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PLASMA PHYSICS

CONTROL OF NONLINEAR SYSTEMS USING PERIODIC
PARAMETRIC PERTURBATIONS
WITH APPLICATION TO A REVERSED FIELD PINCH

Kevin A. Mirus

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**Control of Nonlinear Systems Using Periodic
Parametric Perturbations
with Application to a Reversed Field Pinch**

by

Kevin A. Mirus

A dissertation submitted in partial fulfillment

of the requirements for the degree of

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Abstract

In this thesis, the possibility of controlling low- and high-dimensional chaotic systems by periodically driving an accessible system parameter is examined. This method has been carried out on several numerical systems and the MST Reversed Field Pinch. The numerical systems investigated include the logistic equation, the Lorenz equations, the Rössler equations, a coupled lattice of logistic equations, a coupled lattice of Lorenz equations, the Yoshida equations, which model tearing mode fluctuations in a plasma, and a neural net model for magnetic fluctuations on MST. This method was tested on the MST by sinusoidally driving a magnetic flux through the toroidal gap of the device.

Numerically, periodic drives were found to be most effective at producing limit cycle behavior or significantly reducing the dimension of the system when the perturbation frequency was near natural frequencies of unstable periodic orbits embedded in the attractor of the unperturbed system. Several different unstable periodic orbits have been stabilized in this way for the low-dimensional numerical systems, sometimes with perturbation amplitudes that were less than 5% of the nominal value of the parameter being perturbed. In high-dimensional systems, limit cycle behavior and significant decreases in the system dimension were also achieved using perturbations with frequencies near the natural unstable periodic orbit frequencies. Results for the MST were not this encouraging, most likely because of an insufficient drive amplitude, the extremely high dimension of the plasma behavior, large amounts of noise, and a lack of stationarity in the transient plasma pulses.

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Well, it has taken me eight years to get to this point, so I see no need to be brief now. I would like to thank my advisor, Clint Sprott, for his interminable patience, and for giving me leeway to pursue all sorts of crazy ideas and interests. I'd also like to thank Stewart Prager for the excellent job he does at keeping the MST program alive and kicking, and for allowing me the latitude to pursue this line of research within the MST group.

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Table of Contents

Abstract	i
Acknowledgements	ii
Table of Contents	iv
1.Introduction	1
1.1.Motivation	1
1.2.Plasmas and Chaos	4
2.A Review of Chaos Control Techniques	10
2.1.Overview	10
2.2.Closed Loop Control	11
2.2.1.Linear Control	12
2.2.2.OGY Method	13
2.2.3.OPF Method	20
2.2.4.Pyragas Method	21
2.2.5.Other Variations	22
2.3.Open Loop Control	23
2.3.1.Entrainment	23
2.3.2.Stochastic Drives	26
2.3.3.Chaotic Drives	28
2.3.4.Periodic Drives	29
2.4.Control of Chaos in Plasmas	35
2.5.Summary	37
3.Periodic Parametric Perturbations Applied to Numerical Systems	50
3.1.Analysis Methods	52
3.1.1.The Lyapunov Exponent Spectrum	52
3.1.2.Unstable Periodic Orbits	55
3.2.The Logistic Equation	57
3.3.The Lorenz Equations	70
3.4.The Rössler Equations	87
3.5.Coupled Logistic Equations	94
3.6.Coupled Lorenz Equations	97
3.7.The Yoshida Equations	105
3.8.Neural Net Model for MST	122
3.9.Summary	128

4.Periodic Perturbations Applied to MST	133
4.1.Experimental Apparatus	135
4.2.Low Current Results and Mode Locking	140
4.3.High Current Results	142
4.3.1.Unstable Periodic Orbits	143
4.3.2.Correlation Dimension	147
4.3.3.Stochastic Resonance	150
4.4.Summary	151
5.Conclusions	155

1. Introduction

There has been an explosion of interest and research in the last few years in the "new" field of nonlinear dynamics and chaos. Much of the impetus in this field has come from studying fluid dynamics, whose nonlinear Navier-Stokes equation has been known since the early nineteenth century. Poincaré showed in the late nineteenth century that the three-body gravitational problem could not be solved in terms of known analytic functions. Thus, the field of "nonlinear dynamics" is not very new. One of the main reasons for the recent increased interest in the field is the phenomenal technological advance in computing that has allowed researchers to numerically solve nonlinear equations on their desktop, and thus rapidly develop intuition and test hypotheses. However, once one has developed a core body of knowledge in chaotic systems, one tends to ask, "What good is it?" This thesis attempts to answer that question.

1.1 Motivation

Many systems in nature display chaotic behavior. Chaos is typified by irregular, apparently random-looking behavior which exhibits sensitive dependence on initial conditions and parameter changes, yet which is generated by governing laws that determine the evolution of the system [1.1]. Chaotic systems thus present an interesting juxtaposition of the understandable and the unknowable, and researchers tried early on to tame such systems using techniques from the arsenal of linear control theory [1.2]. Such attempts were not generally successful, as a chaotic system's nonlinearity and sensitivity

to initial conditions almost guarantees that any error in applying a control scheme will result in continued chaotic behavior. Clearly, some insight into the problem was needed.

Such insight came in a 1990 paper by Ott, Grebogi, and Yorke (OGY), which was simply titled "Controlling Chaos" [1.3]. Their control method has been widely acclaimed because it is based on intuitive geometric grounds that are applicable to a wide variety of chaotic systems. Their method also has the added advantages that it can be used to elicit a wide variety of behaviors from a given chaotic system, and it can be easily adapted to control chaos for long periods of time with almost arbitrarily small control signals. Their intuitive geometric insight was the simple observation that any chaotic trajectory has an infinite number of unstable periodic orbits (UPOs) embedded in it. Thus, various controlled behaviors can be selected by choosing the appropriate UPO to stabilize, while small control signals can be used if one can wait for the trajectory to get suitably close to the UPO that one is trying to stabilize. This is always bound to happen, since chaotic systems are transitive, which means that given any two points on an attractor, the chaotic system comes arbitrarily close to both [1.4]. From Ott, Grebogi, and Yorke's work, four qualities have emerged as the hallmarks of a successful chaotic control scheme: applicability to a wide array of systems, flexibility in the types of allowed controlled dynamics, a minimally applied control signal, and long-term stability.

One important example among the myriad of chaotic/nonlinear systems is that of the fusion plasma. The goal of inexpensive energy from hydrogen fusion plasmas has been delayed by the inability to prevent the anomalously large transport of energy and particles out of fusion devices. In the Madison Symmetric Torus (MST) reversed field

pinch (RFP) [1.5], it has been shown that magnetic field fluctuations result in energy [1.6] and particle [1.7] loss from the core of the plasma. Electric field fluctuations in the edge of the plasma have been shown to also result in particle transport [1.8].

In an attempt to control these fluctuations, this thesis looks to OGY for inspiration, while allowing for certain practical compromises in the implementation of a chaotic control scheme. Two main concerns in applying an OGY-type scheme is that the feedback it entails requires extensive calculation, which makes it unsuitable for rapidly evolving, high-dimensional systems. The MST happens to be such a system, and some early, unpublished work on active feedback control of radial magnetic field errors at the poloidal gap of MST showed that such an implementation would be nearly impossible. Thus, open-loop control methods were sought, in which the applied control signal is independent of the state of the chaotic system. This method guarantees ease in application, while still allowing for the four hallmarks of an effective control scheme.

The control method settled upon was that of applying periodic perturbations to some system control parameter. Certainly, this method is applicable to a wide variety of systems, and it will be shown to allow for a wide variety of controlled states for long periods of time. Using knowledge of the UPOs of the system to be controlled, one can find very small perturbation amplitudes which result in limit cycle or other reduced-dimension behavior. This is where the work presented here stands apart from other research concerning control of chaotic systems using periodic perturbations: the identification of many different frequencies determined from the chaotic dynamics at

which to apply small-amplitude perturbations, and the successful numerical application to high dimensional systems.

1.2. Plasmas and Chaos

Numerical results presented in this thesis indicate that controlling chaos to a limit cycle is much easier in low-dimensional systems. There are many reports of low-dimensional chaos in plasma systems, but it is questionable how reliable these are. The most reliable reports come from numerical models and experimental devices such as glow discharge plasmas and pulsed machines; reports of low-dimensional chaos in fusion type plasmas are more speculative.

Low dimensional chaos has exhibited itself in low-dimensional ion and drift wave models [1.9-1.12]. Watts *et al.* also found strong evidence for low-dimensional chaos and simple determinism in a numerical simulation of the MST using the DEBS code [1.11,1.12]. Low-dimensional chaos has also been observed in models for tokamak fluctuations [1.13], and plasma glow discharges [1.14].

Numerous researchers have reported low-dimensional chaos in experimental glow discharge and pulsed plasmas [1.15-1.25]. All evidence of chaos in these types of plasmas is based on one of three experimental observations: a period-doubling route to chaos [1.15-1.17,1.22,1.23,1.25], a quasi-periodic route to chaos [1.18-1.20], or a Lyapunov exponent analysis [1.21,1.23]. Many of the workers who report period-doubling routes show Poincaré plots of a system variable, indicating that the chaotic dynamics has a dimension between two and three for this type of system. Indeed, Huang *et al.* [1.24]

verify this value by measuring the Lyapunov exponent spectrum for an undriven, unmagnetized argon plasma maintained between two electrodes, from which they compute a dimension of 2.86. The Lyapunov exponent spectrum will be discussed in Chapter 3.

The final and most suspect claims of low-dimensional chaos are those concerning fusion-type plasmas [1.26-1.33]. The earliest claims seem to be more of a testing ground for the application of nonlinear data analysis techniques. In 1984, Duperrex *et al.* [1.26] argued for the existence of low dimensional chaos in TCA tokamak plasmas simply from the existence of a broadband power spectrum in fluctuating quantities and an appearance of structure in state space plots, such as one sees when the fluctuating density $\tilde{n}$ is plotted against the fluctuating toroidal field $\tilde{b}_\theta$. Later investigations of the TCA tokamak, in which the correlation dimension of fluctuations was measured by a magnetic probe, revealed that the dimension was higher than 10, which led Sawley *et al.* to conclude that there was no evidence of low-dimensional chaos [1.29,1.30]. Contrary to this result, correlation dimensions of between 2.1 and 8.5 have been claimed for the DITE tokamak [1.28], TFTR [1.31], the CHS heliotron/torsatron [1.33], and the HBTX1A RFP [1.27]. Watts has concluded that there is no evidence for low dimensional chaos or simple determinism in the MST RFP [1.11,1.12].

Despite the pessimistic outlook for success in controlling fluctuations on the MST, attempts were undertaken in the hope that even if the plasma could not be controlled to limit cycle behavior, perhaps its dimensionality could be lowered to the point where it

would be measurable. Before these attempts were made, however, several numerical systems were studied.

The rest of this thesis is organized as follows: Chapter two will examine the history and background of controlling chaos, including chaos in plasmas. Chapter three discusses the results of perturbing various chaotic numerical systems with periodic perturbations. Chapter four presents the results for applying such perturbations to MST. Chapter five contains conclusions and directions for future work.

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2. A Review of Chaos Control Techniques

2.1. Overview

The term "controlling chaos" has many different definitions in the literature. Early work focused on forcing a chaotic system to remain near one of its unstable fixed points [2.1], thus eliminating all time dependence of the system. Later techniques emphasized keeping a trajectory on one of its unstable periodic orbits (UPOs) [2.5], thus allowing for more flexibility in the resulting dynamics. In the broadest sense, "controlling chaos" means to elicit some change in a chaotic system's behavior to meet a given goal. Examples of work toward this general goal include forcing a chaotic system to an arbitrary goal dynamics [2.55-2.57], and on forcing it to remain chaotic, but synchronized with another chaotic system [2.67,2.68]. Control objectives such as these will be reviewed in this chapter.

There are other goals, however, that can be considered. For example, reducing the effective dimension of a high dimensional system is often a desirable goal [2.125]. Manipulating the Lyapunov exponents of a system may also be worthwhile, since they are related to transport coefficients such as the diffusion coefficient [2.125] and viscosity [2.126]. Finally, for a system like the MST, merely reducing the rms amplitude of fluctuations would be beneficial. These control objectives are not generally addressed by the literature in the context of chaotic controls, but will be discussed in the numerical results in Chapter 3.

As mentioned in Chapter 1, the OGY technique [2.5] is generally regarded as the prototypical method for controlling chaos. Before their paper, little work had been done

on controlling chaotic systems, but there does exist historical precedent. Broadly speaking, attempts to control chaos can be classified into one of two categories: a chaotic control can either be "closed-loop" or "open-loop". Closed loop control strategies always employ some type of feedback, while changes made to a chaotic system in an open loop scheme are independent of the state of the system. Many of the control methods actually arose out of, or at least were developed in parallel with, studies of how various nonlinear systems change from regular to chaotic behavior. For example, the study of quasiperiodic routes to chaos naturally leads to considerations of how one might control a chaotic system with periodic perturbations. While this observation may seem obvious, it is not generally recognized in the literature. This chapter discusses both closed-loop and open-loop control schemes, with a roughly historical discussion of the various methods employed in each category. For each method, literature that corresponds to the inverse process of control (or making a regular system chaotic) is pointed out where it exists.

2.2. Closed Loop Control

Closed loop control techniques, which employ feedback, broadly fall into five categories. Early attempts at controlling chaos used standard linear control techniques. Then came the major advance proposed by Ott, Grebogi, and Yorke. The remaining three categories are all based on modifications to the OGY technique, and include Occasional Proportional Feedback (OPF), a time delay method due to Pyragas, and a smorgasbord of other techniques that are hybrids of these three techniques.

2.2.1. Linear Control

The first attempt at controlling chaos found in the literature comes from an article by Fowler in 1989 [2.1], which was based on work he did in 1986. Fowler's attempt involved adding a linear control term to one or more of the state space equations of the Hénon-Heiles and Lorenz systems. In determining how much control to add, a Kalman filter [2.2, 2.3] was used to determine the state of the nonlinear system. A Kalman filter is a "recursive, unbiased, minimum-mean-square-error estimator that uses linear models for system dynamics and state-to-measurement transformation" [2.4]. More simply put, a Kalman filter is a procedure for estimating the state of a system using the assumption that a measured signal or signals for that system arise from a linear model that is subject to a random (stochastic) drive. Nonlinear Kalman filters exist in which truly nonlinear systems are locally linearized, and then the Kalman filter is applied to this system. Thus, Fowler attempted to achieve control by treating the unpredictable fluctuations and sensitivity to initial conditions of a chaotic system as arising from stochastic noise.

Fowler was able to stabilize both systems near unstable fixed points using this method. However, large additive control was often necessary, and the systems' sensitivity to initial conditions meant that any error in estimating the states of the systems would have the effect of driving the systems away from their fixed points. Thus, the controlled system had a nonzero probability of becoming chaotic again during any cycle. A second-order Kalman filter, which uses the second-order moment of the noise's distribution, was found to help. For the Hénon-Heiles system, this improvement reduced the probability of

losing control by about a factor of three. For the Lorenz system, control was generally not possible without using this second-order method.

Fowler drew three conclusions from his work: 1) Chaotic nonlinear systems seem to remain chaotic when control signals are applied; 2) A second-order Kalman filter is preferable to improve performance; and 3) Uncertainties in the state estimate generally become large enough so that control can no longer be maintained.

2.2.2. OGY Method

The less than spectacular results Fowler achieved with his linear control indicated that a new method had to be applied that used the qualities inherent in a chaotic system. In 1990, Ott, Grebogi, and Yorke (OGY) developed a method that did just that [2.5]. Their method is rooted in their early studies of intermittency and crises [2.6-2.11]. Intermittency is a phenomenon in a chaotic system that occurs when a parameter p of a chaotic system is near a critical value p_c . When this occurs, the system alternates between periodic behavior and bursts of chaotic behavior. A crisis is a bifurcation event in which a chaotic attractor undergoes a sudden discontinuous change as a parameter p is changed. Examples of crises include an attractor increasing its size in phase space, merging with another attractor, or being destroyed. From this context of understanding how chaotic systems can change due to parameter changes, OGY made the following two observations.

First, chaotic systems possess unstable fixed points and an infinite number of UPOs, each of which gives rise to different dynamics and different operating behaviors.

Thus, one can change the operation of a single system to meet a variety of goals by choosing which of the available UPOs to stabilize. Second, they noted that each of these orbits can be manipulated by applying parametric feedback control. This manipulation is possible because changing a system parameter changes the underlying manifolds that control the dynamics, and thus the trajectory of the system. The attractiveness of this method is that it is easy to conceptualize in these geometric terms.

Figure 2.1. Schematic diagram of OGY control. Stable and unstable manifolds near the unstable fixed point X^* at parameter value p_0 are identified. By adjusting p_0 by an amount Δp , the manifolds are shifted upward, thus resulting in a trajectory that ends up at point $X'(t_1)$ at time t_1 instead of at $X(t_1)$.

Figure 2.1 shows geometrically what occurs during OGY control. A system parameter p is changed from its nominal value of p_0 by an amount Δp when the trajectory wanders near an unstable fixed point X^* . The manifolds are shifted slightly by this perturbation so that the trajectory passes closer to the stable manifold, thus drawing it closer to X^* . The magnitude of the perturbation Δp can be decreased with successive

passes as the trajectory draws nearer to the unperturbed location of X^* . In a formal application, the OGY method consists of four steps:

- 1) Reduce the dynamics to a linear map in the neighborhood of the periodic fixed point: $X(t_{n+1}) = M[X(t_n), p_n]$. For a low dimensional, continuous time system, this step is simplified by making the map from a Poincaré section. Otherwise, this can be done using an analytic model if available, or the map can be constructed from experimental data.
- 2) Determine the proportionality constant g , which describes how the fixed point moves for small changes Δp from the nominal value of the parameter p_0 : $g \equiv \partial X^*(p)/\partial p|_{p=p_0} \equiv X^*(p_0 + \Delta p)/\Delta p$, where X^* is the location of the fixed point.
- 3) Wait until the trajectory falls within some neighborhood of radius R of the desired limit cycle or fixed point. This is guaranteed to happen, since chaotic attractors are ergodic and orbits come arbitrarily close to any given point at some time. This time may be long if R is small or if the system is high dimensional.
- 4) When the trajectory does come within R of X^* , modify p_n so as to bring $X(t_{n+1})$ close to the stable manifold.

Given this recipe, there are a few more details to complete the method. OGY assumed that the change in the manifold is linearly proportional to the displacement from the fixed point. They then showed that this led to a prescription for how to modify the

parameter p when the trajectory passes near X^* : $p_n = \lambda_u(\lambda_u - 1)^{-1}(X^* \cdot f_u)/(g \cdot f_u)$. Here λ_u and f_u are the unstable eigenvalue and contravariant basis vector, respectively. They also showed that a power law gives the average length of time during which chaotic transients persist after the control is activated: $\langle \tau \rangle \sim p_{\max}^{-\gamma}$. Here $p_{\max}$ is the maximum change in the parameter p that can be made. The exponent γ is given by the following relation, and was computed in their earlier work [2.8,2.9] on crises: $\gamma = 1 + \frac{1}{2} \ln|\lambda_u| / \ln|\lambda_s|^{-1}$.

Ott, Grebogi, and Yorke implemented their control on the numerical example of the Hénon map, which is given by $x_{n+1} = A - x_n^2 + By_n$, $y_{n+1} = x_n$, with $B = 0.3$ and $A = 1.29$. The parameter A was chosen to be perturbed, and they achieved control of the fixed point $x^* = y^* = \frac{1}{2}(B - 1) \pm \sqrt{\frac{1}{4}(B - 1)^2 + A}$ using a maximum perturbation of no more than $p_{\max} = 0.2$. They also showed that this method worked in the presence of additive noise, but occasional bursts into regions far from the fixed point happened occasionally.

While the OGY technique is very attractive from a theoretical viewpoint, it does have some practical limitations. First of all, it is somewhat computationally intensive, which means it is difficult to implement on high dimensional systems and systems with fast dynamics. Secondly, there are several constraints on the distance R from the periodic point at which control is initiated. Both the map M and the change in M with parameter p must be linearizable in the vicinity of the periodic point. The size of R is limited by the maximum amount of control that can be applied ($p_{\max}$), and by the amount of noise present. Finally, R is constrained by the relative size of the maximum eigenvalue of the

linearized map; a large eigenvalue implies a highly unstable periodic point, which in turn only allows control very near the point itself. In other words, since the control parameter p is applied at discrete time intervals determined by the Poincaré map, a system with a large Lyapunov exponent will wander away from the control region during the cycle if the cycle time is longer than the inverse of the Lyapunov exponent.

Once this intuitive method had been advanced, it did not take long for experimental verification. The first such example came from Ditto *et al.* in 1990 [2.12] in a paper titled "Experimental Control of Chaos." They employed the OGY method to control a gravitationally buckled magnetostatic ribbon. The ribbon belongs to a class of amorphous magnetorestrictive materials that exhibit very large reversible changes of Young's modulus with the application of small magnetic fields. The ribbon was long enough so that it would buckle under its own weight, and then placed within three mutually orthogonal Helmholtz coils, so that the earth's magnetic field could be nullified, and a time-varying uniform field along the vertical ribbon $H_{\text{vert}} = H_{\text{dc}} + H_{\text{ac}}\cos(2\pi ft)$ could be applied. The parameter $p = H_{\text{dc}}$ was chosen as the control parameter. The magnetic field amplitudes were in the range of 0.1-0.25 Oe. This system is low-dimensional ($2 < d < 3$), and thus they obtained their linear map M from experimental Poincaré sections obtained by sampling the position of the ribbon at a fixed phase of the drive. They were successful in controlling both period one and period two limit cycles for up to 64 hours with a maximum perturbation of $p_{\text{max}} = 0.01$ Oe. The trajectory as shown on the Poincaré

maps stayed within a factor of one third of the dynamic noise of the system to the desired fixed points.

There are three other experiments of note that achieved control of chaotic systems using a straight application of the OGY method. The first experiments include work by In *et al.* [2.13], and by Dressler *et al.* [2.14], who controlled chaotic vibrations of a horizontal bronze ribbon. Starrett *et al.* [2.15] controlled a chaotic, driven pendulum, and Hayes *et al.* [2.16] were able to control a chaotic circuit for communication by encoding information in a sequence of stabilized unstable periodic orbits.

There has also been much work done toward removing some of the limitations of the OGY method. Alsing *et al.* [2.17] addressed the computational problem of temporal restraints in fast systems by computing the controlling perturbation one or more cycles in advance. They called this *prior iterate control*. Dressler and Nitsche [2.18] observed that the OGY method must be modified in situations where the model system is reconstructed from a time series. They concluded that the map M depends on not only the current value of the parameter p_n , but also on its predecessors ($p_{n-1}, p_{n-2}, \dots$), so that the control law must also depend on past values of p . Cazelles *et al.* [2.19] looked at applying the OGY method in the presence of large amounts of noise by combining it with a Kalman filter to estimate the state of the system better. Hubinger *et al.* [2.20] found that systems with large Lyapunov exponents could be controlled by applying changes to the control parameter quasicontinuously, instead of only at certain discrete times. Ott, Grebogi, Yorke, and Auerbach [2.21] extended the OGY technique to high-dimensional systems. In high-D systems, the traditional computations associated with the OGY technique

become cumbersome. Auerbach *et al.* showed that the algebra can be streamlined by defining the state in terms of delay coordinates, which allows for a recursive control sequence without having to reconstruct the state space. They further showed that the control law can be truncated, further simplifying the control without significant loss of control.

Shinbrot and others [2.22,2.23] made another important improvement to the OGY method. They addressed the issue that one has to wait some unspecified time for the trajectory to wander to within a distance R of the desired periodic orbit. The method of directing trajectories, or "targeting", was developed to overcome this limitation. Targeting also takes advantage of a chaotic system's extreme sensitivity to changes in system parameters, and basically involves finding the pre-images of a desired target point subject to some perturbation in a system parameter. Targeting involves three main steps. First, one chooses an arbitrary starting point X_0 , and then follows many trajectories that start in the neighborhood of X_0 , with the control parameter p set to some value $p_0 - p_{\max} < p < p_0 + p_{\max}$ for one cycle of the system. One follows the trajectories until they have spread themselves out along the unstable manifold on a scale comparable to the attractor size. Second, one evolves a neighborhood of points near a target point X_t , backwards in time at the nominal value p_0 until these trajectories also spread out to a size comparable to the attractor. These trajectories will be distributed along the stable manifold of the system. Third, one finds the intersection of the stable and unstable manifolds for these groups of trajectories, since these intersections represent points that start near X_0 , and end up near X_t . The value of the perturbation p to be applied can be obtained in a

straightforward manner to guide the trajectory from these intersections to the target. Shinbrot *et al.* [2.22] used this method of targeting effectively in a magnetoelastic ribbon experiment to rapidly direct orbits from an arbitrary initial state to an arbitrary desired state.

2.2.3. OPF Method

After this success, several more experimental verifications were made, each with a slightly different twist on the OGY method. One of the main successes along this line is a method discovered by Hunt [2.24]. His scheme avoids the OGY problem of a slow, complicated feedback by applying analog linear feedback to the chaotic system. He termed his method Occasional Proportional Feedback (OPF). Instead of trying to find a control law for the parameter changes p_n , Hunt used the *ansatz* that $p_n = A_0(X_{i,n} - X^*) + p_0$. Here X^* is the desired set point, p_0 is the nominal value of the perturbed parameter, $X_{i,n}$ is the i th component of the state X at time n , and A_0 is the proportional gain of the feedback. Thus, one is left to find an appropriate gain and window in which to activate the feedback, but the method is very fast when such parameters are found. Using this method, Hunt was able to stabilize limit cycles in a diode resonator circuit up to 23 drive cycles long. The diode resonator circuit [2.25-2.27] consists of a p - n junction rectifier in series with an inductor, and exhibits low-dimensional chaos when driven by a sinusoidal voltage of sufficient amplitude. This chaotic system is well characterized by a two-dimensional map. Hunt was able to achieve control with changes of less than 0.5% in the drive

amplitude, but he did not offer any analytic method for determining what window size or feedback gain to use.

Building on Hunt's work, Roy and others [2.28-2.30] were able to stabilize a chaotic multi-mode laser using the OPF technique. This work was significant because it allowed the laser to be driven at 15 times higher steady state output power, and it was a successful application to a system that is more than three dimensional. Peng, Petrov, and Showalter have used OPF to control chemical chaos [2.31-2.33]. Others have extended the OPF method by allowing for control laws which are more complicated functions of $(\mathbf{X}_{i,n} - \mathbf{X}^*)$ [2.34-2.37]. Carr and Schwartz [2.37] extended the OPF technique to high-dimensional systems.

2.2.4. Pyragas Method

Another twist to the OPF technique is to allow a temporal delay in the applied control. Pyragas was the first to introduce this technique [2.38], whereby his control law took the form $p_n = A_0(\mathbf{X}_{i,n-\tau} - \mathbf{X}^*) + p_0$. Thus, he was able to stabilize unstable periodic orbits of period τ when he used this method on the Rössler and Lorenz equations. This method has the advantage that the built-in delay in the control allows for the stabilization of a given UPO based only on knowledge of its period, without any knowledge of the existence, location, or properties of these orbits. However, there is no guarantee of effectiveness, nor is there the ability to distinguish between degenerate UPOs that have the same temporal period, but different spatial structure. Nonetheless, others soon

followed with more numerical and experimental verifications of the Pyragas time delay method [2.39-2.43].

2.2.5. Other Variations

There are almost too many other attempts to control chaos to be mentioned, most of which are hybrids of the above methods, and yet which don't fall neatly into any of the previous categories. One of the simplest methods is just to add straight feedback to the system, and then pick the correct proportionality constant for the new dynamical system that produces the desired behavior [2.44]. People have controlled chaos by applying proportional feedback in short pulses to a system variable at fixed time intervals [2.45]. Others have applied control feedback continuously instead of just at discrete Poincaré mapping times [2.46]. Another method involves constructing three different maps of the system at three different parameter values, then picking one's way from map to map to find a desired state [2.47]. Yet another method involves what the authors call a "differential geometric method" whereby they algebraically transform the nonlinear dynamics into a linear one, so that linear control techniques can be applied. [2.48]. Another method is to employ an OGY-like scheme, except the Poincaré plane is coarse-grained so that a symbolic dynamics approach may be used [2.49].

Of these last techniques, there are two notable experimental successes. One experiment involved controlling cardiac alternans (i.e., irregularities in heartbeat timing) in a dissected rabbit heart by changing the delay time for signals between a structure called the HIS bundle and the atrium cardiac tissue [2.50]. Another experimental success

was the control of fluid flow in a thermal convection loop [2.51]. A thermal convection loop consists of a narrow diameter pipe containing fluid, which is bent into a torus, stood upright in a vertical plane, and heated uniformly over the bottom half and cooled uniformly over the top half. By increasing or decreasing the heating by some preset amount whenever the temperature difference between two diametrically opposed points exceeded or dropped below a preset value, they were able to laminarize the flow both experimentally and numerically.

2.3. Open Loop Control

Open loop control methods, which do not employ feedback, broadly fall into four categories. The first category, which predates OGY control, is based on an extension of the simple harmonic oscillator, and is known as entrainment. The next two categories employ stochastic and chaotic drives to control chaos. Finally, the effect of using periodic perturbations to control chaos will be examined.

2.3.1. Entrainment

An early method for controlling chaos was formulated by Hübler *et al.* in 1989 [2.52]. Their method arose from their work in "resonant stimulation" [2.53,2.54]. Resonant stimulation is the term they coined for a method they used to efficiently excite damped nonlinear oscillators from rest.

To obtain a better intuition for what they were trying to do, consider a damped linear oscillator:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0. \quad (2.1)$$

For a system consisting of a mass m attached to a spring of constant k , $\omega_0 = \sqrt{k/m}$, and γ is the damping (or frictional coefficient). The solution of this equation for $\gamma < \omega_0$ is

$$x(t) = x_0 e^{-\gamma t} \sin(\omega' t - \delta), \quad \omega' = \omega_0 \sqrt{1 - \gamma^2 / \omega_0^2}. \quad (2.2)$$

Thus, the natural behavior of the system is an exponentially damped, monochromatic sinusoid. If the system is then subject to a time-dependent drive,

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = F(t), \quad (2.3)$$

a periodic driving force $F(t) = F_0 \cos(\omega t)$ results in the following steady-state solution:

$$x(t) = \frac{F_0}{\sqrt{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2}} \sin(\omega t - \delta'), \quad \tan(\delta') = \frac{\gamma \omega}{(\omega^2 - \omega_0^2)}. \quad (2.4)$$

The response of the system is still sinusoidal, regardless of the frequency of the applied drive. A "resonant stimulation" in this case corresponds to a sinusoidal drive of frequency $\omega = \omega_0$, which results in the largest amplitude motion.

In analogy to the damped linear oscillator, Hübner *et al.* tried to find a driving function $F(t)$ that would result in large amplitude motion for a nonlinear oscillator given by

$$\ddot{x} + \eta \dot{x} + K_1(x) = F(t). \quad (2.5)$$

Here, K_1 is the nonlinear spring force. The response of such an oscillator is generally not monochromatic, and so the challenge lies in finding the special form of the drive $F(t)$ that produces a large-amplitude response, or resonant stimulation.

According to Hübler *et al.* [2.52], the perturbation $F(t) = 2\eta_1 \dot{y}$, where $y(t)$ is a solution of the unperturbed equation

$$\ddot{y} - \eta_1 \dot{y} + K_1(y) = 0, \quad (2.6)$$

is resonant because it is always in phase with the unperturbed system. Hübler *et al.* go on to state that when the explicit form of the deterministic equations are not known, the experimental time series, and thus the drive $F(t)$ can be approximated by a truncated Fourier expansion of the extrema of the dynamics. While they admit that the coefficients may need to be re-calculated continuously, this method can be used to entrain the nonlinear system to any desired dynamics through proper manipulation of the Fourier coefficients of the drive. They also found that large drive amplitudes were sometimes required, and that it was not always possible to control such systems after a sudden change of parameters.

Hübler then teamed up with Jackson in 1990 [2.55-2.57] to extend this method to any general chaotic system. The basic notion is that given a system described by a set of differential equations

$$\dot{\mathbf{X}}(t) = \mathbf{S}(\mathbf{X}(t)) + \mathbf{F}(t), \quad (2.7)$$

then the most general pursuit of open-loop control is to obtain a drive function $\mathbf{F}(t)$ that will make the system obey instead a set of goal equations

$$\dot{\mathbf{Y}}(t) = \mathbf{G}(\mathbf{Y}(t)). \quad (2.8)$$

By making the choice

$$\mathbf{F}(t) = -\mathbf{S}(\mathbf{Y}(t)) + \mathbf{G}(\mathbf{Y}(t)), \quad (2.9)$$

equation (2.8) formally gives a solution, provided $\mathbf{X}(t) = \mathbf{Y}(t)$. The stability of the desired state, $\mathbf{X}(t) = \mathbf{Y}(t)$ depends on the details of the problem and on the particular value of $\mathbf{X}(t)$ when the control is initiated. Note that even identical systems do not always entrain. Thus, while this method allows for a wide range of behaviors, its implementation depends intimately on the system being controlled.

In 1993, Keefe [2.58] compared the entrainment and OGY methods for controlling the Ginzburg-Landau equation. He was able to achieve limit-cycle behavior in this 120.2 dimensional system with both methods using perturbations of less than 1% of the energy of the chaotic motion. However, he was only able to do this when he artificially forced the dynamics close to the UPOs he was trying to stabilize.

2.3.2 Stochastic Drives

Breeden *et al.* [2.59] presented one of the first papers describing the usefulness of noise for controlling chaotic systems. They observed that a system in stable equilibrium is impossible to model. Ideally, one would give a kick to the system and study its transients to get a clear view of the state space. However, this is often impossible to achieve, and they show that the addition of external noise and the transients this noise creates can be useful in increasing the observable state space, which can then be used to enhance one's model of the system. This is directly applicable to any chaotic control technique that requires a fairly good model of the system being controlled, such as the entrainment schemes of Hübner and Jackson [2.52-2.57], as discussed above.

Fahy and Hamann [2.60] and Chen [2.61] showed that the trajectories of interacting particles moving in a spatial potential will eventually converge to a single unique orbit if their velocities are randomly reset at a regular time interval according to a prescribed probability distribution. Maritan and Banavar [2.62] found that different trajectories of identical chaotic systems that are exposed to identical large-amplitude stochastic drives converge to a single trajectory.

Noise can also control chaos by affecting the average dynamics of the system. Molgedey *et al.* [2.63] derived the mean field equations for a chaotic fully connected neural network with random, long-range interactions in the presence of additive external noise to show that the critical value of a parameter at which chaos onset is increased. Along a similar line, Hakim and Rappel [2.64] obtained the mean field equations for a globally coupled version of the complex Ginzburg-Landau equation with additive white noise. They found that small amounts of noise resulted in a pronounced increase in the periodicity of the dynamics, while larger amounts of noise caused a phase transition to a zero mean field.

The closest phenomenon to using noise to induce chaos rather than control it probably lies in the area of stochastic resonance (SR). Stochastic resonance is a phenomenon whereby certain nonlinear systems, subject to weak input signals, have the property that an optimum amount of external noise increases the coherence of the output (i.e., the signal-to-noise ratio) [2.65]. Stochastic resonance is a threshold phenomenon. It is typically illustrated in the paradigm of the weakly oscillating bistable potential. A particle in this potential that does not have enough energy to move from one local

minimum to the other has a probability of gaining enough of a boost from a random kick to clear the local maximum, thereby establishing an oscillation. This paradigm may provide insight into the synchronization of the randomly kicked particles in a potential studied by Fahy, Hamann, and Chen [2.60,2.61].

Another paradigm for SR is that of the limit-cycle to fixed-point (i.e., Hopf) bifurcation. If a controllable parameter for a nonlinear system is at a level that results in fixed point behavior, but is near a bifurcation point, then adding random noise of sufficient amplitude to that parameter can push the system into limit cycle behavior. This second paradigm seems to be tied to the work of Breeden *et al.* [2.59], and has been observed experimentally in a weakly ionized rf magnetoplasma by I and Liu [2.66]. While SR does not focus on forcing stable systems to become chaotic, it does seem to share some phenomenology with controlling chaotic systems with a stochastic drive.

2.3.3. Chaotic Drives

The first work on controlling chaotic systems with chaotic perturbations was done by Pecora and Carroll in 1990 [2.67, 2.68]. Control to them, however, was not obtaining fixed points or limit cycles for a single chaotic system, but rather getting two chaotic systems to phase lock together as a single chaotic system. This is known as synchronization, and it has spawned a deluge of research, since it has applications to secure communications [2.69]. Information can be encoded in one chaotic system (the transmitter), which can then be extracted by another chaotic system (the receiver) via the receiver's synchronization to the transmitter. The signal will look nearly random to

anyone intercepting it in transit, and will be undecipherable unless the interceptor has a suitable chaotic system to synchronize to the signal. Pecora and others [2.70,2.71] also discovered that a pseudoperiodic chaotic drive could be used to eliminate multiple basins of attraction for periodic systems, thus forcing them always to be in phase, regardless of their initial conditions. Liu and Barbosa [2.72] found a coupling between two chaotic Lorenz systems that resulted in limit cycle behavior, even though each Lorenz system was on a different limit cycle. Outside of these notable examples, the literature on synchronization of chaotic systems is beyond the scope of this review, but additional references are given in [2.73-2.89].

2.3.4. Periodic Drives

The last open loop control to be discussed is that of the periodic perturbation. At this point, a distinction needs to be made between entrainment and control achieved by applying periodic perturbations. Entrainment involves choosing an arbitrary goal dynamics, which may not be inherent to the system, and then finding a suitable (usually non-periodic) drive to achieve this goal. With periodic perturbations, the controlled dynamics achieved rarely match the driving term exactly; the goal dynamics are not chosen arbitrarily, but are instead determined by the underlying UPOs of the system.

Applying periodic perturbations to nonlinear systems can both induce chaos and suppress it. Periodic perturbations applied to stable nonlinear systems can result in a quasiperiodic transition to chaos. The quasiperiodic scenario involves a competition, in a heuristic sense, between two or more independent frequencies that characterize the system

[2.128]. This competition can occur in nonlinear systems where a system with a "natural" oscillation frequency is driven by an external periodic force. An example of such a system would be a driven pendulum. The quasiperiodic scenario also occurs in systems that spontaneously develop oscillations at two or more frequencies as some parameter is varied. Many fluid systems exhibit this route to chaos.

A beautiful experimental realization of the quasiperiodic route to chaos was performed by Stavans *et al.* in 1985 [2.90]. They studied a mercury-filled Rayleigh-Bénard cell. A Rayleigh-Bénard cell consists of two parallel plates separated by a fluid, with the bottom plate held at a higher temperature than the top plate. As the temperature difference increases, steady-state conduction gives way to steady-state convective rolls, which can then develop into full turbulence. Stavans *et al.* set the temperature difference to achieve steady-state convective rolls, and then imposed another frequency on the system with an ac current. They observed the predicted result that chaotic behavior occurs when the ratio of the frequency of the steady state roll to the drive frequency, or winding number, was irrational. The winding number is related to the rotational transform (ι) familiar to plasma physicists. An irrational winding number means that the dynamics covers the surface of a 2-torus in the state space, just as an irrational ι indicates that a magnetic field line will cover a flux surface.

Stavans *et al.* found that when the winding number was rational, then the steady-state rolls would mode-lock to the drive frequency. They also verified that the larger the amplitude of the drive, the more the drive frequency could be de-tuned from the system's natural frequency and still maintain a locked state. When the regions of mode-locked

behavior are plotted on a 2-D grid of drive amplitude vs. winding number, so-called Arnold tongues are observed. Figure 2.2 illustrates Arnold tongues for the sine-circle map, a paradigm for quasiperiodicity [2.124]:

$$\theta_{n+1} = (\theta_n + \beta + \frac{K}{2\pi} \sin(2\pi\theta_n)) \bmod 1. \quad (2.10)$$

Figure 2.2. Locked-mode regions (shaded black) in a diagram of external drive amplitude (K) vs. winding number (β) for the sine-circle map.

Periodic perturbations have been used on a number of chaotic systems. Probably the simplest such system is the logistic equation, first proposed by May as a simple model for population dynamics [2.91]: $x_{n+1} = ax_n(1-x_n)$. Since this is a discrete-time system, the unstable periodic orbits have periods corresponding to all natural numbers. This system

was one of the first chaotic systems to be tested for control by periodic perturbations, and several researchers have perturbed the parameter a in this model, or simply added a perturbative term [2.92-2.97]. The perturbed logistic equation will be discussed in greater detail in Chapter 3. Alekseev and Loskutov [2.98] made another early attempt to control chaos in a numerical system. They considered an ODE model of biological population dynamics, but they gave no prescription for how they chose the amplitude or frequency of their perturbation.

Another widely studied numerical model is the Duffing-Holmes oscillator, a driven nonlinear oscillator with a cubic nonlinearity:

$$\ddot{x} - x + \beta x^3 = -\alpha \dot{x} + \cos(\omega t). \quad (2.11)$$

Lima and Pettini showed in 1990 [2.99] that this chaotic system could be controlled by periodically perturbing the parameter β : $\beta \rightarrow \beta(1 + \epsilon \cos(\Omega t))$. They found that for rational winding numbers Ω/ω , chaotic motion was replaced by periodic motion. Qu *et al.* [2.100] later showed that the phase of the control term relative to the drive term can have important effects. Chacon and Bejarno [2.101] tried a more sophisticated method by adding a driving term to the Duffing-Holmes equation that was proportional to a Jacobian Elliptic Function (JEF), since JEFs are the simplest periodic solutions for nonlinear oscillators. However, Rödelberger, *et al.* [2.102] showed that this result was mainly due to the fundamental frequency in a Fourier expansion of the JEF, and that the higher harmonics of this function were unimportant, which meant that Chacon and Bejarno were only changing the amplitude of an effective harmonic driving force. Kivshar *et al.* [2.103] showed that the Duffing-Holmes oscillator could be controlled with a high frequency,

non-resonant perturbation. They showed that averaging the slow nonlinear dynamics over the fast perturbation results in a stable solution. Experimental verification for control of the Duffing-Holmes model came from Fronzoni *et al.* [2.104]. They controlled the chaotic behavior of a magnetoelastic ribbon, which is well modeled by the Duffing-Holmes equation. They periodically perturbed both the damping and nonlinearity coefficients at rational multiples of the drive frequency.

Braimin and Goldhirsch [2.105] studied the effect of adding a weak periodic control force to a damped, driven pendulum:

$$\ddot{\Theta} + G\dot{\Theta} + \sin\Theta = I + A \sin(\omega t) + \alpha \cos(\beta \omega t). \quad (2.12)$$

This equation is a good model for the dynamics of a Josephson junction. The voltage across the junction is given by $V = \langle \dot{\Theta} \rangle$, the time average of $\dot{\Theta}$, in this interpretation. They found that V depends non-monotonically on the winding number β , and that V can be drastically reduced, which in some cases corresponds to a periodic solution. Their results show that values of $\beta \approx 1/3$ and $2/3$ are especially effective, but they do not make any comparisons to the quasiperiodic Arnold tongues.

In 1994, Chacon [2.106] used a novel method for controlling a chaotic system with a periodic perturbation. He found that periodic pulses of varying amplitude, period, shape, and phase, when applied to a coupled pendulum-harmonic oscillator model, could also control chaos.

The control of laser systems with periodic perturbations has received considerable attention, in addition to the closed-loop methods described before. In 1994, Meucci *et al.* [2.107] demonstrated that chaos could be controlled in a CO₂ laser by perturbing the

amplitude of an intracavity electro-optic modulator. They perturbed the modulator's amplitude with frequencies that were rational fractions of the modulator's frequency. Simonet *et al.* [2.108] also did a very nice experiment in 1994 on the experimental and numerical control of a ruby NMR laser with periodic perturbations. They found that both harmonic and square wave modulation of the Q factor at rational multiples of the laser output frequency resulted in periodic behavior. They also observed large deviations from the Arnold tongues that are observed in the standard 1-D circle map. In 1995, Liu *et al.* [2.109] showed that a numerical model for a semiconductor laser could be controlled by periodically driving the diode injection current at a frequency determined from mode analysis of the output intensity.

Azevedo and Rezende [2.110] presented one of two other notable examples of controlling chaos in an experimental system. They were able to control chaos in spin wave instabilities in a microwave-pumped yttrium iron garnet (YIG) sphere by periodically modulating the biasing magnetic field surrounding the sphere. They found that effective frequencies were rational multiples of the auto-oscillation frequency of spin waves in the sphere established just before the onset of chaos due to sufficient microwave power. Vorha *et al.* [2.111] put a slight twist on controlling and inducing chaos with periodic perturbations by perturbing a magnetoelastic ribbon at subharmonics of the frequency corresponding to the power spectral peak (which also happened to be the frequency of the chaos-inducing drive).

An interesting twist to periodic perturbations involves physically rotating the environment that contains the system being studied. Zik *et al.* [2.112] showed that

rotating a cylinder containing granules of different diameters would cause the granules to separate. Shan *et al.* [2.113] showed that rotating a cylinder containing a current-carrying magnetofluid could stabilize the fluid against helical magnetohydrodynamic activity.

2.4. Control of Chaos in Plasmas

With a background in the methodologies of chaos control, one may examine the attempts at controlling plasmas more objectively. There has not been much work on controlling chaos in plasmas, most likely because the methods outlined above are most effective on low dimensional systems. However, there have been a few numerical and experimental attempts.

One attempt at control by Gang and Kaifen [2.114] was performed on a one dimensional drift wave model. By a one dimensional model, it is meant that the model describes dynamics in one spatial direction only; the model itself is a partial differential equation, which is infinite dimensional. However, Gang and Kaifen found that thirteen spatial Fourier modes could adequately describe the dynamics, which left 26 ordinary differential equations. They found that by adding a negative feedback term to one of the modes, they could elicit limit cycle behavior from the model. They noted that in real space, this form of feedback corresponds to launching a monochromatic wave into the system. In an experimental application, one would have to monitor the system at a number of spatial points to determine the mode structure and the proper wavelength. They did not describe how they picked the parameters that yielded their controlled case.

Buti [2.115] examined a model for magnetoplasmas that possess chaotic Alfvén waves due to external driving, such as are found in the interplanetary medium, planetary bow shocks, and cometary environments. He found that the addition of a small amount of dust grains, on the order of 10^{-8} of the proton density, caused the chaos to disappear due to inertial stabilization by the dust grains.

Sätherblom [2.116] performed an interesting numerical study in 1997 on controlling chaos in an RFP modeled by the DEBS code [2.117]. He used a Pyragas method of control by applying a driving term that constitutes the difference between an external periodic signal and the actual toroidally applied electric field. Thus, he was modulating the loop voltage for a desired plasma current. The DEBS code is set up to keep the toroidal field and the pinch parameter $\Theta = B_\phi(a)/\langle B_\phi \rangle = aI_\phi/2I\Phi_\phi$ approximately constant by adjusting the toroidal loop voltage according to:

$$\frac{\partial \langle E_\phi(a) \rangle}{\partial t} = -C_0(I_\phi - I_0). \quad (2.13)$$

Sätherblom added a drive term to (2.13) of the following form:

$$F(t) = k[\cos(2\pi f_g t) - E_\phi(a)]. \quad (2.14)$$

He examined amplitudes up to $k=4.0$, and frequencies f_g up to three times the frequency at which the fluctuations of the toroidal loop voltage had their peak power. He used 161 radial grid points, poloidal mode numbers from $m=0$ to 2, toroidal mode numbers from $n=-21$ to 21, and an on-axis Lundquist number of 10^4 for a pinch parameter of $\Theta \approx 1.7$. He did not find any cases where limit cycle behavior was achieved, but he did find that the

magnetic helicity increased, while the magnetic energy remained unchanged. This result is related to recent work done by the MST group, which has ongoing experiments in helicity injection for improving confinement using current injected by plasma guns [2.118-2.120], and pulsed poloidal current drive [2.121,2.122]. Stherblom also noticed that the helicity increased in a stepwise manner when the loop voltage underwent a π phase shift. Interestingly, Braimin and Goldhirsch [2.105] found that the variable Θ in their model of a Josephson junction also experienced stepwise jumps when the drive underwent a total phase shift of 2π .

Control of chaos in actual physical plasmas has only been achieved in glow discharges, which are inherently low dimensional. Ding *et al.* [2.123] performed small periodic modulations to the discharge voltage of an unmagnetized argon plasma. The power spectrum of fluctuations in the discharge current showed a peak at $f_0=1.8$ kHz, and when they modulated the discharge voltage at f_0 and $2f_0$, they found that the chaos disappeared for modulation amplitudes greater than 200 mV, which was about 1% of the unperturbed discharge voltage. Weltmann *et al.* [2.124] examined chaotic ionization waves in a periodically driven neon glow discharge column, which they determined to have a dimension of approximately 3.5. They used Hunt's OPF technique by applying a proportional feedback to the amplitude of the plasma-sustaining drive whenever the local axial electric field was within a specified window of its value k drive cycles in the past. They were able to stabilize periodic orbits in this way for values of k from 3 to 16.

2.5. Summary

Several methods for controlling chaos have been reviewed. Early attempts using linear feedback were unsuccessful due to the nonlinearity and sensitive dependence to initial conditions of chaotic systems. Ott, Grebogi, and Yorke [2.5] proposed a novel method for chaotic control based on intuitive geometric concepts that used structures inherent in chaotic systems. Their method was widely applicable to a variety of systems, flexible in the types of controlled dynamics it allowed, used a minimally applied control signal, and showed long-term stability. Several workers subsequently refined and improved the OGY method [2.17-2.23], most notably using occasional proportional feedback [2.24-2.37] and time-delayed feedback [2.38-2.44].

The open-loop control method of entrainment [2.52-2.58] holds promise for forcing a chaotic system to follow any desired dynamic, but it frequently requires large forcing, and extensive analysis of the basin of attraction (i.e., the set of valid initial conditions) for which the control will be effective. Control by stochastic drives [2.59-2.64] has mainly been studied as a limited illustration of principle, while control by chaotic perturbations [2.67-2.89] seems to have applicability mostly for synchronization of two chaotic systems.

The control of chaotic systems with periodic perturbations [2.93-2.124] is attractive because it is easy to implement on a wide variety of systems, and previous work has shown that the amplitude of the perturbation can be small. However, much of the previous work has focused on systems in which the chaos is present due to an external driving, which provided an obvious clue that control could be achieved by choosing a perturbation frequency that is a rational multiple of the drive frequency. Azevedo and

Rezende [2.110] identified suitable perturbation frequencies by measuring the frequency of the oscillation of spin waves before they came chaotic, while Stherblom [2.116] and Ding *et al.* [2.123] used multiples of some dominant fluctuation frequency. However, it is unclear what to do if the system is not chaotic due to an external drive, the route to chaos can not be observed, or the power spectrum is essentially broad-band. What is required is a return to the geometric underpinnings of OGY by identifying the underlying unstable structures in the chaotic system.

Unstable periodic orbits can be identified from experimental data without any knowledge of the system's dynamics. Once these structures have been identified, their natural frequency can be determined, which will give an indication of what perturbation frequencies will be the most successful. The next chapter will undertake this task and show that perturbations applied at these frequencies can indeed control low dimensional chaotic systems. For high dimensional systems (i.e., $D \geq 5$, which have not been addressed in general in the literature), it is shown that the most effective frequencies do not always correspond to dominant fluctuation frequencies.

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3. Periodic Parametric Perturbations Applied to Numerical Systems

Controlling chaotic systems with a periodic perturbation seems like a simple enough proposition, until one needs to choose the frequency of the perturbation. The problem can be especially vexing if there are no obvious clues, such as a periodic drive that initiated the chaos in the first place, or a reproducible route to chaos that can be explored in the laboratory by varying some system parameter. For example, the natural frequencies in a period doubling route to chaos are a good starting point [3.1]. Peaks in the power spectrum of a fluctuating quantity also suggest worthy candidates [3.2,3.3], but as will be shown for the high dimensional case of coupled Lorenz equations, this predictor is not always reliable.

In this chapter, controlling the chaos manifested in a number of low and high dimensional numerical systems will be illustrated. For each case, it will be shown that the optimum perturbation frequencies are rational multiples of the frequencies of unstable periodic orbits (UPOs) inherent in the system dynamics. These UPOs can be extracted directly from the time series of any state space variable, and thus no model for the system dynamics is required to predict optimum perturbation frequencies.

There were two main motivations in the choice of numerical models selected for study here. First, familiar low dimensional models were sought on which intuition could be developed. Second, early work on applying periodic perturbations to plasma boundary conditions in the DEBS model for the MST proved unfruitful, because nearly one month of wall time elapsed just to generate a few "milliseconds" worth of data. This was found to be unacceptable since hundreds of perturbation cases were typically desired for an

appropriate demonstration of the method. Thus, other plasma models and high dimensional systems were sought that would still yield valuable insight, while remaining numerically manageable. Thus, the numerical models presented here are: the logistic equation (a one-dimensional quadratic map), the Lorenz equations (a three-dimensional quadratic flow), the Rössler equations (another three-dimensional quadratic flow), coupled logistic equations (a 100-dimensional quadratic map), coupled Lorenz equations (a 96-dimensional quadratic flow), the Yoshida equations, which model magnetic fluctuations in a plasma fusion device (a six-dimensional and nine-dimensional polynomial flow), and a neural net model for poloidal field fluctuations in the MST (a 64-dimensional nonlinear map).

Diagnosing the effect of perturbations in high dimensional systems can be difficult, but it is not impossible for numerical systems, where the Lyapunov exponent spectrum can be calculated. This diagnostic was used for these systems, and it is described below, along with three methods for finding UPOs.

In addition to the resultant dimension of a perturbed system, two other measures of "control" were examined that have relevance to applications on MST. The first such measure requires a closer examination of the Lyapunov exponents.

Recent work has established relationships between Lyapunov exponents and transport coefficients. This can be understood heuristically by recalling that the Lyapunov exponent measures the rate at which a system "forgets" its initial conditions. The transport coefficients are those functions of the system that also measure "forgetting." For example, scattering erases a particle's memory of its original velocity, and so gives rise to

a finite diffusion coefficient. Barnett *et al.* [3.4] have shown that the leading Lyapunov exponent for an N -body dilute gas is proportional to the cube root of the diffusion coefficient. They verified this scaling in a numerical model for a one-component plasma. Another example is Gaspard and Nicolis [3.5], who found a connection between the diffusion coefficient of a Lorentz gas, its positive Lyapunov exponents and the Kolmogorov entropy. Evans *et al.* [3.6] found a relation between viscosity and the maximum and minimum Lyapunov exponents for a Lennard-Jones potential in a nonequilibrium molecular dynamics simulation. Cohen [3.7] extended this result to express the viscosity of a sheared fluid in a nonequilibrium stationary state in terms of the two maximal Lyapunov exponents. While the numerical systems examined here do not model many-body dynamics, the change of the leading Lyapunov exponent as a result of an applied perturbation will be examined.

The other definition of control arises from considering transport in fusion plasmas. Loss of confinement of particles and energy in a plasma is due to coherent fluctuations between various quantities, such as density and electrostatic potential. If the magnitude of these fluctuations could be diminished, the transport would subsequently decrease. The fluctuation magnitude of the controlled numerical systems will also be examined, and they will be seen to have direct application in the Yoshida equations, which model magnetic field fluctuations due to tearing mode activity.

3.1. Analysis Methods

3.1.1. The Lyapunov Exponent Spectrum

Eliminating chaos in a high dimensional system is difficult unless the system is artificially started extremely close to the UPO that is to be stabilized [3.2, 3.9]. A fully controlled chaotic system is easy to diagnose, since any time series will be periodic, but it is difficult to discern any effects if the time series does not become periodic. Decreasing the dimension by a significant amount can be a substantial improvement, since it opens the door for other closed-loop techniques [3.9], but it is harder to discern just by looking at a time series. This is where the Lyapunov exponent spectrum becomes an indispensable tool.

The Lyapunov exponent spectrum measures the exponential expansion or contraction of the principal axes of an ellipsoid centered on the trajectory of a chaotic system. Wolf *et al.* [3.10] gave an enlightened discussion of the Lyapunov exponent spectrum and its computation for both numerical systems and experimental time series in 1984. Their method has been used in this work. Previous discussions focused only on computing the Lyapunov exponent spectrum from numerical models [3.11-3.14]. A recent and speedier computation of the exponent spectrum is given by Rangarajan *et al.* [3.15].

There are N Lyapunov exponents ($\lambda_i, i=1, \dots, N$) for an N -dimensional system, and they are computed by iterating the Jacobian of the system on each of N basis vectors, which are continuously re-orthonormalized as the ellipsoid contorts while it follows the system's trajectory. The trajectory-averaged logarithm (in base e here) of the expansion or contraction of each of the basis vectors is the Lyapunov exponent (LE) for that direction:

$$\lambda_i = \left\langle \log \left| \frac{\partial f_i(x_i)}{\partial x_i} \right| \right\rangle, i=1,2,\dots,N. \quad (3.1)$$

Figure 3.1 shows this process graphically.

Figure 3.1. A schematic representation for the computation of the Lyapunov exponent spectrum. The exponential expansion and contraction of the axes of the ellipsoid are measured after some time dt , at which time they are re-set to a unit sphere. The axis corresponding to the LLE is merely re-set in magnitude, while the axes corresponding to successively smaller LEs have components parallel to directions corresponding to larger LEs subtracted out.

Once the Lyapunov exponent spectrum is computed, the exponents are arranged in decreasing order $\lambda_i > \lambda_{i+1}$. The first LE, λ_1 , is called the leading Lyapunov exponent (LLE). The LLE must be positive for a system to exhibit chaos. This sorting of the LEs is done as a prelude to using them to compute the fractal dimension of the system. The dimension of a chaotic system is obtained by applying the Kaplan-Yorke conjecture to the LE spectrum [3.16,3.17]. The Kaplan-Yorke conjecture states that the information dimension of the system is related to the Lyapunov exponent spectrum by

$$D_{KY} = j + \frac{\sum_{i=1}^j \lambda_i}{|\lambda_{j+1}|}, \quad (3.2)$$

where j is defined by the condition that

$$\sum_{i=1}^j \lambda_i > 0, \text{ and } \sum_{i=1}^{j+1} \lambda_i < 0. \quad (3.3)$$

D_{KY} is alternatively known as the Kaplan-Yorke dimension, or the Lyapunov exponent dimension. One should note that applying a periodic perturbation increases the dimension of the system by one, because an extra state space variable is added to the system. The Lyapunov exponent associated with this new state space variable is always zero, because it causes neither exponential expansion nor exponential contraction. It is not necessarily obvious that adding an external frequency to a system with an already broadband frequency spectrum will result in a decrease of its chaotic behavior, especially when one recalls that the presence of three incommensurate frequencies in a system can generically lead to chaos [3.18].

3.1.2. Unstable Periodic Orbits

The complicated trajectory of a chaotic system, when viewed in its state space, or a suitable embedding space [3.19], contains the vestiges of regular behavior. These are the UPOs of the system, and a chaotic trajectory spends its time wandering among them. A general procedure to identify the UPOs of a strange attractor was advanced by Lathrop and Kostelich [3.20]. It begins by re-constructing an attractor from a time series using the method of lags [3.19]. Then a point on the reconstructed attractor is picked, and its

subsequent images are observed until the reconstructed trajectory comes within some pre-defined ε of the starting point. The number of steps taken is recorded as m , and the initial point is called an (m, ε) recurrent point. The procedure is then repeated for many initial points on the attractor. When the collection of $\{m\}$ is histogrammed, it is often found that the m values cluster in discrete, integer multiples. The lowest common denominator of these discrete bunchings corresponds to a period-one limit cycle, and the temporal period of this and higher-period UPOs can then be obtained from knowledge of the time step associated with m . That is to say, a period two limit cycle for data sampled at a rate dt has a temporal period of $T_2 = 2m \cdot dt$. Section 3.3 shows some UPOs and an (m, ε) histogram obtained in this way for the Lorenz equations.

A simpler method to identify UPOs in low dimensional systems is to use a Poincaré plot. A Poincaré plot can be generated by plotting the value of a state space variable versus a prior value at certain specific times, such as when a second state space variable passes through a pre-determined value. A 45° line intersects the Poincaré plot at the spatial location of the UPOs. Thus, if values of x_{n+1} are plotted against x_n , the intersection with a 45° line marks the location of a period one UPO. By keeping track of the time between points that lie near the 45° line, the temporal period can be extracted. Higher order UPOs can be obtained by making Poincaré plots of x_{n+k} versus x_n . This method is illustrated in sections 3.3 and 3.4.

A final method to identify UPOs is a multivariate approach, in which all state space variables are used to find (m, ε) recurrent points, instead of relying on a

reconstructed attractor from a univariate time series, as was done in the first method. This method is especially good when a sliding window covering the last few thousand data points is maintained while the equations are iterated to identify the UPOs in real time. For low dimensional attractors, the UPOs obtained in this way bear a striking resemblance to the attractor. Experimentally, this method has the advantage of providing more data for systems in which long univariate time series may be impossible to obtain, provided that the various signals sampled have some degree of mutual independence. Sections 3.2 and 3.4 also have pictures of UPOs obtained with this method.

3.2. The Logistic Equation

The logistic equation is a well-known one-dimensional map with the following form [3.21]:

$$x_{n+1} = ax_n(1 - x_n). \quad (3.4)$$

Control was attempted by perturbing the parameter a as follows:

$$x_{n+1} = (a_0 + a_1 \cos(2\pi n/T + \phi))x_n(1 - x_n). \quad (3.5)$$

Here, a_1 is the amplitude, T is the period, and ϕ is the phase of the drive.

To compute the Lyapunov exponent spectrum for the perturbed map, one can use one of two methods to make it "autonomous." The first method is to split the map into two maps, with "time" represented by the second map. Letting $x_n(1)$ and $x_n(2)$ denote the two state space variables at time n , the maps take the following form:

$$\begin{aligned} x_{n+1}(1) &= (a_0 + a_1 \cos(2\pi x_n(2)/T + \phi))x_n(1)(1 - x_n(1)) \\ x_{n+1}(2) &= x_n(2) + 1. \end{aligned} \quad (3.6)$$

When the two Lyapunov exponents are calculated for this pair of maps, the second map results in a zero exponent (it neither expands nor contracts exponentially), and so, for computational expediency, it may be omitted altogether. Proof of this step follows from explicit calculation of the Lyapunov exponent spectrum:

$$J_n = \begin{bmatrix} \frac{\partial x_{n+1}(1)}{\partial x_n(1)} & \frac{\partial x_{n+1}(1)}{\partial x_n^{(2)}} \\ \frac{\partial x_{n+1}(2)}{\partial x_n(1)} & \frac{\partial x_{n+1}(2)}{\partial x_n(2)} \end{bmatrix}$$

$$= \begin{bmatrix} \left(a_0 + a_1 \cos\left(\frac{2\pi x_n(2)}{T} + \varphi\right) \right) (1 - 2x_n(1)) & -\frac{2\pi a_1}{T} \sin\left(\frac{2\pi x_n(2)}{T} + \varphi\right) x_n(1) (1 - x_n(1)) \\ 0 & 1 \end{bmatrix}.$$
(3.7)

The eigenvalues of this Jacobian are:

$$\left(a + a_1 \cos\left(\frac{2\pi x_n(2)}{T} + \varphi\right) \right) (1 - 2x_n(1)), \quad \text{and } 1. \quad (3.8)$$

The first eigenvalue leads to a LE of

$$\lambda_1 = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \log \left| \left(a + a_1 \cos\left(\frac{2\pi x_n(2)}{T} + \varphi\right) \right) (1 - 2x_n(1)) \right|, \quad (3.9)$$

while the second eigenvalue leads to a Lyapunov exponent of

$$\lambda_2 = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \log |1| = 0. \quad (3.10)$$

When the logistic map is in a chaotic regime, λ_1 is positive, and thus it is the LLE when ordering according to the Kaplan-Yorke conjecture. This first eigenvalue will thus be called the LLE for the remainder of this section, even when an applied perturbation

results in a periodic solution (for which $\lambda_1 \leq 0$). It is interesting to note that the LLE (λ_1) is identical in form to the Lyapunov exponent for the unperturbed logistic equation:

$$\lambda = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \log |a(1 - 2x_n)|. \quad (3.11)$$

A second method for making the perturbed logistic equation autonomous, which is only useful for rational period perturbations, is to break the perturbed logistic equation into p maps. Each map is made by taking p compositions of the logistic equation with p different parameter values. Here p is the period, or the numerator of the period ($T = p/q$; p, q integer with no common divisors); thus the perturbation term takes on p different values in one cycle of the perturbation: $\{a_1 \cos(2\pi qn/p + \phi)\}$. To simplify the definition of these p new maps, first define the parameters $A_1, \dots, A_p$:

$$\begin{aligned} A_1 &= a_0 + a_1 \cos(2\pi/T + \phi) \\ A_2 &= a_0 + a_1 \cos(4\pi/T + \phi) \\ &\vdots \\ A_p &= a_0 + a_1 \cos(2\pi p/T + \phi). \end{aligned} \quad (3.12)$$

A complete cycle of the perturbation is represented by the following p iterations:

$$\begin{aligned} x_{pn+1} &= A_1 x_{pn} (1 - x_{pn}) && \equiv f_1(x_{pn}) \\ x_{pn+2} &= A_2 x_{pn+1} (1 - x_{pn+1}) && \equiv f_2(x_{pn+1}) \\ &\vdots \\ x_{pn+p} &= A_p x_{pn+p-1} (1 - x_{pn+p-1}) && \equiv f_p(x_{pn+p-1}). \end{aligned} \quad (3.13)$$

Letting $x(1), \dots, x(p)$ denote the p state space variables, the following p maps are obtained:

$$\begin{aligned} x_{n+1}(1) &= f_1(x_n(1)) \circ f_p(x_n(1)) \circ f_{p-1}(x_n(1)) \circ \dots \circ f_2(x_n(1)) \\ x_{n+1}(2) &= f_2(x_n(2)) \circ f_1(x_n(2)) \circ f_p(x_n(2)) \circ \dots \circ f_3(x_n(2)) \\ &\vdots \\ x_{n+1}(p) &= f_p(x_n(p)) \circ f_{p-1}(x_n(p)) \circ f_{p-2}(x_n(p)) \circ \dots \circ f_1(x_n(p)). \end{aligned} \quad (3.14)$$

In general, the p maps can be summarized as:

$$x_{n+1}(i) = f_i(x_n(i)) \circ f_{i-1}(x_n(i)) \circ \dots \circ f_{i-p+1}(x_n(i)), \quad (3.15)$$

where the functional subscripts i are cyclic in $(1, \dots, p)$. Thus, these equations, when taken together, take p initial points through one complete period of the perturbation. Note that each composed map is independent of all the other maps, with their only connection being their initial conditions:

$$x_0(i) = f_{i-1}(x_0(i-1)), \quad i > 1. \quad (3.16)$$

When the maps are in a chaotic regime, the information on the initial conditions is rapidly lost, so that even this connection dissolves. While this method of eliminating non-autonomy from the perturbed logistic map is useful for analytic work [3.22-3.27], such as identifying crises [3.24,3.25] and for predicting effective perturbation amplitudes [3.22,3.26,3.27], it rapidly results in very high-order polynomial functions (of order 2^p) for larger periods that are cumbersome to manipulate. Hence, the first method has been used in the numerical portion of this work.

The choice of the period of the perturbation is simple, since the logistic map has unstable periodic orbits of all integer periods, provided the parameter a is large enough. In fact, there are 2^p unstable periodic orbits for any UPO of periodicity p when $a=4$ [3.28]. Most of the literature has focused on low-period perturbations. Here we will illustrate that control can be achieved for period two, three, and five perturbations. Then a non-integer drive period will be examined, which is equivalent to a very high-period drive. Finally, an irrational period drive and perturbations with two driving frequencies will be examined.

For a drive of period two ($T=2$), the perturbed logistic equation takes the following form:

$$x_{n+1} = (a_0 + a_1 \cos(\pi n + \varphi))x_n(1 - x_n). \quad (3.17)$$

The first method for providing autonomy involves the two maps:

$$\begin{aligned} x_{n+1}(1) &= (a_0 + a_1 \cos(\pi x_n(2) + \varphi)) x_n(1) (1 - x_n(1)) \\ x_{n+1}(2) &= x_n(2) + 1, \end{aligned} \quad (3.18)$$

while the second method involves constructing the following two composed maps:

$$\begin{aligned} x_{n+1}(1) &= -A_1 A_2^2 x_n(1)^4 + 2A_1 A_2^2 x_n(1)^3 - A_1 A_2 (A_2 + 1) x_n(1)^2 + A_1 A_2 x_n(1) \\ x_{n+1}(2) &= -A_1^2 A_2 x_n(2)^4 + 2A_1^2 A_2 x_n(2)^3 - A_1 A_2 (A_1 + 1) x_n(2)^2 + A_1 A_2 x_n(2), \end{aligned} \quad (3.19)$$

where $A_1 = (a_0 + a_1 \cos(\varphi))$ and $A_2 = (a_0 + a_1 \cos(\pi + \varphi))$. Figure 3.2 shows the effect of a $T=2$ perturbation on the logistic equation for $\varphi=0$. It is interesting to note that in the bifurcation diagram for the $T=2$ perturbed logistic equation, there seem to be two overlapping attractors with a subsequent enhancement of even period windows. These two attractors are resolved by the two composed maps $x_n(1)$ and $x_n(2)$.

Figure 3.2. Logistic map bifurcation diagram for a $T=2$ perturbation with $a_1=0.05$ and $\varphi=0$. The bifurcation diagram for the unperturbed map is shown in the top plot, while the bifurcation diagram for the perturbed map is directly underneath it. The LLE for the "first method" is plotted next, followed by the two maps obtained by the "second method," with their LEs plotted beneath them.

A period three drive results in the following two equivalent autonomous systems:

$$\begin{aligned} x_{n+1}(1) &= (a_0 + a_1 \cos((2\pi/3)x_n(2) + \varphi))x_n(1)(1 - x_n(1)) \\ x_{n+1}(2) &= x_n(2) + 1, \end{aligned} \quad (3.20)$$

$$\begin{aligned} x_{n+1}(i) &= -A_i A_{i+1}^4 A_{i+2}^2 x_n(i)^8 + 4A_i A_{i+1}^4 A_{i+2}^2 x_n(i)^7 \\ &\quad - 2A_i A_{i+1}^3 A_{i+2}^2 (3A_{i+1} + 1)x_n(i)^6 + 2A_i A_{i+1}^3 A_{i+2}^2 (2A_{i+1} + 3)x_n(i)^5 \\ &\quad - A_i A_{i+1}^2 A_{i+2} (A_{i+1}^2 A_{i+2} + 6A_{i+1} A_{i+2} + A_{i+2} + 1)x_n(i)^4 \\ &\quad + A_i A_{i+1}^2 A_{i+2} (A_{i+1} A_{i+2} + A_{i+2} + 1)x_n(i)^3 \\ &\quad - A_i A_{i+1} A_{i+2} (A_{i+1} A_{i+2} + A_{i+1} + 1)x_n(i)^2 + A_i A_{i+1} A_{i+2} x_n(i), \end{aligned} \quad (3.21)$$

where the A_i are cyclic on (1,2,3) and $A_1=(a_0+a_1\cos(\varphi))$, $A_2=(a_0+a_1\cos(2\pi/3+\varphi))$, and $A_3=(a_0+a_1\cos(4\pi/3+\varphi))$. Figure 3.3 shows the bifurcation diagram for the $T=3$ case. In the $T=3$ case, the period three limit cycle window, which for the unperturbed case starts at $a = 1 + \sqrt{8}$, has been triplicated, and the range of a_0 for which limit cycles with periods which are multiples of three have been extended. The three period three windows correspond to fixed point solutions of the three composed maps.

Figure 3.3. Logistic map bifurcation diagram for a $T=3$ perturbation with $a_1=0.05$ and $\varphi=\pi/2$, showing the unperturbed bifurcation diagram, the perturbed map, its three composed maps, and the LE.

Figure 3.4 shows the effect of a $T=5$ perturbation. The five composed maps were not computed analytically, but one can still see how they contribute to the dynamics and the enhancement of the period five (and its multiples) windows. This tendency to stabilize UPOs of higher periodicity is an effect that will be seen in other systems.

Figure 3.4. Logistic map bifurcation diagrams for a $T=5$ perturbation with $a_1=0.05$ and $\varphi=\pi/2$.

Figure 3.5 answers the question of whether a perturbation can decrease the overall chaos for a nonlinear system over a broad range of parameters. It shows the effect of perturbing the logistic map with a $T=2$ and with a $T=3$ perturbation with amplitude

$0 < a_1 < 0.2$ for two different phases φ . Cases of the logistic map with $3.5 < a < 4$ were examined. Black indicates periodic regions, white indicates chaotic regions, and gray indicates unbounded regions. For $T=2$ and $\varphi=0$, the percentage of periodic cases was increased from 22.3% to 24.9% over this entire range of a and a_1 . The $T=2$, $\varphi=\pi/3$ case results in an overall decrease in the periodic solutions from 22.3% to 12.8%. This difference due to the phase can be understood when one realizes that different phases result in different values for the parameters in the composite maps, which ultimately control the behavior of the system. A difference due to phase can also be seen in the $T=3$ cases. For $\varphi=\pi/2$, the number of periodic cases is increased to 26.9%, while for $\varphi=0$, the number is decreased to 16.2%. The effect of the phase φ is also manifested by the loss of the "middle periodic ray" when $\varphi=0$. This is due to the fact that for $\varphi=0$, the logistic map is only being perturbed with two different net perturbation amplitudes: a_1 and $-a_1/2$.

Figure 3.5. Summary of effects of $T=2$ and $T=3$ perturbations on the logistic map over a range of nominal parameter values a_0 and perturbation amplitudes a_1 for different phases. Clockwise, from upper left: $T=2$, $\varphi=0$; $T=2$, $\varphi=\pi/3$; $T=3$, $\varphi=\pi/2$; $T=3$, $\varphi=0$. Black indicates periodic solutions, white chaotic solutions, and gray unbounded solutions.

The result of perturbing the logistic map with a non-resonant period is shown in the bifurcation diagram in Figure 3.6, where a period of $T=3.1$ is used. This period corresponds to a frequency of $f = 1/3.1 = 10/31$. Thus, this perturbation tends to stabilize a period-31 UPO, which is not readily observed in the unperturbed bifurcation diagram. However, when the limit-cycle windows of Figure 3.6 are minimally magnified, such high-period orbits are easily seen. The left-hand magnification shows half of two period-62 limit cycles, while the left-hand magnification shows a period-31 limit cycle. This

observation suggests that high-order perturbations might be used as a “probe” to find regions in parameter space where high-order unstable periodic orbits exist.

Figure 3.6. Effect of a $T=3.1$ perturbation on the logistic map. Notice the enhancement of period-62 (left-hand magnification) and period-31 windows (right-hand magnification).

As a last comment, it would seem that the only “off-resonance” perturbations that can be applied are those with irrational periods, or periods that at least seem irrational to the limits of computational precision. Figure 3.7 shows that the case for the maximally irrational period of $T=3+(\sqrt{5}-1)/2$ (the golden mean) with $a_1=0.05$ destroys the periodic windows.

Figure 3.7. Effect of a $T=3+(\sqrt{5}-1)/2$ perturbation for $a_1=0.05$; all periodic windows are destroyed.

Another way to perturb the system is to apply two perturbations at different frequencies:

$$x_{n+1} = \left(a + a_1 \left(\cos(2\pi n/T_1 + \varphi_1) + \cos(2\pi n/T_2 + \varphi_2) \right) \right) x_n (1 - x_n). \quad (3.22)$$

The effect of such a double perturbation is readily understood if the perturbation is re-written as follows:

$$x_{n+1} = \left(a + 2a_1 \cos \left(\pi n \frac{T_1 + T_2}{T_1 T_2} + \frac{1}{2} (\varphi_1 + \varphi_2) \right) \cos \left(\pi n \frac{T_2 - T_1}{T_1 T_2} + \frac{1}{2} (\varphi_1 - \varphi_2) \right) \right) x_n (1 - x_n). \quad (3.23)$$

Thus, a double perturbation has the same effect as applying a single perturbation with a period the same as the least common multiple of T_1 and T_2 . This case is illustrated in the bifurcation diagram in Figure 3.8 where $T_1=2$ and $T_2=3$. The effect should be similar to applying a $T=6$ perturbation, which is evidenced by the widening of the $T=6$ windows.

To conclude this section, applying a periodic perturbation to the logistic equation has effects that will be seen in other systems. Small-amplitude perturbations can create windows of periodic behavior when the period of the perturbation is equal to the period of an inherent UPO in the system. Periodic perturbations at UPO frequencies can also decrease the overall amount of chaos over a range of unperturbed parameter values. Perturbations at frequencies that do not correspond to any UPO in the system eliminate periodic behavior for all system parameter values.

Figure 3.8. Effect of a jointly applied $T=2$ and $T=3$ perturbation on the logistic map.

3.3. The Lorenz Equations

The Lorenz equations [3.29] specify a three-dimensional flow given by the following ordinary differential equations:

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= x(r - z) - y \\ \dot{z} &= xy - bz.\end{aligned}\tag{3.24}$$

Here, the parameters $\sigma=16$, $r=45.92$, and $b=4$ were used. In this and the remaining cases, except where noted, the equations were integrated with a fourth-order Runge-Kutta method. Lorenz derived these equations to model heat convection in a Bénard convection

cell, which consists of two horizontal, infinite plates with a temperature difference of ΔT across the fluid between the plates. After truncation of the fluid equations and renormalization of several of the fluid quantities, the parameter r is proportional to the temperature difference. Thus, in the spirit of perturbing a physically accessible parameter that is likely to have a noticeable affect on the system, the parameter r was chosen with a perturbation of amplitude r_1 and angular frequency ω . The system of equations thus becomes:

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= x(r_0 + r_1 \sin(\phi) - z) - y \\ \dot{z} &= xy - bz \\ \dot{\phi} &= \omega.\end{aligned}\tag{3.25}$$

Note that a fixed point solution is not possible, since $\dot{\phi}$ is never zero (unless $\omega=0$).

Once the parameter to be perturbed has been selected, the next step is to choose a perturbation frequency. Since there is no external drive frequency apart from the added control, and it is assumed that r can not be varied over a wide enough parameter range to study the route to chaos, the perturbation frequency is chosen by identifying the UPOs for the system.

Using the method of Lathrop and Kostelich [3.20], a time series is generated by recording values of the state space variable x at every integration time step. Then the attractor is reconstructed in a three-dimensional state space using the method of lags [3.19]. From these reconstructed points, one can measure the number of time steps it takes for the attractor to return to within a distance ε of any given point. Figure 3.9 shows a reconstructed Lorenz attractor and the resulting histogram of the number of occurrences

of UPOs with recurrence time m for $\varepsilon=0.08$. A suitable value for ε is found to be roughly a factor of two larger than the average state-space distance between successive points on the reconstructed attractor. For the example shown here, 8,000 data points (which corresponds to roughly 60 orbits around the unstable fixed points) were required to find 200 (m, ε) points in the reconstructed data. The (m, ε) recurrent points are clustered in integer multiples of $\Delta m=124 \pm 9$, which for a time step of 1/256 seconds corresponds to 0.48 ± 0.04 seconds.

Figure 3.9. Clockwise, from upper left: The Lorenz Attractor; reconstructed Lorenz attractor from time lags of $x(t)$; histogram of number of UPOs with recurrence time m (the discrete clusters occur at $\Delta m=124$); two period-two ($m=238, 239$) UPOs.

Since the Lorenz attractor is low dimensional ($D_{KY} \cong 2.07$ for these parameters), a Poincaré map can also be used to determine the periods of the UPOs. Both of the values of the state space variables x and $\phi (= \omega t)$ are recorded when the attractor trajectory crosses the plane $z = r - 1$, which contains two of the fixed points of the unperturbed attractor. If the elements of the time series for x that results are denoted as x_n , the intersection of the 45° line with plots of x_{n+k} vs. x_n denote UPOs of period $k/2$, and the temporal value of this period can be obtained from ϕ .

Figure 3.10 shows various Poincaré plots for the Lorenz equations obtained in this way. It is interesting to note that the plot for x_{n+2} vs. x_n appears to intersect the 45° line, indicating the presence of a period one UPO. The two apparent intersections happen to coincide with the location of two unstable fixed points for the Lorenz attractor. This means the curves only approach the line asymptotically without intersecting it, because these two fixed points are spiral repellers. Thus, there is no period one UPO for the Lorenz equations. If this Poincaré method were used on an experimental time series, one could determine whether a curve intersected the 45° line or only approached it asymptotically by looking at the time evolution of the Poincaré plot. If the subsequent images of points near the 45° line are repelled from the intersection, then the curve only approaches the line asymptotically. One could also construct a linearized mapping for points near the intersection and compute the eigenvalues. The value for a period-one UPO, which does not exist, but which serves as the multiple for higher-order UPOs,

obtained by this Poincaré method is 0.46 ± 0.03 seconds, which agrees well with the value of 0.48 ± 0.04 seconds obtained by the previous method.

Figure 3.10. Poincaré maps of the x variable for the Lorenz equations. The intersection of the plots with the 45° line indicate the location of (clockwise, from upper left corner): a $T=1$ UPO (no intersection here); a $T=2$ UPO; a $T=4$ UPO; a $T=3$ UPO.

The final method to identify UPOs is the multivariate approach, in which the three state space variables (x,y,z) are used to find (m,ϵ) recurrent points. This method yields $\Delta m = 238 \pm 10$, which for a time step of $1/512$ seconds corresponds to a fundamental period of 0.46 ± 0.02 seconds. This value is also in excellent agreement with those obtained from

the previous two methods. Figure 3.11 shows the UPOs uncovered using this scheme, with each periodicity separated in terms of the number of transits around each fixed point. Note that for any given period T , there are $T-1$ distinct UPOs. Figure 3.12 shows the (m, ϵ) histogram obtained using this method

Figure 3.11. The Lorenz attractor and its first six UPOs.

Figure 3.12. Histogram of number of UPOs with recurrence time m for multi-variate method. The vertical lines are spaced at $\Delta m = 238$.

To understand the stabilizing effect of periodic perturbations on the Lorenz equations, numerous perturbation frequencies and amplitudes were examined. Frequencies ranging from 0.5Hz to 5.0Hz were used, which corresponds to just less than the frequency of a period-four UPO to four times the frequency of a period-two UPO. Amplitudes up to 66% of the unperturbed value of the parameter r were used. For σ and b held constant as prescribed above, the Lorenz equations are chaotic from $r \approx 31.375$ to $r > 100$. Thus, the largest amplitude perturbations ($r_1 = 30.25$) bring the perturbed parameter $r = 45.92 + r_1 \sin(\omega t)$ into a periodic regime for part of the drive period. However, many

stabilizing amplitudes were small enough that they did not do this. Combinations of driving amplitude r_1 and frequency $f = \omega/2\pi$ were sampled in a grid-like fashion, and the results were classified according to the Kaplan-Yorke dimension of the resulting perturbed attractor (as determined by the Lyapunov exponent spectrum). Figure 3.13 shows the results of these calculations. Of the 300,000 cases examined, $37.4\% \pm 0.6\%$ of the combinations resulted in periodic solutions, $26.0\% \pm 0.4\%$ resulted in a dimension between 2 and 3 (as compared to the unperturbed dimension of 2.07), while $36.7\% \pm 0.6\%$ resulted in a higher dimension between 3 and 4. Note that the dimension of the system immediately increased by 1.0 when perturbations of arbitrarily small amplitude r_1 were applied.

Figure 3.13. Results of perturbing the Lorenz equations with various perturbations of frequency f and amplitude r_1 . Arrows denote various UPO frequencies (i.e., T_2 corresponds to a period two UPO) and their harmonics. Black $\rightarrow$ periodic, Dark gray $\rightarrow 2 < D_{KY} < 3$, and Light gray $\rightarrow 3 < D_{KY} < 4$.

Figure 3.14 shows the distribution of Kaplan-Yorke dimensions. There is an interesting peak at $D_{KY}=2.0$, which corresponds to 2-tori solutions. In order to extract this peak, a slight modification to the Kaplan-Yorke conjecture was used. A limit cycle trajectory has all negative LEs, except for the LLE, which is zero. A two-torus has all negative LEs except for the first two, which are also zero. Numerical noise results in the fact that the LEs which should be zero never exactly are. Thus, one needs to define a "fuzzy zero", in which an LE is taken to be zero if it is within some range ε of zero. For

Figure 3.14, an LE was taken to be zero if it was within the range $(-0.01, 0.01)$. Note that the Kaplan-Yorke conjecture essentially fits a straight line to the two points between which the sum of the LEs crosses zero. Thus, it was thought that fitting a parabola to the three points nearest to where the sum of the LEs is zero would result in a better value for the fractal dimension. However, this method resulted in spuriously high results for the dimension.

Figure 3.14. Histogram showing the distribution of Kaplan-Yorke dimensions for the perturbed Lorenz equations.

Figure 3.15 shows the LLE with gray scales for each case. In almost every case, the LLE was decreased by the perturbation. Figure 3.16 shows some of the UPOs that were stabilized with small amplitude perturbations. The "fluctuation amplitudes" are decreased for cases controlled with small amplitude perturbations, since the attractor is confined to a smaller region of state space. For large amplitude perturbations, however, the "fluctuation amplitudes" are increased, since the attractor size scales roughly linearly with the parameter r .

Figure 3.15. LLE results for perturbing the Lorenz equations. Black corresponds to a LLE of 0; lighter colors indicate higher LLEs. The arrows indicate the same frequencies as in Figure 3.13.

Figure 3.16. Various stabilized UPOs for the perturbed Lorenz equations. Clockwise, from upper left corner: Period 2, $(r_1, f) = (4, 2.412)$; Period 3, $(r_1, f) = (3.375, 2.072)$; Period 6, $(r_1, f) = (23, 0.712)$; Period 4, $(r_1, f) = (13.1, 1.108)$.

One surprising discovery was that the frequency of the perturbation that stabilizes a given UPO is rarely the natural frequency of that UPO. Thus, a period-two UPO with a natural frequency of $1/(2 \times 0.46 \text{ sec}) = 1.1 \text{ Hz}$ is commonly stabilized by a perturbation frequency of $\sim 2.2 \text{ Hz}$. This result is typical in that the smallest frequency usually required to stabilize a given UPO is twice its natural frequency or more. This may be due to the fact that the period-one UPO for the Lorenz equations does not exist. However, the

stabilized $T=3$ UPO in the upper right hand corner of Figure 3.16 provides one counter-example, since a period-three UPO is stabilized by a frequency that is closest to that of the non-existent period-one UPO. Another typical occurrence is that a given perturbation frequency may stabilize several different UPOs, an effect here dubbed "aliasing". Figure 3.17 shows four such cases, in which a period-one frequency aliases a period-four and period-eight UPO ((a) and (b)), a period-two frequency aliases a period-sixteen UPO ((c)), and twice a period-one frequency aliases a period-twelve UPO ((d)). It is also interesting to note that the width of effective perturbations in frequency space gets narrower as higher-order UPOs are stabilized, and that these higher-order UPOs require larger perturbation amplitudes to stabilize them. Once any of the UPOs were stabilized, however, they remained controlled for as long as the periodic perturbation was on.

Figure 3.17. Some “aliased” UPOs. Clockwise, from upper left corner: Period 4, $(r_1, f) = (4.25, 2.204)$; Period 8, $(r_1, f) = (4.125, 2.156)$; Period 12, $(r_1, f) = (18.625, 4.624)$; Period 16, $(r_1, f) = (22.750, 0.992)$.

Another unanticipated result is that as the amplitude of the perturbation is increased at a fixed frequency, the system can pass through regimes of control and chaos. Control of chaos with small amplitude perturbations happens only near natural frequencies of various UPOs, but as the amplitude is increased, the system again becomes chaotic, and as the amplitude is increased more, the system again becomes periodic. This second band of periodicity (in the region $1 < f < 2.5$ and $r_1 > 15$ of Figure 3.13) corresponds to a situation where the amplitude of the drive is so great that the system is mode locked to the

perturbation. In this and similar cases, the limit cycles no longer resemble any unperturbed UPO, but resemble something akin to the top two plots in Figure 3.18. This increasing distortion of the underlying attractor with higher-amplitude perturbations is a common effect even for frequencies that stabilize a natural UPO, as can be seen in (e) of Figure 3.18.

Figure 3.13 is also interesting because the controlled areas do not have the typical Arnold tongue structure that one might expect. This is especially noticeable in the band with its minimum near $f=2.4$, where controlled cases increase in a solid band-like structure toward higher perturbation frequency and amplitude. Simonet *et al.* also saw similar departures from the standard Arnold tongue picture in their laser experiment [3.31].

The integration was started for each case with the same initial conditions of $(x_0, y_0, z_0, \phi_0) = (1, 1, 1, 0)$, and the equations were integrated for roughly 12 seconds, or about two dozen transits around a fixed point. The choice of initial condition can lead to differing results, as is seen in the bottom two plots of Figure 3.18. On the bottom right hand side, the different initial condition of $(x_0, y_0, z_0, \phi_0) = (1, 1, 10, 0)$ leads to the left-handed side of the trajectory being "larger" than in the bottom left hand side. Although the specific details of a trajectory like this can be different for different initial conditions, no cases were found in which different initial conditions led to totally different behavior, such as the stabilization of different UPOs.

Figure 3.18. Top two plots: Limit cycle achieved through mode locking; $(r_1, f) = (25, 2.176)$ (left), $(r_1, f) = (22, 1.2)$ (right). Bottom two plots: Slightly different period two UPOs both stabilized with $(r_1, f) = (4.75, 2.328)$ but different initial conditions: $(x_0, y_0, z_0, \phi_0) = (1, 1, 1, 0)$ (left); $(x_0, y_0, z_0, \phi_0) = (1, 1, 10, 0)$ (right).

In addition to these results, some long-period limit cycles and unique strange attractors were also seen, a few of which are shown in Figure 3.19. Frequently, when a perturbation is applied, a chaotic attractor results, except that the chaos is limited to a thin band around some UPO, as is seen in Figure 3.19, where the chaos seems to be centered around a period-four and period-two UPO. This effect was termed "noisy periodicity" by Lorenz in 1980 [3.30], and if there is room for some tolerance in the solution that is

sought, then these "noisy periodic" solutions may be "good enough" from a control standpoint.

Figure 3.19. Some unique attractors. Clockwise from upper left corner: A period-18 limit cycle $(r_1, f) = (19.625, 4.548)$; A noisy period four limit cycle for $(r_1, f) = (4.375, 2.124)$; a mutant cousin of Lorenz's butterfly $(r_1, f) = (23.375, 0.532)$; a noisy, band-like limit cycle $(r_1, f) = (22.75, 0.984)$.

An attempt to predict the perturbation amplitude that would stabilize the Lorenz equations for a given perturbation frequency was unsuccessful. This attempt involved constructing linearized mappings for points in the Poincaré map near the vicinity of the UPO that was to be stabilized. As the perturbation amplitude was increased from zero in

small increments, a smooth dependence of the eigenvalues on perturbation amplitude was sought, which might have lead to an extrapolation to a value of r_1 that would yield control. However, no such smooth dependence was found. Thus, when it comes to selecting the perturbation amplitude, no better method is known than to increase the amplitude until the desired limit cycle is stabilized.

To conclude this section, small-amplitude periodic perturbations were found to stabilize the Lorenz equations when applied at frequencies near the natural frequencies of UPOs. Once a limit cycle was achieved, it remained stable for as long as the perturbation was on. Large-amplitude perturbations that stabilize the Lorenz equations do so by mode locking (stabilizing a highly distorted UPO).

3.4. The Rössler Equations

The Rössler equations are [3.32]:

$$\begin{aligned}\dot{x} &= -(y + z) \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c).\end{aligned}\tag{3.26}$$

Here, the parameters $a=b=0.2$, and $c=5.7$ were used. Rössler derived these equations in an attempt to create a nonlinear system that was simpler than the Lorenz equations, in the sense that the Rössler equations have only one quadratic nonlinearity, and the flow that they generate has only a single spiral (see attractor in Figure 3.20). Thus, the equations and their parameters have no physical significance, and so the parameter c was arbitrarily chosen to be perturbed: $c \equiv c_0 + c_1 \sin(\omega t)$. Thus the system of equations becomes:

$$\begin{aligned}
 \dot{x} &= -(y + z) \\
 \dot{y} &= x + ay \\
 \dot{z} &= b + z(x - (c_0 + c_1 \sin(\phi))) \\
 \dot{\phi} &= \omega.
 \end{aligned}
 \tag{3.27}$$

Two methods were used to identify UPOs and their corresponding frequencies. Figure 3.20 shows the Rössler attractor and its first eight UPOs, as determined using the (m, ε) method on multivariate data. Figure 3.21 shows the histogram that results from identifying 500 UPOs in this manner. Here, $\Delta m = 1488 \pm 24$, which for a time step of $1/256$ seconds, corresponds to a time of 5.81 ± 0.09 seconds. Note the prominence of the $T=3$ UPO. Figure 3.22 shows Poincaré plots made by plotting successive local maxima and minima in the time series of $x(t)$. Since both maxima and minima are plotted, the intersection of the 45° line with a plot of x_{n+k} vs. x_n indicates a UPO of period $k/2$. By identifying points whose subsequent images are within a neighborhood of ε of the intersection points, the period-one UPO was determined to have a period of 5.86 ± 0.02 seconds, in good agreement with the previous result.

Figure 3.20. The Rössler attractor and its first eight UPOs.

Figure 3.21. Histogram of number of UPOs with recurrence time m for multi-variate method. The vertical lines are spaced at $\Delta m = 1488$.

Figure 3.22. Poincaré maps for the Rössler equations. The intersection of the plots with the 45° line indicate the location of (clockwise, from upper left corner): a $T=1$ UPO; a $T=2$ UPO; a $T=4$ UPO; a $T=3$ UPO.

In order to assess the effect of periodic perturbations on the Rössler equations, perturbation amplitudes and frequencies in the range $0 < c_1 < 3.1$ and $0.05 < f < 0.4$ were examined. This range corresponds to amplitudes up to 54% of the unperturbed value of c and frequencies of just less than that of a period-three UPO to more than twice the frequency of a period-one UPO. For a and b held constant as prescribed above, the Rössler equations are chaotic from $c \approx 4.20$ to $c > 13$. Thus, perturbation amplitudes greater than $c_1=1.5$ bring the perturbed parameter $c=5.7+c_1\sin(\omega t)$ into a periodic regime

for part of the drive period. However, as was the case with the perturbed Lorenz equations, many stabilizing amplitudes were smaller than the $c_1=1.5$ limit. Figure 3.23 summarizes the results of perturbing the Rössler equations with various perturbation amplitudes and frequencies. Of the approximately 40,000 cases examined, $8.5\%\pm 2.8\%$ of the combinations thus tried resulted in periodic solutions, $13.3\%\pm 0.4\%$ resulted in a dimension between 2 and 3 (as compared to the unperturbed dimension of 2.01), while $78.2\%\pm 2.3\%$ resulted in a higher dimension between 3 and 4. The large uncertainty in the percentage of periodic cases found is due to the fact that applying a perturbation commonly resulted in a chaotic attractor with the trajectory confined to a thin band around an UPO. Since the LLE for the unperturbed Rössler equations is very small to begin with, the LLE for these cases was sometimes mistakenly identified as zero, and thus classified as a limit cycle. Figure 3.24 shows some of the UPOs that were stabilized with small amplitude perturbations, and some chaotic attractors confined to the neighborhood of an UPO.

Figure 3.23. Results of perturbing the Rössler equations. The arrows show the frequencies of various UPOs and their harmonics. Black $\rightarrow$ periodic, Dark gray $\rightarrow 2 < D_{KY} < 3$, and Light gray $\rightarrow 3 < D_{KY} < 4$.

Figure 3.24. Various attractors arising from the perturbed Rössler equations: The Rössler Attractor; a noisy $T=3$ limit cycle, $(c_1, f)=(1.600, 0.182)$; a $T=2$ limit cycle, $(c_1, f)=(1.200, 0.250)$; a $T=3$ limit cycle, $(c_1, f)=(0.100, 0.114)$; a $T=4$ limit cycle, $(c_1, f)=(0.750, 0.272)(1.35, 0.086)$; a $T=5$ limit cycle, $(c_1, f)=(0.450, 0.106)$; a $T=6$ limit cycle, $(c_1, f)=(0.100, 0.286)$; a $T=7$ limit cycle, $(c_1, f)=(1.050, 0.356)$; a $T=8$ limit cycle, $(c_1, f)=(1.050, 0.168)$.

3.5. Coupled Logistic Equations

To determine the robustness of controlling chaotic systems with periodic perturbations, high dimensional systems were sought to test the method. Sprott showed that for a wide collection of polynomial maps and flows, the average dimension of the attractor scales approximately as the square root of the number of state space variables [3.33]. Albers *et al.* also showed a similar scalling for large, untrained, feedforward

neural networks with randomly chosen weights and feedback [3.34]. These results provide a background for how large of a system to consider, but they do not guarantee chaos for a high dimensional system chosen at random (except in the neural network case [3.34], where it was found that the probability of chaos approached unity as the network size increased).

One successful way to generate well-behaved high dimensional systems is to couple many low dimensional systems together. This is the idea behind Coupled Map Lattices (CMLs). The most frequently studied CML consists of a one-dimensional lattice of coupled logistic equations [3.35-3.41]. Here, each site i is coupled to $2N_c$ nearest neighbors with a coupling strength ε :

$$x_{n+1}(i) = (1 - \varepsilon)ax_n(i)(1 - x_n(i)) + \frac{\varepsilon}{2N_c} \sum_{j=-N_c, j \neq i}^{N_c} ax_n(i+j)(1 - x_n(i+j)). \quad (3.28)$$

The perturbation was globally applied to the parameter a : $a \rightarrow a_0 + a_1 \cos(2\pi n/T + \phi)$. An $N=100$ dimensional CML was studied with coupling to $N_c=2$ nearest neighbors on each side for an unperturbed parameter value of $a_0=3.9$, and a coupling constant of $\varepsilon=0.2$. Five cases of perturbation amplitude $a_1=0.02, 0.04, 0.06, 0.08$, and 0.10 were examined for a perturbation frequency of $T=3$. The unperturbed case was found to have a Lyapunov dimension of $D_{KY}=26.5$, in reasonable agreement with Sprott's prediction of $D_{KY} \approx 10$. The only perturbation applied that had a "controlling" effect was for the amplitude $a_1=0.04$; the other four perturbations increased the dimension, with the perturbation of $a_1=0.10$ raising the dimension to 100.4 , thus exciting the CML into using all 101 state space variables. Figure 3.25 shows spatiotemporal plots for three cases. These plots were made

by coloring a pixel corresponding to a site i black if the value for x_n was in the interval $(0,0.5)$, and gray if $x_n \in (0.5,1)$. The Lyapunov exponent spectra are shown in figure 3.26 for all five cases. More cases than this were not examined since computing the LE spectrum is computationally expensive for high dimensional systems, and so it was decided to invest the time in a slightly more physical model.

Figure 3.25. Spatiotemporal plots of the behavior of a logistic equation CML perturbed with a $T=3$ perturbation and $a_0=3.9$, $\varepsilon=0.2$, and $N_c=2$. Each site was colored black or gray depending on which part of the unit interval it occupied. The perturbation amplitudes

correspond to $a_1=0 \rightarrow$ no perturbation, $D_{KY}=26.5$; to $a_1=0.04 \rightarrow D_{KY}=21.4$; to $a_1=0.8 \rightarrow D_{KY}=100.3$.

Figure 3.26. LE spectrum for the five perturbed cases ($T=3$) of the logistic equation CML. The perturbation amplitude $a_1=0.04$ decreases the dimension of the system to 21.4 from its unperturbed value of 26.5. The other five cases resulted in dimensions of: $a_1=0.02 \rightarrow D_{KY}=33.9$; $a_1=0.06 \rightarrow D_{KY}=90.1$; $a_1=0.08 \rightarrow D_{KY}=100.3$; $a_1=0.10 \rightarrow D_{KY}=100.4$.

3.6. Coupled Lorenz Equations

Coupled Lorenz models are lattices with each site occupied by a set of Lorenz equations. Lawandy and Plant have studied pairs of coupled Lorenz systems in order to

find an improved stability regime for a single-mode laser by coupling two lasers together [3.42]. Malkus [3.43] and Yorke and Yorke [3.44] showed that the Lorenz equations, which were derived as an approximation of the Raleigh-Bénard system, exactly describe the dynamics of a thermal convection loop. A thermal convection loop, or thermosyphon, consists of a narrow diameter pipe containing fluid, which is bent into a torus, stood upright in a vertical plane, and heated uniformly over the bottom half and cooled uniformly over the top half. Davis and Roppo studied the effect of thermally coupling two thermosyphons, which is equivalent to studying two adjacent rolls in a Bénard convection layer [3.45]. Jackson and Kodogeorgiou extended this work by studying long (up to 128 lattice sites), periodic chains of thermosyphons [3.46]. They included both viscous and thermal coupling between adjacent vortices in their model equations, but only examined the case with viscous coupling. Lorenz himself studied the effect of coupling seven Lorenz-like systems together to illustrate that global weather systems most likely do not possess low-dimensional attractors [3.47]. Here we consider the coupled Lorenz cell model studied by Jackson and Kodogeorgiou.

The Lorenz equations are coupled in this model in the following way [3.46]:

$$\begin{aligned}\dot{x}_i &= \sigma(y_i - x_i) - \mu(x_{i+1} + x_{i-1} + 2x_i) \\ \dot{y}_i &= -y_i - x_i z_i + r x_i - \kappa(y_{i+1} + y_{i-1} + 2y_i) \\ \dot{z}_i &= x_i y_i - b z_i\end{aligned}\tag{3.29}$$

Here, i denotes the lattice site ($i = 0, 1, \dots, N-1$), μ is the viscous coupling constant, and κ is the thermal coupling constant. Each lattice site is coupled to one neighbor on each side. This system is taken to have periodic boundary conditions: $x_N = x_0$, $y_N = y_0$, $z_N = z_0$. The

parameter values used here were $\sigma = 10$, $\mu = 3$, $r = 45$, $\kappa = 0$, $b = 1$, and $N = 32$. In the thermosyphon paradigm, x corresponds to the average fluid velocity around the loop, y corresponds to the temperature difference between points at "12 o'clock" and "6 o'clock", and z corresponds to the horizontal temperature difference. The authors computed that the system has stable fixed points for $r < r_c = 17.5$, and it was found that the system exhibits chaotic behavior for r down to ~ 23 .

Computing the Lyapunov exponent is computationally expensive for high-dimensional chaotic systems, since the memory and time required both scale as N^2 . For this reason, a lattice with only $N = 32$ sites (and thus $3 \times 32 = 96$ variables) was chosen to be perturbed. In the absence of a perturbation, this 96-dimensional system has an attractor with a Kaplan-Yorke dimension of 65.8. The uncoupled Lorenz equations for these parameters was measured to have $D_{KY} = 2.06 \pm 0.005$, which yields an expected dimension for the coupled system of $32 \times 2.06 = 65.9$. This is in good agreement with what was measured for the coupled system, although it would not have been surprising if the coupling had reduced the dimension to some extent. As before, the perturbed parameter is $r = r_0 + r_1 \sin(\omega t)$. To do a reasonably accurate calculation of D_{KY} for even this modest lattice size requires on the order of a day of computation for one perturbation case. Thus, there are no extensive perturbation amplitude-frequency scans as before, but Figure 1 does show the Kaplan-Yorke dimension and LLE as a function of perturbation amplitude for seven different perturbation frequencies. Remarkably, a small perturbation amplitude of $r_1 = 4$, which is only 9% the unperturbed parameter value, resulted in limit cycle behavior for a perturbation frequency equal to the frequency of a period one UPO.

Figure 3.27. Results of perturbing a coupled Lorenz system as evidenced by the Kaplan-Yorke dimension and the magnitude of the LLE. The solid horizontal lines indicate values for the unperturbed system. Note that the perturbation frequency $f=1.014$ results in a limit cycle case for $r_1=4.0, 10.0,$ and 12.0 .

Figure 3.28. Recurrence plot histogram for the coupled Lorenz system. The period one peak occurs at $m=252$.

The frequencies were chosen once again by identifying the frequencies of the UPOs for the coupled system. Figure 3.28 shows the recurrence plot histogram for this system. The period of the period one UPO was found to be 1.014 ± 0.050 seconds, which is roughly twice the period one UPO for a single Lorenz system. Thus, the global dynamics of the coupled system seem to evolve more slowly than the fluctuations in any given state space variable. This case proved interesting, since the power spectral peak occurs at roughly 2Hz, which corresponds to the period one UPO time for a single Lorenz. Thus, this is a system where the power spectral peak frequency is misleading when trying

to predict a useful perturbation frequency. The slower global dynamics are shown in spatiotemporal plots in Figure 3.29, which shows the value of x_i , which corresponds to the average fluid velocity for each site vs. time. The perturbed case on Figure 3.29 shows the limit cycle achieved with the small-amplitude period one frequency perturbation. Figure 3.30 shows a spatiotemporal plot meant to show the "turbulence", as defined by Jackson and Kodogeorgiou [3.46], of the coupled Lorenz equations. A turbulent flow was defined as a site having the flow of one of its neighbors in the same direction, meaning that the fluid in these adjacent cells would "bump into" each other. The limit cycle case for the perturbed case shows that the turbulence is eliminated.

Figure 3.29. Spatiotemporal plot of x_i vs. time for the coupled Lorenz equations. Left: unperturbed; right: perturbation amplitude of $(r_1, f) = (4, 1.014)$. The perturbed case is a limit cycle.

Figure 3.30. Spatiotemporal plot showing "turbulence" à la Jackson in the coupled Lorenz equations. Left: unperturbed; right: perturbation amplitude of $(r_1, f) = (4, 1.014)$. A site was shaded black if both its neighbors were rotating in opposite directions and white if one of its neighbors was rotating in the same direction. The applied perturbation eliminates turbulence.

The frequency corresponding to the period one UPO for the coupled Lorenz system is $f = 1.014 \pm 0.050$ Hz. As can be seen from Figure 3.27, this perturbation frequency does result in the best performance by stabilizing a limit cycle. Perturbations of frequency $f = 2.028$ (which is twice the period-one UPO frequency) and $f = 1.595$ (which is about three times the period-two UPO frequency) significantly decreased the dimension of the system. Even if a perturbation does not result in limit cycle behavior, drastically

decreasing the dimension of a high dimension system is a significant step toward controlling such systems [3.9].

Surprisingly, perturbations of frequency corresponding to a period-two UPO ($f=0.507$), a period-three UPO ($f=0.338$), and a period-four UPO ($f=0.254$) did not decrease the dimension of the system. Not surprisingly, a perturbation frequency of $f=2.412$, which was successful at eliminating chaos for the single Lorenz system, also did not decrease the dimension of the system. Another surprise was that none of the unsuccessful frequencies increased the dimension by more than one, which was seen to be the case when perturbing coupled logistic equations. Frequencies that are farther off-resonance may need to be identified to see an "excitation" of additional, latent degrees of freedom. This could have important applications in biological and other high-dimensional systems, where it is sometimes desirous to "uncontrol" chaos [3.48].

3.7. The Yoshida Equations

The Yoshida equations are a set of ordinary differential equations derived by Yoshida *et. al.* [3.49] from a 1-D (i.e., functions have a radial dependence only) set of partial differential equations meant to model the nonlinear interactions of magnetic islands in a low-beta tokamak plasma. The method for reducing the (infinite-dimensional) set of PDEs to ODEs involves the application of the force-free condition $\nabla \times \mathbf{B} = \lambda \mathbf{B}$ to approximate the structure of the magnetic field $\mathbf{B}$. This relation accomplishes such a task by reducing the action of spatial derivatives to the multiplication by a scalar function λ . The justification for the force-free structure of the field is that the instability growth rate is

sufficiently small that the current perpendicular to the magnetic field is negligible, which is indeed the case for a tokamak plasma in which the parallel current is significantly larger than any perpendicular currents. In the case of a reversed field pinch plasma confined by a perfectly conducting toroidal shell, the relaxed state is defined by the force-free condition [3.50].

There are other simplifying assumptions in this model. One is that the magnetic field be decomposed as $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, where $\mathbf{b}$ is the fluctuating component and $\mathbf{B}_0$ is the equilibrium field. Here, $b \ll B_0$, and $\mathbf{B}_0$ is harmonic: $\nabla \times \mathbf{B}_0 = 0$. From this relation it follows that $\nabla \times \mathbf{b} = \lambda \mathbf{B}$. The toroidal plasma is approximated as a cylinder. Thus, $\mathbf{B}_0$ is parallel to ∇z . Neglecting the poloidal component of $\mathbf{B}_0$ may not be a valid approximation for a pinch-type plasma. Magnetic fluctuations parallel to $\mathbf{B}_0$ are neglected, since the plasma has a low beta, which means that $\mathbf{b} \cdot \mathbf{B}_0 = 0$. The plasma is also assumed to have a steady and uniform Hall velocity $\mathbf{V}$ parallel to $\mathbf{B}_0$. This assumption does not drive any instability in this model, but it does cause Mirnov oscillations in observed fluctuating quantities. By taking the divergence of the force-free condition, one sees that $\mathbf{B} \cdot \nabla \lambda = 0$. This requires that $\nabla \lambda = 0$ in the neighborhood of each magnetic island, or resonance, location, because these are the only places in which the field $\mathbf{B}$ has any radial component. Consequently, λ may be considered as an eigenvalue that characterizes the field structure in each resonant region. It is interesting to note that the flattening of λ parallels the flattening of the current density $\mathbf{j}$ near a magnetic island, because $\mathbf{j} = \lambda \mathbf{B} / \mu_0 \approx \lambda \mathbf{B}_0 / \mu_0$. Using the fact that $\nabla \lambda = 0$ in the neighborhood of each resonance, one finds that $\mathbf{B} = \lambda \mathbf{A}$ in the Coulomb gauge, because $\lambda \mathbf{B} = \nabla \times \lambda \mathbf{A} - \nabla \lambda \times \mathbf{A} = \nabla \times \mathbf{B}$.

Tearing mode turbulence results in an additional term in Ohm's law when the magnetic field is averaged over the tearing mode fluctuations [3.51]. This effect is called hyper-resistivity, since it provides an effective coefficient of electric current viscosity in addition to the regular (Spitzer) resistivity. Hyper-resistivity enters into a field-aligned component of Ohm's law as follows:

$$E_{\parallel} = \eta j_{\parallel} - \nabla \cdot (\eta^{(2)} \nabla j_{\parallel,0}). \quad (3.30)$$

Here, η is the resistivity, which is assumed to be constant, and $E_{\parallel}^{(2)} \equiv -\nabla \cdot (\eta^{(2)} \nabla j_{\parallel,0})$ is the hyper-resistive term. The hyper-resistive term represents an internal electric field $\langle -\mathbf{v} \times \mathbf{B} \rangle$, which will be modeled with a transport coefficient approximated from the following expression for $\eta^{(2)}$, and not by solving the momentum equations to estimate $\mathbf{v}$ for $\langle -\mathbf{v} \times \mathbf{B} \rangle$.

The quasi-linear hyper-resistivity is given by [3.52].

$$\frac{\eta^{(2)}}{\mu_0} = \frac{1}{2} \sum_k \frac{\gamma_k}{(\partial_r k_{\parallel})^2} |b_{r,k}|^2 \ln \left| \frac{r_k^2}{(r - r_k)^2 + \gamma_k^2 / (\partial_r k_{\parallel})^2} \right|, \quad (3.31)$$

where the subscript k indicates a Fourier component of the fluctuation that is resonant with the ambient magnetic field at $r = r_k$, γ_k is the growth rate, and $k_{\parallel}$ is the parallel wavenumber.

The reduction of the MHD equations to a set of ordinary differential equations begins by examining the time evolution of the magnetic field using an Ohm's law description for the electric field. Here the algebra from Yoshida *et al.* is stated explicitly:

$$\begin{aligned} \partial_t \mathbf{B} &= -\nabla \times \mathbf{E} \\ \partial_t (\mathbf{B}_0 + \mathbf{b}) &= -\nabla \times (\eta \mathbf{j} - \mathbf{v} \times \mathbf{B} + E_{\parallel}^{(2)} \nabla z); & \mu_0 \mathbf{j} &= \nabla \times \mathbf{B} = \nabla \times \mathbf{b} \\ \partial_t \mathbf{b} &= -\eta / \mu_0 \nabla \times \nabla \times \mathbf{b} + \nabla \times (\mathbf{V} \times \mathbf{B} - E_{\parallel}^{(2)} \nabla z). \end{aligned} \quad (3.32)$$

The individual terms are addressed as follows:

$$\begin{aligned}
\nabla \times \nabla \times \mathbf{b} &= \nabla \times \lambda \mathbf{B} \\
&= \lambda \nabla \times \mathbf{b} - \nabla \lambda \times \mathbf{B} \\
&= \lambda^2 \mathbf{b}.
\end{aligned} \tag{3.33}$$

$$\begin{aligned}
\mathbf{v} \times \mathbf{B} &= \mathbf{V} \times (\mathbf{B}_0 + \mathbf{b}) \\
&= \mathbf{V} \times \mathbf{b}.
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
\nabla \times (\mathbf{V} \times \mathbf{B}) &= \nabla \times (\mathbf{V} \times \mathbf{b}) \\
&= \mathbf{V}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{V}) + (\mathbf{b} \cdot \nabla) \mathbf{V} - (\mathbf{V} \cdot \nabla) \mathbf{b} \\
&= \mathbf{V} \times (\nabla \times \mathbf{b}) - (\nabla \mathbf{b}) \cdot \mathbf{V} \\
&\approx -(\nabla \times \mathbf{b}) \times \mathbf{V} \\
&= -\lambda(\mathbf{b} + \mathbf{B}_0) \times \mathbf{V} \\
&= -\lambda \mathbf{b} \times \mathbf{V}.
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
\nabla \times (E_{\parallel}^{(2)} \nabla z) &= \partial_r E_{\parallel}^{(2)} \nabla \varphi \\
&= \partial_r (-\nabla \cdot (\eta^{(2)} \nabla j_{\parallel,0})) \nabla \varphi \\
&= \partial_r (-\eta^{(2)} \nabla^2 j_{\parallel,0} - \nabla \cdot \eta^{(2)} \cdot \nabla j_{\parallel,0}) \nabla \varphi; \quad j_{\parallel,0} \approx \lambda B_0 / \mu_0 \\
&= -(B_0 / \mu_0) (\partial_r (\eta^{(2)}) \partial_r^2 \lambda + \eta^{(2)} \partial_r^3 \lambda \\
&\quad + \partial_r (\eta^{(2)}) \partial_r^2 \lambda + \partial_r^2 (\eta^{(2)}) \partial_r \lambda) \nabla \varphi \\
&= -2B_0 \partial_r (\eta^{(2)} / \mu_0) \partial_r^2 \lambda \nabla \varphi.
\end{aligned} \tag{3.36}$$

If $p \equiv b_r$, and $q \equiv b_\varphi$, then the magnetic field evolution equation near a resonance is specified by the following two equations:

$$\partial_t p = -\eta / \mu_0 \lambda^2 p - V \lambda q \tag{3.37}$$

$$\partial_t q = -\eta / \mu_0 \lambda^2 q + V \lambda p + 2B_0 \partial_r (\eta^{(2)} / \mu_0) \partial_r^2 \lambda. \tag{3.38}$$

The evolution of λ is found using the helicity transport equation. The helicity density is defined as $h \equiv \mathbf{A} \cdot \mathbf{B} = \lambda A^2$. Thus,

$$\begin{aligned}
\partial_t h &= 2\lambda \mathbf{A} \cdot \partial_t \mathbf{A} + A^2 \partial_t \lambda \rightarrow \\
\partial_t \lambda &= A^{-2} (\partial_t h + 2\mathbf{B} \cdot \mathbf{E}).
\end{aligned} \tag{3.39}$$

The helicity transport can be recast as:

$$\begin{aligned}
\partial_t h &= \mathbf{B} \cdot \partial_t \mathbf{A} + \mathbf{A} \cdot \partial_t \mathbf{B} \\
&= -\mathbf{E} \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{E}) \\
&= -\mathbf{E} \cdot \mathbf{B} - \mathbf{E} \cdot (\nabla \times \mathbf{A}) - \nabla \cdot (\mathbf{E} \times \mathbf{A}) \\
&= -2\mathbf{E} \cdot \mathbf{B} - \nabla \cdot \mathbf{F}.
\end{aligned} \tag{3.40}$$

Here, the helicity flux is defined as $\mathbf{F} \equiv \mathbf{E} \times \mathbf{A} + \phi \mathbf{B}$, where ϕ is the scalar potential. Thus, the time evolution of λ takes a simple form:

$$\partial_t \lambda = -A^{-2} \nabla \cdot \mathbf{F}. \quad (3.41)$$

The helicity flux arises from two distinct contributions. There is an external term which corresponds to the external driving field for the plasma, and an internal term which parallels the hyper-resistivity effect. The external term, which arises from transformer induction, is a harmonic electric field defined by $\mathbf{E}_h = -\partial_t \mathbf{A}_h$; $\nabla \times \mathbf{E}_h = 0$. Thus,

$$\begin{aligned} \nabla \cdot \mathbf{F}_e &= \nabla \cdot (\mathbf{E}_h \times \mathbf{A}) \\ &= -\mathbf{E}_h \cdot \nabla \times \mathbf{A} + \mathbf{A} \cdot \nabla \times \mathbf{E}_h \\ &= -\mathbf{E}_h \cdot (\mathbf{B}_0 + \mathbf{b}) \\ &= -E_h B_0. \end{aligned} \quad (3.42)$$

The internal term is just given by the hyper-resistivity effect:

$$\begin{aligned} \nabla \cdot \mathbf{F}_i &= -B_0 \nabla \cdot (\eta^{(2)} \nabla j_{\parallel,0}) \\ &= -B_0 (\nabla \eta^{(2)} \cdot \nabla j_{\parallel,0} + \eta^{(2)} \nabla^2 j_{\parallel,0}) \\ &\approx -B_0^2 (\eta^{(2)} / \mu_0) \partial_r^2 \lambda. \end{aligned} \quad (3.43)$$

Using the approximation that $A^{-2} = \lambda^2 / B^2 \approx \lambda^2 / B_0^2$, the evolution of λ is specified by:

$$\partial_t \lambda = \lambda^2 ((\eta^{(2)} / \mu_0) \partial_r^2 \lambda + E_h / B_0). \quad (3.44)$$

The term $\eta^{(2)} / \mu_0$ occurs in both the λ and q evolution equations. In view of the form that this term takes as given by Strauss, it is written as $\eta^{(2)} / \mu_0 = C \langle p \rangle^2 / B_0^2$, where C is of order $\gamma (\partial_r k_{\parallel})^2$. Yoshida *et al.* estimate that $\partial_r k_{\parallel} \approx k_{\perp} \partial_r B_{\phi} / B_0 = k_{\perp} \mu_0 j_{\parallel,0} / B_0$. Thus, C is of order $\gamma \Delta^2 \lambda^2$, where Δ is a gradient scale length. Combining these results, the tearing mode turbulence evolution equations for this model are given by:

$$\partial_t p = -\eta / \mu_0 \lambda^2 p - \nabla \lambda q$$

$$\begin{aligned}
\partial_r q &= -\eta/\mu_0 \lambda^2 q + V \lambda p + (2C/B_0) \partial_r (\langle p \rangle^2) \partial_r^2 \lambda \\
\partial_r \lambda &= \lambda^2 ((C/B_0^2) \langle p \rangle^2 \partial_r^2 \lambda + E_h/B_0).
\end{aligned}
\tag{3.45}$$

The last step is to discretize these equations into N radial grid points. If f_n denotes the local value of a function at a radial grid point indexed by n , and Δ_n is the distance between the $n-1$ and $n+1$ grid points, then by approximating the radial derivatives by finite differences one obtains:

$$\begin{aligned}
\dot{p}_n &= -\eta/\mu_0 \lambda_n^2 p_n - V \lambda_n q_n \\
\dot{q}_n &= -\eta/\mu_0 \lambda_n^2 q_n + V \lambda_n p_n + \frac{2C}{B_0} \frac{\langle p_{n+1} \rangle^2 - \langle p_{n-1} \rangle^2}{\Delta_n} \frac{\lambda_{n+1} + \lambda_{n-1} - 2\lambda_n}{\Delta_n^2} \\
\dot{\lambda}_n &= \lambda_n^2 \left(\frac{C}{B_0^2} \langle p_n \rangle^2 \left(\frac{\lambda_{n+1} + \lambda_{n-1} - 2\lambda_n}{\Delta_n^2} \right) + \frac{E_h}{B_0} \right).
\end{aligned}
\tag{3.46}$$

Yoshida et. al. used these equations to model evenly spaced two-mode interactions ($N=2$) in a tokamak plasma. This case corresponds to a linear safety factor profile (q profile) ranging from an on-axis value of one to a value of four at the plasma wall ($r = a$). The safety factor is defined as $q = rB_z/aB_\phi$. Magnetic field lines are periodic on a toroidal surface at resonance locations, and a given field line closes in on itself after an integer number m of toroidal transits around the resonant surface. The safety factor takes on rational values $q = m/n$ on resonant surfaces, where n corresponds to the number of poloidal transits a field line makes before it closes on itself. Thus, Yoshida's two-mode interaction corresponds to an $m=2$, $n=1$ mode interacting with an $m=3$, $n=1$ mode. Physically, the geometry is shown in Figure 3.31:

Figure 3.31. Safety factor profiles for a tokamak plasma. The straight line represents the Yoshida approximation. The dashed line represents a typical q profile for a tokamak. The magnetic islands are located at $a/3$ and $2a/3$ for the Yoshida approximation.

For the $N=2$ case, the Yoshida equations are equivalent to the following six ordinary differential equations:

$$\begin{aligned}
 \dot{p}_1 &= a_1 \lambda_1^2 p_1 + a_2 \lambda_1 q_1 \\
 \dot{q}_1 &= a_1 \lambda_1^2 q_1 - a_2 \lambda_1 p_1 + a_3 (p_2^2 + q_2^2) (\lambda_2 + a_4 - 2\lambda_1) \\
 \dot{\lambda}_1 &= \lambda_1^2 (a_5 (p_1^2 + q_1^2) (\lambda_2 + a_4 - 2\lambda_1) + a_6) \\
 \dot{p}_2 &= a_1 \lambda_2^2 p_2 + a_2 \lambda_2 q_2 \\
 \dot{q}_2 &= a_1 \lambda_2^2 q_2 - a_2 \lambda_2 p_2 + a_3 (a_8 - p_1^2 - q_1^2) (a_7 + \lambda_1 - 2\lambda_2) \\
 \dot{\lambda}_2 &= \lambda_2^2 (a_5 (p_2^2 + q_2^2) (a_7 + \lambda_1 - 2\lambda_2) + a_6).
 \end{aligned} \tag{3.47}$$

Here two additional assumptions were made. First, the radial edge and all axis magnetic fluctuations are zero; i.e., $p_3 = p_0 = q_0 = 0$. Second, $\langle p_n \rangle^2$ was approximated by $\frac{1}{2} (p_n^2 + q_n^2)$, because p_n and q_n are approximately $\frac{1}{2} \pi$ out of phase for sufficiently large V . The parameters a_1 through a_8 are summarized in Table 1, along with typical order-of-magnitude ranges for both tokamak and pinch (i.e., the MST reversed field pinch) plasmas. To simplify a comparison to real MST data, the above equations are normalized so that magnetic fields are in units of Gauss, and the time is measured at a convenient

digitizing rate, which here was taken to be $5\mu\text{s}$. Thus, the equations for $\dot{p}_i$ and $\dot{q}_i$ are multiplied by $0.05(\text{G}\cdot\text{sec}/\text{T}\cdot 5\mu\text{s})$, while the equations for $\dot{\lambda}_i$ are multiplied by $5\times 10^{-6}(\text{sec}/5\mu\text{s})$.

Table 1. Parameters for the Yoshida equations.

Parameter	Physical parameter	Tokamak ranges	Pinch ranges
a_1	$-0.05\eta/\mu_0 =$ $-414Z\ln\Lambda/T^{3/2}(\text{eV})$	$-(0.05\rightarrow 0.5)$	$-(0.5\rightarrow 5)$
a_2	$-0.05V$	$-(0.5\rightarrow 50)$	$-(0.5\rightarrow 50)$
a_3	$(0.05)C/(B_0\Delta^3)\approx$ $(0.05)\gamma\lambda^2/(B_0\Delta)$	1-100	10-1000
a_4	λ_0	$.1\rightarrow 1$	$5\rightarrow 15$
a_5	$(5\times 10^{-6})C/(2B_0^2\Delta^2)\approx$ $(5\times 10^{-6})\gamma\lambda^2/(2B_0^2)$	$0.1\rightarrow 1$	$1\rightarrow 100$
a_6	$(5\times 10^{-6})E_H/B_0$	$10^{-6}\rightarrow 10^{-4}$	$10^{-4}\rightarrow 10^{-2}$
a_7	λ_3	$0.1\rightarrow 1$	0
a_8	q_3^{-2}	0	1-100

The Yoshida equations are not chaotic for all values of parameters in the specified range. Yoshida et. al. found a case in which lowering the resistivity by an order of magnitude changed the behavior of the equations from a limit cycle (coherently interacting modes) to chaotic. Figure 3.32 shows the effect of perturbing this chaotic case

by oscillating the transformer field $E_h = E_{h,0} + E_{h,1}\sin(\omega t)$ (parameter α_6 in equations 3.47). Such an oscillating toroidal electric field would produce a poloidal magnetic field, but in the spirit of neglecting small poloidal fields, this effect was not added to the $\dot{q}_i$ terms. The units shown in Figure 3.32 are different from the ranges shown in Table 1 for a tokamak because Yoshida *et al.* normalized their fields to the background field B_0 , distance to the plasma minor radius, and time to Alfvén units $t_A = a/c_A$. An Alfvén time unit t_A is approximately $1\mu s$ for fusion-type plasmas. In these dimensionless units, the tearing mode growth rate γ is in the range $\gamma \approx 10^{-3}$ - 10^{-4} , which corresponds to $\gamma \approx 10^3$ - 10^2 sec^{-1} in MKS units. There appear to be two preferred frequencies of 2.5×10^{-5} and 3.1×10^{-5} at which the perturbed Yoshida equations show limit cycle behavior; these correspond to periods of 4 and 5 times t_A , respectively. Of the nearly 6,000 combinations of perturbation amplitude and frequency examined, 7% of the cases resulted in limit cycle behavior. While this is not a large fraction of the total cases, it is interesting to note that 71% of the combinations tried decreased the Lyapunov dimension from its unperturbed value of $D_{KY} = 3.75$. The remaining 22% of the cases resulted in an increased Lyapunov dimension. No cases were found in which the Lyapunov dimension was elevated above $D_{KY} = 5$. One note on solving these equations is that it was impossible to replicate the results of Yoshida *et al.* when integrating their equations using a fourth order Runge-Kutta method. Their results could only be replicated using a first-order Euler method integration scheme, and this numerical method was used to generate the results shown in Figure 3.32.

Figure 3.32. Effect of perturbing the chaotic Yoshida equations modeling a tokamak. Black indicates limit cycle cases. Dark gray indicates $2 < D_{ky} < 3$; light gray indicates $3 < D_{ky} < 3.75$; Medium gray indicates $D_{ky} > 3.75$.

When Yoshida *et al.* applied their equations to a tokamak, they used the physical assumption that there were only two evenly-spaced interacting modes. While this assumption is valid for a tokamak, it omits some important physics for an RFP like MST. In this case, one should take into account the fact that bispectral analysis reveals a three-wave coupling, whereby two modes couple to a third with $k_3 = k_1 + k_2$ [3.53]. In MST, magnetic fluctuations tend to be dominated by a few $m=1$ modes; namely $(m,n)=(1,6)$ and $(1,7)$. These two modes couple strongly to the $(0,1)$ and $(2,13)$ modes. The $(0,1)$ mode is

much more dominant than the (2,13) mode, but it is located at the reversal surface of the plasma, where the toroidal magnetic field vanishes. This reversal clearly violates the assumptions of Yoshida *et al.*, who assumed the toroidal field was nearly constant across the plasma. However, the (2,13) mode lies between the (1,6) and (1,7) modes, which means the toroidal field is roughly constant across these three modes. Thus, the (1,6), (2,13), and (1,7) modes were chosen to model the magnetic fluctuations in the MST. It should also be noted at this point that the Yoshida equations have two flaws as a general model for tearing modes. First, the assumption of a curl-free equilibrium field implies that there is no equilibrium current, which is un-physical, and usually considered necessary for tearing mode activity. Second, the different time scales for helicity and energy conservation are not properly separated in Yoshida *et al.*'s model.

The (1,6), (2,13), and (1,7) modes roughly reside at $r/a = 0.3$, 0.36 , and 0.42 , respectively. Thus, this assumption yields the following equations for this $N=3$ model.

$$\begin{aligned}
 \dot{x}_1 &= a_1 x_1 x_3^2 + a_2 x_2 x_3 \\
 \dot{x}_2 &= a_1 x_2 x_3^2 - a_2 x_1 x_3 + a_3 (x_4^2 + x_5^2) (a_4 - 2x_3 + x_6) \\
 \dot{x}_3 &= x_3^2 [a_5 (x_1^2 + x_2^2) (a_4 - 2x_3 + x_6) + a_6] \\
 \dot{x}_4 &= a_1 x_4 x_6^2 + a_2 x_5 x_6 \\
 \dot{x}_5 &= a_1 x_5 x_6^2 - a_2 x_4 x_6 + 0.833 a_3 (x_7^2 + x_8^2 - x_1^2 - x_2^2) (x_3 - 2x_6 + x_9) \\
 \dot{x}_6 &= x_6^2 [a_5 (x_4^2 + x_5^2) (x_3 - 2x_6 + x_9) + a_6] \\
 \dot{x}_7 &= a_1 x_7 x_9^2 + a_2 x_8 x_9 \\
 \dot{x}_8 &= a_1 x_8 x_9^2 - a_2 x_7 x_9 + 0.714 a_3 (a_8 - x_4^2 - x_5^2) (a_7 + x_6 - 2x_9) \\
 \dot{x}_9 &= x_9^2 [a_5 (x_7^2 + x_8^2) (a_7 + x_6 - 2x_9) + a_6]
 \end{aligned} \tag{3.48}$$

The parameters used in the MST application of the Yoshida equations were chosen randomly from values close to those in Table 1. Candidates were chosen from several sets of parameters that yielded chaotic solutions according to how closely $\dot{x}_8$ reproduced the autocorrelation function of a signal from a single poloidal field pickup coil on the wall of MST. New sets of parameters were then selected from these candidate neighborhoods, and the process was repeated until a sufficiently good match to the autocorrelation function was found. Since the autocorrelation function is the Fourier transform of the power spectrum, this method of fitting guarantees that the model is chaotic and captures the dominant frequencies of the experiment. The parameters thus chosen in this manner were: $\{a_i\} = \{-0.57813, -1.73391, 9.78195, 0.66810, 5.46544, 0.66125, 0, 3.63998\}$. The parameter $a_7=0$ because it corresponds to the parallel current at the wall, which is zero. A comparison of the time series for MST data and the model equations are shown in Figure 3.33. The period one UPO was found to have a temporal period of 15.8 ± 0.5 time units. This corresponds to $79 \pm 2.5 \mu\text{s}$ in the MST, which is a frequency of 12.7 ± 0.4 kHz. The (m, ε) recurrence point histogram is shown in Figure 3.34. The results of perturbing this model are shown in Figure 3.35.

Figure 3.33. A comparison of poloidal field fluctuations at the wall of MST and from the outermost magnetic island in the $N=3$ Yoshida model. The bottom plot shows the autocorrelation of both signals. This plot shows 50 points out of the 2000 point time record that was used for the fit.

Figure 3.34. An (m, ε) recurrence histogram to find the UPOs of the $N=3$ Yoshida equations modeling the MST. The vertical lines are spaced at $\Delta m=158$.

Figure 3.35. Effect of perturbing chaotic Yoshida equations modeling the MST. The unperturbed parameter value was $E_{hl} \approx 0.66125$. Black indicates limit cycle cases. Dark gray $\rightarrow 2 < D_{KY} < 3$; Light gray $\rightarrow 3 < D_{KY} < 4$; White $\rightarrow 4 < D_{KY} < 5$. The unperturbed case has a dimension of $D_{KY} = 4.1$. The dark arrows indicate the frequency of the period one UPO and its harmonics. The light gray arrows indicate the power spectral peak frequency and its first harmonic. The frequency is in units of 200 kHz in an MST conversion.

The results of this calculation can be used to determine if any of these perturbations has a positive effect on transport. Rechester and Rosenbluth [3.54] showed that the electron heat transport in a tokamak plasma with destroyed magnetic surfaces is given by:

$$\chi_r(r) = \pi R v \sum_{m,n} \frac{|b_{mn}(r)|^2}{B_z^2} \frac{l_0}{L_c} \exp\left(-\left(\frac{m}{q(r)} - n\right)\right) \quad (3.49)$$

Here, χ_r is the diffusion coefficient, R is the minor radius of the plasma, v is electron thermal velocity, $b_{mn}(r)$ is the radial field fluctuation of a mode (m,n) , B_z is the toroidal magnetic field, $q(r)$ is the safety factor, and $l_0 \exp(\dots)$ is the width of an area associated with a magnetic field surface that gets distorted, and L_c is the radial correlation length. Since the Yoshida equations explicitly describe radial and poloidal magnetic field fluctuations, one can determine if applied perturbations decrease the theoretical heat transport in this model. Figure 3.36 shows χ_r for a range of perturbations, with black indicating a decrease in the diffusion coefficient, and light gray indicating an increase. It is apparent from Figure 3.36 that perturbations near the UPO frequency and its harmonics increase the fluctuation amplitude at low perturbation amplitude. This counterintuitive result may be because these resonant frequencies are most able to feed energy into this nonlinear system. It is remarkable, however, that at the optimal frequency of 50.8 kHz, one can achieve both limit cycle behavior and a reduced amplitude. Figure 3.37 shows the diffusion coefficient as a function of perturbation amplitude at the optimal perturbation frequency of $f=50.8\text{kHz}$.

Figure 3.36. Electron heat diffusion coefficient as calculated using the Rechester-Rosenbluth formula normalized to the diffusion without perturbation. Black indicates a decrease in the diffusion, while light gray indicates an increase. The frequency is in units of 200 kHz in an MST conversion.

Figure 3.37. Rechester-Rosenbluth normalized heat transport at the optimal frequency of 50.8 kHz for the (1,6) mode ($\times$), the (2,13) mode (Δ), the (1,7) mode ($\square$), and their sum ($\circ$).

3.8. Neural Net Model for MST

Another way to study prospects for controlling fluctuations in MST is to construct an artificial neural network model of the magnetic dynamics, and then to add periodic perturbations to the model in a numerical simulation. The neural net chosen has N neurons, with the input of each neuron consisting of the superposition of the weighted sum of the output of all the neurons (including itself). The neurons are represented by hyperbolic tangent squashing functions that provide the nonlinearity necessary for chaos:

$$x_i(t+1) = \tanh\left(s \sum_{j=1}^N W_{ij} x_j(t)\right). \quad (3.50)$$

Neural networks are universal approximators and are automatically bounded. The parameter s is a scale factor on the weights and is initially chosen as $s = 1$. The $N \times N$ matrix of weights W_{ij} is adjusted by a variant of simulated annealing so that the N outputs of the neurons $x_i(t+1)$ at T successive time steps fits an $N \times T$ matrix of training points obtained from an array of N poloidal field magnetic pickup coils (measuring dB_p/dt at the wall). The training set used has $N = 64$ and $T = 255$, with a sample time of $5 \mu s$ (200 kHz sample rate) and was rescaled so that the data fall in the range -1 to +1 as shown in Figure 3.38.

Figure 3.38. Spatiotemporal plot of poloidal field fluctuations at the wall of MST. The dominant feature is a rotating resistive tearing mode with a frequency of about 14 kHz.

After training the net for several days of 200 MHz CPU time, the output of the network on the same scale as above has the waveform shown in Figure 3.39.

Figure 3.39. Spatiotemporal plot of a neural net trained to model MST poloidal field fluctuations.

The simulation does capture some sense of the rotating mode seen in the experiment, with a frequency of about 11 kHz. Note that coil 26 from the top appears to have its polarity reversed in the experimental data, and this anomaly was captured by the neural net. The model is not chaotic, however, as evidenced by calculation of the LLE, which is indistinguishable from zero. Also, the solution appears to be a limit cycle. To obtain a chaotic solution, the parameter s was increased to 1.2, giving the result in Figure 3.40.

Figure 3.40. Spatiotemporal plot of a neural net trained to model MST poloidal field fluctuations with the parameter s modified to yield chaotic solutions.

This case has a largest Lyapunov exponent (base-e) of 0.028/iteration and appears to be slightly more chaotic than the experiment. In retrospect, this case should probably have been done with a smaller value of s , but about a week of computation is required to train the net and another week to produce high quality perturbation plots. The rotating oscillation persists at a frequency of about 10 kHz. A perturbation $s = s_0 + s_1 \sin(2\pi ft)$

with $s_0 = 1.2$ was added to this case, and the perturbation amplitude was scanned over the range $0 < s_1 < s_0$, while the frequency was scanned over the range $1 \text{ kHz} < f < 100 \text{ kHz}$. The largest Lyapunov exponent was calculated for each of 19,200 combinations of s_1 and f . The result is plotted in Figure 3.41 with a gray scale in which the darkest color indicates an exponent of zero (chaos completely suppressed) and the lightest color indicates an enhancement of the chaos (approximate doubling of the Lyapunov exponent):

Figure 3.41. Result of perturbing the chaotic MST neural net. Black indicates limit cycle behavior. White indicates the LLE was approximately doubled.

The general result is a decrease in chaos for increasing perturbation amplitude,

with certain frequencies such as 13 kHz being especially effective. The striations in the upper left are caused by truncating the calculation at a time (500 iterations) that is not an integer number of perturbation cycles. Since the Lyapunov exponent generally increases with s , chaos is enhanced when the final value of s is positive and diminished when it is negative.

3.9. Summary

Periodic perturbations have been applied to several standard chaotic numerical systems by modulating some system parameter. In every case, the optimum frequency to obtain control with a minimum perturbation amplitude was found to be the frequency of the UPOs obtained from the dynamical fluctuations of the system. One interesting result was that plots of controlled cases on a perturbation frequency- amplitude plane bore some strong dissimilarities to the Arnold tongues usually seen in systems that undergo a quasiperiodic route to chaos. This difference has been observed in at least one other experiment, however [3.31]. Also, a given frequency can stabilize more than one UPO. Another interesting discovery was that UPOs can have vastly different time scales than the dominant fluctuations in a given state space variable, which means that frequencies containing the most power are not always the best frequencies to perturb.

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4. Periodic Perturbations Applied to MST

The Madison Symmetric Torus is a relatively large reversed field pinch (RFP) [4.5]. An RFP is a prototypical fusion device that is characterized by a toroidal field that decreases from its on-axis value to zero at what is known as the reversal surface. Outside the reversal surface, the toroidal field is in the opposite direction of its on-axis direction. This device differs from a tokamak, whose toroidal field is much stronger than its poloidal field, and consequently does not change direction. MST has a major radius of 1.50m, a minor radius of 0.52m, and is ohmically heated by a 2V-sec. transformer. MST has many innovations to reduce deleterious magnetic field errors. It has a 5 cm-thick aluminum shell, which serves as both a vacuum vessel and a toroidal field winding. This shell greatly reduces toroidal field ripple associated with discrete coils. It also has a large number of small vacuum portholes to reduce the radial field error associated with large holes in the wall. It can achieve plasma currents up to 700kA, and has typical loop voltages of about 10V, density of about $1 \times 10^{19} \text{m}^{-3}$, and electron temperatures greater than 600eV. Typical confinement times are in the range of 2 to 5 ms.

Once it was decided to attempt to control chaos on MST with periodic perturbations, a parameter to perturb had to be chosen. There are several choices that could have been made which fall into one of three broad categories. First, one could perturb the plasma density. This could be achieved by oscillating the main fueling puff valves or the plasma gun fueling. However, it takes a few milliseconds for the fuel to reach the core of MST, which is too slow for this experiment. Also, it is uncertain whether the puff valves could be reliably driven at kilohertz frequencies.

Second, one could perturb the electrostatic potential of the plasma. Almagri [4.1] found that the plasma rotation could be enhanced or halted by the application of an electrostatic bias to a flux surface. Craig *et al.* [4.2] have seen similar results by applying a bias with up to 16 plasma guns.

Finally, one could manipulate one of the many magnetic fields. For instance, the primary current in the ohmic heating transformer, which establishes the plasma current and thus the poloidal field, could be perturbed. Stherblom [4.3] found this method increases magnetic helicity in a DEBS code simulation of an RFP. Recall from Chapter 3, where the Yoshida equations were adapted to model three tearing modes in the MST, that chaos can be eliminated for perturbations at integer multiples of the UPO frequency of 12.7 kHz. However, this circuit has extremely high power, and it would require major modifications to MST. The toroidal field is another large magnetic field that could be perturbed, but there are implementation problems to this high power circuit as well.

There exist magnetic fields in MST that are more manageable, and yet can still have a significant effect on the plasma behavior. These are the radial magnetic fields at the poloidal and toroidal gaps. These field errors have a magnitude of a few tens of Gauss, but they can have a large effect on plasma behavior, mostly through mode locking and wall interactions [4.4].

The poloidal gap of MST allows flux to enter the machine from the poloidal field transformer. Significant care was taken in the design of MST to minimize field errors due to this gap [4.4,4.5], but radial field errors of about 20 Gauss can cause the plasma to lock to the wall at this gap and thus stop rotating. This locking has the effect of reducing

confinement as particles follow the field lines out and collide with the wall, also resulting in impurity injection. Extensive sets of field error correction coils exist at the poloidal gap to minimize this effect. There are eight main field error coils in parallel with the primary current, so that the waveform of the correction afforded by these coils matches the plasma waveform. There are also static $m=1$ vertical and horizontal field error correction coils, and a set of $m=4$ correction coils. Thus, much time, energy, and effort has been invested in controlling the magnetic field at the poloidal gap.

The other gap in the machine is the toroidal gap. It allows for the poloidal circulation of current in the shell of MST, which generates the low-ripple toroidal field. The toroidal gap was chosen as the place to insert a periodic perturbation to MST. The motivation for this choice was that it was an easily accessible system parameter, and that it allowed for a complementary set of experiments in the prevention of mode locking and the enhancement of plasma rotation with a rotating magnetic perturbation.

As mentioned above, one main goal in the experiment is to eliminate locking of global tearing modes to stationary field errors, since such locking generally results in increased impurity influx and reduced confinement. One way to avoid this locking is to impose a rotating field error on the plasma [4.6-4.14]. Heuristically, this process works by having the plasma lock onto the perturbation, which is at rest in the rotating reference frame, meaning that it will continue to rotate in the lab frame.

4.1. Experimental Apparatus

The experimental apparatus used to affect the rotating modes consisted of two coil sets threaded through leak-checking holes in the toroidal gap. One set was a pair of coils that generated an $n=6$ perturbation. This case was chosen because the (1,6) mode is dominant in the magnetic fluctuation spectrum of MST. The other coil set consisted of a pair of coils that generated an $n=1$ perturbation. This was chosen because the (1,6) and (1,7) modes strongly couple to the (0,1) mode, which is resonant at the reversal surface. Thus, it was hypothesized that these two spatial structures would have the best chance of effecting mode rotation. Figure 4.1 and 4.2 show the $n=6$ and $n=1$ coils, respectively.

Figure 4.1. Top view of the $n=6$ coil set. Drawing courtesy of Mark Thomas.

Figure 4.2. Top view of the $n=1$ coil set. Drawing courtesy of Mark Thomas.

Calculations indicated that the coils have a fairly clean mode spectrum[4.15]. The $n=6$ coils produce a toroidal mode spectrum that has slightly over 80% of its power in the $n=6$ mode, with the majority of the remaining power residing in the $n=18$ mode. The $n=1$ coil set has in excess of 95% of its power in the $n=1$ mode. The coils squeeze about 8 Gauss of field through the gap for every 100A of drive current. This value is verified by

the fact that sawtooth oscillations, which force toroidal flux out through the gap, generate transient current spikes in the coils on the order of the drive current.

The coils were driven by two different methods. The first method is illustrated in Figure 4.3, and was used for initial low-current tests of the circuit. Here, the coils are coupled by step-down transformers to separate branches of a parallel LC ($L_p C_p$) circuit. L_p and C_p are adjusted to give the proper 45° phase shift given the drive frequency for each branch, which results in the total phase shift of 90° required for a rotating perturbation. R_m is adjusted to match impedances across the transformer for maximum current drive.

Figure 4.4 shows the circuit set-up for a higher-current scheme. In this case, drive was achieved by allowing current to be drawn through the coils by fast opening and closing transistor switches in series with the coils and a large DC power supply. Currents up to 500A peak-to-peak could be obtained in this way. This system was also used with the switches continuously closed in order to generate a large "DC" current pulse in the coils to see if locking to the stationary field error thus generated could be achieved.

Figure 4.3. The circuit used to drive small-amplitude rotating magnetic perturbations. $L_{1,2}$ and $R_{1,2}$ are the inductance and resistance for each of the coils that makes up the $n=1$

or $n=6$ coil pair. $R_{m1,2}$ are the matching resistances for the transformers, and $C_{t1,2}$ are the tuning capacitors for each coil set. L_p and C_p are the phasing inductor and capacitor. The drive V_d was supplied by a Hewlett Packard digital function generator whose signal was amplified by an ENI Model 1140LA power amplifier, producing 1000 Watts, with $Z_{out}=50\Omega$.

Figure 4.4. The circuit used in the high-current experiments. $L_{1,2}$ and $R_{1,2}$ are the inductance and resistance for each of the coils that makes up the $n=1$ or $n=6$ coil pair. $S_{1,2}$ are high-power Insulated Gate Bipolar Transistor (IGBT) switches driven by a TC4422CPA chip and HP function generator. The R_c are crowbar resistors that allow current in the coils to dissipate when the switch is opened. V_+ is an adjustable DC voltage supply (up to 1kV). R_s is a shunt resistor that was required to keep the power supply "lit" when the coils were driven with a rotating perturbation. The power supply, which used ignitron switches, must be in conduction at all times. At certain times for the rotating perturbation, both S_1 and S_2 are open, which would extinguish the power supply.

4.2. Low Current Results and Mode Locking

Initial results using low current, rotating perturbations were somewhat ambiguous, since only about 80A peak-to-peak could be driven in the coils. The $n=6$ induction motor seemed to increase the rotation frequency of the $m=1$, $n=5,6,7,8$ modes by about 5%, but this is not statistically significant within the bounds of the noise. A 23 kHz $n=6$ perturbation that rotated in the same direction as the plasma rotation increased the overall number of unlocked shots by about 10%, while a 10.9 kHz $n=6$ and a 10.0 kHz $n=1$

counter-rotating perturbation decreased the number of locked shots by about 5%. For six shots, it was found that the plasma would suddenly lock or unlock if co- or counter-rotating perturbations were gated on and off during the middle of the plasma current flat-top time. Figures 4.5 and 4.6 show this effect for perturbations gated on at 15ms and off at 25ms. While this effect may not be statistically significant, it was never seen in control cases with the perturbation off. With these somewhat encouraging results, the higher power circuit in Figure 4.4 was implemented.

Figure 4.5. Anomalous unlocking of the $n=6$ mode rotation when a co-rotating perturbation was gated on at 15 msec and off at 25 msec.

Figure 4.6. Anomalous locking of the $n=6$ mode rotation when a counter-rotating perturbation was gated on at 15 msec and off at 25 msec.

4.3. High Current Results

Results using the second, higher power circuit are discussed here. In this case, a stationary (i.e., oscillating, but non-rotating) perturbation was applied to maintain some parallelism with numerical results and because the rotating perturbation had smaller amplitude current due to a shunt resistor required to keep the DC power supply from extinguishing. The data are analyzed according to the methods used in Chapter 3:

identification of UPOs, computation of the power spectra, and computation of the correlation dimension. Evidence for stochastic resonance, as discussed in Chapter 2, was also sought. The main signals analyzed were the outputs of the sixty-four toroidal array coils, which measure the poloidal field at the wall, because they give a lot of data that can be used for more accurate dimension calculations.

4.3.1. Unstable Periodic Orbits

The obvious choice for a perturbation frequency is about 12 to 15 kHz, which corresponds to the tearing mode frequencies. However, as was seen in identifying successful perturbation frequencies in Chapter 3, one needs to look for the UPOs in the system. The UPOs were identified in MST using data taken from sixty-four toroidal array coils spaced uniformly around the lower, inner side wall of MST (poloidal angle of 241°). To determine the UPO period, each coil was treated as a state space dimension, which allowed for a larger embedding dimension than Watts used in his search for chaos and simple determinism in MST [4.16,4.17]. Once the state space was defined in this way, Euclidian distances were constructed for each point relative to later temporal points. Data from 10 to 30 milliseconds were used, which is during the flat-top of the plasma shot. This interval gives the best chance for stationarity in the data.

Two slightly different ways of looking at these distances were employed. First, for each point in the 64-dimensional space, the first time at which the trajectory returned to within an ε neighborhood of the starting point was recorded. A histogram of recurrence points obtained in this way is shown in Figure 4.7 (m is in units of $10 \mu\text{sec}$).

Figure 4.7. (m, ϵ) recurrence plot for MST data for shot 117 on 25-May-1998. The large peak at $m=1$ is due to close-in-time cases. The other peak is at $m=11$, which corresponds to a frequency of 9.1 kHz, while the smaller peak at $m=28$ corresponds to a frequency of 3.6 kHz

A second method was used as an extension to the first method, in which subsequent nearby state space points were identified, and the time between these points

recorded. A histogram using this method is shown in Figure 4.8. Figure 4.9 shows the power spectrum for one of the wall coils for the shot corresponding to Figures 4.7 and 4.8.

Figure 4.8. Recurrence plot using all temporally subsequent points for a given initial point for shot 117 on 25-May-1997. The peak is at $m=14$, which corresponds to a frequency of 7.1 kHz. The small second and third peaks are at $m=27$ and $m=36$, which correspond to frequencies of 3.7 and 2.8 kHz, respectively.

Figure 4.9. Power spectrum of a toroidal array pickup coil for shot 117 on 25-May-1997. The two peaks are at 10.5 and 14.8 kHz, roughly twice the frequency of the identified period one UPO at 7.1 kHz. Higher order UPOs have frequencies that are subharmonics of 7.1 kHz.

The UPOs identified in this manner for unperturbed control case shots on 25 and 26-May 1997 indicate that the period one UPOs have a temporal period of 0.140 ± 0.013 ms, which is 7.1 ± 0.7 kHz. This frequency is roughly half the tearing mode frequency. Thus, we see that the global dynamics identified by the dominant UPOs are on a slower time scale than the peak power spectral frequency, which is in agreement with numerical results for high-dimensional cases like the coupled Lorenz equations and the Yoshida equations. Figure 4.10 shows a spatiotemporal plot for a period one UPO. This

plot was made by finding the UPO with the smallest ε and then repeatedly plotting the sequence of m points. For this shot $m=9$, which corresponds to an 11.1 kHz frequency.

Figure 4.10. A period-one UPO for shot 117, 25-May-1998. This UPO has a recurrence time of 0.09 msec, which corresponds to a frequency of 11.1 kHz.

4.3.2. Correlation Dimension

In Chapter 3, one main diagnostic of the effect of a periodic perturbation was to measure the fractal dimension of the perturbed system. Watts *et al.* [4.16,4.17] looked extensively at evidence for the existence of low dimensional chaos and simple

determinism in MST. They concluded that the state space dimension of MST is larger than 8, and probably larger than 15. Here we examine whether the applied perturbations lowered the dimension of the dynamics of MST to a measurable level.

The primary diagnostic used by Watts *et al.* was the correlation dimension. The correlation dimension was proposed by Grassberger and Procaccia [4.18] to provide a measure of the fractal dimension of an attractor. It is defined in terms of the limit of a quantity known as the correlation sum $C(R)$:

$$C(R) = \frac{1}{N(N-1)} \sum_{i=1}^N \sum_{j=1, j \neq i}^N \Theta(R - |x_i - x_j|). \quad (4.1)$$

Conceptually, the correlation sum measures the probability that two randomly chosen points on the attractor are within some distance R of each other. Here, $\Theta(x)$ is the Heaviside step function:

$$\Theta(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases} \quad (4.2)$$

The correlation dimension is thus:

$$D_c = \lim_{R \rightarrow 0} \frac{d \log C(R)}{d \log R}. \quad (4.3)$$

Watts *et al.* were not able to search embedding dimensions higher than seven, because of the paucity of data. For the correlation dimension to have enough statistics in each bin size at smaller separations R , one must have a minimum number of data points N ,

given by Tsallis [4.19]: $N=10^{2+0.4D}$. Here, this problem was circumvented somewhat by generating a time series that consisted of the time series from all 64 coils interleaved to form a giant time series 64 times as long as any one coil data record. If all time points for all 64 coils were used, it would take a year to compute the correlation dimension. Thus, data for 64 coils were used that yielded a time series 30,000 elements long, which used 2.5 msec of data from each coil. Figure 4.11 shows the correlation dimension for embeddings up to 15 for this extended time series method. The result indicates that MST is still of an immeasurably high dimension with the perturbation applied.

Figure 4.11. Correlation dimension for shot 46, 26-May-1998 for an $n=6$ perturbation at 3.5kHz (a period-two UPO frequency). An embedding dimension up to 15 was used on a 30,000 element time series formed from interleaving the time series from 64 toroidal array coils. This result is typical for all perturbed and unperturbed shots examined.

4.3.3. Stochastic Resonance

As mentioned in Chapter 2, stochastic resonance (SR) is a phenomenon whereby a periodic signal is enhanced in a nonlinear system when an optimal amount of noise is present [4.20]. A typical SR study involves examining the signal-to-noise ratio (SNR) for a particular peak in a power spectrum for various amounts of background noise. A system that exhibits SR shows a distinct peak when the SNR is plotted as a function of the background noise. I and Liu [4.21] have found evidence for SR in a low-dimensional radio-frequency magnetoplasma.

Figure 4.12 shows the SNR vs. noise baseline for a single poloidal field pickup coil on the wall of MST for a number of shots that were perturbed at 10.9 kHz with co- and counter-rotating magnetic field perturbations. The noise baseline was computed by fitting an exponential decay to the coil's power spectrum at all frequencies except where peaks occurred. The SNR was taken to be the logarithm of the peak in the power spectrum in a 1.0 kHz window centered on 10.9 kHz divided by the noise baseline at the peak frequency in the window. The lack of a distinct peak in Figure 4.12 indicates that MST does not appear to exhibit stochastic resonance.

Figure 4.12. Plot showing lack of evidence of stochastic resonance for co-rotating (o) and counter-rotating (+) 10.9 kHz perturbations. The "noise" is taken to be the baseline in the power spectrum of a poloidal field pick-up coil at the wall at 10.9 kHz. The SNR is the logarithm of the peak value of the power spectrum within a ± 0.5 kHz window of 10.9 kHz divided by the baseline noise level at the peak frequency.

4.4. Summary

Periodic radial magnetic field perturbations were applied to MST plasmas by driving a set of $n=1$ and $n=6$ coil sets that pierced the toroidal gap. Calculations show that the UPOs for MST have a frequency of about 7 kHz, which is about half the peak power spectral frequency due to tearing modes. This result of the UPO having a slower time

scale than is indicated by power spectra is consistent with numerical results for high dimensional systems.

There is no evidence that the small periodic perturbations applied decreased the dimension of the system to a level that could be measured using the correlation dimension. However, this result does not mean that they had no effect; rather that the effect could not be properly diagnosed within the limits of the finite amount of noisy data available. Higher power experiments with better-tuned perturbations in the future may show some positive effects, but it is uncertain if the perturbation can be properly tuned for each plasma shot, since UPO analysis indicated slight differences in the peak of the (m, ϵ) histogram from shot to shot. Also, numerical results suggest that the perturbation frequency has to be very close to the frequency of the UPO, and this frequency may change somewhat during and between shots.

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5. Conclusions

In this thesis, the effect of applying periodic perturbations to some accessible system parameter of various numerical chaotic systems and the Madison Symmetric Torus was studied. This method of controlling chaos was initially chosen because it is widely and easily applicable to a number of systems. It was found that it can elicit a wide variety of behaviors in a given system depending on the perturbation frequency and amplitude. The most effective frequencies are those of the underlying unstable structures, or unstable periodic orbits (UPOs) in the dynamics. For perturbation frequencies that are rational multiples of the UPO frequencies, control can be achieved for quite low perturbation amplitudes, on the order of a few percent of the nominal value of the perturbed parameter. In low dimensional systems, "control" means the attainment of stable limit cycles. For high dimensional systems, "control" means a significant reduction in the dimension of the system or the largest Lyapunov exponent (LLE), even if stable limit cycles are not achieved.

It was found that in high dimensional systems the UPO frequency does not always correspond to the peak power spectral frequency. In the high dimensional systems examined here, the UPOs always had lower frequencies than what might be surmised from looking at a power spectrum. Thus, the overall global dynamics tend to oscillate more slowly than any single state space fluctuation.

A connection was also found between the LLE and the dimension of controlled high dimensional systems; a system that has had its dimension lowered by a periodic perturbation tends to have a smaller LLE than the unperturbed system. Controlled

systems may also have smaller fluctuation amplitudes, which has important implications for plasma confinement.

This method has not succeeded at controlling chaos in MST, because the experiment was probably on the margin of not having enough power. Also, the dynamics in MST are high dimensional and noisy. A UPO analysis in a sixty-four dimensional state space of MST poloidal field fluctuations indicate that future work should use a frequency of around 7 kHz. Perturbations should be at least 20% of the unperturbed parameter value. Numerical results indicate that success might be achieved if the perturbation were applied to the poloidal field transformer. Other good candidates for perturbation include the plasma guns, the toroidal field transformer, and magnetic field errors at the poloidal gap. Continued studies of a perturbative field applied through the toroidal gap at higher power might not only decrease the chaos in MST, but also give added insight into mode locking and improve plasma confinement.