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Identification of a Reflection Boundary Coefficient in an Acoustic Wave Equation by Optimal Control Techniques *

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IDENTIFICATION OF A REFLECTION BOUNDARY COEFFICIENT IN AN ACOUSTIC WAVE EQUATION BY OPTIMAL CONTROL TECHNIQUES

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ABSTRACT: We apply optimal control techniques to find approximate solutions to an inverse problem for the acoustic wave equation. The inverse problem (assumed here to have a solution) is to determine the boundary reflection coefficient from partial measurements of the acoustic signal. The sought reflection coefficient is treated as a control and the goal - quantified by an approximate functional - is to drive the model solution close to the experimental data by adjusting this coefficient. The problem is solved by finding the optimal control that minimizes the approximate functional. Then by driving the "cost of the control" to zero one proves that the corresponding sequence of optimal controls represents a converging sequence of estimates for the solution of the inverse problem. Compared to classical regularization methods (e.g., Tikhonov coupled with optimization schemes), our approach yields: (i) a systematic procedure to solve inverse problems of identification type and (ii) an explicit expression for the approximations of the solution.

1. Introduction

Over the last two decades, parameter identification, i.e. reconstruction of model properties from observed data has become one of the most active and work intensive areas of applied mathematics. Traditionally, identification techniques have been associated to model (retro)fitting and validation, reverse engineering, and signal detection and interpretation. With the advent of high power computers, new standards of performance have become the

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required norm in modern fields such as communication systems, computer networks, astrophysics, bioengineering, or sophisticated military command and control systems. These standards, especially in domains where on-line responses are crucial, make it necessary to have a much better understanding of the quantitative models involved and warrant the continuing interest in the development of new robust and rigorous identification methods.

From a formal mathematical standpoint, the parameter identification problem is an inverse problem that consists of two separate albeit related subproblems, namely: (a) the identifiability problem and (b) the parameter estimation problem. To understand more precisely the nature of these inverse problems we first describe briefly the direct problem. In the direct problem, a physical system is described (modeled) by the state function, u , which satisfies the abstract operator state equation with data F :

$$(1.1) \quad A(u(\xi); F(\xi)) = 0.$$

The state of the system and the data may be scalar or vector. The independent variables of the problem, ξ - which for evolution problems contain the time, t - take values in the domain $\Omega \subset \mathbb{R}^n$, with sufficiently regular boundary, $\partial\Omega$. The operator A may be rather general, including nonlinear equations of higher order, ordinary differential equations, hybrid systems, etc.

If the system (1.1) accurately describes a realistic physical situation, the direct problem is well-posed, i.e. the data F (parameters, functions, coefficients, sources, initial and boundary values, etc.) determine uniquely a regular solution u [4,14]. The inverse problem is usually ill-posed in the sense that existence, uniqueness, and/or regularity cannot be expected in general [1,5,8].

The identifiability problem consists in studying the well-posedness of the parameter identification problem. Identifiability is loosely defined as the injectivity of the mapping between the sought parameter and their output within the used model that is supposed to be completely and accurately known. The identifiability problem consists in determining whether one can uniquely recover (a part of) F from the model solution, u . The parameter estimation problem consists in finding an estimated value of the unknown parameter from the data within the admissible set that is consistent with identifiability. In the following, we shall assume that the parameters are uniquely identifiable. The remaining problem is then related to regularity, explicit characterization, and approximation of the parameters.

More precisely, we shall tackle the following identification problem corresponding to the direct problem (1.1): "Given partial observations, $B\tilde{u}$, of the true solution $\tilde{u}$, in a subdomain Ω' of the phase space, $\Omega' \subset \Omega$, and a known part, ϕ , of the data F , we seek to determine the unknown part of the data, f ."

The standard methods of solving parameter estimation problems are based on Tikhonov's regularization. In this approach, one constructs - starting from actual observations, $B\tilde{u}$, - an approximate functional:

$$(1.2) \quad J_\beta(f) = \frac{1}{2} \|Bu - B\tilde{u}\|^2 + \frac{\beta}{2} \|f\|^2, \quad \beta > 0$$

where u is the solution of (1.1) for the data $F = (\phi, f)$. The exact form of the cost functional and the types of norms involved depend on the concrete problem that has to be solved. Tikhonov's approach [28] and its variants seek to minimize the functional $J_\beta(f)$ - for a fixed β - over the set of unknown data. It assures - in principle - that the observation Bu of the model solution, $u(f)$, obtained with parameter f , will approach the actual observation, $B\tilde{u}$. The functions f_β which achieve the absolute minimization of $J_\beta(f)$ represent the approximate solution of the parameter identification problem. The disadvantages of this approach are:

- when β is small, the problem is unstable;
- when β is large, the solution is not accurate;
- there is no systematic procedure for finding the absolute minimum;
- there are no systematic means to evaluate the approximations.

To eliminate most of the disadvantages above, we propose a different approach to the inverse problem of identification, based on optimal control for operator equations as developed by J.-L. Lions [17,20].

Our idea is to consider a family of functionals (1.2) for $\beta \geq 0$. For each β strictly positive one considers the unknown data, f , as a control which belongs to a certain bounded set, U ; the control has to be adjusted in order to minimize the functional $J_\beta(f)$. The minimum of the cost functional over f is attained at the optimal control, $f = f_\beta$:

$$J_\beta(f_\beta) = \inf_{f \in U} J_\beta(f).$$

Letting the sequence of β tend toward zero, one can verify that the sequence f_β converges in an appropriate sense to an element of the control set, $f^* \in U$. This element represents the (sometimes unique) solution of the parameter identification problem.

In this paper, we shall specifically apply the formalism to a hyperbolic (acoustic wave) equation. The paper is structured as follows. In Section 3, we shall briefly review the existence of a weak solution to the state problem stated in Section 2, and of an optimal control (for a more detailed analysis see Ref 12). In Section 4, we derive the necessary conditions that an optimal control and its corresponding state must satisfy. Uniqueness of the optimal control for sufficiently short times is proved in Section 5. Section 6 illustrates the application of this optimal control problem to determining an estimate of the reflection coefficient from partial measurements of the solution.

2. Statement of the Problem

Given a bounded domain $D \subset \mathbb{R}^2$ with C^1 boundary, define the spatial domain

$$\Omega = \{(x, y, z) | (x, y) \in D, u(x, y) < z < 0\},$$

where $u : D \rightarrow (-\infty, 0)$ is a C^2 function. Assume the region Ω contains a certain medium (like water in a section of the ocean) with a known velocity tensor E . Let $K > 0$ be a finite constant and define the control set

$$U_K = \{\sigma \in L^\infty(D) | 0 \leq \sigma(x, y) \leq K\}.$$

Given a control $\sigma \in U_K$, we consider the solution $w = w(\sigma)$ of the acoustic wave equation:

$$\begin{aligned}
 (2.1) \quad & w_{tt} - \nabla(E\nabla w) = f \quad \text{in } Q \\
 & w = 0 \quad \text{on } \Sigma \times (0, T), \quad \text{sides of spatial domain} \\
 & \frac{\partial w}{\partial \nu} = 0 \quad \text{on } D \times \{z = 0\} \times (0, T), \quad \text{top of spatial domain} \\
 & \frac{\partial w}{\partial \nu} + \sigma w = 0 \quad \text{on } \Gamma \times (0, T), \quad \text{bottom of spatial domain} \\
 & w = g_1, \quad w_t = g_2 \quad \text{on } \Omega \times \{0\},
 \end{aligned}$$

where

$$\begin{aligned}
 \Gamma &= \{(x, y, u(x, y)) | (x, y) \in D\} \\
 \Sigma &= \{(x, y, z) | (x, y) \in \partial D, \quad u(x, y) < z < 0\} \\
 Q &= \Omega \times (0, T)
 \end{aligned}$$

and

$$\frac{\partial w}{\partial \nu} = E\nabla w \cdot \eta \quad \text{with } \eta, \text{ the outward unit normal vector at the boundary.}$$

The objective functional $J_\beta(\sigma)$ is defined by

$$(2.2) \quad J_\beta(\sigma) = \frac{1}{2} \int_{G \times (0, T)} (w - h)^2 dx dy dz dt + \frac{\beta}{2} \int_\Gamma (\sigma(x, y))^2 ds$$

where $G \subset \Omega$ with positive measure. The first term in $J_\beta(\sigma)$ drives w close to the target h on $G \times (0, T)$ and the second term is the cost of the control. We seek to characterize σ_β such that

$$J_\beta(\sigma_\beta) = \inf_{\sigma \in U_K} J_\beta(\sigma).$$

After this characterization has been completed, we let the parameter β go to zero to approximate a solution to an inverse problem. The inverse problem is to identify σ from observations h of a solution w on $G \times (0, T)$, resulting from a signal source f . Assuming that this inverse problem has a unique solution, then for β small, the optimal control determined from J_β , σ_β , will approximate it reasonably well.

This bilinear optimal control problem is new for wave equations. See [3] for a controllability result for wave equation using a velocity damped control term. See [14] for a similar bilinear boundary optimal control problem in the parabolic case.

The approach of using optimal control techniques with adjoint equations to approximate solutions to inverse problems of identification type is different from traditional approaches which couple Tikhonov regularization (with a functional like ours) with an optimization

algorithm [1]. Our approach has the advantage of an explicit characterization of the approximation σ_β . In a series of recent papers [21,22,29], Puel and Yamamoto obtain uniqueness, stability, and reconstruction results for the inverse problem of identifying the source in a wave equation. Their work is based on exact controllability results and the Hilbert Uniqueness Method [18,19,20] that apply to *linear* equations only. Moreover, the identification is realized only for special types of observed data.

The other approaches which use control theory for solving inverse problems are limited to particular cases. Indeed, Russell [7, 23-26] developed the control theory for the wave equation, including several controllability and stability results. However, our framework is applicable to more general equations. Lasiecka and Triggiani [10,11] further extended control theory for wave equations, in particular developing better trace estimates and the Riccati equation framework for linear-quadratic problems. Our control problem cannot be treated within this scheme since it is bilinear - σw - in the boundary condition. Liang [15] analyzes a simpler bilinear optimal control problem for a wave equation with the control in the state equation itself.

Finally, ideas more closely related to the present approach have been developed by Tikhonov [28], Chavent [4], James and Sepulveda [9] and by the authors [12,13].

Tikhonov introduces the regularization to stabilize an optimal control problem where the objective functional does not depend explicitly on the control [28].

Chavent [4] has investigated the stability of applying optimal control techniques to identification problems and conditions to guarantee the identifiability of the parameters. Barbu and Pavel [2] recently solved an optimal control problem, which approximates the inverse problem of identifying the acoustic impedance function in a one dimensional wave equation.

James and Sepulveda [9] solve the parameter identification problem by treating it as a constrained optimization problem. They formally tackle the latter by the Lagrangian method and solve it numerically without establishing any rigorous result related to its solvability.

The results contained in this paper further the application of our general formalism [13] to the parameter identification problem for the acoustic wave equation [12].

3. Existence of an Optimal Control

To define the solution space for the state problem (2.1), let

$$V = \{v \in H^1(\Omega) | v = 0 \text{ on } \Sigma\}$$

with norm

$$\|v\|_V = \left(\int_{\Omega} |\nabla v|^2 dx dy dz \right)^{\frac{1}{2}}.$$

Note that this norm on V is equivalent to the usual H^1 norm due to zero boundary

conditions on Σ and Poincaré's inequality. We make the following assumptions:

$$(3.1) \quad E \in C(\overline{\Omega}; \mathbb{R}^{3 \times 3}) \text{ and } E(x, y, z) \geq \delta I \\ \text{for all } (x, y, z) \in \overline{\Omega}, \text{ for some } \delta > 0$$

$$(3.2) \quad f, f_t \in L^2(Q)$$

$$(3.3) \quad G \subset \Omega \text{ with positive Lebesgue measure}$$

$$(3.4) \quad h \in L^2(G \times (0, T))$$

$$(3.5) \quad u \in C^2(D), \quad u(x, y) < 0$$

$$(3.6) \quad g_1 \in H^2(\Omega) \cap V, \quad g_2 \in H^1(\Omega).$$

We introduce the bilinear form:

$$B[w, v; t] = \int_{\Omega} E \nabla w \cdot \nabla v(x, y, z, t) dx dy dz + \int_{\Gamma} \sigma w v(x, y, u(x, y), t) ds$$

for $w, v \in L^2(0, T; V)$, $0 \leq t \leq T$.

Definition: A function $w \in L^2(0, T; V)$ with $w_t \in L^2(0, T; L^2(\Omega))$ and $w_{tt} \in L^2(0, T; V')$ is a weak solution of (2.1) if

- (i) $\langle w_{tt}, v \rangle + B[w, v; t] = \int_{\Omega} f(t) v dx dy dz$ for each $v \in V$ and for a.e. $0 \leq t \leq T$,
- (ii) at $t = 0$, $w = g_1$, $w_t = g_2$.

Here $\langle \cdot, \cdot \rangle$ denotes the V', V duality pairing. Note that we also have $w \in C([0, T]; L^2(\Omega))$ and $w_t \in C([0, T]; V')$ (see [3]), so (ii) makes sense.

The weak solution to the state problem (2.1) is constructed by applying Galerkin's method of finite dimensional approximations [6]. Unlike alternative methods used to prove existence here (e.g. semigroups), the Galerkin method clearly displays the regularity and the dependence on σ . The detailed derivation of these results is contained in [12].

Theorem 3.1. For $\sigma \in U_K$, the state problem has a unique weak solution w that satisfies the following regularity estimate:

$$\sup_{0 \leq t \leq T} (\|w(t)\|_V + \|w_t(t)\|_{L^2(\Omega)}) + \|w_{tt}\|_{L^2(0, T; V')} \leq C(\|g_1\|_V + \|g_2\|_{L^2(\Omega)} + \|f\|_{L^2(Q)}).$$

Sketch of the Proof. One considers a sequence of approximations w_n to the solution w

$$(3.7) \quad w_m(t) = w_m(x, y, z, t) = \sum_{k=1}^m d_k^m(t) \psi_k, m \in \mathbb{N}$$

$\{\psi_k\}_{k \in \mathbb{N}}$ is a basis for V

and

$\{\psi_k\}_{k \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega)$,

where $\{\psi_k\}$ are eigenvectors (corresponding to the eigenvalue λ_k) of the following Sturm-Liouville problem:

$$\begin{aligned} \frac{\partial \psi}{\partial \nu} + \sigma \psi &= 0 & \text{on } \Gamma \\ \psi &= 0 & \text{on } \Sigma \\ \frac{\partial \psi}{\partial \nu} &= 0 & \text{on } D \times \{z = 0\}. \end{aligned}$$

The functions $d_k^m(t)$ satisfy

$$(3.8) \quad d_k^{m''}(t) + \sum_{\ell=1}^m B[w_\ell, w_k; t] d_\ell^m(t) = \int_{\Omega} f(t) w_k d\Omega$$

and

$$(3.9) \quad d_k^m(0) = (g_1, \psi_k)_{L^2}, \quad d_k^{m'}(0) = (g_2, \psi_k)_{L^2}, \quad k = 1, \dots, m.$$

Using the orthogonality of $\{\psi_k\}$ in $L^2(\Omega)$ and the definition of w_m , one shows that w_m approximately satisfies the definition of the weak solution for $v = \psi_k$ and thus for $v = w_m$. One can then derive a priori estimates for w_m from which, by passing to the limit, one obtains convergences on subsequences. In turn, these convergences are sufficient to show that the limit w of the sequence w_m is the unique weak solution of the equation (2.1). $\square$

Additional regularity on the state solution is needed to derive the necessary conditions that an optimal control and corresponding state must satisfy. In fact, one can prove [12]:

Proposition 3.2. *The weak solution of (2.1) has additional regularity:*

$$\sup_{0 \leq t \leq T} \left[\int_{\Omega} (w_{tt})^2(x, y, z, t) d\Omega + \int_{\Omega} |\nabla w_t|^2(x, y, z, t) d\Omega + \int_{\Gamma} \sigma (w_t)^2(x, y, u(x, y), t) ds \right] \leq C$$

with C depending only on $\|f\|_{L^2}$, $\|f_t\|_{L^2}$, $\|g_1\|_{H^2}$ and $\|g_2\|_{H^1}$.

Remarks: 1. We do not get the $L^2(0, T; H^2(\Omega))$ estimate on w since we have a Robin boundary condition on the bottom with only L^∞ regularity on the coefficient σ . See [6] for this estimate for Dirichlet boundary conditions.

2. Note that we also obtain

$$\sup_{0 \leq t \leq T} \int_{\Gamma} w_t^2(x, y, u(x, y), t) ds \leq C$$

from the bound on

$$\int_{\Omega} |\nabla w_t|^2(x, y, z, t) d\Omega,$$

and Poincaré's Inequality, since

$$\begin{aligned} \int_{\Gamma} (w_t)^2(t) ds &\leq C \int_{\Omega} |\nabla w_t|^2(t) d\Omega + \int_{\Omega} (w_t)^2(t) d\Omega \\ &\leq C_1 \int_{\Omega} |\nabla w_t|^2(t) d\Omega \end{aligned}$$

and $w_t = 0$ on $\Sigma \times \{t\}$.

Based on the existence and regularity results contained in Theorem 3.1 we can now prove the existence of an optimal control.

Theorem 3.3: *There exists a control σ_{β} in U_K that minimizes the functional $J_{\beta}(\sigma)$ over U_K .*

Proof. Let $\{\sigma^n\}$ be a minimizing sequence, i.e.

$$\lim_{n \rightarrow \infty} J_{\beta}(\sigma^n) = \inf_{\sigma \in U_K} J_{\beta}(\sigma).$$

Let $w^n = w(\sigma^n)$ be the corresponding state solution. By estimates in Theorem 3.1,

$$\|w^n\|_{L^2(0,T,V)} + \|w_t^n\|_{L^2(0,T;L^2(\Omega))} + \|w_{tt}^n\|_{L^2(0,T,V')} \leq C_1$$

where the constant C_1 is independent of n . Then there exists $\sigma_{\beta} \in U_K$, $w_{\beta} \in L^2(0,T;V)$ such that, on a subsequence,

$$\begin{aligned} w^n &\rightharpoonup w_{\beta} \text{ weakly in } L^2(0,T;V) \\ w_{tt}^n &\rightharpoonup (w_{\beta})_{tt} \text{ weakly}^* \text{ in } L^2(0,T;V') \end{aligned}$$

and

$$\sigma^n \rightharpoonup \sigma_{\beta} \text{ weakly on } L^2(\Gamma).$$

Using a compactness result [27],

$$w^n \rightarrow w_{\beta} \text{ strongly in } L^2(0,T, H_{\{0\}}^{\frac{1}{2}+\varepsilon}(\Omega))$$

where the subscript $\{0\}$ means zero on ∂D , and $0 < \varepsilon < \frac{1}{2}$. By trace results,

$$w^n \rightarrow w_{\beta} \text{ strongly in } L^2(0,T; \partial\Omega),$$

which is needed for convergence of the term

$$\int_{\Gamma \times (0, T)} \sigma^n w^n \phi ds$$

in the weak form of solution w^n . Thus we can pass to the limit and conclude that $w_\beta = w(\sigma_\beta)$.

The objective functional $J_\beta(\sigma)$ is lower semi-continuous with respect to L^2 weak convergence and hence the minimum of $J_\beta(\sigma)$ is attained by σ_β . $\square$

4. Necessary Conditions

To derive necessary conditions that an optimal pair σ_β , $w_\beta = w(\sigma_\beta)$ must satisfy, we differentiate the map

$$\sigma \rightarrow J_\beta(\sigma)$$

with respect to σ . Since $J_\beta(\sigma)$ also depends on w , we first differentiate the map

$$\sigma \rightarrow w(\sigma).$$

We give the two theorems from [12] deriving the necessary conditions and briefly summarize the proofs.

Theorem 4.1. *The map*

$$\sigma \in U_K \rightarrow w(\sigma) \in L^2(0, T; V)$$

is differentiable in the following sense:

$$\frac{w(\sigma + \varepsilon \ell) - w(\sigma)}{\varepsilon} \rightarrow \xi \quad \text{in } L^2(0, T; V)$$

as $\varepsilon \rightarrow 0$, for $\sigma + \varepsilon \ell$, $\sigma \in U_K$, $\ell \in L^\infty(D)$. Furthermore, ξ is a weak solution of

$$\begin{aligned} (4.1) \quad & \xi_{tt} - \nabla(E \nabla \xi) = 0 \quad \text{in } Q \\ & \xi = 0 \quad \text{on } \Sigma \times (0, T) \\ & \frac{\partial \xi}{\partial \nu} = 0 \quad \text{on } D \times \{z = 0\} \times (0, T) \\ & \frac{\partial \xi}{\partial \nu} + \sigma \xi = -\ell w \quad \text{on } \Gamma \times (0, T) \\ & \xi = \xi_t = 0 \quad \text{on } \Omega \times \{0\} \end{aligned}$$

where $w = w(\sigma)$.

Proof: Using the notation of $w = w(\sigma)$, $w^\varepsilon = w(\sigma + \varepsilon \ell)$ and noting the added regularity of w and w^ε from Proposition 3.2, we choose $\left(\frac{w^\varepsilon - w}{\varepsilon}\right)_t$ as test function in the weak solution form of the $\frac{w^\varepsilon - w}{\varepsilon}$ equation. On $Q_s = \Omega \times (0, s)$, $\Gamma_s = \Gamma \times (0, s)$, $0 < s \leq T$, we obtain

$$\begin{aligned} & \int_{Q_s} \left(\frac{w^\varepsilon - w}{\varepsilon}\right)_{tt} \left(\frac{w^\varepsilon - w}{\varepsilon}\right)_t d\Omega dt + \int_{Q_s} E \nabla \left(\frac{w^\varepsilon - w}{\varepsilon}\right) \cdot \nabla \left(\left(\frac{w^\varepsilon - w}{\varepsilon}\right)_t\right) d\Omega dt \\ & + \int_{\Gamma_s} \sigma \left(\frac{w^\varepsilon - w}{\varepsilon}\right) \left(\frac{w^\varepsilon - w}{\varepsilon}\right)_t d\Gamma dt = - \int_{\Gamma_s} \ell w^\varepsilon \left(\frac{w^\varepsilon - w}{\varepsilon}\right)_t d\Gamma dt. \end{aligned}$$

Integrating by parts and using Cauchy-Schwarz one obtains

$$\begin{aligned} & \int_{\Gamma} \left(\left(\frac{w^\varepsilon - w}{\varepsilon}\right)_t\right)^2(s) d\Omega + \int_{\Omega} \left|\nabla \left(\frac{w^\varepsilon - w}{\varepsilon}\right)\right|^2(s) d\Omega + \int_{\Gamma} \sigma \left(\frac{w^\varepsilon - w}{\varepsilon}\right)^2(s) d\Omega \\ & \leq C \int_{\Gamma_s} (w_t^\varepsilon)^2 d\Gamma dt + C_\delta \int_{\Gamma} (w^\varepsilon)^2(s) d\Gamma + \int_{\Gamma_s} \left(\frac{w^\varepsilon - w}{\varepsilon}\right)^2 d\Gamma + \delta \int_{\Gamma} \left(\frac{w^\varepsilon - w}{\varepsilon}\right)^2(s) d\Gamma. \end{aligned}$$

Using trace estimates and Poincaré's Inequality on the right hand side, we have

$$\begin{aligned} & \int_{\Omega} \left(\left(\frac{w^\varepsilon - w}{\varepsilon}\right)_t\right)^2(s) d\Omega + (1 - \delta C) \int_{\Omega} \left|\nabla \left(\frac{w^\varepsilon - w}{\varepsilon}\right)\right|^2(s) d\Omega + \int_{\Gamma} \sigma \left(\frac{w^\varepsilon - w}{\varepsilon}\right)^2(s) d\Gamma \\ & \leq C + \int_0^s \int_{\Omega} \left|\nabla \left(\frac{w^\varepsilon - w}{\varepsilon}\right)\right|^2(t) d\Omega dt. \end{aligned}$$

Finally, choosing δ small so that $1 - \delta C > 0$ and applying Gronwall's Inequality [6] gives

$$\sup_{0 \leq t \leq T} \int_{\Omega} \left(\left(\frac{w^\varepsilon - w}{\varepsilon}\right)_t\right)^2(s) d\Omega + \int_{\Omega} \left|\nabla \left(\frac{w^\varepsilon - w}{\varepsilon}\right)\right|^2(s) d\Omega + \int_{\Gamma} \sigma \left(\frac{w^\varepsilon - w}{\varepsilon}\right)^2(s) d\Gamma \leq C_1.$$

As in the proof of Theorem 3.1, this estimate implies

$$\left\| \frac{w^\varepsilon - w}{\varepsilon} \right\|_{L^2(0, T; V')} \leq C$$

and on a subsequence, as $\varepsilon \rightarrow 0$,

$$\begin{aligned} \frac{w^\varepsilon - w}{\varepsilon} & \rightarrow \xi \quad \text{in } L^2(0, T; V) \\ \left(\frac{w^\varepsilon - w}{\varepsilon}\right)_{tt} & \rightarrow \xi_{tt} \quad \text{in } L^2(0, T; V') \\ \left(\frac{w^\varepsilon - w}{\varepsilon}\right)_t & \rightarrow \xi_t \quad \text{in } L^2(0, T; L^2(\Omega)) \\ \frac{w^\varepsilon - w}{\varepsilon} & \rightarrow \xi \quad \text{in } L^2(0, T; L^2(\Gamma)). \end{aligned}$$

Also ξ satisfies (4.1) in the weak sense:

$$\langle \xi_{tt}, v \rangle + B[\xi, v; t] = - \int_{\Gamma} \ell w v(t) d\Gamma, \quad v \in V, \quad \text{a.e. } t. \quad \square$$

Next, we state the necessary conditions:

Theorem 4.2. *Given an optimal control σ_{β} and a corresponding state $w_{\beta} = w(\sigma_{\beta})$, there exists an adjoint solution p in $L^2(0, T; V)$, solving in a weak sense the adjoint system:*

$$(4.3) \quad \begin{aligned} p_{tt} - \nabla(E \nabla p) &= (w_{\beta} - h) \chi_G \quad \text{in } Q \\ p &= 0 \quad \text{on } \Sigma \times (0, T) \\ \frac{\partial p}{\partial \nu} &= 0 \quad \text{on } D \times \{z = 0\} \times (0, T) \\ \frac{\partial p}{\partial \nu} + \sigma_{\beta} p &= 0 \quad \text{on } \Gamma \times (0, T) \\ p = p_t &= 0 \quad \text{on } \Omega \times \{T\}. \end{aligned}$$

Furthermore,

$$\sigma_{\beta}(x, y) = \min \left\{ \left(\frac{1}{\beta} \int_0^T w_{\beta} p(x, y, u(x, y), t) dt \right)^+, K \right\}.$$

Proof: Since $(w_{\beta} - h) \chi_G \in L^2(Q)$, we can obtain the existence and uniqueness of the weak solution p by arguments like in Theorem 3.1.

Suppose $\sigma_{\beta} + \varepsilon \ell \in U_K$ for ε small. Consider the directional derivative of J with respect to σ at σ_{β} in the direction ℓ :

$$\begin{aligned} 0 &\leq \lim_{\varepsilon \rightarrow 0^+} \frac{J_{\beta}(\sigma_{\beta} + \varepsilon \ell) - J_{\beta}(\sigma_{\beta})}{\varepsilon} \\ &= \int_{G \times (0, T)} \xi (w_{\beta} - h) dG dt + \beta \int_{\Gamma} \sigma_{\beta} \ell d\Gamma \\ &= \int_0^T \langle p_{tt}, \xi \rangle dt + \int_0^T B[p, \xi; t] dt + \beta \int_{\Gamma} \sigma_{\beta} \ell d\Gamma \\ &= - \int_{\Gamma \times (0, T)} \ell w_{\beta} p d\Gamma dt + \beta \int_{\Gamma} \sigma_{\beta} \ell d\Gamma \\ &= \int_{\Gamma} \ell(x, y) \left(- \int_0^T w_{\beta} p(x, y, u(x, y), t) dt + \beta \sigma_{\beta}(x, y) \right) d\Gamma \end{aligned}$$

by using the adjoint equation (4.3) and then using the equation (4.1) satisfied by ξ . Standard control arguments give

$$\sigma_\beta(x, y) = \min \left\{ \left(\frac{1}{\beta} \int_0^T w_\beta p(x, y, u(x, y), t) dt \right)^+, K \right\}. \quad \square$$

5. Uniqueness of the Optimal Control

Uniqueness was not addressed in [12]; we now prove the following result:

Theorem 5.1. *For T sufficiently small, the solution of the optimality system (OS) is unique:*

$$\begin{aligned} w_{tt} - \nabla(E\nabla w) &= f && \text{in } Q \\ p_{tt} - \nabla(E\nabla p) &= (w - h)\chi_G && \text{in } Q \\ w = p &= 0 && \text{on } \Sigma \times (0, T) \\ \frac{\partial w}{\partial \nu} = \frac{\partial p}{\partial \nu} &= 0 && \text{on } D \times \{z = 0\} \times (0, T) \\ w = g_1, \quad w_t &= g_2 && \text{on } \Omega \times \{0\} \\ p = p_t &= 0 && \text{on } \Omega \times \{T\} \\ \frac{\partial w}{\partial \nu} + \min \left\{ \frac{1}{\beta} \left(\int_0^T wp(x, y, u(x, y), t) dt \right)^+, K \right\} w &= 0 && \text{on } \Gamma \times (0, T) \\ \frac{\partial p}{\partial \nu} + \min \left\{ \frac{1}{\beta} \left(\int_0^T wp(x, y, u(x, y), t) dt \right)^+, K \right\} p &= 0 && \text{on } \Gamma \times (0, T). \end{aligned}$$

Proof. Suppose w, p and $\bar{w}, \bar{p}$ are solutions of (OS) system. Let $\hat{w} = w - \bar{w}$, $\hat{p} = p - \bar{p}$. Consider the $\hat{w}$ equation on $\Omega \times (0, t)$, $0 < t \leq T$, and obtain

$$(5.1) \quad \int_\Omega [(\hat{w}_t(t))^2 + |\nabla \hat{w}(t)|^2] d\Omega + \int_\Gamma \sigma \hat{w}(t)^2 d\Gamma = -2 \int_0^t \int_\Gamma (\sigma - \bar{\sigma}) \bar{w} \hat{w}_t d\Gamma d\tau$$

where

$$\sigma(x, y) = \min \left\{ \frac{1}{\beta} \left(\int_0^T wp(x, y, u(x, y), t) dt \right)^+, K \right\}$$

and

$$\bar{\sigma}(x, y) = \min \left\{ \frac{1}{\beta} \left(\int_0^T \bar{w} \bar{p}(x, y, u(x, y), t) dt \right)^+, K \right\}$$

Similarly on $\Omega \times (t, T)$, $0 < t \leq T$, in the adjoint case, we obtain

$$(5.2) \quad \begin{aligned} & \int_{\Omega} [(\hat{p}_t(t))^2 + |\nabla \hat{p}(t)|^2] d\Omega + \int_{\Gamma} \sigma(\hat{p}(t))^2 d\Gamma \\ &= -2 \int_0^T \int_G \hat{w} \hat{p}_t dG d\tau + 2 \int_t^T \int_{\Gamma} (\sigma - \bar{\sigma}) \bar{p} \hat{p}_t d\Gamma d\tau. \end{aligned}$$

Integration by parts gives

$$(5.3) \quad \int_t^T \int_{\Gamma} (\sigma - \bar{\sigma}) \bar{p} \hat{p}_t d\Gamma d\tau = - \int_t^T \int_{\Gamma} (\sigma - \bar{\sigma}) \bar{p}_t \hat{p} d\Gamma d\tau - \int_{\Gamma} (\sigma - \bar{\sigma}) \bar{p} \hat{p}(t) d\Gamma.$$

Continuing to estimate, we have

$$\begin{aligned} & \int_t^T \int_{\Gamma} |\sigma - \bar{\sigma}| |\bar{p}_t| |\hat{p}| d\Gamma d\tau \\ & \leq \frac{1}{2} \int_t^T \int_{\Gamma} (\sigma - \bar{\sigma})^2 d\Gamma d\tau + \frac{1}{2} \int_t^T \int_{\Gamma} |\bar{p}_t|^2 |\hat{p}|^2 d\Gamma d\tau \\ & \leq C \int_t^T \int_{\Gamma} \int_0^T \bar{w}^2 \hat{p}^2 + p^2 \hat{w}^2 d\tau_1 d\Gamma d\tau + \frac{1}{2} \int_t^T \left(\int_{\Gamma} (\bar{p}_t)^4 d\Gamma \right)^{\frac{1}{2}} \left(\int_{\Gamma} \hat{p}^4 d\Gamma \right)^{\frac{1}{2}} d\tau \\ & \leq C \int_t^T \int_0^T \left(\int_{\Gamma} \bar{w}^4(\tau_1) d\Gamma \right)^{\frac{1}{2}} \left(\int_{\Gamma} \hat{p}^4(\tau_1) d\Gamma \right)^{\frac{1}{2}} d\tau_1 d\tau \\ & \quad + \int_t^T \int_0^T \left(\int_{\Gamma} p^4(\tau_1) d\Gamma \right)^{\frac{1}{2}} \left(\int_{\Gamma} \hat{w}^4(\tau_1) d\Gamma \right)^{\frac{1}{2}} d\tau_1 d\tau + C_2 \int_t^T \left(\int_{\Gamma} \hat{p}^4 ds \right)^{\frac{1}{2}} d\tau \end{aligned}$$

where we used that $\bar{p}$ satisfies the additional regularity like in Proposition 3.2 and

$$\bar{p}_t(t) \in H^1(\Omega) \hookrightarrow L^4(\Gamma).$$

If a similar estimate on $\bar{w}, p$ is used, one obtains

$$\int_t^T |\sigma - \bar{\sigma}| |\bar{p}_t| |\hat{p}| d\Gamma d\tau \leq CT \sup_{0 \leq t \leq T} \left(\|\nabla \hat{p}(t)\|_{L^2(\Omega)}^2 + \|\nabla \hat{w}(t)\|_{L^2(\Omega)}^2 \right).$$

For $\epsilon > 0$, estimate the other type of boundary term in (5.3)

$$\begin{aligned} & \int_{\Gamma} |\sigma - \bar{\sigma}| |\bar{p}| |\hat{p}|(t) d\Gamma \\ & \leq C_{\epsilon} \int_{\Gamma} (\sigma - \bar{\sigma})^2 d\Gamma + \epsilon \int_{\Gamma} |\bar{p}|^2 |\hat{p}|^2(t) d\Gamma \\ & \leq \hat{C}_{\epsilon} T \sup_{0 \leq t \leq T} \left[\|\nabla \hat{p}(t)\|_{L^2(\Omega)}^2 + \|\nabla \hat{w}(t)\|_{L^2(\Omega)}^2 \right] + \epsilon \|\nabla \hat{p}(t)\|_{L^2(\Omega)}^2, \end{aligned}$$

using similar imbeddings involving $L^4(\Gamma)$.

Taking the supremum over t in the adjoint estimate, gives

$$\begin{aligned} & \sup_{0 \leq t \leq T} \left(\|\hat{p}_t(t)\|_{L^2(\Omega)}^2 + \|\nabla \hat{p}(t)\|_{L^2(\Omega)}^2 + \int_{\Gamma} \sigma \hat{p}^2(t) d\Gamma \right) \\ & \leq \bar{C}_\epsilon T \sup_{0 \leq t \leq T} \left(\|\nabla \hat{p}(t)\|_{L^2(\Omega)}^2 + \|\nabla \hat{w}(t)\|_{L^2(\Omega)}^2 + \|\hat{w}_t(t)\|_{L^2(\Omega)}^2 + \|\hat{p}_t(t)\|_{L^2(\Omega)}^2 \right) \\ & \quad + \epsilon \sup_{0 \leq t \leq T} \left(\|\nabla \hat{p}(t)\|_{L^2(\Omega)}^2 + \|\nabla \hat{w}(t)\|_{L^2(\Omega)}^2 \right). \end{aligned}$$

We have a similar estimate for $\hat{w}$.

Adding the supremum estimates for $\hat{p}, \hat{w}$ and taking ϵ small enough and then T small so that

$$2(\bar{C}_\epsilon T + \epsilon) < 1,$$

we obtain

$$\sup_{0 \leq t \leq T} \left(\|\hat{p}_t(t)\|_{L^2(\Omega)}^2 + \|\nabla \hat{p}(t)\|_{L^2(\Omega)}^2 + \|\hat{w}_t(t)\|_{L^2(\Omega)}^2 + \|\nabla \hat{w}(t)\|_{L^2(\Omega)}^2 \right) \leq 0$$

which gives uniqueness. $\square$

6. Approximate Solution of the Parameter Identification Problem

Having solved the optimal control problem, we now use a sequence of optimal controls to approximate the solution of the parameter identification problem. We view h as an observation of the solution w on $G \times (0, T)$ and seek to identify a reflection coefficient σ_0 such that the solution $w = w(\sigma_0)$ satisfies

$$w|_{G \times (0, T)} = h.$$

Assuming the existence of a solution of this inverse problem, we will approximate σ_0 by the optimal control σ_β as β gets small.

Theorem 6.1. *Suppose the inverse problem has a solution, i.e. there exists $\sigma^* \in U_K$ such that $w^* = w(\sigma^*)$ satisfies $w^* = h$ a.e. on $G \times (0, T)$. Then there exists $0 \leq \sigma_0 \leq K$ such that on a subsequence of $\beta \rightarrow 0$,*

$$\begin{aligned} \sigma_\beta & \rightarrow \sigma_0 \quad \text{in } L^2(\Gamma) \\ w_\beta = w(\sigma_\beta) & \rightarrow w_0 \quad \text{in } L^2(0, T; V) \end{aligned}$$

and

$$w_0 = h \quad \text{a.e. on } G \times (0, T).$$

Proof. Notice that $\sigma^* \in U_K$ and let σ_β be an optimal control that minimizes $J_\beta(\sigma)$. Then we have

$$J_\beta(\sigma_\beta) \leq J_\beta(\sigma^*)$$

and

$$(6.1) \quad \frac{1}{2} \int_{G \times (0, T)} (w_\beta - h)^2 dG dt + \frac{\beta}{2} \int_{\Gamma} \sigma_\beta^2 d\Gamma \leq \frac{\beta}{2} \int_{\Gamma} (\sigma^*)^2 d\Gamma.$$

Using the above a priori estimates on w_β , which are independent of β , for a (sub)sequence $\beta \rightarrow 0$, we have

$$\begin{aligned} w_\beta &\rightharpoonup w_0 \quad \text{in } L^2(0, T; V) \\ (w_\beta)_{tt} &\rightharpoonup (w_0)_{tt} \quad \text{in } L^2(0, T; V') \\ (w_\beta)_t &\rightharpoonup (w_0)_t \quad \text{in } L^2(0, T; L^2(\Omega)) \\ \sigma^\beta &\rightharpoonup \sigma_0 \quad \text{in } L^2(\Gamma) \end{aligned}$$

and

$$w_0 = w(\sigma_0).$$

Letting $\beta \rightarrow 0$ in (6.1), we also obtain

$$\int_{G \times (0, T)} (w_0 - h)^2 dG dt = 0.$$

Thus w_0 has the desired property, i.e., $w = h$ a.e. on $G \times (0, T)$, and σ_0 is the coefficient that is "identified". If the solution of the inverse problem, σ^* , is unique, then $\sigma_0 = \sigma^*$. $\square$

7. Discussion

Optimal control techniques applied to an acoustic wave equation provide a systematic method to approximate a boundary reflection coefficient, σ , that yields a solution consistent with the actual observations. By the same procedure, we also have an explicit characterization of the approximations σ_β .

In the previous discussion we did not take into account causality effects that are due to the finite speed of propagation of signals in hyperbolic systems. Indeed, if we suppose that: (i) at $u = u_t = 0$ at $t = 0$, (ii) the source is localized, and (iii) the receptors are far from the source, then there will be a time interval during which both the measured signal and the model solution are zero at the receptor location. Thus, for that time interval (whose length will depend on the signal speed and the actual geometric configuration), the first

term in the functional (1.3) is zero, which means that the minimum of J is realized for $\sigma = 0$, independently of the value of the actual reflection coefficient at the boundary.

In order to obtain meaningful results, the time interval should be large enough to allow signals reflected by the boundary to actually reach the receptors and overcome the spurious threshold effects. Of course, this may hamper the uniqueness result which, as noted, is valid only for short times.

Another source of meaningless results is related to model inadequacies and/or the presence of noise. Indeed, suppose the model does not predict any (reflected) signal at the receptor, i.e. $u = 0$, while the actual measured signal, h , is different from zero (or vice-versa). Again the minimizer is given by the (most likely wrong) result $\sigma = 0$, independently of its actual value or functional form. This spurious result occurs because our method assumes the model is completely and correctly known. The knowledge of the model is paramount to this parameter identification method and any model imperfection will be reflected - sometimes in the form of severe anomalies - in the identified parameters.

Finally, we expect the quality of the recovery to increase with the amount and quality of the available information. In other words, the larger G is the better the results are expected to become. However this relationship cannot be expressed by a simple monotonic dependence.

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