TITLE: ONE—AND TWO-DIMENSIONAL WAVE FRONTS IN DIFFUSIVE
SYSTEMS WITH DISCRETE SETS OF NONLINEAR SOURCES

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JUN 18 1998
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SUBMITTED TO Conference Proceedings on 18th Annual International Conference on
“Predictability: Quantifying Uncertainty in Models of Complex
Phenomena” on May 11-15, 1998 at Los Alamos, NM

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One- and Two-Dimensional Wave Fronts in Diffusive Systems with Discrete Sets of Nonlinear Sources

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(September 1, 1998)

We study the dynamics of one- and two-dimensional diffusion systems with sets of discrete nonlinear sources. We show that wave fronts propagating in such systems are pinned if the diffusion constant is below a critical value which corresponds to a saddle-node bifurcation of the dynamics. In two dimensions we find that the dissipation is enhanced and moving plain and circular fronts are stable with respect to any perturbations.

Keywords: discrete source, burst waves, front propagation, bistable system

PACS: 47.54.+r, 87.22.As, 82.40.Ck, 47.20.Ky

I. INTRODUCTION

Discrete nonlinear sources are found in many dynamical systems including crystal dislocations [1], nonlinear crystal lattices [2], chains of Josephson junctions [3], charge density waves in one-dimensional strongly correlated electron systems [3-6], calcium waves in living cells [7,8], *etc.* One of the most important questions for this type of systems is propagation of field fronts and their pinning. Models describing field continuously diffusing between localized sources (see, *e.g.*, Ref. [4]) have been usually replaced by fully discrete ones, neglecting the field dynamics between sites [3,5]. This simplification, however, does not allow for explanation of the mechanism of one-dimensional front instability. The field structure between sites is also necessary for understanding of the dissipation effect on two-dimensional structures.

In the present paper we study propagation and pinning of one- and two-dimensional diffusive fronts using a model which consists of linear diffusion equation with discrete set of nonlinear sources (sites):

$$u_t = D \nabla^2 u + \alpha d^N \sum_i \delta(\mathbf{x} - \mathbf{x}_i) f(u), \quad (1)$$

where u is a dimensionless field, D is the diffusion coefficient, α is the inverse time scale of the source dynamics, and d is the distance between adjacent sites. Points $\mathbf{x}_i$'s represent the locations of the sites, and $N = 1, 2$ is the space dimension. The nonlinear source dynamics are given by the function $f(u)$. For propagation of fronts separating one stable phase from another we used bistable source dynamics (see, *e.g.*, [9]) which in the simplest case is given by

$$f(u) = -u(u - u_0)(u - 1), \quad (2)$$

with two stable fixed points $u = 0, 1$, and one unstable fixed point $u = u_0$.

II. ONE-DIMENSIONAL FRONT DYNAMICS

For one-dimensional system we rescale $x = \tilde{x}d$, $t = \tilde{t}\alpha^{-1}$, and obtain from Eq. (1) (after dropping tildes),

$$u_t = \beta u_{xx} + \sum_i \delta(x - i) f(u), \quad (3)$$

with the effective dimensionless diffusion coefficient $\beta = D/\alpha d^2$. The source dynamics potential $F(u)$ is defined as $f = -dF/du$. The potential $F(u)$ for bistable dynamics possesses two minima (stable fixed points), separated by a maximum (unstable fixed point). For the cubic forcing (2) the two minima are at $u = 0, 1$, and the maximum is at $u = u_0$. The front propagation is determined by the difference between the depths of the minima which, for (2), is parameterized by $\gamma = 1/2 - u_0$.

The dimensionless equation (3) reaches its continuous limit when the effective diffusion coefficient $\beta \rightarrow \infty$:

$$u_t = \beta u_{xx} + f(u). \quad (4)$$

Eq. (4) possesses traveling-wave kink solution propagating from locally stable minimum to globally stable minimum [9] which, for small γ , reads as

$$u = \frac{1}{2} \left[1 - \tanh \left(\frac{x - Ct}{2\sqrt{2\beta}} \right) \right] + O(\gamma), \quad (5)$$

where the propagation velocity C scales, for small γ , as [9-11]

$$C \sim \gamma \sqrt{\beta}. \quad (6)$$

The subject of our interest in this work is the dynamics in the discrete limit, $\beta \ll 1$. We have solved Eq. (3) numerically using the following technique. First, we have multiplied the equation by $G(x, x')$, the Green-function of one-dimensional Laplace operator with no-flux boundary condition $u_x = 0$ at $x = 0$ and Dirichlet boundary condition $u = 0$ at $x = L$, which has the form

$G(x, x') = (x - x')H(x - x') + x' - L$, with H being a step-function. We then integrated the product over the entire domain length L . The resulting transformed equation is

$$U_i(x, t) = \beta u(x, t) + \sum_i G(x, i) f(u_i), \quad (7)$$

where $U(x, t) = \int_0^L G(x, x') u(x', t) dx$ and $u_i = u(x_i)$. Eq. (7), unlike Eq. (3), is well-defined. At each time step we integrated Eq. (7) implicitly to find function U , and then used the inverse Green-function G^{-1} to calculate the original field u , which was then substituted back to the equation. This method was checked on the convergence with spatial grid refinement.

We have found that, for small β , the front dynamics result in burst waves, see Figure 1. Such a wave is characterized by the time periodicity in the frame moving along with the front. Four successive profiles in the figure represent instant shots of the solution $u(x)$ during one bursting period. We have shown that, if β is below some critical value β_c , the wave no longer propagates and relaxes to a stationary kink. This pinned solution of Eq. (3) obtained numerically is shown in Fig. 2.

FIG. 1. Instant profiles $u(x)$ in the burst-wave regime obtained by numerical solution of Eq. (7). Parameters of our model are $\beta = 0.017$, $\gamma = 0.3$ ($u_0 = 0.2$). System length is $L = 20$, number of grid-points is 1000. Time step $dt \approx 0.007$. The simulation runs within one bursting period, $T = 35.28$.

Let us analyze the stationary solution and its stability. The stationary solution of Eq. (3) shown in Fig. 2 is a piece-wise linear curve connecting the stable states $u = 1$ and $u = 0$ (see Fig 2), since the solution of one-dimensional Laplace equation must be linear and second derivative at the sites must have the following form $(u_i)_{xx} = \sigma_i \delta(x - i)$, where σ_i is a gap of u_x at site i . This reduces the problem to a set of algebraic equations on the values u_i at the sites

$$\beta(u_{i+1} + u_{i-1} - 2u_i) + f(u_i) = 0. \quad (8)$$

The front site in Fig. 2 is the first site from the left for which $\sigma_i > 0$, with the site number k . For $\beta \ll 1$ it can be shown that u_i 's approach 1 or 0 exponentially as the site number decreases or increases away from the front, respectively: $u_{k+i} \sim \beta^i$ and $1 - u_{k-i} \sim \beta^i$.

FIG. 2. Stationary (pinned) kink solution of Eq. (3). The diffusion constant is $\beta = 0.011$. The rest of the parameters are the same as in Fig. 1.

To the first order in β , Eq. (8) for the front site reads

$$g(u_k) \equiv u_k[u_k^2 - u_k(u_0 + 1) + u_0 + 2\beta] - \beta = 0. \quad (9)$$

In Figure 3 we analyze the solution of Eq (9). As we have already mentioned, $\sigma_k > 0$ and, therefore, $u_k < u_0$ [see Eq. (8)]. Then only two roots, $u_k = u^\pm$ in the figure, could be the front value. The value of β , for which these roots merge at the graph maximum $u = u_{max}$, is the critical value of the diffusion coefficient, β_c . For $\beta > \beta_c$ no stationary solution exists.

To find the stability (to the first order in β) of a stationary kink solution of Eq. (3), u_i^* ($i = 1, 2, \dots$), shown in Fig. 2, one has to linearize system (8) for slightly perturbed stationary solution $u_i = u_i^* + \delta u$. The obtained maximum eigenvalue then reads as

$$\lambda_{max} = f_u(u_k) - 2\beta + O(\beta^2). \quad (10)$$

This eigenvalue, when taken at two roots of Eq. (9), $u_k = u^-$ and $u_k = u^+$, is negative and positive, respectively. Since all remaining eigenvalues are negative, $u = u^+$ is a saddle and $u = u^-$ is a node (see Fig. 3). If we substitute $\beta = \beta_c$ to Eq. (10), we will have, using Eq. (9), that $\lambda_{max} = 0$. This implies that the boundary of stationary solution stability coincides with that of its existence, that is the depinning instability mechanism is a saddle-node bifurcation. We can, therefore, assume that the average

velocity C of moving front slightly above the instability threshold scales with the bifurcation parameter $\beta - \beta_c$ as $\sqrt{\beta - \beta_c}$. In Fig. 4 we plot the velocity C as a function of $\beta - \beta_c$ obtained numerically from Eq. (7). We clearly see $C \sim \sqrt{\beta - \beta_c}$ scaling for $\beta \ll 1$ which, after a transient domain of β , turns into the continuous limit scaling (6).

FIG. 3. A graphical representation of $g(u_k)$ used to solve the stationary problem (8) to the first order in β . Parameter $\gamma = 0.1$ ($u_0 = 0.4$). Stable (unstable) front solution corresponds to $u_k = u^-$ (u^+).

FIG. 4. Wave propagation velocity C vs $\beta - \beta_c$ (solid line). Dashed lines with slope 1/2 show the scalings $C \sim \sqrt{\beta - \beta_c}$ and $C \sim \sqrt{\beta}$ for discrete and continuous domain, respectively. Dashed-dotted lines illustrate the boundaries of continuous, discrete, and transient domains of β . Parameters are $\gamma = 0.1$, $\beta_c \approx 0.0617$.

We have studied analytically the dynamics of bursting waves in the space of parameters β and γ of our model.

We have obtained the bifurcation line of the pinning-depinning instability by solving system (8) to second order in β . This was done similarly to the first order calculations described above. Thus we do not present here the details of this straightforward but tedious procedure. We compare the analytical bifurcation line with that obtained by numerical simulations of Eq. (7) in Figure 5. As one can see from the Figure, the prediction of the theory is in good quantitative agreement with the simulations, even *beyond* the applicability of the perturbation theory about small β (far from the discrete limit). For $\gamma \rightarrow 0$ the energy source of wave motion vanishes and consequently β_c diverges.

FIG. 5. Bifurcation line $\beta = \beta_c(\gamma)$ for pinning-depinning instability in the plane $\beta - \gamma$ of our model parameters. In the region above the line burst waves propagate and below the line they are pinned.

III. TWO-DIMENSIONAL FRONTS

We have studied the dynamics of two-dimensional fronts and their stability. In two dimensions Eq. (1) can be rescaled, using the same scaling as in one dimension ($x = \tilde{x}d$, $t = \tilde{t}\alpha^{-1}$), to the following analogue of Eq. (3)

$$u_t = \beta \nabla^2 u + \sum_{i,j} \delta(x - i) \delta(y - j) f(u). \quad (11)$$

We have performed numerical simulations of Eq. (11) using technique similar to that described above, for simulating Eq. (3). As for the one-dimensional case, we have also found the pinning effect of the front. The obtained values of $\beta_c(\gamma)$ for two-dimensional plain fronts appear to be larger than the one-dimensional $\beta_c(\gamma)$, in the range of the parameter γ for which $\beta_c \ll 1$, i.e., in the discrete limit. In the continuous limit, $\beta_c \gg 1$, the one- and two-dimensional critical diffusion constants are virtually equal. This can be explained by the fact that for

the discrete dynamics the two-dimensional diffusion is “stronger” and, therefore, two-dimensional fronts need more energy to propagate, while in the continuous limit the propagation of a two-dimensional plain front is equivalent to that of a one-dimensional front.

FIG. 6. The successive two-dimensional plots of initially perturbed plain front obtained by numerical simulations of Eq. (11). Black corresponds to $u = 1$ and white to $u = 0$. Parameters of the simulations are $\beta = 0.08$, $\gamma = 0.3$, $L = 40$, $dt = 0.001$. Number of grid-points is 400. Times of the plots are (a) 0; (b) 50; (c) 100; (d) 200.

In addition to a completely plain front, we have studied the dynamics of various two-dimensional patterns. First, for a perturbed plain front, we have found that the perturbation always decays with time, that is the front tends to restore its plain shape. In Fig. 6 we show the evolution of a perturbed plain front. We have also simulated the dynamics of a closed island of phase $u = 1$. We have started with various initial shapes but the island always relaxed to a circle, see Fig. 7. Such behavior implies that fronts are stable with respect to a perturbation of all nonzero harmonics. This is similar to the dynamics of continuous bistable systems with global constraints (see [12] and references therein).

IV. CONCLUSIONS

We have studied the dynamics of wave fronts in diffusive systems with discrete bistable sources. We have demonstrated the pinning effect of the fronts and found that the mechanism of depinning instability is a saddle-

node bifurcation. The moving one-dimensional fronts represent burst waves which are periodic in properly selected moving reference frame. For the two-dimensional system we have found that the pinning effect is enhanced and the propagating plain and circular fronts are stable with respect to geometric perturbations. In a future work we are planning to pursue the analytic study of two-dimensional fronts and their stability.

FIG. 7. The successive two-dimensional plots of initially noncircular closed island of phase $u = 1$. The diffusion is $\beta = 0.04$. The rest of the parameters are the same as in Fig. 6. Times of the plots are (a) 0; (b) 40; (c) 100; (d) 200.

We have obtained all our results for one-variable bistable systems. We expect these results to hold qualitatively for bistable systems with more variables. However, for discrete source systems with more complicated dynamics, the front behavior could be different. For instance, for “fire-diffuse-fire” model of calcium dynamics [8] no pinning effect was found. Our intention is to generalize the approach developed in the present work to other discrete source systems with more dynamic variables.

Acknowledgments. We are thankful to I. Aranson, K. Kladko, and J. E. Pearson for clarifying discussions.

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