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HYDRODYNAMIC INSTABILITIES AND COHRERENT STRUCTURES

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This project has had two related parts:

1. Nonlinear behavior of wavy film flows, and
2. Linear and nonlinear stability of space-time-periodic flows.

The principal accomplishments of the project are reported below under these two headers.

My collaborators in this work included (i) my postdoctoral associates, Dr. V. Kerchman (postdoc for 1992–93) and Dr. K. Indireskumar (postdoc for 1994–96); (ii) my graduate student (Ph.D. in Applied Mathematics, 1995), Dr. X. Zhang; (iii) another graduate student (Master of Arts in Applied Mathematics, 1994), Mr. N. Raghuram. There were also a number of Graduate Research Assistants paid by the grant for purely technical tasks of the project, most usefully I. Yakushin (Ph.D. in Physics, University of Alabama, 1997).

Our work was published in the papers marked in the References section, and reported in many scientific meetings, most frequently the Annual Meetings of the Division of Fluid Dynamics of the American Physical Society. Some of it was discussed in recent books, such as those by Joseph and Renardy 93 and by Frisch 95. My contribution was recognized in the Acknowledgments sections of such papers as the recent review paper by Oron, Davis, and Bankoff 97, and the experimental paper by Liu and Gollub 94. I have also received personal invitations to give talks at prestigious meetings (e.g. AMS-IMS-SIAM Joint Summer Research Conference on Multi-Fluid Flows and Interfacial Instabilities, Seattle 1995) and was invited, among a group of internationally selected researchers, to contribute a chapter to a book, which has resulted in the review article Frenkel and Indireskumar 96. The latter, in particular, gives details of many accomplishments briefly listed in the next section.

1. NONLINEAR BEHAVIOR OF WAVY FILM FLOWS

The full dynamical problems of three-dimensional (3D) incompressible fluid flows are mathematically very complicated: they involve nonlinear partial differential equations with four independent variables. When there are many dynamically important scales of motion, these Navier-Stokes (NS) equations are beyond the capabilities of even the most powerful modern computers. Therefore, such systems for which it is possible to present an approximate description reducing the dimensionality of the problem to at most two independent spatial coordinates (plus time) are of great interest. This point of view unifies this project's studies of flows of liquid layers on solid surfaces.

On its other side, such a liquid layer is assumed to be limited by either a free surface or an interface with another fluid. One example is a free-surface film flow down an inclined plane or a vertical cylinder. Another example is the so-called core-annular flow in a pipe where a liquid film flows between the pipe wall and another, "core", fluid.

There are known exact solutions of NS equations which are time-independent and correspond to idealized flows of the above types, having higher than usual degree of symmetry. However, frequently, they are unstable to small disturbances (which are always present in the environment), and thus are not realized. The disturbances may lead to *wavy* film flows. Typically the wavelengths are much larger than the film thickness. This yields a small ("longwave") parameter which was used, starting with the work of Benney 66, to asymptotically reduce the description of the wave dynamics to a single evolution equation governing the film thickness dependence on time and the two spatial coordinates, those in the downstream and the spanwise directions along the film. With those early theories, because of the highly formal character of the mathematical procedure employed in their derivations, it would be impossible to a priori establish whether the theory would be applicable to a given concrete experiment.

In this project, we have developed novel methods of perturbative derivations of evolution equations governing film flows. In particular, these methods allow one to obtain a priori parametric regions of validity for such approximate descriptions of the nonlinear film dynamics. Our methods yield the most general evolution equations, thus eliminating the need for a new derivation every time a new parametric domain is considered (corresponding to one choice of the infinite number of possible different order-of-magnitude relations between the system parameters). New concepts were introduced to advance our understanding of these matters. New evolution equations were derived, and their numerical solutions were worked out. The latter uncovered new and unexpected types of nonlinear dynamical phenomena, including particle-like interactions of localized coherent structures having emerged spontaneously and, under certain conditions, proceeding to self-organized process of formation of patterns with unprecedented (at least in fluid dynamics) complexity of order (in an intuitive sense). Exciting agreements were obtained with recent important real-life experiments performed by different groups of researchers.

In summary, in my view, the conceptual foundations of the field are in considerably better shape now than before this project started, although much work remains to be done (see Frenkel and Indireskumar 96, section 4). Also, many important concrete results and agreements with real-life experiments have been obtained.

The main details of these accomplishments of the project are briefly listed below. More details can be found in our recent self-review paper, Frenkel and Indireskumar 96 and in references therein, including other papers appended to this report.

1. We (Frenkel and Indireskumar 96) have classified the evolution-equation based theories of film flow as (i) long-wave ones (such as in Benney 66), (ii) single-parameter theories (e.g. Schlang and Sivashinsky 82) and (iii) the multiparametric approach (first formulated in Frenkel 88). The understanding of differences between these types of theories was facilitated by our introducing the notions of, on the one hand, *local* (in time) validity of an evolution equation and, on the other hand, all-time, or "uniform in time", or "global" (in time) validity. We (Frenkel 93) suggested that a physical factor which, judging by the literature, was not appreciated before by the researchers in the field, was the following fact: Due to the dissipative character of the film evolution, the system essentially *forgets* its initial conditions as it evolves to a time-asymptotic regime (corresponding to a dynamical attractor in its phase space). Therefore, the NS terms which were negligible thanks to a favorable choice of initial conditions, can become important at some later stage in the course of evolution, thus making *invalid* the derivation which used the negligibility of those terms. If this actually happens, the evolution equation derived in this way has only a *time-limited* validity. The conditions of time-limited validity involve local (in time), time-dependent parameters such as the lengthscale of the film waves; whereas the conditions of global validity are formulated in terms of the time-independent, internal parameters of the problem. Therefore, the global validity conditions can be given a priori, even before any numerical simulation of the evolution equation (provided it is known that the solutions are bounded). For the longwave theories, even their perturbation parameter is based on the time-dependent lengthscale, hence they cannot render any global validity conditions. The single-parameter theories are an improvement since their (single) perturbation parameter is based on time-independent system parameters. However, the global validity conditions they lead to are unnecessarily restrictive due to the artificial dependences the derivation imposes on the (intrinsically independent) internal parameters. The multiparametric approach avoids these drawbacks. As a result, it gives the least restrictive conditions of global validity (see. e.g., Frenkel 91b; this work is also discussed in the book of Joseph and Renardy 1993) and the most general single (partial differential) evolution equation for the film thickness valid for all time: Any other equation of this type (which approximates the system evolution for some domain in the space of system parameters) is necessarily a subset of the most general equation, the subset that remains after discarding some number of terms of the general equation.

We (see Frenkel and Indireskumar 96) also introduced another concept we found useful to classify the evolution equations, that of the *large-amplitude* evolution equations, i.e. those which govern the surface waves with amplitudes as large as the average film thickness, as against the *small-amplitude* equations, such as the Kuramoto-Sivashinsky one. In contrast to the latter, the large-amplitude equations are highly nonlinear.

2. In this connection, we have shown (Frenkel 93) that no large-amplitude equation for *inclined-plane* film flow (such as the celebrated Benney 66 equation) can be globally valid. This is due to the fact that the instability of the time-independent, uniform-thickness solution of the NS problem is due to *inertial* NS terms.

3. In contrast, flows of sufficiently low Reynolds numbers in systems where the destabilization of the guiding symmetric solution is caused by a physical factor other than film inertia (e.g. (i) gravity in a horizontal planar film with a denser immiscible liquid on top of it, or (ii) the transverse part of the surface tension in cylindrical films) may be well approximated by a *large-amplitude* equation valid *for all times*.

4. An example of the latter is the *large-amplitude equation I* (Frenkel 92) derived to describe the elegant experimental results of Queré 90 for wavy films draining down a *vertical fiber*.

5. With my postdoc at that time, Dr. V. Kerchman, we developed an appropriate computer code to simulate the evolution of the fiber film.

6. These long-time simulations on spatially extended intervals yielded an *excellent agreement* with Queré's experiments. (Kerchman and Frenkel 94).

7. As an unexpected bonus, we discovered very interesting nonlinear dynamical phenomena. Starting from a small-amplitude white-noise surface, we observed self-organization of localized coherent bulges of the surface ("pulses"), and then fascinating particle-like interactions of the neighboring pulses, most notably their *coalescences*. This was the first (as far as we know) instance of a theory leading to coalescences of film pulses, . Such coalescences were observed experimentally (in a somewhat different context, see Liu and Gollub 94) and are thought to play an important role in the film dynamics.

8. The process of a pulse growth by a *cascade of coalescences* discovered in Frenkel and Kerchman 94 and in Kerchman and Frenkel 94 (see also Kerchman 95), was recently confirmed by Chang and Demekhin 97 (although previously Kalliadasis and Chang 94 asserted a different type of pulse growth, which was criticized in Kerchman and Frenkel 94, pp. 250-251). This lead them to an interesting attempt to explain this phenomenon by using some recently advanced mathematical ideas.

9. By using the multiparametric perturbation approach discussed above, we (Frenkel and Kerchman 1994) developed *large-amplitude* evolution equations for both *horizontal* and *vertical core-annular* flows. Simulations with these equations showed results similar to the case of the fiber film flow mentioned above, including the pulse coalescence cascades (see also Kerchman 1995).

10. Based on these results, we suggested an explanation of Aul's puzzling experimental observations on core-annular flow in capillaries (see Aul 89). However, an earlier hope (that I expressed in my 1993 proposal) to explain the "bamboo waves" (discovered by Joseph and his collaborators in vertical core annular flows in *pipes* rather than Aul's capillaries) was not born out: it turned out that their parameters (of the Reynolds number type) were too large for any description by a single evolution equation to be valid.

11. In the mid 1990s, new important experimental results on the *inclined-plane* films (by J. Gollub and his collaborators) turned our attention to planar films. By using a new, refined version of our multiparametric perturbation approach, we (with my second postdoc, Dr. K. Indireskumar) derived the appropriate all-time valid evolution equation (necessarily a small-amplitude one, in accordance with subsection 2. above).

12. We worked out a 2D generalization of our earlier 1D pseudo-spectral code. The simulations

of the inclined-film equation on 2D spatially extended domains showed a remarkable *agreement* with the *experimentally* observed phenomena: period-doubling transition to checkerboard patterns for higher frequencies of flow forcing, synchronous instability for intermediate frequencies, and solitary waves for low frequencies of forcing [see Frenkel and Indireskumar 1996, section. 3.1.5. This agreement was a pleasant surprise since their parameters were *outside* of our validity region (though not by far.)]

13. We (Indireskumar and Frenkel 96, section 3.2; Frenkel and Indireskumar 96, section 3.1.5) performed the analysis of stability of 2D inclined-film waves to 3D infinitesimal disturbances. We used our methods developed for the stability studies of space-time-periodic flows (which topic, as was mentioned above, was the other leg of this project). As a result, we found that there is a stability window in the space of wavenumbers for inclined films which is absent for the vertical case. These analytical predictions were confirmed by our numerical experiments.

14. We have also shown (Indireskumar and Frenkel 96, Frenkel and Indireskumar 97) that our inclined-film evolution equation (which has a dispersive, third-order-derivative term in addition to the dissipative, even-order-derivative terms of the Kuramoto-Sivashinsky type), is the *most general* equation possible, in the sense mentioned above in subsection 1.

15. We (Frenkel 96; Indireskumar and Frenkel 1997) carried out numerical simulation of this equation for the case of a vertical-plane film with *small* dissipation, when the equation becomes a 2D generalization of the Korteweg-deVries equation with additional small dissipative terms of the Kuramoto-Sivashinsky type. It showed a remarkable many-stage process in which two populations of 2D localized, soliton-like structures emerge spontaneously and are gradually self-organizing into *ordered patterns* of unprecedented complexity. These patterns seem to be of a completely *novel type* and, in our view, may be of a considerable interest for the general fields of pattern formation and complexity studies. Our animation movies of different stages of this fascinating evolution (including the final, time-asymptotic attractor regime) are available on videotapes from this author; they can also be enjoyed at <http://euler.math.ua.edu:1997>.

16. A proper cylindrical-case analog of the planar-film Benney equation was derived (Frenkel 93). The reason that earlier attempts by other researchers were unsuccessful is that they failed to appropriately utilize the assumption of a small thickness-to-radius ratio. (Also, a second-order correction to the leading-order equation was obtained by Lukin and Frenkel 97, unpublished.)

17. We derived a dissipative-dispersive (small-amplitude) equation governing the evolution of a film flowing down a vertical fiber. We indicated the range of parameters where the new dispersive term, overlooked by the earlier researchers (e.g., Shlang and Sivashinsky 1982), will be important (see Frenkel and Indireskumar 96, section 3.2.2).

18. Recently, we (Frenkel and Halpern 97) derived a film thickness evolution equation for a two-layer horizontal system with a *time-oscillating* planar Couette basic flow. It has the form of a Kuramoto-Sivashinsky equation modified with a time-dependent factor in the nonlinear term. Its analytical and numerical studies, which are being continued, show a remarkable qualitative change of the time-asymptotic regime at a certain critical frequencies.

2. LINEAR AND NONLINEAR STABILITY OF SPACE-TIME-PERIODIC FLOWS

Large-scale instability of *space*-periodic flow has been studied in connection with such intriguing phenomena in hydrodynamic turbulence as inverse cascade of energy to—and negative eddy viscosity at—large scales of motion and self-organization of large-scale coherent structures. Also, certain spatially periodic flows may model large-scale zonal currents in planetary atmospheres, such as those on Earth and Jupiter. Other applications are found in the Introduction of the appended paper, Zhang and Frenkel 98.

Certain results can be obtained by considering a statistical ensemble of flows such as in

Lipscombe, Frenkel and ter Haar 91. However, the main thrust in this project has been the study of instability of *deterministic* viscous flows sustained by a periodic external forcing.

Whereas time-independent flows were considered before, we included into consideration the periodic dependence on time; it stands to reason that flows periodic in *both* space and time are more appropriate for turbulence modeling. The conceptual foundation for such studies was worked out in Frenkel 91a. That approach, reducing a differential eigenvalue problem to an algebraic one, proved in the course of our studies very useful in clarifying the question of generality of previous conclusions even for steady flows, where only very special, having high symmetry flows had been considered before. We also emphasized the importance of intermediate-scale instabilities (which cannot be captured by the alternative, differential-equation approach popular in the field), and developed the first nonlinear theory of such instability. (As a connection to the other part of this project, similar methods were used, as was mentioned in section 1, to study the stability of certain travelling-wave solutions of film evolution equations.)

In summary, we have achieved significant expansion of the field in some important directions. Some of our work was discussed in the recent book on turbulence, Frisch 95. Some analytical predictions of Frenkel 91a found confirmation in direct numerical simulations (see Murakami and Watanabe 94). The Ph.D. Dissertation Zhang 95 was awarded a Best Dissertation prize of the College of Arts and Sciences of the University of Alabama. The principal accomplishment of this part of the project are briefly listed below (more details can be found in the appended papers):

1. The foundation for an algebraic approach to the linear stability problem for flows periodic in both space and time has been suggested (Frenkel 91a).

2. An analog of Squire's theorem was established for unidirectional time-periodic flow, which allows one to restrict the consideration to only two-dimensional disturbances when determining the critical Reynolds number of instability (Frenkel 91a).

3. A number of new analytical results were obtained for oscillating sinusoidal Kolmogorov flows, as well as for steady Kolmogorov flows first considered in Meshalkin and Sinai 61 (Frenkel 91a; see the remark above on the independent confirmation of our predictions.)

4. The rather long-standing popular belief that the isotropic eddy viscosity cannot be negative was proved to be wrong by demonstrating the case of a time-oscillating flow where the large-scale disturbances have *both* isotropy and *negative* eddy viscosity. (Sivashinsky and Frenkel 1992; see also the book of Frisch 95).

5. By using Bloch-type modes to capture all possible disturbances, new linear-stability results were obtained for *unidirectional steady* periodic flows, significantly more general and rigorous than those available in the previous literature (Zhang and Frenkel 98).

6. Stability of a general class of time-dependent, oscillating generalized Kolmogorov flows was considered for the first time. Floquet-Bloch-type modes were used for this as was first suggested in Frenkel 91a. The expression for eddy-diffusivity coefficient was obtained and analyzed. The critical parameters of instability were determined as functions of the fundamental frequency. The most interesting result is that any flow of this type becomes *absolutely stable* for sufficiently large, supercritical frequencies. In general, it turns out, the critical Floquet-Bloch disturbance does not have the same periodicity in the transverse direction as the basic flow, that is the instability is "non-transverse". However, the instability becomes transverse at the critical value of the frequency (Zhang and Frenkel 1998).

7. For *multi-directional* basic flows, a new classification of instabilities was suggested. A new type, "basic-scale" instability, which does not exist for unidirectional flows, was pointed out (Zhang 1995).

8. For the first time, a *nonlinear* theory of saturation of an intermediate-scale instability (i.e. one in which the lengthscale of the critical mode is comparable to that of the basic flow) was constructed. In contrast to previously known nonlinear theories (for the saturation of *large-scale*

instabilities) which were of the Kuramoto-Sivashinsky type, our nonlinear theory is of the *Landau-equation* type. This is remarkable because, in contrast to previously known instances in different branches of physics where the Landau equation was obtained, the leading mode in our case is *not* monochromatic, but rather has infinitely *many* Fourier components. The coefficients of the equation, "the Landau constants", were obtained in terms of the basic flow (Zhang 95; Frenkel and Zhang, to be published).

9. Before, only *large-scale* instabilities were considered for time-dependent periodic flows. We (Rudenko and Frenkel 97, unpublished) found *intermediate-scale* instability for an oscillating triangular-eddy flow. Moreover, in certain domains of parameters we find two well separated groups of unstable modes, one having large lengthscales and the other intermediate lengthscales. This may lead to interesting effects in the nonlinear theory of this instability.

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