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Geometry in the Large and Hyperbolic Chaos

Brosl Hasslacher* and Ronnie Mainieri

Abstract

This is the final report of a three-year, Laboratory Directed Research and Development (LDRD) project at the Los Alamos National Laboratory (LANL). We calculated observables in strongly chaotic systems. This is difficult to do because of a lack of a workable orbit classification for such systems. This is due to global geometrical information from the original dynamical system being entangled in an unknown way throughout the orbit sequence. We used geometrical methods from modern mathematics and recent connections between global geometry and modern quantum field theory to study the natural geometrical objects belonging to hard chaos—hyperbolic manifolds.

Background and Research Objectives

Hyperbolic systems have become fashionable in physics since the discoveries of Feigenbaum in period doubling. These systems arise in a large class of problems including the study of finite-dimensional dynamical systems, elliptic equations, coding problems, lattice gases, and the statistical mechanics of spin systems. Within these classes the majority are hyperbolic and share two features: a dense set of periodic orbits and exponential difficulty in computing their properties. There has been an enormous literature in physics about chaotic systems, but very little of it has addressed the hard problems of the field. That is, most of the analysis has been local and based on computer experiments. We have already successfully applied our proposed methods to a few problems. We feel that the methods we are using can change the approach used to study hyperbolic systems.

If we look at a small patch of orbits in a hyperbolic system they form a saddle-like surface. These orbits will not remain together for very long; they diverge exponentially quickly. This exponential growth is the main difficulty in the study of chaotic dynamical systems and extends beyond simple ones. Many problems in physics and mathematics that require a global solution have this hyperbolic behavior. The ingredients for our proposed approach come from modern tools in theoretical physics and mathematics: algebraic topology, braid and knot theory, and path integration. The basic idea is that progress in the study of hyperbolic systems can only be achieved by global techniques, and these are geometrical. A lesson learned from algebraic topology, which is borne out by our results,

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is that to gain geometrical insight it is necessary to equip the system with additional mathematical structures.

In the study of a hyperbolic system the technical difficulty is to compute an average or to evaluate the trace of a Green's function. To solve this technical problem we bring two new insights that, although not traditional in the field of chaotic systems, are the heart of most modern mathematics: fiber (gauge field) attachment and dimensional lifting. We add gauge fields to the dynamical system and let the dynamics transport it. This way we can encode the global features of the dynamics into the gauge.

Dimensional lifting avoids the complications of chaotic dynamics by examining the dynamics in higher dimensional spaces. The central idea is to find a larger and more natural space for the dynamics, a space where the averages we are trying to compute are constants. This is against current trends in chaotic systems that try and compress the entire dynamics into a one-dimensional shift sequence. By compressing the dynamics, global data from the original dynamical system becomes encoded in a complex way in the sequence. Dimensional lifting is presently a technically demanding technique that allows one to study systems that could not otherwise be analyzed.

Dimensional lifting has been successfully applied to some chaotic systems. Spin systems with long range interactions have been geometrized, and a new technique for semiclassical quantum mechanics has been developed. Central to dimensional lifting is a novel and rigorous technique for the computation of path integrals. Dynamical systems can easily be expressed as path integrals. When hyperbolic systems are so written, the method shows its power. The new technique makes path integration as simple to perform as a finite dimensional integral. In this new method the action in the integral controls the dimension of the space where the integration takes place. It realizes in a very explicit form what Feynman thought to be the most important feature of path integration. Unlike other methods for quantum mechanics, Feynman claimed that a path integral chooses the space (meaning the Hilbert space) for you. One writes down the path integral, and the formalism fixes the space. The current formalism for path integration lacks this feature. In our technique, the space is directly dictated by the dynamics that the action encodes. In a typical case the resulting space will have a higher dimension than the original one, hence the name dimensional lifting.

The technique of dimensional lifting in path integration can be directly used in the study of chaotic systems. It leads, in the case of chaotic systems, to rapidly convergent cycle expansions; and in general to path integrals that do not have to be regularized. This rigorous method leads only to well defined quantities.

The objective of the research is to develop our techniques for a few hyperbolic systems into general methods. We developed the technique of attaching a fiber (gauge field) to a dynamical system to impose global constraints, into a method that can be applied to a large class of systems. We also developed the method of hyperbolic groups to enumerate the orbits that are accessible to a dynamical system.

Importance to LANL's Science and Technology Base and National R&D Needs

Many of the problems of slow convergence in scientific computing can be traced back to the exponential difficulties in studying hyperbolic systems. (The case beyond hyperbolicity is intermittency, which we hope will fall under our general methods.) If our methods become more general we hope that they will have an impact in the efficiency of scientific computing, by resolving the exponential difficulties. The end result would be a method of general utility such as multi-grid or Monte Carlo.

Scientific Approach and Accomplishments

To understand our approach to the problem, one has to compare it to a more traditional approach. A typical problem is to study a geodesic flow on a hyperbolic surface and compute the average of some observable of the system (average velocity, for example). The flow is studied by enumerating its periodic orbits. This is done by creating a code for each periodic orbit (based on a tessellation of the manifold). If the code is one-to-one then it is possible to compute the averages needed. But if not all codes correspond to an orbit—if there are gaps—then the only way out is to enumerate the orbits on a computer. This is difficult for the number of orbits grows exponentially. The gaps appear because an orbit, with all its folds and turns, has been projected down onto one dimension (its code). This projection is the methods power, as the entire arsenal of combinatorics is available, but it is also its fatal flaw. (This difficulty does not arise in toy examples of chaotic systems—that is why they are toy problems—it is a generic feature). In our method one does not compress the problem into one dimension.

We studied hyperbolic systems by finding a natural setting in which global structure is transparent and is effectively encoded into the topological invariants belonging to the manifold or a structure on the manifold. (The only topological invariant of a Riemann manifold is its genus, so we need to endow it with additional structure.) The technique is to lift the symbol sequence in dimension until one arrives at a natural manifold in which global conditions are transparent. In simple cases the geodesic flow on this manifold encodes an induced dynamical system which contains all the information one can extract

about observables. The effects of the gaps are contained in the topological invariants of the manifold, which can now be computed without the use of computers. These invariants are used to guide a numerical procedure for extracting metric properties. This results in a splitting theorem. One part contains a topological piece (encoding the gaps in the code) and the other part is an entire function, which can be calculated to any accuracy by known methods. This fusion, of frontier methods in mathematics with physics, has been applied with great success in high energy physics, as the body of work centered around Witten's results shows. We are using an analogous program for dynamical systems.

We applied this method to a few problems, so we know it is feasible. We have solved two classes of problems. In one, we have shown how to construct a chaotic dynamical system from an exactly solved spin system. This gives us a laboratory in which to try many ideas and rapidly links us to the tools of modern mathematics. We have also developed a rigorous and new method for path integration. This gives a powerful tool for developing the formalism. At the heart of this new method for path integration is a scheme where the dynamics fixes the natural number of space dimensions for the problem.

The problem undertaken here is one of the most difficult in the mathematics of chaotic systems and is really a conjecture about a unification problem. The central notion is whether there is a deep connection between exactly solvable statistical mechanical systems, and therefore soliton systems on the one hand, and hard hyperbolic chaotic dynamics on the other. This is a question in pure mathematics which has profound consequences for both conceptual understanding of both exact and chaotic systems and important implications for our ability to do applied calculations with very chaotic dynamics. So far, besides the usual tools developed over the past twenty years or so, there is only the method of cycle expansions available for attacking questions on hard hyperbolic systems, and it has considerable computational obstructions for any but the simplest setups. Further progress in chaotic dynamics has hit a conceptual block, so one has to take a whole different approach to the problem.

The method we chose was a global attack, concentrating on topological classification and constraints on global objects as opposed to local structure. This leads immediately to the theory of knot invariants, generalizations of the topological field theories that control them and the image of the Kauffman skein relations in a hyperbolic setting. Knot invariants, more generally Vassiliev invariants, control all of the classification information in the exactly integrable or soliton sector. A relatively recent and difficult result of Thurston's shows that the complement of every hyperbolic space is a knot and that this construction is a smooth operation.

This immediately opens a possible avenue of connection between exactly integrable and hard hyperbolic systems: how are the skein relations deformed in passing from the knot to its complement space, and how the usual knot polynomials are distorted in a hyperbolic system. If this map is smooth and the resulting functional equations have solutions, one has an immediate method to carry over all the powerful tools developed for the soliton sector to the extreme opposite of hyperbolic dynamics. All of this requires the development or extension of complex tools developed by pure mathematicians for other problems in foliation theory, Thurston's theory of train tracks, Dehn surgery techniques, Drinfeld quantum groups etc. The list is long and all these concepts are intertwined.

Although the principal investigator (PI) is well versed in the theory of knot invariants and soliton systems, having published extensively in these fields, the hyperbolic sector remained an enigma for a long time due to the lack of a coherent discussion of intricate tools, which were available only in the mathematics literature in obscure and clumsy formulations. Recently Thurston's unpublished notes on hyperbolic complement constructions became available, and these provide the correct slant on completing an approach to carrying over the skein relations, which is available no other way. It took a long time to digest Thurston's tools, which were the product of fifteen years of work by a number of outstanding mathematicians who were not looking at the skein problem. It now appears that patching together simplices in hyperbolic space, in a non-trivial way that is even difficult to visualize, but can be algorithmically formulated, will carry the skein relations over to the hyperbolic sector in a minimally deformed way. One ends up working in a Teichmüller space, the space of all conformally non-equivalent Riemann manifolds of fixed genus, but this has to be approached by simplicial construction, so as not to lose control over the functional equations describing the skein relations. These are at the heart of the Yang-Baxter equations in a more general framework and carry with them the entire machine of quantum groups, which plays a crucial role. This work is in progress with several world class mathematicians.

There is a great deal of local structure work that was tried with Ronnie Mainieri and Sergei Nechaev to make simpler descriptions of chaotic dynamics at the price of working on a more complex manifold. This is called dimensional lifting and met with mixed success, consuming a large percentage of the PI's time, since these calculations are both conceptually and mathematically difficult. We tried looking at monodromy groups on many-punctured planes in an attempt to recover train track descriptions, but this was intractable beyond the first few cases. The Ising model, considered as two interconnected dynamical systems, was also worked out in detail, but led to a very complex four-

dimensional manifold and a nasty incarnation of the pruning front problem, which stopped us for a long time.

This unification conjecture does not lend itself to partial results. It is either true or not. It appears true now, but still difficult to access by direct construction and then only by using global methods. But this research gave us tremendous insight into the intricate structure of the problem and led to many beautiful tools and novel outlooks that we are now using.