On Constraints for Heavy-Meson Form Factors

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November 1992

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*This article will be published in Physics Letters B.
†Operated by the Universities Research Association, Inc., for the U.S. Department of Energy under Contract No. DE-AC35-89ER40486.

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On Constraints for Heavy-Meson Form Factors

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We examine the recent work of de Rafael and Taron where model-independent bounds on the Isgur-Wise function are presented. We first argue that the bounds cannot hold in as much generality as implied. We show that the effects of resonances omitted in their discussion (such as heavy-heavy "onium" states below threshold) modify the bound. The resulting bound is much weaker but may be useful where the size of the additional contribution may be computed or estimated.

November 1992

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1. Introduction

In the theoretical limit of very large charm and bottom masses, $\Lambda_{\text{QCD}} \ll m_c \ll m_b$, approximate symmetries of QCD allow one to express all six form factors relevant to semileptonic decays of $B$ mesons into $D$ and $D^*$ mesons in terms of the $B$-number form factor of the $B$ meson, $F(q^2)$:

$$\langle \bar{B}(p')|\bar{b}\gamma_{\mu}|B(p)\rangle = F(q^2)(p + p')_{\mu}$$  \hspace{1cm} (1.1)

where $q = p - p'$. $B$-number conservation implies $F(0) = 1$. Thus the six form factors for $B \rightarrow D^{(*)}\ell\nu$ are known, in this limit, at one kinematic point. This is remarkable because it allows in principle for the extraction of the mixing angle $\theta_c$ from experiment without the need for a phenomenological model of the hadrons involved. In practice, however, the differential decay rate vanishes at the kinematic point where the form factors are predicted, so an extrapolation is necessary. The extracted value of the mixing angle depends on the nature of the extrapolation. Different models suggest different functional dependence of $F(q^2)$, and we are stuck again with a model dependent extraction of the mixing angle.

De Rafael and Taron[2] have recently argued that the form factor $F(q^2)$ must satisfy some rather restrictive model-independent conditions in the form of bounds:

$$F_-(q^2) < F(q^2) < F_+(q^2)$$  \hspace{1cm} (1.2)

where $F_{\pm}(q^2)$ are known functions which depend on $F(0)$. If these bounds were correct, they would rule out some models which have been used to extract $V_{cb}$ from experiment, making the determination of this mixing angle far more precise.

Our discussion is organized as follows. In section 2 we review the derivation of Ref. [2]. In section 3 we note that their conclusion conflicts with the known physical behavior when applied to some other systems. Section 4 identifies the omission of Ref. [2]: part of the states contributing to the analytic structure of $F(q^2)$ were neglected. We note why this omission is more severe for heavy-heavy quark states. We show how the bound may be modified to include additional contributions. The resulting bound is useful, however, only if one can estimate or compute the couplings of additional states. We explain how this works in models, and consider in particular the case of the 't Hooft model. In the last section we suggest some avenues of research that may yield somewhat model dependent but useful bounds.

2. Derivation of the Bound

Our derivation of the bounds follows closely that of de Rafael and Taron, except that we avoid strictly taking the limit $m_b \rightarrow \infty$ and we work with arbitrary dimensions of spacetime, $D$. Consider the two point function of the heavy current $J_\mu = \bar{b} \gamma_\mu b$:

$$(q_{\mu}q_{\nu} - q^2 g_{\mu\nu})\Pi(q^2) = i \int d^Dx e^{i q x} (T J_\mu(x) J_\nu(0))$$  \hspace{1cm} (2.1)

In QCD, for $D < 6$ it satisfies a once subtracted dispersion relation:

$$\chi(q^2) = -\frac{\partial \Pi}{\partial q^2} \bigg|_{q^2 = -Q^2} = \frac{1}{\pi} \int_0^\infty dt \frac{\text{Im} \Pi(t)}{(t + Q^2)^2}.$$  \hspace{1cm} (2.2)

The absorptive part, $\text{Im} \Pi(q^2)$, corresponds to all the contributions to the right-hand side of Eq. (2.1) with a real (on shell) state between the two current insertions. This is a sum of positive definite terms, so one can obtain strict inequalities by neglecting some contributions to the sum. Concentrating on the term with intermediate states of $B\bar{B}$ pairs, one finds:

$$\frac{1}{\pi} \text{Im} \Pi(q^2) \geq \frac{1}{(4\pi)^{D-1}/2 \Gamma(D/2+1)} \frac{(q^2/4 - M_B^2)^{(D-1)/2}}{q^2/2} |F(q^2)|^2 \theta(q^2/4 - M_B^2).$$  \hspace{1cm} (2.3)

where the function $F$ is the analytic continuation of the form factor in Eq. (1.1) for spacelike momentum transfer. The dispersion relation can be used to extend this inequality into the timelike region:

$$\chi(q^2) \geq \frac{1}{(4\pi)^{(D-1)/2} \Gamma(D/2+1)} \int_0^\infty dt \frac{(t/4 - M_B^2)^{(D-1)/2}}{t^3/2(t + Q^2)^2} |F(t)|^2.$$  \hspace{1cm} (2.4)
For large b-quark mass the two point function can be reliably computed from perturbative QCD on the timelike region, and in particular at $q^2 = 0$. With $N_c$ colors, it is easy to verify that, to leading order in $a_s$, 

$$\chi(Q^2) = \frac{2N_c}{(2\pi)^2} \Gamma(3-D/2) \int_0^1 dx x^2(1-x)^2(m_b^2 + x(1-x)Q^2)^{D/2-3}. \quad (2.5)$$

Setting $Q = 0$, changing variables to $y = t/4M_B^2$ and recalling that $M_B/m_b = 1 + O(Q_{\text{CD}}/m_b)$, one has, to leading order in $Q_{\text{CD}}/m_b$,

$$\int_1^\infty dy k(y)|F|^2 \leq 1, \quad (2.6)$$

where we have introduced

$$k(y) = Cy^{-7/2}(y-1)^{(D-1)/2} \quad (2.7)$$

and

$$C = \frac{15\sqrt{\pi}}{2^{D/2+4}N_c\Gamma(D+1)\Gamma(5-D)}. \quad (2.8)$$

Next we argue\cite{3,4} that any function $F$ satisfying (2.6) and analytic except for a cut on the real axis for $y \geq 1$ is bounded as in (1.2), and we will explicitly construct the functions $F_\pm$ for any $D < 6$. To this end we map the complex $y$-plane onto the unit disk $|z| \leq 1$ by the transformation

$$\sqrt{y-1} = \frac{1+z}{1-z} \quad (2.9)$$

The cut $y \geq 1$ is mapped into the unit circle $z = e^{i\theta}$. (The two edges of the cut are mapped into the upper and lower semicircles.) The timelike region $y \leq 0$ is mapped into the segment of the real axis $0 \leq z < 1$. In terms of this new variable the inequality (2.6) is

$$\frac{1}{2} \int_0^{2\pi} d\theta w(\theta)|F|^2 < 1, \quad (2.10)$$

where

$$w(\theta) = k(y(\theta)) \frac{dy}{d\theta} = C \cos^D \theta/2 \sin^{5-D} \theta/2. \quad (2.11)$$

As we will see shortly, the derivation of the inequality uses a function $\phi(z)$ analytic in $|z| < 1$ such that $|\phi(e^{i\theta})|^2 = w(\theta)$. The construction of such a function is not unique — one may always multiply one such function by a power of $z$. We make the choice that reproduces the result of de Rafael and Taron:

$$\phi(z) = C^{1/2}z^{-5/2}(1+z)^{D/2}(1-z)^{(5-D)/2}. \quad (2.12)$$

With ref.\cite{4}, let us define an inner product on the space of complex functions of a real variable $\theta$, with $0 \leq \theta < 2\pi$, by

$$(f,g) \equiv \frac{1}{2\pi} \int_0^{2\pi} d\theta f^*(\theta)g(\theta). \quad (2.13)$$

Next let

$$f_1(\theta) = \phi(e^{i\theta})F(e^{i\theta})$$

$$f_2(\theta) = \frac{1}{1-z_0e^{i\theta}}$$

$$f_3(\theta) = 1$$

With this, we have

$$I \equiv (f_1, f_1) = \frac{1}{2\pi} \int_0^{2\pi} d\theta w(\theta)|F|^2 < \frac{1}{\pi} \quad (2.15)$$

Using Cauchy's theorem we can evaluate the other inner products. For example, with a contour $C$ given by the unit circle $|z| = 1$,

$$g(z_0) = \frac{1}{2\pi i} \int_C dz \frac{g(z)}{z-z_0} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{g(e^{i\theta})}{1-z_0e^{-i\theta}} = (f_2, f_1) \quad (2.16)$$

From the positivity of the inner product we have that the $3 \times 3$ matrix $(f_1, f_1)$ has positive determinant. This gives

$$\det \begin{pmatrix} I & g(z_0) & g(0) \\ g^*(z_0) & 1 & 0 \\ g^*(0) & 0 & 1 \end{pmatrix} \geq 0. \quad (2.17)$$

The inequality (1.2)

$$F_-(q^2) < F(q^2) < F_+(q^2)$$
follows, with

$$F_{\pm}(z) = \frac{g(0)}{\phi(z)} \left[ 1 \pm \frac{|z|}{\sqrt{1 - |z|^2}} \sqrt{\frac{1}{\pi |g(0)|^2 - 1}} \right]$$

(2.18)

The result can be used to obtain a bound on the charge radius,

$$\langle r^2 \rangle = 2(D - 1) \frac{dF}{dq^2}(0) \leq \frac{D - 1}{8 M_B^2} \left[ D - \frac{5}{2} + \sqrt{\frac{25}{\pi C} - 1} \right]$$

(2.19)

De Rafael and Taron phrase their result as a bound on the Isgur-Wise function, by taking the infinite mass limit literally. We refrain from taking the limit $$\nu u' = 0$$ since for $$\nu u' > 1$$ this sends the form factor to zero. The Isgur-Wise function is not a physical observable, e.g., it depends on the renormalization point $$\mu$$. In fact, the form factor $$F(q^2)$$ is related to the Isgur-Wise function $$\xi(\nu u', \mu)$$ through

$$F(q^2 = 2M_B^2(1 - \nu u')) = \left( \frac{\alpha_s(m_b)}{\alpha_s(\mu)} \right)^{a_L(\nu u')} \xi(\nu u', \mu)$$

(2.20)

in the limit $$m_b \to \infty$$. In this limit only the leading log term needs be retained. In Eq. (2.20) the anomalous dimension is

$$a_L(\nu u') = \frac{8}{33 - 2n_f} (\nu u' r(\nu u') - 1)$$

(2.21)

where $$n_f = 4$$ is the number of active flavors and

$$r(x) \equiv \frac{1}{\sqrt{x^2 - 1}} \ln \left( x + \sqrt{x^2 - 1} \right).$$

(2.22)

Notice that, for $$\nu u' > 1$$, the factor $$(\alpha_s(m_b)/\alpha_s(\mu))^{a_L(\nu u')}$$ vanishes as $$m_b \to \infty$$.

3. Trouble on the horizon

De Rafael and Taron point out that some models of the form factor violate the bound (1.2); they view this as a shortcoming of such models.

One can turn this sort of observation around to cast doubt on the validity of the bound by trying to apply it first to simpler physical cases than the intricate heavy-light systems which are of primary interest. In particular, one may argue that the bound Eq. (2.19) cannot be correct because the charge radius of hydrogen atom-like wave-functions for the $$B$$ meson must become arbitrarily large in the weakly bound case of the small coupling limit.

Isgur[5] has pointed out how to refine this objection further. The derivation above holds not just for the form factor of the $$B$$ meson, but also for that of any particle that can be pair produced from the vacuum by the current $$J_\mu$$. Thus, the bound must hold for the form factor of the $$B_c$$ meson, and even for that of the $$\eta_b$$. Now, these are well known to be well described by weakly bound non-relativistic wave-functions. Moreover, the charge radius is enormous compared to the Compton wavelength of the $$b$$-quark, so that the violation of the bound is rather dramatic (as opposed to the rather marginal violation in the case of heavy-light systems).

Clearly there is trouble, and a satisfactory explanation of what is wrong with the bound should account for the different behavior of form factors of heavy-light and heavy-heavy systems.

4. The demise of the bound.

Let us analyze in detail the assumptions and manipulations that lead to the form factor bound Eq. (1.2). A number of physical considerations lead to the intermediate inequality (2.6), which plays a crucial role. Is this inequality valid? As far as we can tell there is nothing wrong with the derivation that leads to it. In fact, the inequality is rather weak because one expects the absorptive part of the two point function, $$\text{Im} F$$, to be dominated by resonant intermediate states ($$T$$, $$T'$$, etc). Is the perturbative QCD calculation of $$\chi$$ reliable at $$Q^2 = 0$$ rather than in the more familiar limit $$Q \to \infty$$? We believe it is. The onset of the physical cut is at spacelike momentum and scales with the square of the large mass $$m_b$$. Since this is much larger than the hadronic scale one expects QCD to be a very good approximation for timelike (and lightlike) momentum.
Consider therefore the second half of the derivation (following Eq. (2.10)) in which a positivity condition was derived for the function $F(y)$. One of the key assumptions about the analytic structure of $F$ — that $F(q^2)$ have no singularities below the $B$-threshold cut at $q^2 \geq 4M_B^2$ — is in fact violated for good physical reason. This assumption is used to deduce $(f_2, f_1) = g(z_0)$ in Eq. (2.16), and in computing $(f_2, f_1) = g(0)$. Without this condition, the contours cannot be suitably deformed. We will argue that the form factor $F(q^2)$ has singularities for $q^2 \leq 4M_B^2$, and that these modify the bounds in (1.2) even in the large $m_b$ limit.

The singularities we are referring to correspond to processes in which an 'onium' state $T_n$ is created by the current out of the vacuum with amplitude $f_n$, and then couples to the $B\bar{B}$ pair with amplitude $g_{nBB}$. For simplicity, let us take these states to be stable so they contribute poles rather than cuts to the form factor. The modification replacing poles by cuts and residues by integrals over discontinuities is straightforward and described below. Since several of these states may lie below threshold, $F(q^2)$ has a term of the form

$$
\sum_n \frac{f_n g_{nBB}}{q^2 - M_n^2}.
$$

(4.1)

When the $q^2$ plane is mapped into the circle $|y| \leq 1$, these poles land in the interior of the circle and hence in the interior of the contour of Eq. (2.16). Now the naive scaling of these quantities for large $m_b$ is a factor of $\sqrt{m_b}$ for each heavy meson in an amplitude, so

$$
f_n \sim m_b^{1/2} \quad \text{and} \quad g_{nBB} \sim m_b^{3/2},
$$

(4.2)

and of course $M_n \sim m_b$. Therefore the contribution (4.1) can be rewritten as

$$
\sum_n \frac{f_n g_{nBB}}{y - M_n^2},
$$

(4.3)

where $f_n = (2M_B)^{1/2}f_n$, $g_{nBB} = (2M_B)^{3/2}g_{nBB}$, $M_n^2 = 4M_B^2 M_n^2$ and, as above, $y = q^2 / 4M_B^2$. Clearly this contribution cannot be ignored even in the large mass limit.

When these poles are taken into account Eq. (2.16) is no longer valid. Instead they generate an additional contribution:

$$
(f_2, f_1) = g(z_0) + \tilde{g}(z_0),
$$

(4.4)

The explicit form of $\tilde{g}$ for the case of pole singularities is

$$
\tilde{g}(z) = \sum_n \left( \frac{f_n \tilde{g}_{nBB}(1 - z_n)^2 \phi(z_n)}{M_n^2 z_n - z_n + 1} \right) \frac{1}{z - z_n},
$$

(4.5)

where $z_n$ are the roots of $z^2 M_n^2 - 2z M_n^2 + 4z + M_n^2 = 0$, with $|z_n| < 1$.

A new bound is still obtained in this fashion, but without a priori knowledge of the quantities $f_n$, $g_{nBB}$ and $M_n$ it is hardly very useful. To obtain interesting constraints, one needs to first estimate the size of these contributions and second, to work with a system in which they can hopefully be shown to be small.

Figure 1 shows the $z$-plane with the disc $|z| \leq 1$ corresponding to the complex $q^2 = 4M_B^2 y$ plane as per Eq. (2.9). The heavy line depicts the cut from some branch point below threshold out to infinity. It is clear from the figure that the angular integral in Eq. (2.16) and the contour integral cannot be made to agree, since the contour cannot be taken to cross the cut. An appropriate choice of contour differs from the angular integral by an integral over the discontinuity across the cut for $|z| \leq 1$. Hence, again, Eq. (2.16) must be replaced as in Eq. (4.4), $(f_2, f_1) = g(z_0) + \tilde{g}(z_0)$, where now, in more generality,

$$
\tilde{g}(z_0) = -\int_{-1}^{+1} dz \ \text{Im} \frac{g(z)}{z - z_0}.
$$

(4.6)

Here $z_T$ stands for the image under (2.9) of the location of the first branch point, $y_T = M_T^2 / 4M_B^2$ (discontinuities associated with light particles, e.g., $\pi\pi$, are present but decouple in the large $m_b$ limit).

We can now understand why the putative bound Eq. (2.19) was more severely violated for heavy-heavy mesons than for heavy-light mesons. As above we can phrase the question in terms of how large the slope of the form factor at $q^2 = 0$, the charge radius, may be. In ref. [6] and ref. [7] it was shown that
the anomalously large radius of a heavy-heavy QCD bound state arises from the presence of many closely spaced singularities with significant residues. It is because the effect of such states was missed that de Rafael and Taron obtained an overly stringent bound on the charge radius of the heavy-heavy states.

Furthermore, if one of these quarks is taken to be increasingly light, some of these singularities migrate above threshold until, for a heavy-light system, one is left with only a few resonances below threshold (recall that the $\Gamma^{(n)}$ is already above threshold). Therefore, the violation of the naive bound Eq. (1.2) is less severe in these cases. Finally, we note that resonances continue to migrate above threshold for light-light systems, so the objections do not apply to the arguments of ref. [4] (although the applicability of perturbative QCD in that case may cast a doubt on those results).

Finally, we have examined all of these considerations in a relevant solvable toy model. We investigated the validity of these bounds in $1 + 1$ dimensions in the large-$N_c$ limit. This is an ideal testing ground for these issues since, on the one hand the quantities $g_{nBB}$ and indeed the full singularity structure of the form factors can be computed exactly; and on the other hand much of the dynamics does in fact parallel 3+1 dimensional heavy quark physics in regimes where they can be compared. It is well known that this is a good approximation for $N_c \geq 4$. The bound in eq. (1.2), now with $D = 2$, holds for heavy-light systems provided $N_c \geq 4$, and is violated for $N_c \leq 3$. (This can be seen in Fig. 2 of ref. [8].) The bound is violated for heavy-heavy systems, even for rather large values of $N_c$, recovering the observation above (but now without recourse to non-relativistic hadronic models)[7].

In this model[9] the absorptive part is saturated by resonances, so its form is the form of an infinite sum over single poles on the positive $q^2$ axis, so that

$$\chi(Q^2) = \sum_n \frac{f_n}{(Q^2 + m_n^2)^2}.$$  \hspace{1cm} (4.7)

The coefficients $f_n$ and the masses $m_n$ can be calculated numerically. We have checked, numerically, that for $Q^2 \geq 0$ the perturbative expression, Eq. (2.5) with $D = 2$, is in excellent agreement with the sum in Eq. (4.7).

5. Conclusions

We have shown the form factor bounds on $F(q^2)$ for heavy-light mesons claimed by de Rafael and Taron do not hold because their derivation neglects contributions to the form factor below the $B$-threshold, for $q^2 < 4M_B^2$. While the model-independent bound on form factors is lost, we can still derive a bound on form factors if we are willing to use a model to estimate the discontinuity across the cut in $F(q^2)$ for $q^2 < 4M_B^2$. For example, a good approximation may be a sum of poles corresponding to the upsilon resonances below threshold, as in Eq. (4.3). If one can either measure or estimate the constants $f_n$ and $g_{nBB}$ that determine the residues, then a bound on the form factor is obtained. By including the effect of these very resonances on the right hand side of (2.3) an interestingly tight bound may follow. This new ‘model’ of the form factor seems worth pursuing.

It is also worth pointing out that nontrivial information about the form factors for the decay of heavy-light mesons into light-light mesons may possibly be obtained through similar methods. The argument is similar to that presented above, but now one considers the two-point function of a heavy-light current, $J_{\mu} = g_{\mu\nu}Q$, and its hermitian conjugate. The disadvantage in this case is that one no longer has knowledge, a priori, of the value of the form factor at one value of $q^2$. On the other hand, there is only one resonance below threshold, $M_B^2 < q^2_{\text{thres}} = (M_B + M_\pi)^2$. Thus, if one is willing to use, say, the constituent quark model\footnote{Alternatively, some or all of these quantities may eventually be obtained from Monte Carlo simulations of lattice QCD, but a computation of the form factor over this wide momentum range may prove difficult on the lattice.} to calculate both $F$ at $q^2_{\text{max}} = (M_B - M_\pi)^2$, and the two constants $f_{B^*}$ and $g_{B^*B^*}$, then one may obtain a bound on the form factor for all momentum in the physical region for the decay $B \rightarrow \pi\nu\bar{\nu}$. With any luck this bound will be tight enough that the form factor is determined to good accuracy.

Acknowledgments. We are grateful to A. Falk, M. Luke and M. Wise for communicating their results of a similar investigation prior to publication[10].
We would like to thank N. Isgur for many interesting discussions. The research of B.G. is funded in part by the Alfred P. Sloan Foundation. This work is supported in part by the Department of Energy under contracts DE-AC35-89ER40486 and DE-AC02-76-ER03130.

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Figure Captions

Fig. 1. $z$-plane with the disc $|z| \leq 1$ corresponding to the complex $q^2 = 4M^2 \gamma^2$ plane as per Eq. (2.9). The heavy line depicts the cut from some branch point below threshold out to infinity.