).

The nonlinear terms $\tilde{X}^T N \tilde{X}$ are integrated explicitly by a predictor-corrector scheme. Although the explicit part has a simpler form, in the implementation it requires special considerations for numerical stability. The Courant-Friedrichs-Lewy

(CFL) condition [28] for the pressure diffusion equation, Eq. (3.17), is given by

$$\Delta t \leq \frac{S(\Delta\rho)^2}{2\kappa}. \quad (3.33)$$

Here S is the Lundquist number, and κ is the diffusion coefficient. The smaller the radial grid spacing $\Delta\rho$, the smaller one needs to take the time step Δt .

Another important stability index is the Magnetic-cell-Reynolds number. In the vicinity of the mode rational surface, the vorticity tends to be localized radially.

To resolve small scale eddies correctly, one needs to satisfy the condition

$$\Delta t \leq \frac{\Delta\rho S}{v\rho}. \quad (3.34)$$

3.2.3 Boundary conditions

The boundary conditions are subject to gauge conditions for the vector potential and the vorticity. The boundary conditions at the magnetic axis are determined by requiring regular solutions there. As $\rho \rightarrow 0$, the harmonics $\psi_{m/n}, \phi_{m/n}, p_{m/n}, \alpha_{m/n}$ must approach a power series in ρ of the form

$$X_{m/n} = \rho^{[m]} \sum_{k=0}^{2k} a_{2k} \rho^{2k}, \quad (3.35)$$

where m, n stand for poloidal, toroidal components of Fourier expansion. Here, the boundary condition $\alpha|_{\rho=0} = 0$ has been chosen at the axis. A perfectly conducting wall condition is taken at the plasma edge $\rho = a$:

$$B_\rho|_{\rho=a} = 0. \quad (3.36)$$

Since fluid elements do not cross the wall, we also must satisfy

$$v_\rho|_{\rho=a} = 0. \quad (3.37)$$

Equations (3.36) and (3.37) require that

$$\psi_{m/n}|_{\rho=a} = \phi_{m/n}|_{\rho=a} = 0. \quad (3.38)$$

The pressure is also assumed to vanish at the wall:

$$p|_{\rho=a} = 0. \quad (3.39)$$

3.3 Energy Conservation

The energy conservation law for the reduced MHD equations is given by [29]

$$\begin{aligned} \frac{d}{dt}(E_B + E_K + E_p) &= \frac{d}{dt} \int dV \left[\frac{B^2}{2} + \frac{v^2}{2S^2} + P \right] \\ &= \frac{d}{dt} \int dV \left[\frac{(\nabla\psi)^2}{2R^2} + \frac{R^2(\nabla\phi)^2}{2S^2} + P \right] \\ &= -Q_{\text{Poynting}} - Q_{\text{Joule}} - Q_{\text{pres}}. \end{aligned} \quad (3.40)$$

The equation above indicates that the rate of change of the total plasma energy is equal to the sum of the Poynting flux through the wall ($Q_{\text{Poynting}} \equiv \int dS \cdot \mathbf{E} \times \mathbf{B}$), the Joule heating ($Q_{\text{Joule}} \equiv \int dV \eta j^2$), and the heat conduction through the wall ($Q_{\text{pres}} \equiv \int dS \cdot \mathbf{q}_h$). Here, the heat flux is given by Fourier's law of conventional thermodynamics [30]

$$\mathbf{q}_h = -\kappa \nabla p. \quad (3.41)$$

Note that the ∇ operator in Eq. (3.41) includes both the parallel and the perpendicular contributions (which reflects anisotropic nature of plasma parameters). Since a boundary condition $\mathbf{B} \cdot \mathbf{n} \equiv 0$ is imposed on the wall, the heat flux loss comes from the perpendicular component. Here, we have The quantity

$$\left[\dot{E}_B + \dot{E}_K + \dot{E}_p - Q_{\text{Poynting}} - Q_{\text{Joule}} - Q_{\text{pres}} \right] / \left(\dot{E}_B + \dot{E}_K + \dot{E}_p \right) \quad (3.42)$$

is monitored as a numerical test of energy conservation.

Chapter 4

Linear And Nonlinear Resistive

Kink Modes

In this chapter the theoretical background for an $m = 1$ kink/tearing mode in a tokamak plasma will be discussed to provide a basic understanding of the sawtooth phenomenon.

In the first section, the linear growth of the $m = 1$ ideal kink mode and its nonlinear saturation level as derived by Rosenbluth et al. [31] will be discussed. In the next section, the growth rate of the $m = 1$ tearing mode, i.e., the $\gamma \sim S^{-1/3}$ law (here, S is the Lundquist number), derived by Coppi et al. [32, 33] will be discussed. Furthermore, a heuristic model proposed by Kadomtsev [2, 34] will be illustrated. The Kadomtsev model is a combination of two stages, a rigid shift of the internal core and the resistive evolution of an $m/n = 1/1$ magnetic island. The first three sections are based on an analysis of tokamak plasmas in the cylindrical approximation.

The discussion will be extended to toroidal geometry to introduce the $1/R$ (here, R is the major radius) variation of the toroidal magnetic field. Due to the

geometry, the $m/n = 1/1$ mode drives modes of incommensurate helicity, and the interaction between these modes can lead to stochasticization of the magnetic field lines. In the last section, numerical simulation results with modest (low) β values ($\beta \leq 0.5\%$) will be presented.

4.1 Linear growth and nonlinear saturation of the ideal $m = 1$ kink mode

As suggested in Chapter 2, an ideal $m = 1$ kink mode plays an important role, especially in the initial phase (onset) of a sawtooth crash. In this section, the linear growth rate of the $m = 1$ kink mode [31] will be derived. This will be obtained from matching an interior solution (a solution within the $q = 1$ singular layer) to an exterior solution. The notation for variables in this section will be based on the cylindrical tokamak analysis [31, 35]. Consider a perfectly conducting plasma column in a cylindrical coordinate system (r, θ, z) with pressure $p(r)$, axial magnetic field B_z , and azimuthal magnetic field B_θ . The energy integral [36] with radial displacement $\xi(r) \cos(m\theta - k_z z)$ is found to be [35]

$$W = \int_0^a dr \left[f \left(\frac{d\xi}{dr} \right)^2 + g\xi^2 \right], \quad (4.1)$$

where

$$f = \frac{r^3(\mathbf{k} \cdot \mathbf{B})^2}{m^2 + k_z^2 r^2}, \quad (4.2)$$

$$g = \frac{(m^2 - 1)r(\mathbf{k} \cdot \mathbf{B})^2}{m^2 + k_z^2 r^2} + \frac{k_z^2 r^2}{m^2 + k_z^2 r^2} \left(\frac{dp}{dr} + r(\mathbf{k} \cdot \mathbf{B})^2 + \frac{2k_z^2 r^2 B_z^2 - m^2 B_\theta^2}{m^2 + k_z^2 r^2} \right). \quad (4.3)$$

Here, a is the minor radius, and $\mathbf{k}$ is the wave vector. In the long wavelength limit ($k_z r \ll 1$), $m = 1$ causes the first term in Eq. (4.3) to vanish and reduces g to a term of order $k_z^2 r^2 \ll 1$.

The internal kink mode is distinguished from other modes by the fact that $\xi \sim \cos \theta$ implies a radial displacement, where the magnetic flux surface moves together in a single direction. The $m^2 - 1$ contribution in the energy integral represents the dominant term for $m \neq 1$; the fact that it vanishes for $m = 1$ indicates the near marginal stability condition of the $m = 1$ mode.

To the order of $k_z^2 r^2$, the total energy W is minimized by a step function

$$\xi(r) = \begin{cases} \xi_a = \text{const} & \text{for } r < r_s, \\ 0 & \text{for } r > r_s. \end{cases} \quad (4.4)$$

Here, r_s is the mode rational surface where $\mathbf{k} \cdot \mathbf{B} = (B_z/R)(1 - q)$ vanishes. The form of the trial function given by Eq. (4.4) is based on the nature of the internal mode; it is localized inside the rational surface. With this trial function, the minimized total energy can be written as

$$W_{\min} = \xi_a^2 \int_0^{r_s} dr g_1, \quad (4.5)$$

where

$$g_1 = k_z^2 r \left(2rp' + 3k_z^2 r^2 B_z^2 + 2k_z r E_z B_z - B_z^2 \right). \quad (4.6)$$

For monotonically decreasing profiles of pressure and current, g_1 will be negative and an instability takes place since then the variation of the total energy is negative: $\delta W < 0$.

Let us now derive the form of the interior and exterior solutions. For the inertia term, we only consider the contribution from perpendicular components of ξ

or the velocity v . Assuming perpendicular incompressibility ($\nabla \cdot v_\perp = 0$), the velocity components can be written in terms of the growth rate γ_{ideal} and the displacement ξ as

$$v_r = \gamma_{ideal} \xi \cos(\theta + k_z z), \quad (4.7)$$

$$v_\theta = -\gamma_{ideal} \frac{d(r\xi)}{dr} \sin(\theta + k_z z). \quad (4.8)$$

Defining a Lagrangian density $L = \rho_m v^2/2 + g_1$, where ρ_m is the mass density, the Euler-Lagrange equation $d/dr (\partial L / \partial \dot{\xi}) - \partial L / \partial \xi = 0$ will be given by

$$\frac{d}{dr} \left(\left[\rho_m \gamma_{ideal}^2 + (\mathbf{k} \cdot \mathbf{B})^2 \right] r^2 \frac{d\xi}{dr} \right) - g_1 \xi = 0. \quad (4.9)$$

In the exterior region, neglecting the inertia term in Eq. (4.9), we obtain

$$\frac{d\xi}{dr} = \begin{cases} \xi_a / [r^3 (\mathbf{k} \cdot \mathbf{B})] \int dr g_1 & \text{for } r < r_s, \\ \text{const} / [r^3 (\mathbf{k} \cdot \mathbf{B})] & \text{for } r > r_s. \end{cases} \quad (4.10)$$

By expanding $\mathbf{k} \cdot \mathbf{B}$ in the vicinity of the mode rational surface, we obtain an analytical solution for the displacement,

$$\xi = \frac{\xi_a}{2} \left\{ 1 - \frac{2}{\pi} \arctan \left[\frac{x(\mathbf{k} \cdot \mathbf{B})'}{\gamma_{ideal} \sqrt{\rho_m}} \right] \right\}, \quad (4.11)$$

and its first derivative

$$\frac{d\xi}{dx} = -\frac{\xi_a}{\pi} \frac{\gamma_{ideal} \sqrt{\rho_m}}{(\mathbf{k} \cdot \mathbf{B})} \frac{1}{x^2}. \quad (4.12)$$

Here, we have defined $x \equiv r - r_s$. The character of Eq. (4.11) is shown in Fig. 4.1. This eigensolution is known as a *top-hat* function exhibiting a rigid shift of the core plasma inside the $q = 1$ surface.

Figure 4.1: The behavior of the analytical solution for the internal column displacement ξ .

Equating Eq. (4.12) for $x \rightarrow -\infty$ and Eq. (4.10) for $r \rightarrow r_s^-$, or matching the solutions, the growth rate is given by

$$\gamma_{ideal} = -\frac{\pi^{1/2}}{\rho_s^{1/2} k \cdot B \tau_s^3} \int_0^r dr g_1. \quad (4.13)$$

This is a quantity of the order of $k_z^3 \tau_a^3 \sim (r/R)^3 1/\tau_a$, or a factor of $\varepsilon^3 \equiv (a/R_0)^3$ smaller than the characteristic Alfvén frequency. Here, $v_a = B_0/\sqrt{\rho_m}$ and $\tau_a = a/v_a$ are the Alfvén velocity and the poloidal Alfvén transit time, respectively.

Next, the nonlinear equilibrium of the $m = 1$ kink mode will be discussed. The nonlinear equilibrium is obtained through the formation of a thin current sheet at the $q = 1$ surface. It was shown by Rosenbluth et al. [31] that growing modes of this type would produce negative voltage spikes and inward shifts in major radius. Based on the conservation of helical magnetic flux or the incompressibility, the amplitude

of the displacement ξ_s in a helically saturated state has been shown [31] to be of the order of $\varepsilon^2 \sim (r/R)^2 \ll 1$, which suggests the existence of a marginally stable state.

In actuality, the formation of a current spike is non-physical since finite plasma resistivity will relax a current singularity. An ideal MHD plasma displacement of a millimeter will produce a local current density spike as large as the equilibrium toroidal current density. Seen from a more practical point of view, the nonlinear saturation gives an upper limit on the amplitude at which for small, but finite, resistivity, reconnection of the magnetic field lines that form the current sheet become important, which then modifies the growth rate of the mode. From Ampere's law, we obtain [37]

$$\nabla^2 \psi = j_z(\psi), \quad (4.14)$$

where ψ is the helical flux function and j_z is the toroidal current density. Equation (4.14) together with the incompressibility condition is solved in the vicinity of the resonant surface $r = r_s$ to match to the exterior solution and determine the amplitude of the displacement ξ_s . Within the thin layer, due to rapid radial variation, we can take $\nabla^2 \sim \partial_r^2$. Then, Eq. (4.14) can be integrated

$$\left(\frac{d\psi}{dr}\right)^2 = f(\psi) - g(\theta), \quad (4.15)$$

where $f(\psi)$ and $g(\theta)$ are unknown functions to be determined. Note that the helical flux ψ is not a good flux-surface label for unconnected flux surfaces since there are two distinct flux surfaces with the same value of ψ . We eliminate ψ by the Taylor expansion $\psi = x^2 \psi''/2$. Since the displacement is given by $\xi = r - x$, we obtain

$$\xi(x, \theta) = \int_0^x dx \left(\frac{x'}{\sqrt{f-g}} - 1 \right). \quad (4.16)$$

Furthermore, incompressibility (equivalent to magnetic helicity conservation) requires

that the cross-sectional area enclosed by a flux surface be invariant:

$$\int_{\psi_i} r dr d\theta = \int_{\psi_f} r dr d\theta, \quad (4.17)$$

where ψ_i, ψ_f are the initial, final state of the helical flux function, respectively. Combining Eq. (4.16) and (4.17), we obtain

$$\oint \frac{d\theta}{\pm \sqrt{f-g}} = 1. \quad (4.18)$$

The sign of the square root must be chosen so as to satisfy Eq. (4.15). We choose the positive sign for $x > 0$ and the negative sign for $x < 0$. A change in the sign of $\sqrt{f-g}$ in general implies a jump of the helical field $d\psi/dr$ and hence of B_θ , which suggests the existence of a current sheet at the mode rational surface.

To smoothly match the internal solution, Eq. (4.16), to the exterior solution of Eq. (4.4), we rewrite it in the following asymptotic forms

$$\xi(x, \theta) = \begin{cases} \int_0^\infty dx' (x'/\sqrt{f-g-1}) + g(\theta)/2x & \text{for } x \rightarrow \infty, \\ -\int_0^\infty dx' (x'/\sqrt{f-g-1}) + g(\theta)/2x & \text{for } x \rightarrow -\infty. \end{cases} \quad (4.19)$$

Eliminating x by the relation given by Eq. (4.18), we obtain an integral equation for $g(\theta)$:

$$\int_0^\infty d\theta \left(\oint \frac{d\theta}{\sqrt{f-g}} \right)^{-3} \left(\oint \frac{d\theta}{(\sqrt{f-g})^3} \right) \left(\frac{1}{\sqrt{f-g}} - \oint \frac{d\theta}{\sqrt{f-g}} \right) = -\xi_a \cos \theta. \quad (4.20)$$

Multiplying Eq. (4.20) by $g(\theta)$ and integrating over θ , the resultant variational principle indicates [31] that there is a minimum value for the displacement ξ_a :

$$\xi_a = -13 \int_0^{r_s} \frac{dr g_{11}}{r_s^3 k \cdot B^2} \sim \epsilon^2. \quad (4.21)$$

The following properties can be noted from Eq. (4.21). It was shown in Ref. [31] that $g \sim (\pi - \theta)^2$ near $\theta = \pi$ and $g \sim \cos^8 \theta / 2$ near $\theta = 0$. At $x = 0$, the flux

surfaces are densely packed near $\theta = 0$, whereas they are spread apart on the opposite side near $\theta = \pi$. In terms of an $m/n = 1/1$ magnetic island structure, the O-point is located at $\theta = 0$, while the X-point is at $\theta = \pi$. Such an ideal MHD structure will rapidly bifurcate into a magnetic island structure once finite resistivity is added.

Now, the effect of finite resistivity on the current sheet will be discussed. The effect has been examined by Waelbroeck [38], allowing for a new concept of "Y-points." Solving for $\partial\psi$ in Eq. (4.15) discussed above, one obtains

$$\partial_r \psi = \pm \sqrt{f(\psi) - g(\theta)}. \quad (4.22)$$

In Eq. (4.22), the arbitrary constant $g(\theta)$ already indicates the existence of a magnetic island. The surfaces can be *trapped* in the poloidal direction when $f(\psi) < g(\theta)$ is satisfied. These trapped surfaces correspond to reconnected flux surfaces inside the $m=1$ magnetic island [38].

Here, as mentioned above, $x = r - r_s$ is the flux surface label in real space which can take on multiple values for the same ψ . From magnetic flux conservation and the incompressibility one obtains

$$r \frac{dr}{d\psi} = \frac{r}{\sqrt{f-g}} = \text{const} \quad (4.23)$$

or

$$\frac{1}{r} = \left(\oint d\theta \frac{H(f-g)}{\sqrt{f-g}} \right),$$

where H is the Heaviside step function.

The magnetic topology of Eq. (4.22) can be readily understood by a $x - f(\psi)$ diagram (see Fig. 4.2). Each curve in the diagram represents a magnetic configuration as a function of $g(\theta)$. Curve A represents the case with an island, and curve B represents the case for a concentric equilibrium. As one moves along the curve A in

Figure 4.2: The relation between x and the function $f(\psi)$. Curve B added on top of the analysis by Waclbroeck [38].

the positive f direction, the f value can jump from f_i to f_e without a change in x . This jump in " f " (flux surfaces whose labels have the same ψ) represents the current sheet singularities.

This seemingly simple algebraic discussion has a significance in understanding that the magnetic topology in that the the outermost flux surface in the island (just adjacent to the separatrix) does not rotate 2π in the poloidal direction. The analysis above provides an explanation for the formation of the two "Y-points." Between the two Y-points there exists a current sheet (hereafter Y-ribbon), which consists of topologically isolated magnetic field lines. As we discuss later in Chapter 5, the Y-ribbon can play a role in the magnetic shear reduction and thus minimization of

the line bending energy. Later in this chapter the magnetic field line behavior in the vicinity of the current sheet will be investigated.

4.2 Linear layer reconnection theory

As described in the previous section, the ideal internal kink mode is marginal — either weakly unstable or weakly stable. Thus, finite resistivity can be expected to have a significant effect. The linear growth rate of the $m/n = 1/1$ tearing mode obtained by Coppi et al. [32, 33] is described in this section. Note that the analysis will be different from that for constant ψ , $m \geq 2$ tearing modes [39, 40]. The solution to the Euler-Lagrange equation will be obtained in the form of Eq. (4.4)

$$\xi(r) = \begin{cases} \xi_a(1 - c/x) & \text{for } x \rightarrow 0^- \\ \xi_a - c/x & \text{for } x \rightarrow 0^+ \end{cases} \quad (4.24)$$

where c is given as a small constant. To include the effect of finite resistivity, we write the reduced MHD equations [41] in the following dimensionless forms:

$$\hat{\gamma}^2 \frac{d^2 \xi}{dx^2} = x \frac{d^2 \psi}{dx^2}, \quad (4.25)$$

$$\psi = -x\xi + \frac{\hat{\eta}}{\hat{\gamma}} \frac{d^2 \psi}{dx^2}. \quad (4.26)$$

The first equation is the vorticity equation while the second is Ohm's law. Here, $\hat{\gamma} \equiv \gamma\tau_a$ and $\hat{\eta} \equiv S^{-1}$ are the normalized growth rate and resistivity, respectively. In the presence of the finite resistivity, all terms in the reduced MHD equations are of the same order, which yields

$$\hat{\psi}/x \sim x \sim \hat{\gamma} \sim \hat{\eta}^{1/3} \quad (4.27)$$

One uses the interior solutions obtained from Eqs. (4.25) and (4.26). Matching this interior solution to the exterior solution yields a dispersion relation for the normalized ideal growth rate $\lambda_H = \gamma_{ideal} \tau_a$ of

$$\lambda_H = \frac{8\hat{\gamma}}{\zeta^{9/4}} \frac{\Gamma\left[\left(\zeta^{3/2} + 5\right)/4\right]}{\Gamma\left[\left(\zeta^{3/2} - 1\right)/4\right]} \quad (4.28)$$

where $\zeta \equiv \hat{\gamma}/\hat{\eta}^{1/3}$. The detailed mathematical derivation of the dispersion relation is slightly complicated [32].

Under marginal stability conditions, the argument of the Γ -function in the denominator should vanish, corresponding to the first pole $\zeta = 1$. This yields

$$\hat{\gamma} = \hat{\eta}^{1/3} \sim S^{-1/3} \quad (4.29)$$

The interior layer solution is given by

$$\xi = \frac{1}{2} \xi_a \left(1 - \operatorname{Erf} \left(\frac{x}{\delta} \right) \right) \quad (4.30)$$

for the displacement, and

$$\psi = \frac{1}{2} \xi_a \left[x \operatorname{Erf} \left(\frac{x}{\delta} \right) - x + \frac{1}{2} \delta^2 \frac{d}{dx} \operatorname{Erf} \left(\frac{x}{\delta} \right) \right] \quad (4.31)$$

for the poloidal magnetic flux function ($\psi \sim x\xi$, an opposite parity, across the layer). Here, δ is the normalized layer width and $\operatorname{Erf}(x) \equiv (2/\sqrt{\pi}) \int_0^x e^{-t^2} dt$. Figure 4.3 compares the eigenfunctions in the resistive layer for the tearing and the kink mode for the $m = 1$ case. As we can see, the internal kink completely resides well within a distance of δ of the $q = 1$ rational surface, whereas the eigenfunctions of the resistive kink mode leak out of the $q = 1$ surface. These properties will be reproduced in the numerical simulation in a later section.

Figure 4.3: The behavior of eigenfunctions ξ (thicker solid line) and ψ (thinner solid line). The dashed lines indicate the ideal kink mode behavior. The solid lines indicate the resistive kink mode behavior.

4.3 Kadomtsev helical flux reconnection model

Based on the $m/n = 1/1$ resistive mode, Kadomtsev [2, 34] proposed a model to describe the sawtooth phenomenon (for the ST-like tokamak). The basic idea of the model is as follows. On the $q = 1$ magnetic surface, due to the finite resistivity, magnetic field lines reconnect and form a topologically isolated system, or a magnetic island. As the $m/n = 1/1$ magnetic island grows in amplitude, the hot core of the plasma column will be pushed in the opposite direction of the island, leading to a disruption or a sawtooth crash. Though the model is simple, this was an important working hypothesis for sawtooth modeling.

A key component to explain the magnetic reconnection model is the auxiliary

(or helical) magnetic field B_*

$$B_* = B_L - \frac{r B_z}{R} e_\theta \quad (4.32)$$

where B_L is the component transverse to the toroidal magnetic field and R is the major radius. Here, B_L is the magnetic component seen from the coordinate moving along a field line on the $q = 1$ rational surface.

The evolution of the magnetic field in this model is described in Fig. 4.4. Figure 4.4(a) shows the configuration of this magnetic field in the initial state. We see that B_* vanishes at the resonant surface, $q = 1$, which is the dashed circle. On both sides of the surface, however, the field lines B_* have slopes of opposite sign corresponding to oppositely directed fields. When the internal region shifts out radially, oppositely directed fields face one another and begin to reconnect via resistivity where magnetic field lines are pinched and a current sheet occurs. In the Kadomtsev picture, this reconnection does not stop until the entire internal region is reconnected with the external field.

Therefore, after the reconnection, q becomes larger than unity throughout the plasma. That is the current is weakened or becomes redistributed from the center radially throughout and somewhat beyond the $q = 1$ surface. After reconnection, the $m = 1$ kink free energy is dispersed, and relaxed to an energetically favorable state.

The time scale for the column shift is estimated in the same manner as the Sweet-Parker reconnection model [3, 4]. The equation of motion for the ejected mass, which is pushed by the magnetic pressure, can be written as

$$\rho_m \frac{dv}{dt} = -\nabla \left(\frac{B^2}{2} \right), \quad (4.33)$$

where ρ_m is the mass density ($\mu_0 = 1$ as before). Thus, the velocity of the ejection of

the mass from the thin layer can be written as

$$v = v_a = \sqrt{\frac{B^2}{\rho_m}}. \quad (4.34)$$

Assuming incompressibility within the thin layer formed in region **ab** of Fig. 4.4(c), the relation $r_s \delta = \text{const}$ holds, since the area of the thin layer pushed by the column is preserved [3, 4]. Applying the mass continuity condition (see Fig. 4.5), we obtain

$$v_c \simeq v_e \frac{\delta}{r_s}, \quad (4.35)$$

where v_c is the speed of the column shift, $v_e \sim v_a$ is the ejection speed from the tip of the layer, and δ is the resistive skin depth,

$$\delta \equiv c \sqrt{\frac{\tau_s \eta}{\pi v_a}}. \quad (4.36)$$

Finally, the sawtooth crash phase time, which is equivalent to the time scale for the column to shift radially by a distance r_s , will be given by

$$\tau_c \simeq \frac{r_s}{v_c} = \frac{r_s^2}{\delta v_a} \sim (v_a \tau_s \eta)^{-1/2} \sim (\tau_a \tau_r)^{1/2}, \quad (4.37)$$

where τ_r is the resistive time scale. This time scale agreed reasonably well with the experimental results in the ST-tokamak. However, as described in the Chapter 2, sawteeth in large tokamak devices with high plasma temperatures of several keV crash faster than implied by Eq. (4.37) and exhibited other discrepancies with the Kadomtsev model.

4.4 Mode coupling in toroidal geometry

Up to now, most theories have been restricted to cylindrical geometries with magnetic perturbations of a single helicity. The existence of good magnetic flux surfaces were

assumed by virtue of the helical symmetry. In reality, a tokamak is toroidal. Because of the $1/R$ variation of the toroidal magnetic field, in the toroidal coordinate system (ρ, θ, ζ) ,

$$B \sim \frac{1}{R} \approx \frac{1}{R_0} \left(1 - \frac{\rho}{R_0} \cos \theta \right) \approx \frac{1}{R_0} (1 - \epsilon \cos \theta). \quad (4.38)$$

Thus modes with the same toroidal numbers (n) are coupled geometrically; this induces a modulation of Fourier components.

Nonlinear mode coupling can be important as well, when the amplitude of perturbations become large. The nonlinearities in the dynamics equations excite modes by the mode coupling process

$$(m \pm m, n \pm n) = (m, n) \cdot (m, n). \quad (4.39)$$

The magnitude of the driven modes can be simply estimated by multiplications of the driving modes. For example, applying an ordering similar scheme to that used by Hicks et al. [29], (that is $\psi_{e1} \sim 1$, $\psi_{11} \sim \epsilon$), $(1, 1)$ and $(1, 0)$, mode amplitudes $\sim \epsilon$ produces a $(2, 1)$ mode whose amplitude scales as ϵ^2 .

Although the ordering schemes have been successful in predicting the modes which are important for numerical calculations, they do not guarantee that other modes are not important (especially for the nonlinear calculations). Note that all the schemes mentioned above only apply for the linear phase of the instability evolution. The number of modes needed for a given simulation must be tested by making runs with different selections of modes.

It is commonly believed that the $m/n = 1/1$ and $m/n = 2/1$ tearing modes are particularly significant in tokamak plasmas [42]. The interactions between these modes as a result of toroidal coupling may well play a role in sawtooth oscillations and disruptions in a tokamak. At low aspect ratio, the coupling of the $m = 2$ and

$m = 1$ modes become particularly significant. Interestingly, since the tearing-mode eigenfunction $\psi_{2/1}$ is located mainly inside the radius of the $1/1$ resonant surface, the effect of a $(2, 1)$ mode on the $(1, 1)$ mode is stronger than the inverse effect.

The interaction of modes of incommensurate helicity affects the growth of each mode and can possibly lead to destruction of magnetic flux surfaces and stochasticization of the magnetic field lines. The coupling effects can be important especially for ballooning mode studies. The helically ballooning-mode-induced twisted hot spot and its finger structure (see Chapter 5) have a prominent toroidal asymmetry on the bad curvature side. This toroidal asymmetry drives magnetic islands of different helicity, primarily outside the $q = 1$ surface. Thus, the stability analysis should include many different modes simultaneously.

For example, here we consider two different tearing eigenmodes ψ_1 and ψ_2 with two mode rational surfaces ρ_{s1} and ρ_{s2} . Instead of a single Δ' value [39], four parameters, including off-diagonal elements, are needed for the stability analysis. These are defined by [43]

$$\begin{pmatrix} \Delta'_{11} & \Delta'_{21} \\ \Delta'_{12} & \Delta'_{22} \end{pmatrix} = \begin{pmatrix} [\psi'_1(\rho_{s1} + \delta) - \psi'_1(\rho_{s1} - \delta)]/\psi_1(\rho_{s1}) & -\psi'_1(\rho_{s2} - \delta)/\psi_1(\rho_{s1}) \\ \psi'_2(\rho_{s1} + \delta)/\psi_2(\rho_{s2}) & [\psi'_2(\rho_{s2} + \delta) - \psi'_2(\rho_{s2} - \delta)]/\psi_2(\rho_{s2}) \end{pmatrix},$$

where the resistive layer width is represented by δ . The growth rate for the tearing mode is then obtained by solving the dispersion relation

$$[\Delta'_{11}(\gamma) - \Delta'_{11}][\Delta'_{11}(\gamma) - \Delta'_{11}] - \Delta'_{12}\Delta'_{21} = 0. \quad (4.40)$$

The roots for γ will give the growth rate of the modes [43]. Here, $\Delta'(\gamma)$ is the function of the properties of the inner resistive layer.

4.5 Low β simulation results

The numerical results for a resistive internal kink mode will be shown in this section. Emphasis will be placed on the magnetic field line behavior in a toroidal plasma. The equilibrium magnetic field is assumed to be given by $B = \nabla\psi \times \nabla(q\theta - \zeta)$. Hereafter, the straight field line toroidal coordinate system (ρ, θ, ζ) , is employed. Two different series of simulations are conducted: one with a rounded q profile [40] ($q \sim q_0(1 + \rho^4)^{1/2}$), and the other with a peaked profile [40] ($q \sim q_0(1 + \rho^2)$). The pressure profile has the form of $p \sim \psi_q^2$ and is zero at the boundary $\rho = a$ for both cases.

We first calculate the rounded q profile case, with equilibrium profiles similar to Holmes et al. [44] so as to compare and check the validity of our new results. With the rounded profile we obtain a large volume $q \leq 1$ region, since the magnetic shear is small near the magnetic axis for this case. The profile is numerically suitable for investigating the nature of the microscopic magnetic field structure in the vicinity of the $q = 1$ island separatrix. On the other hand, the peaked q profile has features much closer to tokamak discharges such as those seen in TFTR and TEXTOR experiments.

Common parameters used in the calculations were: major radius $R = 5$ m, minor radius $a = 1.25$ m (thus, the inverse aspect ratio $\epsilon = 1/4$), Lundquist number $S = 10^5$. The Lundquist number S is taken to be relatively low (it is $\sim 10^9$ for TFTR supershot parameters) to satisfy the requirements for numerical stability from the cell magnetic Reynolds number [24]. A total of 500 equally spaced mesh points were used in the radial direction. A nonuniform mesh was not employed since we do a global multi-mode calculation in this work.

4.5.1 Rounded q profile ($\lambda_{rs} = 2$) case

Here, the safety factor was taken to be in the range of $0.8 \leq q \leq 2.1$, so that a large $q \leq 1$ region can be obtained. The central q value was taken to be 0.81 to avoid the resonance and formation of a large $m/n = 4/5$ island near the magnetic axis. Poloidal and toroidal mode numbers were taken to be $1 \leq n \leq 6$ and $n - 3 \leq m \leq n + 3$, respectively. A total of only 43 modes were taken, since high- n pressure-driven modes are absent typically in the low β cases. (This is justified by checking against numerical runs with higher n and m mode numbers.)

First, we briefly review the results from the linear simulation results. The linear growth of an $m/n = 1/1$ resistive kink mode has been calculated and compared with the analytic prediction. As in Eq. (4.29), the growth rate is predicted [32] to be proportional to $S^{-1/3}$.

The solid line in Fig. 4.6 is the linear growth of the mode versus the Lundquist number S ; there is good agreement with the solid line that represents the theory. Figure 4.7 show the radial profile of the eigenfunctions ψ , ϕ , p , and U^i for the $m/n = 1/1$ resistive kink mode at $t = 900\tau_e$ (in the linear stage). The vertical dashed line represents the mode rational surface where $q = 1$. The behavior of $\phi_{1/1}$ in the $\rho < 0.4$ region suggests a rigid shift of the central column ($\xi \simeq \phi/\rho \sim \text{const}$ for $\rho \leq 0.56$). The eigenfunctions of the resistive kink mode extend beyond the $q = 1$ surface, in contrast to the ideal internal kink mode which resides almost completely inside the $q = 1$ surface [31] (except for toroidicity effects).

Figure 4.8 show the radial profile of the eigenfunctions in the nonlinear developed phase. The presence of the $m/n \neq 1/1$ modes can be observed in the $\rho \leq 0.56$ region of the U^i profile.

Magnetic field line trajectories have been obtained by integrating magnetic field

line equation using a fourth order Runge-Kutta-Gill method [45] (See Appendix). As a reminder, the Runge-Kutta-Gill method uses a correction to the round-off error. It adds up the accumulated round-off error after each integration step. A bicubic spline method was applied to obtain each Fourier component of the magnetic field. The numerical step size was chosen so that regions inside and outside the separatrices of the short wave-length magnetic islands were completely separated.

Figures 4.9 and 4.10 show Poincaré plots of field line trajectories in a poloidal cross section. Poincaré plots include information from all the magnetic field Fourier components (unless toroidally localized magnetic reconnection takes place). In Fig. 4.9, the abscissa $X = (R - 1)/\epsilon$ and the ordinate Z are the horizontal and vertical dimensionless minor radius coordinates, respectively. Imagine that the center line is located at the middle of the page and the bad curvature side is on the margin sides of the paper.

The magnetic field line trajectories are followed for up to 1000 revolutions around the torus with 2000 integration steps in each toroidal rotation. Initial positions are spaced equally on the X-axis. In the presence of a single magnetic perturbation, the trajectories form an $m/n = 1/1$ magnetic island with well formed flux surfaces. The formation of lines in Poincaré mappings suggest the existence of an integral of motion, which, in this case, is the local flux function.

As shown in Figs. 4.9 and 4.10, in the presence of the higher mode number toroidal harmonics, the trajectories form chains of sideband islands exhibiting poloidally mode coupled structures. In the region where the islands do not overlap, the number of islands in each chain correspond to the poloidal mode number of the perturbation. According to Chirikov's overlap criterion [46], if the unperturbed separatrices become tangent to each other, stochastic motion of the magnetic field lines

can occur.

In the island *overlapped region*, due to the nonlinear interaction of the magnetic perturbations of incommensurate helicity, the magnetic field lines become stochastic and the flux surfaces are no longer invariant. Magnetic field line stochasticity stems from overlapping between sidebands rather than mode coupling of $m/n = 1/1$ and $(m \pm 1)/m$ modes [47]. The distance between mode rational surfaces is proportional to $1/m^2$ for the nonlinear collection of the magnetic islands of incommensurate helicity (the distance between $m/n = 1/1$ and $(m \pm 1)/m$ is proportional to $1/m$). The incommensurate helicity island overlap process accounts for the field line stochasticity generation outside the $m = 1$ magnetic island. Note that the free energy source for the $m/n \neq 1/1$ modes comes from a combination of the nonlinear mode coupling and the poloidal mode coupling. Macroscopic modes rearrange the entire plasma column and induce microscopic fluctuations. In the equilibrium we employed, it is only the $m/n = 1/1$ mode and the $m/n = 2/2$ mode that are linearly unstable. A quantitative evaluation of magnetic field line stochasticity is given in Appendix B.

Stochasticity generation inside the $m = 1$ magnetic island is slightly different. Inside the magnetic island, in the region sufficiently distant from the separatrix, each secondary island is well separated and the trajectories are stable. On the other hand, near the separatrix of the $m = 1$ magnetic island, slowly rotating trajectories can resonate with high Fourier mode numbers of sideband islands. (*Fourier modes* here are not m/n modes but the modulation of m/n modes which have an infinitely wide spectrum instead of monochromatic ones.) Thus, secondary islands are densely produced, they easily overlap, and the trajectories become stochastic.

One can say that there is a mirror-like effect; $m/n = 1/1$ induces, say, $m/n = 2/1$ and the modulation of $m/n = 2/1$ induces secondary islands inside the $m/n =$

1/1 magnetic island. Due to this mechanism, island separatrices are always broken; KAM theory [48] ("perturbation under the threshold preserves invariant tori") is not applicable on the separatrix. The results shown in Figs. 4.9 and 4.10 suggest the existence of well formed magnetic surfaces in the vicinity of the magnetic axis of the magnetic island. These results are consistent with the recent ECE tomographic reconstruction experiments in TFTR [5]. The analysis of near axis field line behavior is shown in the appendix to account for the nonresonant feature, where the magnetic shear becomes zero.

Kadomtsev-type full reconnection can be seen at $t = 1113t_0$ (Fig. 4.9(a)); this state relaxes back to a nearly concentric flux surface configuration at $t = 1800t_0$ (Fig. 4.10(e)). In this specific case, $\beta = 0.3\%$ ($\beta_p = 0.19$) was taken to suppress the effect of the Shafranov shift. Interestingly, residual stochasticity can be observed outside the $q = 1$ inversion radius, even after the hot core region is pushed out.

Figure 4.11 shows the spatial profile of the Lyapunov exponents of field lines (measured for 200 toroidal revolutions) that start at different initial positions on the X axis of Figs. 4.9(a),(c),(e) and 4.10(a),(c),(e). These provide quantitative proof of field line stochasticity. Relatively small values suggest the existence of a stable region. Some exponents converge to a finite value after many more revolutions around the torus; they may be simply staying in the vicinity of a KAM surface.

One can observe fractal structures as well. For example, at $t = 1133$, at the center of Fig. 4.9(a), a dense black circle was produced on the top half of the figure. The trajectories are trapped by the higher order islands and cannot move to the bottom half of the poloidal cross section. As a reminder, the MHD formulation here relies on the up-down symmetry. This trapping is sometimes referred to as the stickiness [49] of trajectories. In Fig. 4.10(a), we can count an $m = 19$ island chain.

However $m = 19$ perturbation is not present in the MHD mode selection. As we discussed in the previous section, distorted flux surfaces are susceptible to a wide spectrum of modulated Fourier harmonics.

Toroidal asymmetry is almost absent throughout the Kadomtsev-type full reconnection phase. This is the result of both the small Shafranov shift and the small pressure drive.

4.5.2 Peaked q profile ($\lambda_{fs} = 1$) case

Here, the safety factor was taken to be in the range of $0.81 \leq q \leq 2.7$. The q profile corresponds to that of a peaked current profile, $\lambda_{fs} = 1$, in Furth et al. [40]. Poloidal and toroidal mode numbers were taken to be $1 \leq n \leq 6$ and $n - 4 \leq m \leq n + 4$, respectively. Figure 4.12 shows the radial profiles of the eigenfunctions. The $m/n = 1/1$ mode rational surface is located at $\rho \sim 0.31$. Figure 4.13 shows the eigenfunctions in the nonlinear stage. The $m/n = 1/1$ mode rational surface is located at $\rho \sim 0.31$.

Figures 4.14 and 4.15 show Poincaré plots of field line trajectories and the flow velocity field in a poloidal cross section (at $\zeta = \pi$, nearly toroidally symmetric due to the small β). Interestingly, there is a very strong flow toward the reconnection X-point, and the flow remains even after full magnetic reconnection. These secondary flows are typically seen in cylindrical sawtooth simulations [50, 19]. The strong flow pulls the helical flux back into the central region. However, these results contradict TFTR experiments [7]. Neither cool nor hot regions appear on the opposite side of the hot spot in any sawtooth crashes on TFTR. As we show later in Chapter 5, the $m/n = 1/1$ flows are damped in high- β plasmas, due to the activity of short wavelength ($n \sim 10$) ballooning-type modes.

Toroidal asymmetry of the $m = 1$ magnetic island

Here, we have taken a q profile as in the previous section but with a relatively large equilibrium β value ($\beta = 5\%$). However, we do not include pressure/curvature effects in the mode evolution, i.e., $\beta_0 = 0$ in Eq. (3.16). Thus, the plasma responds to a Shafranov shift (which is 10% of the minor radius for the equilibrium employed here) through the modified equilibrium geometry metric elements, but dynamical pressure drives are absent. The aim of this part of the calculation is to extract the magnetic field line structure in the vicinity of the reconnection layer, particularly the local magnetic shear which can be relevant to the onset of ballooning modes, which will be presented in the next chapter. Inclusion of a dynamical curvature term, which is needed for ballooning mode studies, generates magnetic stochasticity and makes observation of the magnetic structures very difficult.

Figure 4.16 shows Poincaré plots of field line trajectories on both the good (left) and the bad (right) curvature sides of the plasma. Toroidal asymmetry in the magnetic field structure can be seen. As before, the center line is located at the middle of the page, and the toroidal magnetic field falls off as $\sim 1/R$ toward the edge of the paper. As one can see, the internal column is close to the wall in Figs. 4.16 when the hot core region faces the bad curvature (outboard) side. As compared to Figs. 4.9 and 4.10, the amount of stochasticity is benign. Although $m/n \neq 1/1$ harmonics exist in the $q \leq 1$ region, the amplitude at the $q = 1$ surface is small compared to the rounded q profile, $\lambda_{rs} = 2$ case.

In this series of simulations, an extremely large viscosity value ($\nu = 5.0$) was used so as to kill the strong secondary flow shown previously in Figs. 4.14 and 4.15. As a reminder, the ion viscosity term in Eq. 3.16 has the form of a ∇^2 operator which strongly damps short wavelength modes. This large dissipation terms help in

suppressing stochasticity as well.

By comparing Poincaré mappings of the field line trajectories on the good and bad curvature sides in Fig. 4.16, one can observe the three dimensional structure of the current sheet [31] of the $m = 1$ magnetic island (or the Y-ribbons [38]); the compression of the flux surfaces is much more pronounced at a toroidal angle where the current sheet is located on the bad curvature side of the torus. The central core has an oval shape rather than circular. This oval shape is upright on the bad curvature side, but horizontal on the good curvature side.

From this magnetic structure, one can expect that the resultant pressure-driven ballooning-type instabilities across the $q = 1$ surface will be toroidally localized by their alignment with the magnetic island structure. This mechanism is *not* the same as the localization of the ballooning mode on the bad curvature side. This mechanism arises purely from the geometrical effects or the $1/R$ (here, R is the major radius) dependence of the toroidal magnetic field. It is worth pointing out that the asymmetric structure of the Y-ribbons is found in a highly nonlinear regime of the simulation. This is an important hidden factor which is different from the localization of the ballooning mode on the bad curvature side. The magnetic structure near the island separatrix is shown in the next chapter in connection with the ballooning mode stability analysis.

Figure 4.4: Schematic diagrams showing a rigid shift of the core and the reconnection of an $m = 1, n = 1$ magnetic island, i.e., the Kadomtsev model. Solid lines show the poloidal magnetic field in the plasma. (a) Initial state. (b) Nonlinear saturation after the establishment of an equilibrium with infinite electrical conductivity. (c) During reconnection of the magnetic field lines due to the finite conductivity. (d) After reconnection. (From B.B. Kadomtsev, [2]).

Figure 4.5: A schematic diagram showing the behavior in the thin layer where the magnetic reconnection takes place. (a) Initial state, (b) After the layer was squeezed by the shift of the center plasma column. Here r is the initial length of the layer, δ is the width of the layer, v_c is the velocity of the column shift, and v_e is the velocity of the ejection.

Figure 4.6: Linear growth rate of an $m/n = 1/1$ resistive kink mode versus Lundquist number. The dashed line is from numerical calculations.

Figure 4.7: Eigenmodes of an $m/n = 1/1$ resistive kink mode with a rounded q -profile and $\beta = 0.3\%$ at $t = 900\tau_a$. The dashed line at $\rho = 0.56$ corresponds to the $q = 1$ rational surface: (a) $\psi_{1/1}$, (b) $\phi_{1/1}$, (c) $p_{1/1}$, and (d) $U_{1/1}$ profiles.

Figure 4.8: Eigenmodes of an $m/n = 1/1$ resistive kink mode at $t = 1058\tau_a$. The dashed line at $\rho = 0.56$ corresponds to the $q = 1$ rational surface: (a) $\psi_{1/1}$, (b) $\phi_{1/1}$, (c) $p_{1/1}$, and (d) $U^{\psi_{1/1}}$ profiles.

Figure 4.9: (a), (c), and (e) Poincaré plots of magnetic field line trajectories at a toroidal angle $\zeta = 0$, and (b), (d), and (f) at a toroidal angle $\zeta = \pi$, for nonlinear evolution of an internal kink mode in a $\beta = 0.3\%$ and $S = 10^5$ plasma.

Figure 4.10: Poincaré plots for further time evolution of plots in Figs. 4.9.

Figure 4.11: The values of the Lyapunov exponents after 200 revolutions, versus the initial position at $(z=0)$ of the field line trajectories. Corresponds to Figs. 4.9 and Figs. 4.10: (a) $t=1058$, (b) $t=1113$, (c) $t=1133$, (d) $t=1300$, (e) $t=1300$, and (f) $t=1800$.

Figure 4.12: Eigenmodes of an $m/n = 1/1$ resistive kink mode with a peaked q -profile and $\beta = 0.3\%$, at $t = 3000\tau_a$. The dashed line at $\rho = 0.31$ corresponds to the $q = 1$ rational surface: (a) $\psi_{1/1}$, (b) $\phi_{1/1}$, (c) $p_{1/1}$, and (d) $U^c_{1/1}$ profiles.

Figure 4.13: Eigenmodes of an $m/n = 1/1$ resistive kink mode with a peaked q -profile and $\beta = 0.3\%$, at $t = 3100\tau_a$ (right after complete reconnection). The dashed line at $\rho = 0.31$ corresponds to the $q = 1$ rational surface: (a) $\psi_{1/1}$, (b) $\phi_{1/1}$, (c) $p_{1/1}$, and (d) $U^c_{1/1}$ profiles.

Figure 4.14: (a), (c), and (e) Velocity fields at a toroidal angle $\zeta = \pi$, and (b), (d), (f) Poincaré plots of magnetic field line trajectories at a toroidal angle $\zeta = \pi$, for nonlinear evolution of an internal kink mode in a $\beta = 0.5\%$ and $S = 10^5$ plasma.

Figure 4.15: Poincaré plots and the velocity field for further time evolution of the plots in Figs. 4.14.

Figure 4.16: (a), (c), and (e) Poincaré plots of magnetic field line trajectories at a toroidal angle $\zeta = 0$, and (b), (d), and (f) at a toroidal angle $\zeta = \pi$, for nonlinear evolution of an internal kink mode in a $\beta = 5\%$ and $S = 10^5$ plasma.

Chapter 5

Nonlinear Excitation Of Pressure-Driven Instabilities

This chapter, which is the crux of this dissertation, deals with both the analytical framework and the simulation results for finite β effects during the sawtooth crash phenomenon.

First, the effect of the magnetic well will be described. Since the toroidal magnetic field falls off as $\sim 1/R$, the tokamak has both the good curvature (on the inboard) side and a bad curvature side (on the outboard). The stability index is represented by the average curvature or magnetic well depth.

Second, finite pressure effects (the instability β threshold) on the $m = 1$ internal kink modes in toroidal geometry will be discussed [26, 27]. This threshold is closely related to the precursor signals and thus the onset of the sawtooth crash.

Third, the theoretical framework for the ballooning instability will be discussed. Though conventional ballooning theory is based on topologically concentric equilibria, concepts such as the $s - \alpha$ diagram (a diagram that relates pressure gradient and field

line bending stabilization effects) can still be a useful tool for analyzing the stability in the presence of an $m = 1$ magnetic island.

Finally, simulation results for secondary, high- n ballooning mode in the non-linear stage of sawtooth crash will be presented and analyzed. The typical ballooning mode spectrum and the growth rate will be discussed (mode amplitudes of different m numbers but with the same toroidal mode numbers are correlated with each other). Ballooning stability across the $m = 1$ island separatrix (or Waelbroeck's Y-ribbon) will be discussed (see the previous chapter for the magnetic structures). The magnetic shear is changed locally due to the deformation of magnetic field lines in the vicinity of the magnetic island. Shear reduction stems from the island X-point singularity.

5.1 Suydam and Mercier criteria

Here, we briefly review three stabilizing effects for MHD modes [51]. They are: (1) Magnetic field line bending (for the perturbed magnetic field lines bent along the equilibrium magnetic field require extra energy to be consumed), (2) good magnetic curvature, and (3) compression of magnetic field line lines and the plasma. In the incompressible limit, a competition between (1) and a bad local curvature in (2) determines the stability of pressure-driven modes.

By retaining the pressure gradient term, the minimization of the cylindrical energy integral, Eq. (4.1) gives an Euler-Lagrange equation in the form of

$$\frac{d}{dX} \left(X^2 \frac{d\xi}{dX} \right) + D\xi = 0. \quad (5.1)$$

where the equation is expanded in $X = \rho - \rho_s$ in the vicinity of the mode rational surface. In the presence of finite pressure effects, the exterior equation, Eq. (5.1), has strong singularities with two independent solutions called large (dominant) and

small (subdominant) [52], which exhibits different fractional-power-like asymptotic behavior:

$$\xi = A|X|^h - B|X|^{-h-1}, \quad (5.2)$$

where A and B are constants to be determined from the exterior region. Note that the $h = 0$ limit corresponds to the zero-pressure solution. To obtain the eigensolution one needs to match the asymptotic solutions of Eqs. (5.2) to the exterior solution at the resonant surface.

The stability of the mode can be understood from the coefficient h . The envelope of the eigensolution ξ damps away from the mode rational surface in the case $D \leq 1/4$, while the solution oscillates in the presence of finite imaginary contributions in h when $D > 1/4$. In a cylinder, the necessary condition for instability is given by the Suydam criterion [53]

$$D \equiv \left(\frac{-2q^2}{q^2 B_z^2 x} \right) \left(\frac{dp}{dx} \right) \Big|_{x=x_s} \leq \frac{1}{4}. \quad (5.3)$$

From the $1/q^2$ dependence we see that increased magnetic shear is stabilizing.

In a toroidal geometry, the condition becomes slightly different. A necessary condition for stability to pressure-driven interchange-type modes is given by the Mercier criterion [54] which includes the effect of toroidal curvature in addition to the poloidal one. The rigorous derivation of the indices h and D in toroidal geometry from the energy principle is too complicated to cover here (see Ref. [55, 54] for example). Here we extract the physical essence in a simplified and intuitive manner [54, 56]. In a toroidal geometry the D value of Eq. (5.3) is replaced by the Mercier

index [54, 57]

$$\begin{aligned}
 D &\equiv \left(\frac{-q^2}{q^2 \rho^2} \right) \left(\frac{dp}{d\rho} \right) \left(\frac{1}{g\rho^2} \right) \left(\frac{d}{d\rho} \left(\frac{R^2}{d\rho} \right) \right) \Big|_{\rho=\rho_s} \\
 &\simeq \left(\frac{-2q^2}{q^2 B_z^2 \rho} \right) \left(\frac{dp}{d\rho} \right) (1 - q^2) \Big|_{\rho=\rho_s} \\
 &\leq \frac{1}{4}.
 \end{aligned}$$

The second relation is derived by Shafranov [58] in a large aspect ratio expansion for a circular cross section plasma. The $(dp/dr)q^2$ contribution on the right side represents the stabilizing effect by the average curvature of the toroidal magnetic field [54]. Thus, stability can be obtained for pressure-driven interchange modes by maintaining $q > 1$. (However, at sufficiently high β localized instabilities can become unstable to ballooning modes for $q > 1$). The Mercier criterion is a necessary condition for the stability of localized ideal MHD perturbations in a toroidal plasma. The pressure-driven instability occurs in the presence of a magnetic hill which corresponds to unfavorable average magnetic field curvature.

Figure 5.1 exhibits the straight field line coordinate system — both the equilibrium flux contours and the equi- θ lines. Note that in a cylindrical plasma with a monotonically decreasing current, $d(R^2)/d\rho \geq 0$, while $d(R^2)/d\rho \leq 0$ can occur in a torus since a good curvature region exists in a tokamak. Thus, the D value in a toroidal system can be negative (h can be positive) when q is above unity ($D \geq 0$ or $h \leq 0$ in a cylinder).

Both the interchange and the ballooning modes are analogous to Rayleigh-Taylor instabilities (instability of heavy fluid over a the light fluid); in tokamak plasmas, a curvature-induced centrifugal force is equivalent to the earth's gravity. The concept of a magnetic well in relation to interchange stability is introduced to

Figure 5.1: The equilibrium flux surfaces and θ contour lines in a toroidal geometry with a very large Shafranov shift $\beta = 20\%$. Magnetic field lines spend a longer "time" on the good curvature side than on the bad curvature side.

elucidate an analogy with such stratified fluids. The magnetic well defined by

$$\mathcal{W} = 2 \left(\frac{V}{V'} \right) \left(\frac{B^2/2 + p}{\langle B^2 \rangle} \right) \quad (5.4)$$

represents the stability index. Here, $V = V(\psi)$ is the volume contained within the flux label ψ . (The magnetic well in Eq. (5.3) corresponds to $\mathcal{W}_c = -2q^2/B_z^2 r$ for a cylindrical model of a tokamak.) The average curvature and magnetic well are identical in low-beta closed-field line systems, and give a necessary condition for interchange stability. In analogy with the hydrodynamic Rayleigh-Taylor type modes where $\nabla p \cdot g \geq 0$ for stability, $p'\mathcal{W} \leq 0$ implies stability in a plasma confinement system. Since we are typically dealing with monotonically a decreasing pressure profile ($dp/d\rho \leq 0$) in a tokamak, a positive magnetic well is favorable for stable

plasma confinement.

5.2 Finite pressure effects on internal kink modes

We have shown that the pressure-driven mode can be unstable when $q \leq 1$. In this section, the stability of the $m/n = 1/1$ ideal internal kink mode in a toroidal geometry will be discussed. Bussac et al. [26, 27] have examined the stability of an $m/n = 1/1$ internal mode in toroidal geometry, using a small aspect ratio tokamak expansion with $\beta \sim \epsilon^2$. In the analysis, the energy integral is separated into cylindrical and toroidal contributions:

$$\delta W = \left(1 - \frac{1}{n^2}\right) \delta W_{cyl} + \frac{1}{n^2} \delta W_{tor}. \quad (5.5)$$

The expression for δW_{cyl} in the limit of large mode number is given by

$$\delta W_{cyl} = \frac{n^2}{R^2} \int_0^a r dr \left[\left(\frac{2r}{B^2} \frac{dp}{dr} + \frac{r^2}{R^2} \left(n - \frac{1}{q} \right) \left(3n + \frac{1}{q} \right) \right) \xi^2 \right] = O(\epsilon^4), \quad (5.6)$$

where the first term corresponds to the pressure-gradient drive and the other term corresponds to the field line bending stabilization term. By employing a q -profile, $q - q_0 \sim x^{\lambda_{bus}}$, Bussac deduced the toroidal contribution in the form [26]

$$\delta W_{tor} = \frac{3n^2 \rho_s^2}{R^2} (1 - q_0) \left[\left(\frac{13}{48} \right) \left(\frac{4 - \lambda}{4 + \lambda} \right) - \beta_p^2 \right]. \quad (5.7)$$

By choosing $\lambda_{bus} = 2$ (parabolic current profile), $\beta_p > 0.3$ will be the onset criterion for the internal kink mode [26]. Note that in Bussac's analysis the integration domain for the β_p definition,

$$\beta_p \equiv \frac{r R^2}{n^2 r_s^2 B^2} \int_0^{rs} \frac{dp}{dr} r^2 dr \quad (5.8)$$

is only up to the $q = 1$ mode rational surface. (It will be called the Bussac beta-poloïdal hereafter). The threshold $\beta_p > 0.3$ has been confirmed using a variational principle code such as ERATO [59]. (Initial value simulations do not reproduce this onset criterion, due to the difficulty in simulating the ideal MHD instabilities in initial value codes without finite dissipation). The $n = 1$ internal kink mode is strongly affected by toroidal geometry. While the cylindrical curvature plays a role in the $m = 1$ mode, toroidal curvature has a similar effect on the $n = 1$ mode. As a reminder, the $\beta_p > 0.3$ condition is dependent on the specification of the current profile. In fact, Nave and Wesson [60] later found that the stabilizing effect vanishes with a flat current profile inside the $q = 1$ region.

5.3 Eikonal representation and ballooning transformation

Interchange-type pressure driven modes are stabilized when the appropriate average curvature of the field lines is "good" with respect to the pressure gradient. If the pressure gradient is high, it is possible for the perturbation to be concentrated in the region of destabilizing curvature. These are called ballooning instabilities because they are concentrated in the bad curvature (outboard) region of the tokamak.

In the $k_\perp \rightarrow \infty$ ($k_\perp$ signifies the perpendicular wave number) limit, the kink source term can be neglected, and the stability can be determined by the balance between field line bending and the pressure-drive energies. Dropping the kink-mode free energy term, and minimizing the plasma compressibility term, we obtain the

ballooning δW [61]:

$$\delta W = \int dV [(\nabla \times \xi \times B)^2 - 2(\xi \cdot \nabla p)(\xi \cdot \kappa_c)],$$

where κ_c is the curvature. With conditions $k_{\parallel} \sim 1$ and $k_{\perp} \gg 1$, the Euler-Lagrange equation for the stream function ϕ is given by [62]

$$B \cdot \nabla \left[\frac{\nabla_{\perp}^2}{B^2} (B \cdot \nabla \tilde{\phi}) \right] + B \cdot \kappa_c \times \nabla_{\perp} (B \cdot \nabla p_{eq} \times \nabla_{\perp} \tilde{\phi}) = 0. \quad (5.9)$$

This equation is known as ballooning eigenmode equation [63]. Note that the equation has an operator $\nabla_{\perp}$ which retains gradients in both the ρ and θ directions. Thus, the mode can be localized in either the ρ or θ directions for $k_{\perp} \gg 1$. When $\partial_{\rho} \gg \partial_{\theta}$, we obtain the Mercier criterion as we discussed previously.

For $\partial_{\theta} \gg \partial_{\rho}$, we introduce the WKB approximation [64]. The derivative ∂_{θ} contributes for both $k_{\parallel}$ and $k_{\perp}$, and one needs to separate the perpendicular (eikonal S_{stk}) and the parallel (envelope Φ) behavior:

$$\phi(\rho, \theta, \zeta) = \Phi(\theta) e^{iS_{stk}} \quad (5.10)$$

where $S_{stk} \equiv m\theta + n\zeta$. However, this form alone does not allow periodicity in both ϕ (along the field) and θ . To avoid this problem, Conner et al. [63, 65] have introduced a Fourier transform for $\phi(\rho, \theta, \zeta)$ (sometimes referred to as the ballooning transformation):

$$\phi(\rho, \theta, \zeta) = \sum_m e^{iS_{stk}} \int_{-\infty}^{\infty} dx \Phi(x) e^{-iS_{stk}(x)} \quad (5.11)$$

An important feature of the ballooning mode is the fact that it possesses translational invariance [66]: Fourier modes with different m but the same n are correlated. The amplitude of neighboring poloidal modes (with the same toroidal mode number, say

... 8/10, 9/10, 10/10, 11/10, 12/10 ...), are correlated with each other, and grow at the same rate. Conceptually, the translational property will be described as

$$\phi(\rho, \theta, \zeta) = \sum_m f_m(\rho) \exp(im\theta + in\zeta), \quad (5.12)$$

$$f_m(\rho - \rho_s) = f'_m(\rho - \rho'_s), \quad (5.13)$$

where ρ_s (ρ'_s) is the resonant surface for the m/n (m'/n) mode. The eigensolution to the ballooning equation is known to be a Weber function [67]. The Weber function has a Gaussian radial dependence, so that the mode is localized radially on the mode rational surfaces. The nature of the ballooning equation will be examined quantitatively in the next subsections. Furthermore, the correlated mode structure will be exhibited by a high β simulation.

5.3.1 Ballooning equation

Starting from the reduced MHD equations, by assuming a large aspect ratio and circular flux surfaces, the ballooning equation is obtained in the form of a second order ordinary differential equation:

$$\frac{d}{d\theta} \left([1 + (s\theta - \alpha \sin \theta)^2] \frac{d\phi}{d\theta} \right) + \alpha [\cos \theta - \sin \theta (s\theta - \alpha \sin \theta)] \phi = 0. \quad (5.14)$$

Here, $s \equiv d(\log q)/d(\log r)$ is the magnetic shear, and $\alpha \equiv -(2Rq^2/B^2) dp/dr$ represents the normalized pressure gradient. The sine/cosine dependence stems from the finite β or Shafranov shift effect (variation of the magnetic shear on a flux surface). The solution ϕ can be regarded as function of both θ and $s\theta$. Although this reduced ballooning equation is simple enough, it carries all the important information about the relation between the local magnetic shear and pressure gradient.

The ballooning equation reduces to the form

$$\phi'' + g_1(\theta)\phi' + g_2(\theta)\phi = 0, \quad (5.15)$$

where

$$\begin{aligned} g_1(\theta) &= h_1'/h_1, \\ g_2(\theta) &= h_2/h_1, \\ h_1(\theta) &= [1 + (s\theta - \alpha \sin \theta)^2], \\ h_2(\theta) &= \alpha [\cos \theta - \sin \theta (s\theta - \alpha \sin \theta)]. \end{aligned}$$

The boundary condition $\phi \rightarrow 0$ as $\theta \rightarrow \infty$ is employed. The coefficient $h_1(\theta)$ represents the characteristic connection length $1/s$ from the mode rational surface.

Defining $Y = (y_1, y_2, y_3, y_4) = (\phi, \phi', \partial\phi/\partial A, \partial\phi'/\partial A)$, Eq. (5.14) reduces to a system of simultaneous ordinary differential equations

$$\frac{dY}{dx} = \begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \end{pmatrix} = \begin{pmatrix} y_2 \\ -g_2(x)y_2 - g_1(x)y_1 \\ y_4 \\ -g_2(x)y_2 - g_1(x)y_1 \end{pmatrix} \quad (5.16)$$

with boundary conditions at $\theta = 0$:

$$(y_1, y_2, y_3, y_4) = (1, A, 0, 1). \quad (5.17)$$

The unknown constant A is obtained by iterating until the values $A^{l+1} - A^l = -(\phi)/(\partial\phi/\partial A)$ at the boundary converge. Here, l represents the iteration step (see Ref. [57] for details of a shooting method algorithm). Figure 5.2 shows the eigenfunction obtained from Eq. (5.14). As in Ref. [63], the value $\alpha = 0.25$ is taken. By comparing the $s = 0.1$ and $s = 0.7$ cases, the stronger localization of the mode caused

Figure 5.2: The eigenfunction of the ballooning equation Eq. (5.14).

by larger shear can be observed. Both cases satisfy the Mercier stability condition and the mode amplitude decays as $\theta \rightarrow \infty$. One can see the periodicity $2\pi \sim 6.28$ in the θ direction. (The function is even: it is symmetric about the $\phi = 0$ axis, needless to say).

The boundary ($\theta \rightarrow \infty$ by definition) is artificially set at $\theta = 30$ for $s = 0.7$, and $\theta = 60$ for $s = 0.1$, respectively. The decaying oscillation implies that the ballooning mode can have a node at the $\zeta \rightarrow \infty$ boundary, when the Mercier stability condition is satisfied. Thus, satisfying the Mercier stability criterion is a necessary condition for the existence of unstable ballooning modes.

5.3.2 Ballooning $s - \alpha$ diagram

In the previous subsection, the ballooning equation was solved to see the nature of the eigenfunctions for given magnetic shear s and pressure gradient α parameters. In order to analyze the stability threshold of the ballooning mode, one needs to introduce a variational principle to represent the system.

Andersson and Weiland derived an energy principle (δW) in the form of Eq. (5.14) in the limit of small s and α , expanding up to order α^3 terms:

$$\delta W = \frac{\pi}{4s} \left(s^2 - \frac{3}{2} \alpha^2 s^2 + \frac{9}{32} \alpha^4 + 2\delta\alpha - \frac{5}{2} \alpha e^{-1/s} \right), \quad (5.18)$$

where δ represents the magnetic well discussed previously. Minimizing the integrands gives the stability criteria. Ignoring the exponential term the stability boundary reads

$$s = \frac{1}{2} \left[\frac{3}{2} \alpha^2 \pm \sqrt{\frac{9}{4} \alpha^4 - 8\alpha\delta} \right]. \quad (5.19)$$

The stability boundary Andersson-Weiland obtained is shown in Fig. 5.3. The inclusion of the $e^{-1/s}$ term gives extra stability outside the parabolic $s \sim \alpha^2$ as one seen in Ref.[51]. The negative square root solution contribution represents the so-called second stability region. A stable region can be found in an extremely low shear (s) and high pressure gradient (α) configuration. Although the second stability region provides interesting tokamak operation regime, it is not discussed further in this dissertation.

5.4 High β simulation results

The numerical results for a resistive internal kink mode with emphasis on the pressure driven modes (ballooning modes to be specific) are shown in this section. Parameters

Figure 5.3: The $s - \alpha$ diagram from Eq. (5.19).

used in the calculations were: inverse aspect ratio of $\epsilon = 1/4$, and Lundquist number $S = 10^5$. The toroidal β values were varied in the range of 2% to 5%. A total of 500 equally spaced mesh points was used in the radial direction. The mode spectra was selected as shown in Fig. 5.4, so that it fills the $0.81 \leq q \sim m/n \leq 2.7$ region. A total of 151 modes were included in the simulations.

With higher β (and by including high- n mode numbers), a sudden onset of an Alfvén time scale pressure-driven mode [68] is observed in the final stage of the sawtooth crash. Figures 5.5 and 5.6 show pressure contours in the poloidal cross sections at $\zeta = \pi$ and $\zeta = 0$ (parallel heat conduction is absent in this part of the work). The time scale is denoted by the Alfvén time. (Note that $t = 0$ does not

Figure 5.4: Mode spectrum obtained from the high β simulation. The abscissa and ordinate corresponds to poloidal and toroidal mode numbers, respectively. A total of 151 modes are used in the simulations.

mean the beginning of the simulation. Approximately 1400 Alfvén times had already been simulated by the stage of Fig. 5.5(a) and (b), where the $m = 1$ magnetic island had already grown to a finite width). Figure 5.6 shows the final crash phase which exhibits localized bulges (sometimes referred to as ballooning fingers [69]).

Localization of the mode on the bad curvature side can be seen starting from $t = 30$; the $m/n = 1/1$ magnetic island evolution gives rise to convection of the pressure inside the inversion radius and builds up a steep pressure gradient across the island separatrix or the Y-ribbon [38], and thereby triggers ballooning instabilities below the threshold at the equilibrium. In Figs. 5.5, 5.6, and 5.7, a maximum β of 5% at the equilibrium magnetic axis was used. The pressure behavior is similar

to the ECE and the soft X-ray tomography data of TFTR (see Fig.2.5) in that it shows mode localization on the bad curvature side and significant activity in short wavelength modes (as a reminder, Nagayama et al. 1993 [7] exhibited an unstable $7/6$ mode together with a growing $1/1$ mode and a $3/2$ mode). The pressure contour structures in the poloidal plane seems to be quite similar to those obtained by TFTR signals.

Figures 5.8, 5.9, 5.10, and 5.11 show the mode spectrum for a maximum toroidal mode number $n = 10$, at $t = 0$ when the high- n modes are absent, and at $t = 70$ when the high- n modes are active. Figures 5.8 and 5.10 show the eigenprofiles of the stream function ϕ for the higher harmonics of the $1/1$ mode that have the same helicity: the $1/1$ to $10/10$ mode spectrum. The high n modes are indicated by thick lines (violet, crayon, and magenta). Figure 5.10 shows that at time $t = 70t_a$ the $8/8$ to $10/10$ modes have become significant, while the $4/4$ to $7/7$ modes remained at the same or smaller amplitudes.

Figures 5.9 and 5.11 show eigenprofiles of $n = 10$ modes but with different m (poloidal mode) numbers. These incommensurate helicity modes are observed to grow simultaneously. As we saw previously in the ballooning theory part, different m modes with same n are correlated in the ballooning transformation. Note that each of the modes are localized near their corresponding mode rational surfaces — say $15/10$ at $q = 1.5$ (magenta), $16/10$ at 1.6 (orange), $17/10$ at 1.7 (violet),... , and so on.

An important observation is that these higher mode numbers are not unstable in the axisymmetric equilibrium but are triggered by the change in the pressure profile and magnetic shear in the vicinity of the Y-ribbon [38]. Moreover, these high- n modes are not generated from mode coupling but rather they result from changes in the equilibrium. The key mechanism is the susceptibility of the wavelength of say

8/8, 9/9, 10/10 modes to the current layer length. We hypothesize that the secondary ballooning modes are triggered when the wavelength fits locally within the length of (or shorter than) the Sweet-Parker layer. Thus, the phenomena is entirely different from a mode-coupling energy cascade as one sees in magneto-turbulence.

Figure 5.7 shows the flow velocity field and the corresponding pressure contours for the same $\beta = 5\%$ simulation. Both these figures are located on the bad curvature side of the torus. Interestingly, we can observe small scale eddy structures along the ballooning mode fingers streaming both inward and outward. Note that the significant eddy structures occur outside the $q = 1$ radius, while similar structures can possibly exist inside the $q = 1$ radius; thus, these eddies move plasma substantial distances ($\Delta\rho \sim 0.1$) across the $q = 1$ surfaces.

In an actual plasma discharge, no modes can keep growing forever, to $t \rightarrow \infty$ (unless say, the entire plasma discharge is terminated). Any instability should reach a saturation stage. For example, in Fig 5.6 (e) the ballooning mode fingers are torn and we see a transition to a Kelvin-Helmholtz-type instability. This signifies that numerous modes come into play through nonlinear coupling and destroy the coherent structures. Another key to understand for "tearing" of the ballooning fingers is the finite dissipation. In a dissipative plasma, resistivity as well as viscosity plays a role in damping modes especially the short wavelength modes. As a reminder, the diffusion terms in Eqs. 3.16 and 3.17 are proportional to k^2 (k here stands for a wave-vector).

Figures 5.12 and 5.13 show magnetic field line Poincaré plots and the Kolmogorov entropy during the ballooning mode evolution, in conditions corresponding to those for the pressure contours of Figure 5.6 (b). As one can see, although the magnetic field lines are globally stochasticized, the field lines basically follow pressure surfaces in this case without parallel pressure diffusion effects ($\kappa_{\parallel}$). This is in

dramatic contrast with the work by Kleva and Guzdar [70] where the authors claim that the pressure and the magnetic field completely decouple and the pressure finger evolution does not affect the poloidal magnetic field strength. The later work however is inconsistent with the "frozen flux theorem" of ideal MHD.

Global stochasticity can be seen in the annular region extending to $q \sim 1.5$. In the figures, different colors are employed to specify different initial conditions. One color signifies a single magnetic field line. Though the tip of the finger preserves KAM surfaces which act as a thermal transport barrier, the $q=1$ annular region is stochastic enough to produce rapid radial heat transport through parallel heat transport. The random step size of the field line motion becomes on the order of a tenth of the minor radius ($\Delta\rho \sim 0.1$).

Generation of stochasticity, as before, is due to the interaction between the modes of incommensurate helicity. (This time all the mode comes into play as compared to low beta cases where only low mode number, long wavelength modes were dominant). The degree of field line divergence (stochasticity) is much larger then the case shown before for low β .

With a decrease in β , (5% $\rightarrow$ 2%, see Figs. 5.14, 5.15, and 5.16), the ballooning structures can be still seen, but they are rather modest. The onset of the ballooning instability is delayed for the lower β cases; finger-like structures appear at a much later phase of the $m=1$ island evolution since it takes a longer time for the pressure gradient to build up to the ballooning threshold for low β cases. Another observation is the degree of radial extension of the fingers. For example, in the $\beta = 2\%$ case, the finger extends only up to $q \sim 1.2$ surface.

5.4.1 Magnetic shear reduction near the current sheet

As we saw previously, experimental studies indicate the generation of a steep pressure gradient across the current sheet [38] of the $m = 1$ magnetic island. There is evidence which shows a ballooning type structure in connection with the sawtooth crash and the disruption. Then two questions arise: Is the onset only a function of the pressure gradient? Does the onset of high- n ballooning take place in a concentric equilibrium with a very steep pressure gradient?

The equilibrium we employed is not susceptible to high- n ballooning modes. Only the $1/1$ and the $2/2$ resistive kink mode are linearly unstable at the beginning of simulation. We suggest that the topological change in the magnetic field configuration may be playing a role in the nonlinear stage of the sawtooth crash. In the vicinity of the X-point of the island (or Y-points after the large shift of the central core), the local magnetic shear can be reduced due to the stagnation of the magnetic field lines. This can reduce the field-line bending term and thereby trigger ballooning instabilities below the usual threshold. The effect is analogous to the ballooning mode near the divertor separatrix where the poloidal component of the magnetic field completely disappears.

Hegna and Callen [71] have previously investigated the shear reduction effect on the ballooning mode for constant- ψ , $m \geq 2$ island. The local magnetic shear represented by

$$C(\Theta) = \Omega / \partial_\psi \psi^*,$$

is estimated in an action-angle (ψ^*, Θ) coordinate system, expanding in the vicinity of the single helicity mode rational surface. Here Ω represents the angular frequency. It has been shown that the effect of the magnetic field line bunching or stagnation

and shear reduction is not so significant unless the field lines reach extremely close to the island separatrix, and hence the X-point.

We suspect that the ballooning mode destabilization by such a mechanism can be much more prominent in the presence of an $m = 1$ mode where the hot core shift steepens the plasma pressure near the current shear during the kink evolution. Figure 5.17 shows an expanded field line trajectory in the vicinity of the X-point (this corresponds to the X-point of Fig. 4.16(a)).

5.5 Effect of heat transport along stochastic field lines

The previous simulation results used $\kappa_{\parallel} = 0$. Here, the resultant pressure evolution in response to $\kappa_{\parallel} \neq 0$ and the stochastic magnetic field lines has been investigated. We found the maximum value of the stochasticity induced "radial" thermal diffusion to be $\kappa_{eff} \sim 10$ for the $\kappa_{\parallel} \sim 10^3$ value employed here (see Appendix B.2). Figure 5.18 shows the pressure contour in the presence of finite $\kappa_{\parallel}$. The differences with Fig. 5.5 and 5.6 suggest ergodization (flattening) of the pressure within the stochastic magnetic field line regions.

In Fig. 5.18, toroidal localization of the pressure bulge is much more pronounced than in the $\kappa_{\parallel} = 0$ case. These results suggest that the pressure is more susceptible to toroidal asymmetry (or ballooning modes) when the flux surfaces are destroyed and plasma is not constrained to be on good flux surfaces.

It is also suggested from the magnetic structure of Figs. 5.12 and 5.13 that the edge of the finger structures, where the KAM surface resides, act as a thermal transport barrier. The heat (pressure) moves radially in the stochastic magnetic

region but can not cross the KAM surface and thus can enlarge the discontinuity in the pressure profile (steepen the pressure gradient). Extracting such a mechanism, that is the nonlinear interaction of "almost being broken surface" and the pressure discontinuity, is beyond the scope of this thesis; the exact location of the KAM surface is difficult to resolve numerically, as well as the local pressure gradient. Furthermore, the simulation with $\kappa_{\parallel} \geq 10^3$ is restricted by the cpu-time due to the CFL condition [28]. Thermal transport with larger parallel thermal diffusivity remains to be studied extensively. In the future, the interaction of pressure effects on the secondary island formation and subsequent magnetic stochasticity will be examined to produce a more complete picture of the simultaneous evolution of pressure and magnetic fields.

Figure 5.3: Pressure contour for the $\beta = 5\%$ simulation. The symmetry axis of the tokamak equilibrium is a vertical line through the middle of the page. Thus, the inward side of the torus is located between the two pictures on each row. The outward side of the torus is toward the edges of the page.

Figure 5.6: Pressure contour for the $\beta = 5\%$ simulation. The axis of symmetry of the tokamak plasma is a vertical line down the middle of the page.

Figure 5.7: The velocity field and the corresponding pressure contour for the $\beta = 5\%$ simulation. Both the figures are located on the bad curvature (outboard) side of the torus.

Figure 5.8: Fourier spectrum of the stream functions ϕ for the $\beta = 5\%$ simulation, at $t = 0 t_e$. Same helicity modes, $m/n = 1/1$ to $m/n = 10/10$.

Figure 5.9: Fourier spectrum of stream function ϕ for the $\beta = 5\%$ simulation, at $t = 0 t_e$. Modes with same toroidal mode number $n = 10$.

Figure 5.10: Fourier spectrum of stream function ϕ for the $\beta = 5\%$ simulation, at $t = 70 t_a$. Same helicity modes, $m/n = 1/1$ to $m/n = 10/10$.

Figure 5.11: Fourier spectrum of stream function ϕ for the $\beta = 5\%$ simulation, at $t = 70 t_a$. All modes shown have the same toroidal mode number $n = 10$, but they have incommensurate helicities.

Figure 5.12: Poincaré plot for the $\beta = 5\%$ simulation. To the right is the bad curvature side. These field lines are for the case as that for the pressure contours in Fig. 5.6(b).

Figure 5.13: The values of the Kolmogorov entropy after 200 revolutions, versus the initial position ($\zeta = \pi$) of the field line trajectories. Case corresponds to the Poincaré plot of Fig. 5.12.

Figure 5.14: Pressure contours for the $\beta = 4\%$ simulation. The axisymmetry center line is located at the middle of the page.

Figure 5.15: Pressure contours for the $\beta = 3\%$ simulation. The axisymmetry center line is located at the middle of the page.

Figure 5.16: Pressure contours for the $\beta = 2\%$ simulation. The axisymmetry center line is located at the middle of the page.

Figure 5.17: Poincaré plots of magnetic field line trajectories in the vicinity of the X-point ($\beta = 5\%$ and $S = 10^5$ plasma).

Figure 5.18: Pressure contours for the $\beta = 5\%$ simulation in the presence of finite parallel diffusion. The axisymmetry center line is located at the middle of the page.

ence of overlapping sideband harmonics. A magnetic field line solver with high spatial resolution has been developed and incorporated to investigate the microscopic structure of the magnetic field lines. Generation of the magnetic field line stochasticity was discussed in detail. Field line stochasticity has been evaluated by calculating the Lyapunov exponents.

Secondary reconnection without electron inertia effects [19] has been produced for the first time in a resistive three dimensional MHD simulation. It has been shown that the strong $m = 1$ flow remains along the $q = 1$ surface even after a full Kadomtsev-type reconnection, and pulls q_0 back below unity. (Note that the high- n modes were absent in this low β case).

As a preliminary to the high- n ballooning mode studies, we have simulated the $m/n = 1/1$ island evolution for a large equilibrium β value (and thus a large Shafranov shift) but without the dynamical pressure-drive. By comparing Poincaré mappings of the magnetic field line trajectories on the good and bad curvature sides, we have observed the three dimensional, non-axisymmetric structure of the $m = 1$ magnetic island. It has been shown that the current sheet, or the Y-ribbon [38] is stretched (vertically elongated) on the bad curvature side.

In higher β simulations with a wide selection of Fourier modes, it has been shown that the $m/n = 1/1$ magnetic island evolution gives rise to convection of the pressure inside the inversion radius and builds up a steep pressure gradient across the island separatrix. This change in the equilibrium triggers high- n secondary ballooning modes which cannot be seen in the concentric (non-island) equilibrium. As emphasized in Chapter 5, this is different from a mode-coupling energy cascade from the $m/n = 1/1$ to higher n mode numbers. The ballooning spectrum in the simulation compares favorably with that expected from conventional MHD theory: all the

Chapter 6

Summary and Discussion

In this dissertation, the tokamak sawtooth crash phase has been studied numerically employing a toroidal magnetohydrodynamic initial value simulation. Motivated by experimental results from TFTR, which show dramatic temperature evolution during the crash, special emphasis has been put on the role of high- n pressure driven modes as well as the magnetic field line stochasticity. A series of simulations has been conducted by varying the plasma β value and the plasma current profile. The volume-averaged toroidal β value is varied in the range of $\langle \beta \rangle \sim 0.07\%$ (referred to as low β) to $\langle \beta \rangle \sim 1.4\%$ (referred to as high β).

Low β plasma behavior has been studied with two different q -profiles (peaked and rounded current profiles) and compared with the linear and nonlinear theories of the $m/n = 1/1$ resistive kink instability. It has been shown that the Kadomtsev-type full reconnection takes place in both cases. An interesting observation was the activity of the $m/n \neq 1/1$ modes after the crash phase, which gave rise to global stochasticity. When the current profiles are flat in the vicinity of the magnetic axis, magnetic stochasticity can be generated due to the geometrical coupling in the pres-

modes with different m numbers but with the same n mode numbers are correlated and grow simultaneously. In the crash phase, it is observed that small scale vortices are generated. The results were significant in suggesting the breaking of the symmetric flows, and hence the possibility of pressure profile flattening without complete magnetic reconnection.

Compared to previous work [68, 74] where the helically twisted equilibrium produced by the kink mode was taken as an initial configuration, this is the first MHD-simulation to reveal the activity of the high- n ballooning modes starting from a self-consistent concentric equilibrium that is evolved into a slowly growing $m/n = 1/1$ magnetic island structure. As a reminder, in the simulation studies by Park et al. [68], a dominant $m/n = 1/1$ kink mode in the equilibrium mode couples to the high- n modes and drives ballooning instability. (In their simulation, the plasma does not contain $q = 1$ surface, as well). Another difference is that the simulation by Park et al. [68] was conducted to explain the TFTR β limit disruptions [75] rather than the sawtooth crash; In the β limit disruptions [75], it is observed that the ideal kink modes (instead of resistive modes) give rise to precursor signals and the $1/1$ kink strongly couples to higher m/n components. While the temperature inside the inversion radius completely flattens during the sawtooth, the temperature profiled stays peaked during such high- β (minor) disruptions.

To emphasize again, in our studies, the high- n ballooning activity does not come from the mode couplings. The ballooning modes are suddenly triggered due to the local pressure steepening and the emergence of the magnetic island or the "topological change in magnetic field lines." More significantly, we have shown that ballooning modes can play an important role (the energy in these modes can be as large as the initial perturbation $m/n = 1/1$ modes) and alter the plasma behavior

during the sawtooth crash phase.

Furthermore, with regard to magnetic structures, it has been shown that the toroidal localization of the ballooning mode can break the helical symmetry and thereby induce magnetic stochasticity. In the high β simulations, significant stochasticity has been induced (most of the flux surfaces except for a few peripheral regions are destroyed) by the strong mode couplings between modes of incommensurate helicity, which induces "radial" thermal transport along the field lines. By introducing finite parallel transport effects, it has been suggested that the constant pressure surfaces are more susceptible to the toroidal asymmetry (or ballooning modes) when the flux surfaces are destroyed and plasma is not constrained to be on good magnetic flux surfaces.

The results obtained from our MHD simulation model, which include three-dimensional effects, compares favorably with ECE and the soft X-ray measurements from TFTR as discussed in Chapter 2. The threshold for the secondary ballooning instabilities lies in the range of $\beta_0 \leq 3\%$ (volume average of $\beta \leq 0.755\%$, poloidal beta of $\beta \leq 0.736$). Although there must be an equilibrium current profile dependence for the ballooning onset, as well as geometrical dependence, we believe the β values we estimated for the onset threshold can be applicable to the actual tokamak experiments.

Now, our hypothesis for partial reconnection is discussed.

Both from the experimental and theoretical points of view, there is not much doubt about the mechanism for the temperature crash in a sawtooth: it is a combination of fluid-like kink and/or high- n ballooning instabilities and the magnetic field line and/or particle stochasticity [76] they induce. For halting the uniform growth of the $m=1$ mode, we suggest that the mode conversion (via onset of high- n ballooning

modes) to $m/n \neq 1/1$ higher harmonics and the resultant small scale vortices are playing an important role in the mixing/damping of strong $m=1$ flow. The ballooning modes become unstable because the destabilizing pressure-drive overcomes the field line bending stabilizing contribution on the perturbed magnetic surface. The pressure-drive energy freed by such a mechanism can be far larger than the primary $m = 1$ kink free energy.

As indicated experimentally [16] and suggested by the magnetic field line structure in this paper, the flattening of the temperature profile takes place before the full reconnection of the magnetic field lines. In this case the pressure associated free energy can be reduced before the current profile flattens. It has been suggested by Yamada et al. [10], that in cases where the $m = 1$ internal kink mode is coupled to this pressure driven mode, reconnection may halt due to the expulsion of the free energy related to Bussac's onset condition $\beta_p \geq 0.3$ for the $m/n = 1/1$ mode. It is worth pointing out that Bussac's onset criterion is valuable for a concentric equilibrium. However, there is evidence that the $m = 1$ magnetic island exists throughout the tokamak discharge [77], regardless of the β values.

In contrast, this dissertation has explored the excitation of secondary high- n ballooning modes that become unstable for high enough β and large enough $m/n = 1/1$ magnetic island structures in a toroidal plasma. We are not yet at the stage where we can clarify the exact onset criteria for secondary ballooning instabilities, but this decoupling of the magnetic structures and pressure contours, together with the transition of the dominant $m = 1$ kink mode to high- n ballooning modes gives a scenario for incomplete $m = 1$ magnetic island reconnection in dissipative tokamak plasmas.

Though we were not able to compute the diamagnetic flow frequency stabi-

lization effect [78] (our MHD calculation relies on the single fluid model) it may be possible that the diamagnetic current stabilization effect and the destabilizing ballooning modes are competing with each other. For example, the full temperature crash after the partial one in TFTR [7] and double sawtooth in Alcator-C Mod [20] may fall into this category (see Chapter 2).

Theories has been developed [19] to explain fast reconnection in the thin current layer including nonideal (kinetic) effects. Our numerical simulation results are antithetical to those models in that they suggest that high- n ballooning modes across the current layer can destroy such nonideal effects. If high- n ballooning modes are triggered across the current sheet, it is difficult to imagine that the electron current sheet is not being influenced by the pressure-drive effects (thus causing very strong radial ion flows) and that it can persist in the presence of these instabilities.

Although Bussac's onset criterion for the $m = 1$ internal kink mode is not inherently included in the formulation, the compactness of the reduced MHD formulation, ironically, turned out to help in extracting the important role of secondary high- n ballooning modes during the sawtooth crash phase. Specifically, the simulations in this thesis highlighted the development of high- n modes ($n \sim 10$) and mode-coupling interactions over a wide spectrum of Fourier modes. A full toroidal MHD model, that is one without the small aspect ratio expansion (which would also time advance the poloidal potentials χ and λ) would require three times larger core memory and the number of modes that could be included would be significantly reduced. Note that our compressible, reduced MHD formulation still includes the most important elements such as toroidal geometry effects (via metric elements) and the toroidal curvature (∇R^2), which are the minimum requirements to describe the ballooning effects.

We end this dissertation by mentioning work left for the future. A similar

numerical investigation with the full toroidal model will be carried out (the simulation is being carried out but is not presented in this dissertation because it is not yet complete). More importantly, the plasma and magnetic field line compressibility are thought to be significant in the $m = 1$ reconnection process. The strong flow toward the X-point of the $m = 1$ island separatrix compresses the plasma and what comes out from the layer is not exactly what went into the layer, as is usually depicted in a Sweet-Parker-like reconnection model [3, 4]. If plasma is compressed strongly, sound wave propagation comes into play. This requires implementation of a fully compressible MHD code. The study of the plasma compressibility in a full toroidal MHD simulation is far beyond the scope of this dissertation work.

Appendix A

Magnetic Field Line Behavior

A.1 Magnetic field line equation in a straight field line coordinate system

In the straight field line coordinate system, the magnetic field is given by

$$B = \nabla\theta \times \nabla(\rho\chi_t) + \nabla\zeta \times \nabla\psi_t, \quad (\text{A.1})$$

where $\psi_t(\rho, \theta, \zeta) = \psi_{eq}(\rho) + \psi(\rho, \theta, \zeta)$, $\chi(\rho, \theta, \zeta) = \chi_{eq}(\rho) + \chi(\rho, \theta, \zeta)$. Thus,

$$\begin{aligned} B &= \nabla\theta \times \nabla(\rho\chi_{eq}) + \nabla\theta \times \nabla(\rho\chi) \\ &\quad + \nabla\zeta \times \nabla\psi_{eq} + \nabla\zeta \times \nabla\psi, \end{aligned} \quad (\text{A.2})$$

and its contravariant components are given by

$$\begin{aligned}
 B^\theta &= B \cdot \nabla \theta \\
 &= \nabla \zeta \times \nabla \rho \frac{\partial \psi}{\partial \rho} \cdot \nabla \theta \\
 &= -\frac{1}{\rho R^2} \frac{\partial \psi}{\partial \rho} \\
 &= -\frac{1}{\rho R^2} \frac{\partial \psi}{\partial \rho} + \frac{F_{eq}}{q(\rho) R^2},
 \end{aligned} \tag{A.3}$$

$$\begin{aligned}
 B^\rho &= B \cdot \nabla \rho \\
 &= [\nabla \theta \times \nabla (\rho \chi_{eq}) + \nabla \theta \times \nabla (\rho \chi)] \\
 &\quad + \nabla \zeta \times \nabla \psi_{eq} + \nabla \zeta \times \nabla \psi] \cdot \nabla \rho \\
 &= [\nabla \theta \times \nabla (\rho \chi) + \nabla \zeta \times \nabla \psi] \cdot \nabla \rho \\
 &= -\frac{1}{\rho R^2} \left[\rho \frac{\partial \chi}{\partial \zeta} - \frac{\partial \psi}{\partial \theta} \right] \\
 &= -\frac{1}{R^2} \frac{\partial \chi}{\partial \zeta} + \frac{1}{\rho R^2} \frac{\partial \psi}{\partial \theta},
 \end{aligned} \tag{A.4}$$

$$\begin{aligned}
 B^\zeta &= B \cdot \nabla \zeta \\
 &= [\nabla \theta \times \nabla (\rho \chi_{eq}) + \nabla \theta \times \nabla (\rho \chi)] \cdot \nabla \zeta \\
 &= -\frac{1}{\rho R^2} \left[\frac{\partial (\rho \chi_{eq})}{\partial \rho} + \frac{\partial (\rho \chi)}{\partial \rho} \right] \\
 &= -\frac{F_{eq}}{\rho R^2} + \frac{\chi}{\rho R^2} + \frac{1}{R^2} \frac{\partial \chi}{\partial \rho},
 \end{aligned} \tag{A.5}$$

where the relations

$$\psi_{eq} = - \int_0^\rho d\rho \left[\frac{\rho F_{eq}}{q(\rho)} \right], \tag{A.6}$$

or

$$\frac{\partial \psi_{eq}}{\partial \rho} = - \frac{\rho F_{eq}}{q(\rho)}, \tag{A.7}$$

and

$$\chi_{eq} = - \frac{1}{\rho} \int_0^\rho d\rho [\rho F_{eq}], \tag{A.8}$$

or

$$\frac{\partial (\rho \chi_{eq})}{\partial \rho} = - \rho F_{eq}, \tag{A.9}$$

are used. Thus the magnetic field line equation is given by

$$\frac{d\rho}{d\zeta} = \frac{B^\rho}{B^\zeta} = \frac{-\partial_\zeta \chi + \partial_\theta \psi / \rho}{-F_{eq} + \chi / \rho + \partial_\rho \chi} = \frac{\partial_\theta \psi - \rho \partial_\zeta \chi}{\chi + \rho \partial_\rho \chi - \rho F_{eq}}, \tag{A.10}$$

$$\frac{d\theta}{d\zeta} = \frac{B^\theta}{B^\zeta} = \frac{\partial_\rho \psi / \rho + F_{eq} / q}{-F_{eq} + \chi / \rho + \partial_\rho \chi} = \frac{\partial_\rho \psi + \rho F_{eq} / q}{\chi + \rho \partial_\rho \chi - \rho F_{eq}}. \tag{A.11}$$

These are the simultaneous ordinary differential equation that are integrated numerically. Contributions from the second and third term in the denominator are nominally negligible, though implemented in the code. The equation above reduces to that of reduced MHD if $\chi = 0$.

One of the unknowns in sawtooth studies is the mechanism that induces the incomplete magnetic reconnection process; the central q value (q_0) remains below unity even after the crash of the electron temperature profile [17]. If the magnetic reconnection were complete, the magnetic axis would reconnect with a field line on a $q > 1$ surface and the central q value would be above unity [2]. To check whether the reconnection process is complete or not in MHD simulations, the central q value should be evaluated precisely without an approximation. In the axisymmetric equilibrium, a magnetic axis is located at $\rho = 0$. However, in the presence of an internal kink mode, the q value is no longer a function of the equilibrium flux surfaces. The location of the magnetic axis is shifted and the axis has a helically twisted three dimensional structure. In this case, the q profile can only be calculated from the magnetic field

line pitch integrated over many toroidal transit of the field line. Given the location of the shifted magnetic axis (X_0, Z_0) in a Cartesian coordinate system, a poloidal-like angle is calculated from

$$\Theta = \arctan \frac{Z - Z_0(\zeta)}{X - X_0(\zeta)}. \quad (\text{A.12})$$

Then, the safety factor is calculated by an averaged field line pitch

$$\langle q \rangle = \lim_{\zeta \rightarrow \infty} \frac{\zeta}{\Theta(\zeta) - \Theta(0)} \quad (\text{A.13})$$

based on the ergodization hypothesis. The flux surface averaged $\langle q \rangle$ value in the perturbed state is no more than the ratio between the number of toroidal windings to the poloidal transits. A mode rational surface is only labeled by the $\langle q \rangle$ value where a field line resonates and closes on itself. Figure A.1 shows the q profile in the presence of magnetic island. The solid line represents the equilibrium q profile.

It is also important to analyze inside the magnetic island since this region becomes the central core after the crash if the magnetic reconnection is complete; the island q value will be a touchstone indicating if a Kadomtsev type [2, 34] full reconnection takes place. The O-point of the $m = 1$ magnetic island is obtained from

$$\nabla \psi^* = 0, \quad (\text{A.14})$$

where

$$\psi^* \equiv \int d\psi_{eq} (q(\psi_{eq}) - 1.0) - \psi. \quad (\text{A.15})$$

The algorithm described above provides a tool to investigate the time evolution of the three-dimensional reconnection process. The q profile in the presence of an $m/n = 1/1$ island is shown in Fig. A.1.

Figure A.1: Equilibrium q profile (solid line) and q profile in the presence of a magnetic island (dashed line).

A.2 Magnetic field line equation in a Cartesian coordinate system

We recall the relation between the straight field line coordinates (ρ, θ) and the Cartesian (configuration space) coordinates (X, Z) before discussing the magnetic field line equations. The two can be related through the Lao-Hirshman variational moment solutions [80]:

$$X_{eq}(\rho, \theta) = \sum_m X_m(\rho) \exp(im\theta), \quad (\text{A.16})$$

$$Z_{eq}(\rho, \theta) = (-i) \sum_m Z_m(\rho) \exp(im\theta). \quad (\text{A.17})$$

The inversion of Eqs. (A.16) and (A.17) can be obtained by calculating the Jacobian matrix (this inversion is accurate to the order of 10^{-7} in the numerical computations):

$$\begin{pmatrix} \rho_{n+1} \\ \theta_{n+1} \end{pmatrix} = \begin{pmatrix} \rho_n \\ \theta_n \end{pmatrix} - \begin{pmatrix} \partial X / \partial \rho & \partial X / \partial \theta \\ \partial Z / \partial \rho & \partial Z / \partial \theta \end{pmatrix}^{-1} \begin{pmatrix} X_{eq}(\rho_n, \theta_n) - X_{eq}(\rho, \theta) \\ Z_{eq}(\rho_n, \theta_n) - Z_{eq}(\rho, \theta) \end{pmatrix}, \quad (\text{A.18})$$

where the subscript n stands for the iteration step of the Newton-Raphson method [8]. Based on the inversion relation above, the flux function $\psi(\rho(X, Z), \theta(X, Z), \zeta)$ can be obtained as a function of X , Z , and ζ . This enables one to solve the magnetic field line equation in the Cartesian coordinate system. Magnetic field components are calculated from

$$B^X = B \cdot \nabla X = -J_{XZ} \frac{\partial \psi}{\partial Z} = -J_{XZ} \sum_n \frac{\partial \psi_n(X, Z)}{\partial Z} \exp(in\zeta), \quad (\text{A.19})$$

$$B^Z = B \cdot \nabla Z = J_{XZ} \frac{\partial \psi}{\partial X} = J_{XZ} \sum_n \frac{\partial \psi_n(X, Z)}{\partial X} \exp(in\zeta), \quad (\text{A.20})$$

$$B^\zeta = \frac{-F}{R^2}, \quad (\text{A.21})$$

where $J_{XZ} = (\nabla X \times \nabla Z \cdot \nabla \zeta)$ stands for the inverse of Jacobian of the Cartesian-toroidal coordinate system. Here, the major radius is represented by R . Note that it is necessary to retain the Fourier components in the toroidal direction so that the magnetic field lines follow the three dimensional structure of the perturbed field. The magnetic field line equation to be integrated are

$$\begin{aligned} \frac{dX}{d\zeta} &= \frac{B^X}{B^\zeta}, \\ \frac{dZ}{d\zeta} &= \frac{B^Z}{B^\zeta}. \end{aligned} \quad (\text{A.22})$$

The results corresponding to the straight field line coordinate system are also calculated in the Cartesian coordinate system. The results are equivalent. The equivalence justifies both the algorithm and the calculation. Also, one can learn the nature of three-dimensional reconnection. One needs to retain Fourier components in the toroidal direction of Eq. (A.20) instead of taking $\partial \psi_n(X, Z, \zeta) / \partial X$ or $\partial \psi_n(X, Z, \zeta) / \partial Z$ at a fixed ζ .

Appendix B

Statistical Analysis Of The Magnetic Field Line Stochasticity

Conventional magnetohydrodynamic (MHD) simulation results [68, 47, 72] generate magnetic field line stochasticity in terms of the random motions of field lines observed in Poincaré mappings. However, Poincaré mappings are not a precise tool to measure stochasticity. Even if it seems that the magnetic surfaces are destroyed, they may still produce Kolmogorov, Arnold, and Moser (KAM) surfaces [48] if one follows field lines for an infinite number of revolutions around the torus. In the presence of KAM surfaces, effective perpendicular thermal transport is inhibited, and one will obtain a totally different transport process. As a reminder, various references [82] indicate that "in stochastic or chaotic regions, the distance between two orbits increases rapidly, roughly exponentially."

To extract the stochastic features, the Lyapunov characteristic exponents (sometimes referred to as Kolmogorov-Sinai (KS) entropy) should be calculated. Of some concern is the fact that numerical stochasticity can easily mimic the physical one [83].

In this work, a magnetic field line solver with high spatial resolution is developed (see Appendix A).

B.1 Lyapunov exponents

As mentioned previously, one important feature of stochastic motion is an exponential divergence of field lines from initial conditions. Assuming δ_0 to be an initial small distance between two trajectories at time $t = 0$, and that the distance after time interval t diverges exponentially as

$$\delta(t) = \delta_0 \exp(ht), \quad (\text{B.1})$$

the coefficient h is called the Lyapunov exponent. It measures the rate of the exponential divergence of the distance between the two trajectories [82, 84].

This concept for dynamical systems (time-dependent system) can be applied to the magnetic field line behavior by simply replacing t with the toroidal angle ζ . If one takes the logarithm of Eq. (B.1), $h = (1/\zeta) \log[\delta(\zeta)/\delta_0]$, one sees it is easy to produce computational overflow and consequently numerical errors. To reduce this error, we follow Benettin *et al.* [85]. Thus we follow two field lines, an objective one and a test one, and repeatedly replaced the test one to a position near the objective one after small time intervals. By this replacement, the test trajectory always stays near an objective one and retains the information of the objective one. There is another advantage of this replacement; the position of the test field line is adjusted in the direction of an eigenvector corresponding to the largest eigenvalue [85]. Letting R_j and r_j be the position of the objective one and the test one respectively, we replace

the position of the test one by

$$r_{j+1} = R_{j+1} + \delta_0 \frac{R_j - r_j}{|R_j - r_j|}, \quad (\text{B.2})$$

where the integer j denotes the numerical time step. Thus, Lyapunov exponents h in Eq. (B.1) are computed by

$$h = \lim_{N \rightarrow \infty} \frac{1}{N \Delta \zeta} \sum_{j=1}^N \log \left(\frac{|r_{j+1} - R_{j+1}|}{\delta_0} \right). \quad (\text{B.3})$$

Figure B.1 shows the history of Lyapunov exponents for the field lines in Fig. 4.11(d) calculated using $\delta_0 \leq 10^{-3}$. The colors correspond to the colors of Poincaré plots in Fig. B.2, which distinguished different initial positions of the field line trajectories. On well formed magnetic surfaces ($x_0 = -0.96, 0.86$: greens and reds), the value decreases monotonically, suggesting that the field line trajectory is stable. For the case with destroyed flux surfaces ($x_0 = -0.36, 0.26$: blacks and blues), the value of the Lyapunov exponents ends up in a finite value after a sufficiently long evolution, suggesting that the distance between two field lines diverges exponentially. Also, see Fig. 4.11 for the spatial profile of the Lyapunov exponents of field lines as a function of different initial positions. The smooth oscillation of the green curve in Fig. B.1 suggests the trapping of the field lines in the vicinity of the X-point of the $m/n = 2/1$ magnetic island separatrix.

Note that the Lyapunov exponent is defined in the limit of infinite toroidal revolutions. Therefore, if one follows the field lines for an infinite number of toroidal revolutions, the value of h in the stochastic regions become uniform (ergodization hypothesis), and h in the stable region becomes literally zero. However, it is impossible to demonstrate this in practice since the computation time and numerical accuracy are always limited. We have employed the value of h after a finite number of revolutions

Figure B.1: Toroidal revolution history of the Lyapunov exponents of field line trajectories up to 200 revolutions around the torus. Stable trajectories occur with initial positions $x_0 = -0.96, 0.86$, $Z = 0$ (green and red curves); unstable trajectories occur with initial positions $x_0 = -0.36, 0.26$, $Z = 0$ (black and blue curves). Poincaré plots for these cases are shown in Fig. B.2.

Figure B.2: Poincaré mappings of magnetic field line trajectories. The colors correspond to those of Fig. B.1

(200) around the torus as the value to measure the degree of the stochasticity. This is denoted by $h(200)$ to distinguish it from the h rigorously defined in Eq. (B.3).

The absolute value of the Lyapunov exponents h does not directly give a perpendicular random step size of the field line motion. As a reminder, the h value is an index for the degree of divergence: the physical dimension of h is the inverse of a toroidal angle. Taking a large value of $\delta_0 = 10^{-3}$ (which still gives the same order of magnitude for the h value calculation), the random step size per toroidal revolution around the torus is given as $\delta_0 e^h$, which is on the order of 0.001 to 0.01 of the minor radius for the low β cases, and 0.01 to 0.1 of the minor radius for the high β ballooning cases.

B.2 Spatial correlations and magnetic field line diffusion

In this part of the work, we would like to extract the information representing the diffusion of magnetic field lines. We would also like to develop a method to estimate a physically consistent value of the effective radial thermal diffusivity κ_1 in comparison with the stochastic magnetic field line diffusion $D_{m,st}$. Basically, for islands that grow rapidly on the Sweet-Parker time scale the pressure is fixed to a magnetic surface and moves with the surface as the island grows [86]. Thus, the κ_1 value should be taken to be smaller than the magnetic field diffusion which is unity in our normalization. Rigorously speaking, diffusion constants should be calculated in terms of statistical values, that is, the second order cumulants of the field line motion. However, this method is difficult to apply because of the limited computation time. The $D_{m,st}$ value is estimated by the square of the random radial step size of the magnetic field line motion per toroidal revolution.

For the statistical analysis, the only values of importance are the values derived in intervals long enough for the magnetic field lines to become decorrelated from the initial state. We always need to check whether a Markov process is achieved. Here, an auto-correlation of the velocity is employed (the velocity correlation represents the direct response of the random motion) [87]:

$$C_{vv}(\zeta_{lag}) = \frac{\overline{\delta v(\zeta) \cdot \delta v(\zeta + \zeta_{lag})}}{\overline{\delta v(\zeta)} \cdot \overline{\delta v(\zeta)}}, \quad (\text{B.4})$$

where $\delta v(\zeta) = v(\zeta) - \overline{v(\zeta)}$, and $v \equiv (d\rho/d\zeta, d\theta/d\zeta)$. The bar denotes the time average and ζ_{lag} is a value analogous to the lag time. The revolution of $C_{vv}(\tau)$ versus the lag time is shown in Fig. B.3. One can see the damping of the envelope of the oscillatory behavior. The auto-correlation time ζ_c , which is equivalent to the mean

Figure B.3: Auto correlations.

free time of the collisionless regime [87], was estimated as the logarithmic e-folding time of $C_{av}(\zeta)$. The minimum value of ζ_c in the most stochastic region was given by $\zeta_c \sim 3.5$ toroidal revolutions.

The cross correlation of the velocity defined by

$$C_{av}(\delta, \zeta) = \frac{\delta v(r + \delta, \zeta) \cdot \delta v(r, \zeta)}{\delta v(\zeta) \cdot \delta v(\zeta)}, \quad (\text{B.5})$$

has been calculated as well. Here δ stands for the separation of two trajectories in the X direction in the poloidal plane $\zeta = 0$. As shown in the Fig. B.4 (short range behavior) and Fig. B.5 (long range behavior), the cross correlation decrease in magnitude as either δ or ζ increases. Fig. B.4. Over a long range, the cross correlation decreases algebraically (typically by a power law of ζ , i.e. ζ^{-N} , $N \geq 0$), while in a

Figure B.4: Short range behavior of cross correlation.

short range it decreases exponentially. This power law is due to the existence of the cantori [88] and the field lines persisting near the remnants of the KAM surfaces (stickiness [49] of the magnetic field line motions). Assuming C_{xx} to have the form of $\sim \exp(-\lambda\zeta) \exp(-\Delta r/L)$ (notice $1/\lambda \neq \zeta_c$, see Ref. [87], p.51) the correlation length is given by $L \simeq 0.03$. The value corresponds to the degree of divergence obtained by the Lyapunov exponents. Assuming that the electron guiding center motion moves together with the magnetic field lines, effective radial electron diffusion by the random walks is given by

$$\kappa_{eff} = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\kappa_{\parallel}}} \exp\left(-\frac{\zeta^2}{4\kappa_{\parallel}}\right) \left(\frac{\zeta h}{2\pi R}\right)^2 d\zeta. \quad (\text{B.6})$$

This form includes the effect of fast streaming along the field lines. The value of $\kappa_{\perp}$

Figure B.5: Long range behavior of cross correlation.

should be an order of magnitude smaller than the κ_{eff} value so as to describe the pressure response of the stochastic field lines. In this work $\kappa_{\perp} = 0.1$ was taken. As a reminder, thermal diffusion coefficients are normalized by τ_r (resistive time scale) and the minor radius a .

B.3 Thermal transport term along stochastic field lines

We describe the form of both the parallel heat conduction and the $B \cdot \nabla$ operator in the pressure evolution equation. In Eq. (3.17), $\nabla_{\parallel}^2 p$ is calculated as

$$\begin{aligned}\nabla_{\parallel}^2 p &= \nabla \cdot \nabla_{\parallel} p \\ &= (\beta \cdot \nabla) \cdot \frac{(B \cdot \nabla p)}{B^2}.\end{aligned}\quad (\text{B.7})$$

where

$$\nabla_{\parallel} p = b(b \cdot \nabla p) = \frac{B}{B} \left(\frac{B}{B} \cdot \nabla p \right) = B \left(\frac{B \cdot \nabla p}{B^2} \right)$$

and $B^2 = B \cdot B = F^2(\nabla\zeta)^2 + (\nabla\zeta)^2(\nabla\psi)^2 = F^2/R^2(1 + g^{\phi\rho}\rho^2/q^2)$. For further convenience, $b^2 = F^2(1 + g^{\phi\rho}\rho^2/q^2)$ is defined here. The $B \cdot \nabla p$ term takes account of projections of the pressure gradient along the magnetic field lines. This form is not only applicable to the magnetic field with a single helicity mode but also to the field with multi-helicity (braided, or stochastic) magnetic field lines. In our straight field line coordinate system we have

$$\begin{aligned}\frac{(B \cdot \nabla)}{B^2} &= \frac{(-F\nabla\zeta + \nabla\zeta \times \nabla\psi) \cdot (\nabla\rho\partial_\rho + \nabla\theta\partial_\theta + \nabla\zeta\partial_\zeta)}{B^2} \\ &= \frac{1}{\rho R^2 B^2} \partial_\theta \psi \partial_\rho - \frac{1}{\rho R^2 B^2} \partial_\rho \psi \partial_\theta - \frac{F}{R^2 B^2} \partial_\zeta\end{aligned}\quad (\text{B.8})$$

$$= \frac{1}{\rho b^2} \partial_\theta \psi \partial_\rho - \frac{1}{\rho b^2} \partial_\rho \psi \partial_\theta - \frac{F}{b^2} \partial_\zeta. \quad (\text{B.9})$$

Thus,

$$\begin{aligned}
 (B \cdot \nabla) \left(\frac{B \cdot \nabla p}{B^2} \right) &= \frac{1}{\rho R^2} \partial_\theta \partial_\theta \left(\frac{1}{\rho b^2} \partial_\theta \psi \partial_\theta p - \frac{1}{\rho b^2} \partial_\theta \psi \partial_\theta p - \frac{F}{b^2} \partial_\theta p \right) \\
 &- \frac{1}{\rho R^2} \partial_\theta \psi \partial_\theta \left(\frac{1}{\rho b^2} \partial_\theta \psi \partial_\theta p - \frac{1}{\rho b^2} \partial_\theta \psi \partial_\theta p - \frac{F}{b^2} \partial_\theta p \right) \\
 &- \frac{F}{R^2} \partial_\theta \left(\frac{1}{\rho b^2} \partial_\theta \psi \partial_\theta p - \frac{1}{\rho b^2} \partial_\theta \psi \partial_\theta p - \frac{F}{b^2} \partial_\theta p \right). \quad (B.10)
 \end{aligned}$$

Bibliography

- [1] S. von Goeler, W. Stodiek, and N. Sauthoff, Phys. Rev. Lett. **33**, 1201 (1974).
- [2] B.B. Kadomtsev, Sov. J. Plasma Phys. **1**, 389 (1975).
- [3] P.A. Sweet, Nuovo Cimento Suppl. **8**, 188 (1958).
- [4] E.N. Parker, Astrophys. J. **121**, 491 (1955).
- [5] Y. Nagayama et al., Phys. Plasmas **3**, 1647 (1996).
- [6] J.D. Callen, B.A. Carreras, and R.D. Stambaugh, Phys. Today **45**, 34 (January 1992).
- [7] Y. Nagayama et al., Phys. Rev. Lett. **69**, 2376 (1992).
- [8] H. Soltwisch and H. R. Koslowski, Plasma Phys. and Controlled Fusion **37**, 667 (1995).
- [9] E. D. Fredrickson et al., Phys. Rev. Lett. **65**, 2869 (1990).
- [10] M. Yamada et al., Phys. Plasmas **1**, 3269 (1994).
- [11] A. B. Rechester and M. N. Rosenbluth, Phys. Rev. Lett. **40**, 38 (1978).
- [12] J. D. Callen, Phys. Rev. Lett. **39**, 1540 (1977).
- [13] Y. Nagayama et al., Phys. Rev. Lett. **67**, 3527 (1991).
- [14] R.D. Gill, et al., Nucl. Fusion **32**, 723 (1992).

- [15] T. H. Osborne, R. N. Dexter, and S. C. Prager, *Phys. Rev. Lett.* **49**, 734 (1982).
- [16] A. Weller et al., *Phys. Rev. Lett.* **59**, 2303 (1987).
- [17] H. Soltwisch, W. Stodiek, J. Manickam, and J. Schlüter, in *Plasma Physics and Controlled Nuclear Fusion Research, 1986*, Proceedings of the 11th International Conference, Kyoto (International Atomic Energy Agency, Vienna, 1987) Vol. 1, p. 263.
- [18] H. Soltwisch, *Rev. Sci. Instrum.* **59**, 1599 (1988).
- [19] Ya. I. Kolesnichenko et al., *Phys. Rev. Lett.* **68**, 3881 (1992); D. Biskamp and J. F. Drake, *Phys. Rev. Lett.* **73**, 971 (1994).
- [20] C. L. Fiore et al., *Bull. Am. Phys. Soc.* **42**, 2017 (1997).
- [21] H. R. Strauss, *Phys. Fluids* **20**, 1354 (1977).
- [22] V. E. Lynch et al., *Comput. Phys. Commun.* **24**, 465 (1981).
- [23] L. A. Charlton et al., *J. Comput. Phys.* **63**, 107 (1986).
- [24] L. A. Charlton et al., *J. Comput. Phys.* **86**, 270 (1990).
- [25] M. Azumi et al., in *Proceedings of the U.S.-Japan Workshop on 3-D MHD Simulation*, Nagoya, Japan, 1983.
- [26] M. N. Bussac, et al., *Phys. Rev. Lett.* **35**, 1638 (1975).
- [27] M. N. Bussac and R. Pellat, *Phys. Rev. Lett.* **59**, 2650 (1987).
- [28] C. Canuto, M. Y. Hussaini, A. Quarteroni, T. A. Zang, *Spectral Methods in Fluid Dynamics* (Springer Verlag, Berlin, 1987), P.118.
- [29] H. R. Hicks, et al., *J. Comput. Phys.*, **44** 46 (1981).

- [30] R. B. Bird, W. E. Stewart, E. N. Lightfoot, *Transport Phenomena* (Wiley, New York, 1960), p. 245.
- [31] M. N. Rosenbluth, R. Y. Dagazian, and P. H. Rutherford, *Phys. Fluids* **16**, 1894 (1973).
- [32] G. Ara et al., *Ann. Phys.* **112**, 443 (1978).
- [33] B. Coppi et al., *Sov. J. Plasma Phys.* **2**, 533 (1976).
- [34] B. B. Kadomtsev, in *Plasma Physics and Controlled Nuclear Fusion Research, 1976*, Proceedings of the 6th International Conference, Berchtesgaden (IAEA, Vienna, 1977) Vol. 1, p. 555.
- [35] V. D. Shafranov, *Zh. Tekh. Fiz.* **40**, 241 (1970) [*Sov. Phys.-Tech. Phys.* **15**, 175 (1970)].
- [36] W. A. Newcomb, *Ann. Phys.* **10**, 232 (1958).
- [37] P. H. Rutherford, *Phys. Fluids* **16**, 1903 (1973).
- [38] F. L. Waelbroeck, *Phys. Fluids B* **1**, 2372 (1989).
- [39] H. P. Furth, J. Killeen, and M. N. Rosenbluth, *Phys. Fluids* **6**, 459 (1963).
- [40] H. P. Furth, P. H. Rutherford, and H. Selberg, *Phys. Fluids* **16**, 1054 (1973).
- [41] H. R. Strauss, *Phys. Fluids* **19**, 134 (1976).
- [42] J. W. Connor, R. J. Hastie, and J. B. Taylor, *Phys. Fluids B* **3**, 1532 (1988).
- [43] R. L. Dewar and M. Persson, *Phys. Fluids B* **5**, 4273 (1993).
- [44] J. A. Holmes, B. A. Carreras, and V. E. Lynch, *Phys. Fluids* **25**, 800 (1982).
- [45] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, (Dover, New York, 1970), p. 896.

- [46] B. V. Chirikov, Phys. Reports **52**, 263 (1979).
- [47] H. Baty, J. F. Luciani, and M. N. Bussac, Phys. Fluids B **5**, 1213 (1993).
- [48] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **98**, 527 (1954).
- [49] M. De Rover, N. J. Lopes Cardozo, and A. Montvai, Phys. Plasmas **3**, 4478 (1996).
- [50] B. V. Waddell et al., Nucl. Fusion **16**, 528 (1976).
- [51] J. Wesson, *Tokamaks*, (Cambridge Press, Cambridge, 1987), p. 308.
- [52] J. L. Johnson, J. M. Greene, and B. Coppi, Phys. Fluids **6**, 1169 (1963);
R. D. Hazeltine and J. D. Miss, *Plasma Confinement* (Addison Wesley, New York, 1992), p. 281.
- [53] B. R. Suydam, in *Proceedings of the 2nd United Nations International Conference on the Peaceful Uses of Atomic Energy*, United Nations, Geneva (IAEA, Vienna, 1958) Vol. 311, p. 157.
- [54] C. Mercier, Nucl. Fusion **1**, 47 (1960).
- [55] C. C. Hegna and J. D. Callen, Phys. Plasmas **1**, 2308 (1994).
- [56] R. C. Grimm, J. M. Greene, and J. L. Johnson, Meth. Comput. Phys. **9**, 253 (1976).
- [57] Y. Nishimura, J. D. Callen, and C. C. Hegna, "Tearing mode analysis in tokamaks, revisited," UW-CPTC-21, December 1997.
- [58] V. D. Shafranov and E. I. Yurchenko, Soviet Phys. JETP **26**, 682 (1968).
- [59] W. Kerner, R. Gruber, and F. Troyon, Phys. Rev. Lett. **44**, 536 (1980).
- [60] M. F. F. Nave and J. A. Wesson, Nucl. Fusion **28** 297 (1988).

- [61] J. P. Friedberg, *Ideal Magnetohydrodynamics*, (Plenum Press, New York, 1987), p. 364.
- [62] R. D. Hazeltine and J. D. Miss, Ref. [52], p. 303.
- [63] J. W. Connor, R. J. Hastie, and J. B. Taylor, Phys. Rev. Lett. **40**, 396 (1978).
- [64] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1955), p. 311.
- [65] J. W. Connor, R. J. Hastie, and J. B. Taylor, Proc. R. Soc. London A **365**, 1 (1979).
- [66] L. E. Zakharov, in *Plasma Physics and Controlled Nuclear Fusion Research, 1978*, Proceedings of the 6th International Conference, Innsbruck (IAEA, Vienna, 1979) Vol. 1, p. 689.
- [67] M. Abramowitz and I. A. Stegun, Ref. [45], p. 498.
- [68] W. Park et al., Phys. Rev. Lett. **75**, 1763 (1995).
- [69] S. C. Cowley and M. Aurn, Phys. Reports **283**, 185 (1997).
- [70] R. G. Kleva and P. N. Guzdar, Phys. Rev. Lett. **80**, 3081 (1998).
- [71] C. C. Hegna and J. D. Callen, Phys. Fluids B **4**, 3031 (1992).
- [72] A. Y. Aydemir, Phys. Fluids B **3**, 3469 (1992).
- [73] A. J. Lichtenberg, Nucl. Fusion, **24**, 1277 (1984).
- [74] H. Lütjens and J.-F. Luciani, Phys. Plasmas **4**, 4192 (1997).
- [75] E. D. Fredrickson et al., Phys. Plasmas **2**, 4216 (1995).
- [76] Y. Nishimura and M. Azumi, Phys. Plasmas **4**, 2365 (1997).
- [77] J. Manickam et al., Bull. Am. Phys. Soc. **42**, 1850 (1997).
- [78] L. Zakharov, B. Rogers, and S. Migulinolo, Phys. Fluids B **5**, 2498 (1993).

- [79] R. C. Grimm, R. L. Dewar, and J. Manickam, *J. Comput. Phys.* **49**, 94 (1983).
- [80] L. L. Lao, S. P. Hirshman, and R. M. Wieland, *Phys. Fluids* **24**, 1431 (1981).
- [81] W. W. Hagar, *Applied Numerical Linear Algebra*, 2nd ed. (Prentice Hall, New Jersey, 1988), p. 164.
- [82] A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics*, 2nd ed. (Springer Verlag, Berlin, 1992), p. 294.
- [83] D. F. Duchs, A. Montvai, and C. Sack, *Plasma Phys. Controlled Fusion* **33**, 919 (1991).
- [84] B.V. Chirikov, in *Reviews of Plasma Physics* (Consultants Bureau, New York, 1987) Vol. 13, p.1.
- [85] G. Benettin, L. Galgani, and J.-M. Strelcyn, *Phys. Rev. A* **14**, 2338 (1976).
- [86] C. C. Hegna and J. D. Callen, *Phys. Fluids B* **4**, 4072 (1992).
- [87] R. Kubo, M. Toda, and N. Hashitsume, *Statistical Physics II*, 2nd ed. (Springer Verlag, Berlin, 1991), p. 7.
- [88] J.D. Meiss, *Rev. Mod. Phys.* **64**, 795 (1992).