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FIELD THEORY AND SCHWINGER'S CLOSED
TIME PATH FORMALISM**

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NONEQUILIBRIUM PROBLEMS IN QUANTUM FIELD THEORY AND SCHWINGER'S CLOSED TIME PATH FORMALISM

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INTRODUCTION

We review the closed time path formalism of Schwinger using a path integral approach. We apply this formalism to the study of pair production from strong external fields as well as the time evolution of a nonequilibrium chiral phase transition.

In 1961 in his classic paper "Brownian Motion of a Quantum Particle"¹, Schwinger solved the formidable technical problem of how to use the action principle to study initial value problems. Previously, the action principle was formulated to study only transition matrix elements from an earlier time to a later time. The elegant solution of this problem was the invention of the closed time path (CTP) formalism. This formalism was first used to study field theory problems by Mahanthappa and Bakshi¹.

With the advent of supercomputers, it has now become possible to use this formalism to numerically solve important field theory questions which are presented as initial value problems. Two of these problems we shall review here. They are

1. The time evolution of the quark-gluon plasma².
2. Dynamical evolution of a non-equilibrium chiral phase transition following a relativistic heavy ion collision³.

The basic idea of the CTP formalism is to take a diagonal matrix element of the system at a given time $t = 0$ and insert a complete set of states into this matrix element at a different (later) time t' . In this way one can express the original fixed time matrix element as a product of transition matrix elements from 0 to t' and the time reversed (complex conjugate) matrix element from t' to 0. Since each term in this product is a transition matrix element of the usual or time reversed kind, standard path integral representations for each may be introduced. If the same external source operates in

the forward evolution as the backward one, then the two matrix elements are precisely complex conjugates of each other, all dependence on the source drops out and nothing has been gained. However, if the forward time evolution takes place in the presence of one source J_+ but the reversed time evolution takes place in the presence of a *different* source J_- , then the resulting functional is precisely the generating functional we seek.

Fig. 1

Complex time contour $\mathcal{C}$ for the closed time path propagators.

Indeed

$$\begin{aligned}
 Z_{in}[J_+, J_-] &\equiv \int [\mathcal{D}\Psi] \langle in | \psi \rangle_{J_-} \langle \psi | in \rangle_{J_+} \\
 &= \int [\mathcal{D}\Psi] \langle in | \mathcal{T}^* \exp \left[-i \int_0^{t'} dt J_-(t) \phi(t) \right] | \Psi, t' \rangle \times \\
 &\quad \langle \Psi, t' | \mathcal{T} \exp \left[i \int_0^{t'} dt J_+(t) \phi(t) \right] | in \rangle
 \end{aligned} \tag{1}$$

so that, for example,

$$\left. \frac{\delta W_{in}[J_+, J_-]}{\delta J_+(t)} \right|_{J_+=J_-=0} = - \left. \frac{\delta W_{in}[J_+, J_-]}{\delta J_-(t)} \right|_{J_+=J_-=0} = \langle in | \phi(t) | in \rangle \tag{2}$$

is a true expectation value in the given time-independent Heisenberg state $|in\rangle$. Here $\phi(t) = \phi(x, t)$ and we are suppressing the coordinate dependence and the integration over the spatial volume in what follows for notational simplicity.

Since the time ordering in eq. (1) is forward (denoted by $\mathcal{T}$) along the time path from 0 to t' in the second transition matrix element, but backward (denoted by $\mathcal{T}^*$) along the path from t' to 0 in the first matrix element, thus the name: *closed time path generating functional*. If we deform the backward and forward directed segments of the path slightly in opposite directions in the complex t plane, the symbol $\mathcal{T}_C$ may be introduced for path ordering along the full closed time contour, C depicted in Fig.1.

This deformation of the path corresponds precisely to opposite $i\epsilon$ prescriptions along the forward and backward directed segments, which we shall denote by $C_\pm$ respectively in the following.

If we have an arbitrary initial density matrix ρ then we have instead:

$$\begin{aligned} Z[J_+, J_-, \rho] &\equiv \text{Tr} \left\{ \rho \left(\mathcal{T}^* \exp \left[-i \int_0^{t'} dt J_-(t) \phi(t) \right] \right) \left(\mathcal{T} \exp \left[i \int_0^{t'} dt J_+(t) \phi(t) \right] \right) \right\} \\ &= \int [\mathcal{D}\varphi][\mathcal{D}\varphi'] [\mathcal{D}\psi] \langle \varphi | \rho | \varphi' \rangle \langle \varphi' | \mathcal{T}^* \exp \left[-i \int_0^{t'} dt J_-(t) \phi(t) \right] | \psi \rangle \\ &\quad \times \langle \psi | \mathcal{T} \exp \left[i \int_0^{t'} dt J_+(t) \phi(t) \right] | \varphi \rangle . \end{aligned} \quad (3)$$

Variations of this generating function will yield Green's functions in the state specified by the initial density matrix, *i.e.* expressions of the form,

$$\text{Tr} \{ \rho \phi(t_1) \phi(t_2) \phi(t_3) \dots \} . \quad (4)$$

Introducing the path integral representation for each transition matrix element in eq. (3) results in the expression,

$$\begin{aligned} Z[J_+, J_-, \rho] &= \int [\mathcal{D}\varphi][\mathcal{D}\varphi'] \langle \varphi | \rho | \varphi' \rangle \int [\mathcal{D}\psi] \int_\varphi^\psi [\mathcal{D}\phi_+] \int_{\varphi'}^\psi [\mathcal{D}\phi_-] \times \\ &\quad \exp \left[i \int_0^\infty dt (L[\phi_+] - L[\phi_-] + J_+ \phi_+ - J_- \phi_-) \right] , \end{aligned} \quad (5)$$

where L is the classical Lagrangian functional, and we have taken the arbitrary future time at which the time path closes $t' \rightarrow \infty$.

The double path integral over the fields ϕ_+ and ϕ_- in (5) suggests that we introduce a two component contravariant vector of field variables by

$$\phi^a = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} ; \quad a = 1, 2 \quad (6)$$

with a corresponding two component source vector,

$$J^a = \begin{pmatrix} J_+ \\ J_- \end{pmatrix} ; \quad a = 1, 2 . \quad (7)$$

Because of the minus signs in the exponent of (5), it is necessary to raise and lower indices in this vector space with a 2×2 matrix with indefinite signature, namely

$$c_{ab} = \text{diag} (+1, -1) = c^{ab} \quad (8)$$

so that, for example

$$J^a c_{ab} \Phi^b = J_+ \phi_+ - J_- \phi_- . \quad (9)$$

These definitions imply that the correlation functions of the theory will exhibit a matrix structure in the 2×2 space. For instance, the matrix of connected two point functions in the CTP space is

$$G^{ab}(t, t') = \frac{\delta^2 W}{\delta J_a(t) \delta J_b(t')} \Big|_{J=0} . \quad (10)$$

Explicitly, the components of this 2×2 matrix are

$$\begin{aligned} G^{21}(t, t') &\equiv G_>(t, t') = i \text{Tr} \{ \rho \Phi(t) \bar{\Phi}(t') \}_{con} , \\ G^{12}(t, t') &\equiv G_<(t, t') = i \text{Tr} \{ \rho \bar{\Phi}(t') \Phi(t) \}_{con} \\ G^{11}(t, t') &= i \text{Tr} \{ \rho \mathcal{T}[\Phi(t) \bar{\Phi}(t')] \}_{con} = \theta(t, t') G_>(t, t') + \theta(t', t) G_<(t, t') \\ G^{22}(t, t') &= i \text{Tr} \{ \rho \mathcal{T}^*[\Phi(t) \bar{\Phi}(t')] \}_{con} = \theta(t', t) G_>(t, t') + \theta(t, t') G_<(t, t') \end{aligned} \quad (11)$$

Notice that

$$G^{11}(t, t) = G^{22}(t, t) \quad (12)$$

with the usual convention that

$$\theta(t, t) = \frac{1}{2} . \quad (13)$$

The 2×2 matrix notation originated with Schwinger's classic article in 1960¹.

In what follows we will use an alternative generating functional⁴ using the Complex Path Ordered Form:

$$\begin{aligned} \int [\mathcal{D}\psi] \langle \varphi' | \mathcal{T}^* \exp \left[-i \int_0^\infty dt J_-(t) \phi(t) \right] | \psi \rangle \langle \psi | \mathcal{T} \exp \left[i \int_0^\infty dt J_+(t) \phi(t) \right] | \varphi \rangle \\ = \langle \varphi' | \mathcal{T}_C \exp \left[i \int_C dt J(t) \phi(t) \right] | \varphi \rangle \end{aligned} \quad (14)$$

so that (3) may be rewritten more concisely in the CTP complex path ordered form,

$$\begin{aligned} Z_C[J, \rho] &= \text{Tr} \left\{ \rho \left(\mathcal{T}_C \exp \left[i \int_C dt J(t) \phi(t) \right] \right) \right\} \\ &= \int [\mathcal{D}\varphi^1] \int [\mathcal{D}\varphi^2] \langle \varphi^1 | \rho | \varphi^2 \rangle \int_{\varphi^1}^{\varphi^2} [\mathcal{D}\phi] \exp \left[i \int_C dt (L[\phi] + J\phi) \right] . \end{aligned} \quad (15)$$

This is identical in structure to the usual expression for the generating functional in the more familiar in-out formalism, Only difference – path ordering according to the

complex time contour $\mathcal{C}$ replacing the ordinary time ordering prescription along only $\mathcal{C}_+$. For example, the propagator function becomes

$$\begin{aligned} G(t, t') &= \theta_{\mathcal{C}}(t, t') G_{>}(t, t') + \theta_{\mathcal{C}}(t', t) G_{<}(t, t') \\ &\equiv \theta_{\mathcal{C}}(t, t') G^{21}(t, t') + \theta_{\mathcal{C}}(t', t) G^{12}(t, t') \end{aligned} \quad (16)$$

where $\theta_{\mathcal{C}}$ is the CTP complex contour ordered theta function defined by

$$\theta_{\mathcal{C}}(t, t') \equiv \begin{cases} \theta(t, t') & \text{for } t, t' \text{ both on } \mathcal{C}_+ \\ \theta(t', t) & \text{for } t, t' \text{ both on } \mathcal{C}_- \\ 1 & \text{for } t \text{ on } \mathcal{C}_-, t' \text{ on } \mathcal{C}_+ \\ 0 & \text{for } t \text{ on } \mathcal{C}_+, t' \text{ on } \mathcal{C}_- \end{cases} \quad (17)$$

With this definition of $G(t, t')$ on the closed time contour, the Feynman rules are the ordinary ones, and matrix indices are not required. In integrating over the second half of the contour $\mathcal{C}_-$ we have only to remember to multiply by an overall negative sign to take account of the opposite direction of integration, according to the rule,

$$\int_{\mathcal{C}} dt = \int_0^{\infty} dt - \int_0^{\infty} dt. \quad (18)$$

A second simplification is possible in the form of the generating functional of (15), if we recognize that it is always possible to express the matrix elements of the density matrix as an exponential of a polynomial in the fields⁵

$$\langle \varphi^1 | \rho | \varphi^2 \rangle = \exp \left[R + R_a(t_0) \varphi^a(t_0) + R_{ab}(t_0) \varphi^a(t_0) \varphi^b(t_0) + \dots \right]. \quad (19)$$

Since any density matrix can be expressed in this form, there is no loss of generality involved in expressing ρ as an exponential. If we add this exponent to that of the action in (15), and integrate over the two endpoints of the closed time path φ^1 and φ^2 , then the only effect of the non-trivial density matrix ρ is to introduce source terms into the path integral for $Z_{\mathcal{C}}[J, \rho]$ with support *only* at the endpoints. This means that the density matrix can only influence the boundary conditions on the path integral at $t = 0$, where the various coefficient functions R_a , R_{ab} , *etc.* have the simple interpretations of initial conditions on the one-point (mean field), two-point (propagator), functions *etc.* It is clear that the equations of motion for $t \neq 0$ are not influenced by the presence of these terms at $t_0 = 0$. In the special case that the initial density matrix describes a thermal state, $\rho_{\beta} = \exp\{-\beta H\}$ then the trace over ρ_{β} may be represented as an additional functional integration over fields along the purely imaginary contour from $t = -i\beta$ to $t = 0$ traversed before $\mathcal{C}_-$ in Fig. 1. In this way the Feynman rules for real time thermal Green's functions are obtained⁶. Since we consider general nonequilibrium

initial conditions here we have only the general expression for the initial ρ above and no contour along the negative imaginary axis in Fig. 1.

To summarize, we may take over all the results of the usual scattering theory generating functionals, effective actions, and equations of motion provided only that we

1. substitute the CTP path ordered Green's function(s) (16) for the ordinary Feynman propagators in internal lines;
2. integrate over the full closed time contour, $\mathcal{C}$, according to (18); and
3. satisfy the conditions at $t = 0$ corresponding to the initial density matrix ρ .

Closed time path contour and causality

Rules for evaluating the time integrals using the closed time path [CTP] contour shown in Fig. (1).

The integration path is given by

$$\int_{\mathcal{C}} dt = \int_{0:c_+}^{\infty} dt - \int_{0:c_-}^{\infty} dt. \quad (20)$$

The *causal* Green's functions are given by functions of the form,

$$A(t, t') = \Theta_c(t, t') A_{>}(t, t') + \Theta_c(t', t) A_{<}(t, t'), \quad (21)$$

where $\Theta_c(t, t')$ is defined in eq.(17) These causal Green's functions are symmetric.

$$A_{>}(t, t') = A_{<}(t', t)$$

To prove causality of any graph we need two lemmas.

1. Lemma 1- a loop of two causal functions, such as self energy graph, is another causal function.

$$\begin{aligned} B(t, t') &= \Theta_c(t, t') B_{>}(t, t') + \Theta_c(t', t) B_{<}(t, t') \\ C(t, t') &= \Theta_c(t, t') C_{>}(t, t') + \Theta_c(t', t) C_{<}(t, t'), \end{aligned} \quad (22)$$

the self energy graph

$$A(t, t') = iB(t, t')C(t, t'),$$

then

$$A(t, t') = \Theta_c(t, t') A_{>}(t, t') + \Theta_c(t', t) A_{<}(t, t') \quad (23)$$

where

$$A_{>,<}(t, t') = iB_{>,<}(t, t')C_{>,<}(t, t') \quad (24)$$

2. Lemma 2 -Matrix product of two causal functions is causal

$$A(t_1, t_3) = \int_c dt_2 B(t_1, t_2) C(t_2, t_3) , \quad (25)$$

we find then

$$A(t, t') = \Theta_c(t, t') A_>(t, t') + \Theta_c(t', t) A_<(t, t') , \quad (26)$$

where

$$\begin{aligned} A_{\geq}(t_1, t_3) &= - \int_0^{t_3} dt_2 B_{\geq}(t_1, t_2) [C_{>}(t_2, t_3) - C_{<}(t_2, t_3)] \\ &\quad + \int_0^{t_1} dt_2 [B_{>}(t_1, t_2) - B_{<}(t_1, t_2)] C_{\geq}(t_2, t_3) . \end{aligned} \quad (27)$$

Now consider the product of three causal functions:

$$A(t_1, t_4) = \int_c dt_2 \int_c dt_3 B(t_1, t_2) C(t_2, t_3) D(t_3, t_4) . \quad (28)$$

We can work this case out by applying the second lemma from left to right. That is, we can let

$$E(t_1, t_3) = \int_c dt_2 B(t_1, t_2) C(t_2, t_3) . \quad (29)$$

Then $E(t_1, t_3)$ is causal. We are then left with:

$$A(t_1, t_4) = \int_c dt_3 E(t_1, t_3) D(t_3, t_4) , \quad (30)$$

and so A is also causal.

After doing the integrals sequentially one is left with

$$f(t) = \int_c dt_1 F(t, t_1) = \int_0^t [F_{>}(t, t_1) - F_{<}(t, t_1)] , \quad (31)$$

which explicitly displays the causality (dependence only on earlier times).

To see how these rules work in practice, consider the terms contributing to order $1/N$ to the induced current determining the backreaction on an initially strong electric field⁴. The current is just $\text{tr}\{\gamma^\mu \tilde{G}\}$. Where $\tilde{G}$ is the full fermion propagator to order $1/N$.

Using the above lemmas we obtain that the Maxwell eqs. of motion take the form,

$$\begin{aligned} \partial_\nu F^{\mu\nu}(x) &= \langle j^\mu(x) \rangle = -\frac{ie^2}{2} \text{tr} \{ \gamma^\mu [G_{>}(x, x) + G_{<}(x, x)] \} \\ &+ \frac{2e^2}{N} \int_0^t dt_1 d^3 \vec{x}_1 \int_0^{t_1} dt_2 d^3 \vec{x}_2 \text{Imtr} \left\{ \gamma^\mu [G_{>}(x, x_1) - G_{<}(x, x_1)] \times \right. \\ &\quad \left. [\Sigma_{<}(x_1, x_2) G_{>}(x_2, x) - \Sigma_{>}(x_1, x_2) G_{<}(x_2, x)] \right\} . \end{aligned} \quad (32)$$

Here:

$$\Sigma_{<}(x_1, x_2) = i\gamma^\mu G_{<}(x_1, x_2)\gamma^\nu D_{\nu\mu}<(x_1, x_2). \quad (33)$$

This immediately displays the causality of the result: that is only previous times contribute to the space-time integrations.

TIME EVOLUTION OF THE QUARK GLUON PLASMA

Our model for the production of the quark-gluon plasma begins with the creation of a flux tube containing a strong color electric field. We assume the kinematics of ultrarelativistic high energy collisions results in a boost invariant dynamics in the longitudinal (z) direction where all expectation values are functions of the proper time $\tau = \sqrt{t^2 - z^2}$. We therefore introduce the light cone variables τ and η , which will be identified later with fluid proper time and rapidity. These coordinates are defined in terms of the ordinary lab-frame Minkowski time t and coordinate along the beam direction z by

$$z = \tau \sinh \eta, \quad t = \tau \cosh \eta. \quad (34)$$

The Minkowski line element in these coordinates has the form

$$ds^2 = -d\tau^2 + dx^2 + dy^2 + \tau^2 d\eta^2. \quad (35)$$

Hence the metric tensor is given by

$$g_{\mu\nu} = \text{diag}(-1, 1, 1, \tau^2). \quad (36)$$

For simplicity, here we discuss pair production from an abelian Electric Field and the subsequent quantum back-reaction on the Electric Field. The physics of the problem can be understood for constant electric fields as a simple tunneling process. If the electric field can produce work of at least twice the rest mass of the pair in one compton wavelength, then the vacuum is unstable to tunnelling. This condition is:

$$\frac{eE\hbar}{mc} \geq 2mc^2. \quad (37)$$

The problem of pair production from a constant Electric field (ignoring the back reaction) was studied by J. Schwinger in 1951⁷. The WKB argument is as follows: One imagines an electron bound by a potential well of order $|V_0| \approx 2m$ and submitted to an additional electric potential eEx . The ionization probability is proportional to the WKB barrier penetration factor:

$$\exp[-2 \int_0^{V_0/e} dx \{2m(V_0 - |eEx|)\}^{1/2}] = \exp(-\frac{4}{3}m^2/|eE|) \quad (38)$$

In his classic paper Schwinger was able to analytically solve for the effective Action in a constant background electric field and determine an exact pair production rate:

$$w = [\alpha E^2 / (2\pi^2)] \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \exp(-n\pi m^2 / |eE|). \quad (39)$$

By assuming this rate could be used when the Electric field was slowly varying in time, the first back reaction calculations were attempted using semi classical transport methods. Here we use the *CTP* formalism and perform the field theory calculation. The lagrangian density for QED in curvilinear coordinates gives rise to the action

$$S = \int d^{d+1}x (\det V) \left[\frac{i}{2} \bar{\Psi} \tilde{\gamma}^\mu \nabla_\mu \Psi + \frac{i}{2} (\nabla_\mu^\dagger \bar{\Psi}) \tilde{\gamma}^\mu \Psi - im \bar{\Psi} \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \quad (40)$$

where⁸

$$\nabla_\mu \Psi \equiv (\partial_\mu + \Gamma_\mu - ieA_\mu) \Psi \quad (41)$$

From the action (40) we obtain the Heisenberg field equation for the fermions,

$$(\tilde{\gamma}^\mu \nabla_\mu + m) \Psi = 0, \quad (42)$$

which takes the form

$$\left[\gamma^0 \left(\partial_\tau + \frac{1}{2\tau} \right) + \gamma_\perp \cdot \partial_\perp + \frac{\gamma^3}{\tau} (\partial_\eta - ieA_\eta) + m \right] \Psi = 0, \quad (43)$$

Variation of the action with respect to A_ν yields the Maxwell equations: If the electric field is in the z direction and a function of τ only, we find that the only nontrivial Maxwell equation is

$$\frac{1}{\tau} \frac{dE(\tau)}{d\tau} = \frac{e}{2} \langle [\bar{\Psi}, \tilde{\gamma}^\eta \Psi] \rangle = \frac{e}{2\tau} \langle [\Psi^\dagger, \gamma^0 \gamma^3 \Psi] \rangle. \quad (44)$$

We expand the fermion field in terms of Fourier modes at fixed proper time τ ,

$$\Psi(x) = \int [dk] \sum_s [b_s(\mathbf{k}) \psi_{\mathbf{k}s}^+(\tau) e^{ik\eta} e^{i\mathbf{p} \cdot \mathbf{x}} + d_s^\dagger(-\mathbf{k}) \psi_{-\mathbf{k}s}^-(\tau) e^{-ik\eta} e^{-i\mathbf{p} \cdot \mathbf{x}}]. \quad (45)$$

The $\psi_{\mathbf{k}s}^\pm$ then obey

$$\left[\gamma^0 \left(\frac{d}{d\tau} + \frac{1}{2\tau} \right) + i\gamma_\perp \cdot \mathbf{k}_\perp + i\gamma^3 \pi_\eta + m \right] \psi_{\mathbf{k}s}^\pm(\tau) = 0, \quad (46)$$

We square the Dirac equation by introducing

$$\psi_{\mathbf{k}s}^\pm = \left[-\gamma^0 \left(\frac{d}{d\tau} + \frac{1}{2\tau} \right) - i\gamma_\perp \cdot \mathbf{k}_\perp - i\gamma^3 \pi_\eta + m \right] \chi_s \frac{f_{\mathbf{k}s}^\pm}{\sqrt{\tau}}. \quad (47)$$

The spinors χ_s are chosen to be eigenspinors of $\gamma^0\gamma^3$,

$$\gamma^0\gamma^3\chi_s = \lambda_s\chi_s \quad (48)$$

with $\lambda_s = 1$ for $s = 1, 2$ and $\lambda_s = -1$ for $s = 3, 4$. They are normalized,

$$\chi_r^\dagger \chi_s = 2\delta_{rs}. \quad (49)$$

The sets $s = 1, 2$ and $s = 3, 4$ are two different complete sets of linearly independent solutions of the Dirac equation. Inserting (47) into the Dirac equation (46) we obtain the quadratic mode equation

$$\left(\frac{d^2}{d\tau^2} + \omega_k^2 - i\lambda_s \dot{\pi}_\eta \right) f_{ks}^\pm(\tau) = 0, \quad (50)$$

where now

$$\omega_k^2 = \pi_\eta^2 + \mathbf{k}_\perp^2 + m^2. \quad (51)$$

We obtain

$$\frac{1}{\tau} \frac{dE(\tau)}{d\tau} = -\frac{2e}{\tau^2} \sum_{s=1}^4 \int [dk] (\mathbf{k}_\perp^2 + m^2) \lambda_s |f_{ks}^+|^2, \quad (52)$$

We would like to compare the field theory calculation with a semiclassical transport approach which uses as a source of particle production Schwinger's production rate for time independent fields.

Assuming boost-invariant initial conditions for f , invariance of the Boltzmann-Vlasov assures that the distribution function is a function only of the boost invariant variables $(\tau, \eta - y)$ or (τ, p_η) . The kinetic equation reduces to

$$\begin{aligned} \frac{\partial f}{\partial \tau} + eF_{\eta\tau}(\tau) \frac{\partial f}{\partial p_\eta} &= \pm [1 \pm 2f(\mathbf{p}, \tau)] e\tau |E(\tau)| \\ &\times \ln \left[1 \pm \exp \left(-\frac{\pi(m^2 + \mathbf{p}_\perp^2)}{e|E(\tau)|} \right) \right] \delta(p_\eta). \end{aligned} \quad (53)$$

Turning now to the Maxwell equation, we have that

$$-\tau \frac{dE}{d\tau} = j_\eta = j_\eta^{cond} + j_\eta^{pol}, \quad (54)$$

where j_η^{cond} is the conduction current and j_μ^{pol} is the polarization current due to pair creation⁹.

Fig. 2

Proper time evolution of the electric field $E(\tau)$ and the fermion current $j_\eta(\tau)$ for initial conditions $E(\tau = 1) = 4.0$. The field theory calculation is compared to the semiclassical transport approach with a Schwinger source term.

Thus in (1+1) dimensions we have

$$j_\eta^{cond} = 2e \int \frac{dp_\eta}{2\pi\tau p_\tau} p_\eta f(p_\eta, \tau)$$

$$\begin{aligned}
j_{\eta}^{pol} &= \frac{2}{F\tau\eta} \int \frac{dp_{\eta}}{2\pi\tau p_{\tau}} p^{\tau} \frac{Df}{D\tau} \\
&= \pm[1 \pm 2f(p_{\eta}=0, \tau)] \frac{me\tau}{\pi} \text{sign}[E(\tau)] \ln \left[1 \pm \exp \left(-\frac{\pi m^2}{|eE(\tau)|} \right) \right].
\end{aligned} \tag{55}$$

In figure two we compare the results of a numerically solving the back reaction equations in the field theory with the semiclassical transport approach. We find that for this approximation (no rescattering of quarks), the semiclassical method does reasonable well when compared to a coarse grained in time and momentum field theory calculation.

DYNAMICAL EVOLUTION OF A NON-EQUILIBRIUM CHIRAL PHASE TRANSITION

Recently there has been speculation that disoriented chiral condensates (DCC's) can be formed following a heavy ion collision, and these condensates, formed during a quenched phase transition from the unbroken high temperature phase, could lead to events with a nonequilibrium distribution of charged to neutral pions. To see whether these ideas made sense we studied numerically the time evolution of pions produced following a heavy ion collision using the linear sigma model, starting from the unbroken phase. The quenching (if present) in this model is due to the expansion of the initial Lorentz contracted energy density by free expansion into vacuum. Starting from an approximate equilibrium configuration at an initial proper time τ in the disordered phase we study the transition to the ordered broken symmetry phase as the system expands and cools. We determined³ the proper time evolution of the effective pion mass, the order parameter $\langle \sigma \rangle$ as well as the pion two point correlation function. We studied the phase space of initial conditions that lead to instabilities (exponentially growing long wave length modes) which can lead to disoriented chiral condensates. The model we used to study the chiral phase transition is the σ model in the large N approximation. This model has the correct chiral properties and gives a reasonable description of low energy pion dynamics. The $O(4)$ σ model is described by the Lagrangian:

$$L = \left\{ \frac{1}{2} \partial \Phi \cdot \partial \Phi - \frac{1}{4} \lambda (\Phi \cdot \Phi - v^2)^2 + H\sigma \right\}. \tag{56}$$

The mesons form an $O(4)$ vector

$$\Phi = (\sigma, \pi_i)$$

Introducing the order parameter:

$$\chi = \lambda(\Phi \cdot \Phi - v^2).$$

we have the alternative Lagrangian:

$$L_2 = -\frac{1}{2}\phi_i(\square + \chi)\phi_i + \frac{\chi^2}{4\lambda} + \frac{1}{2}\chi v^2 + H\sigma \quad (57)$$

Perform the Gaussian path integral over the Φ field. Evaluate the remaining χ integral at the stationary phase point . Legendre transforming:

$$\Gamma[\Phi, \chi] = \int d^4x [L_2(\Phi, \chi, H) + \frac{i}{2}N \text{tr} \ln G_0^{-1}]$$

$$G_0^{-1}(x, y) = i[\square + \chi(x)] \delta^4(x - y)$$

$$[\square + \chi(x)]\pi_i = 0 \quad [\square + \chi(x)]\sigma = H$$

$$\chi = -\lambda v^2 + \lambda(\sigma^2 + \pi \cdot \pi) + \lambda N G_0(x, x). \quad (58)$$

As in the quark-gluon plasma problem we consider the kinematics of an ultrarelativistic Heavy Ion Collision which possesses longitudinal Boost invariance as the center of mass energy goes to infinity. Energy densities become function of the proper time only. Natural coordinates are the proper time τ and the spatial rapidity η defined as

$$\tau \equiv (t^2 - x^2)^{1/2}, \quad \eta \equiv \frac{1}{2} \log\left(\frac{t - x}{t + x}\right). \quad (59)$$

We assume that the mean (expectation) values of the fields Φ and χ are functions of τ only (homogeneity in the constant τ hypersurface)

$$\begin{aligned} \tau^{-1} \partial_\tau \tau \partial_\tau \Phi_i(\tau) + \chi(\tau) \Phi_i(\tau) &= H \delta_{i1} \\ \chi(\tau) &= \lambda(-v^2 + \Phi_i^2(\tau) + N \langle \phi^2(x, \tau) \rangle). \end{aligned} \quad (60)$$

On the otherhand the fluctuation fields (which are quantum operators) are functions of both x and t and obey the sourceless equation:

$$(\tau^{-1} \partial_\tau \tau \partial_\tau - \tau^{-2} \partial_\eta^2 - \partial_\perp^2 + \chi(x)) \phi(x, \tau) = 0. \quad (61)$$

$$G_0(x, y; \tau) \equiv \langle \phi(x, \tau) \phi(y, \tau) \rangle. \quad (62)$$

We expand this field in an orthonormal basis:

$$\phi(\eta, x_\perp, \tau) \equiv \frac{1}{\tau^{1/2}} \int [d^3k] (\exp(ikx) f_k(\tau) a_k + h.c.) \quad (63)$$

where $kx \equiv k_\eta \eta + \vec{k}_\perp \vec{x}_\perp$, $[d^3k] \equiv dk_\eta d^2k_\perp / (2\pi)^3$ and the mode functions $f_k(\tau)$ evolve according to (a dot here denotes the derivative with respect to the proper time τ):

$$\ddot{f}_k + \left(\frac{k_\eta^2}{\tau^2} + \vec{k}_\perp^2 + \chi(\tau) + \frac{1}{4\tau^2} \right) f_k = 0. \quad (64)$$

$$\chi(\tau) = \lambda \left(-v^2 + \Phi_i^2(\tau) + \frac{1}{\tau} N \int [d^3k] |f_k(\tau)|^2 (1 + 2 n_k) \right). \quad (65)$$

If we assume the initial density matrix is one of local thermal equilibrium then we have at $\tau = \tau_0$ (the surface of constant energy density and temperature T_0) that:

$$n_k = \frac{1}{e^{\beta_0 E_k^0} - 1}$$

where $\beta_0 = 1/T_0$ and $E_k^0 = \sqrt{k^2 + \chi(\tau_0)}$.

In choosing initial conditions we assumed that the initial value of χ was determined by the equilibrium gap equation for an initial temperature of $200MeV$. The phase transition in this model occurs at a critical temperature of $160MeV$. We chose initial σ and π_i expectation values consistent with the constraint

$$\vec{\pi}^2(\tau_0) + \sigma^2(\tau_0) = \sigma_T^2 \quad (66)$$

where σ_T is the equilibrium value of Φ at the initial temperature T . which we choose to be a temperature of $200MeV$. We varied the value of the initial proper time derivative of the sigma field expectation value and found that there is a narrow range of initial values that lead to the growth of instabilities. Namely

$$.25 < |\dot{\sigma}| < 1.3 \quad (67)$$

Surprisingly when $|\dot{\sigma}| > 1.3$ instabilities no longer occur.

Figures 3-4 summarize the results of the numerical simulation for the evolution of the system (65)-(64). We display the auxiliary field χ in units of fm^{-2} , the classical fields Φ in units of fm^{-1} and the proper time in units of fm^{-1} ($1fm^{-1} = 197MeV$). In Fig. 3 the proper time evolutions of the auxiliary field χ field is presented for four different initial conditions.

We are interested in knowing how our results differ from the case where the system evolves in local thermal equilibrium which is described by two correlation lengths, the inverse of the effective pion mass associated with χ , and the inverse of the proper time evolving effective temperature $T(\tau) = T_0(\frac{\tau_0}{\tau})^{\frac{1}{3}}$ discussed earlier. We see from Fig. 4 that in the case that $\sigma(1) = \sigma_T$, $\pi^i(1) = 0$ and $\dot{\sigma}(1) = -1$, where maximum instability exists, complex structures are formed as contrasted to the local thermal equilibrium

evolution. The interpolating phase space distribution $n(k_\eta, k_\perp, \tau)$ obtained numerically, clearly exhibits a larger correlation length in the transverse direction than the equilibrium one and has correlation in rapidity of the order of 1-2 units of rapidity. We notice that in both directions there is structure which does not lend itself to a simple interpretation. On the other hand the local thermal equilibrium evolution is quite regular apart from the normalization of the distributions that are changing with time due to oscillation in the quantity $\chi(\tau)$ which is damped to its equilibrium value once the system expands sufficiently.

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Fig. 3

Proper time evolution of the χ field for four different initial conditions with $f_\pi = 92.5 \text{ MeV}$.

Fig. 4

Slices of $k_\eta = 0$ and $p \equiv |\mathbf{k}_\perp| = 0$ of the proper time evolution of the interpolating phase space particle number density $n(k_\eta, \mathbf{k}_\perp, \tau)$ for $\sigma(1) = \sigma_T$, $\pi^i(1) = 0$ and $\dot{\sigma}(1) = -1$ compared with the corresponding local thermal equilibrium densities $n_T(k_\eta, \mathbf{k}_\perp, \tau)$.