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TITLE: COMPUTING n-DIMENSIONAL VOLUMES OF COMPLEXES: APPLICATION TO CONSTRUCTIVE ENTROPY BOUNDS

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Computing n-Dimensional Volumes of Complexes: Application to Constructive Entropy Bounds

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Abstract— Constructive bounds on the needed number-of-bits (entropy) for solving a dichotomy problem can be represented by a quotient of two volumes of multidimensional solids. Exact methods for the calculation of these volumes are presented. They lead to a tighter lower bound on the needed number-of-bits than the ones previously known.

I. Introduction

The problem of finding the smallest size neural network which can realise an arbitrary function given by a set of m vectors (i.e. examples or points) in n dimensions is not new. Many results have been obtained for neural networks having a threshold activation function [3]. Probably the first lower bound on the size of a threshold gate circuit for "almost all" Boolean functions was given by Neciporuk: *size* $\geq 2(2^n\ n)^{1/2}$ [12]. Later [11], for *depth* $= 4$, Lupanov has proven a very tight upper bound: *size* $\leq 2(2^n\ n)^{1/2} \times \{1 + \Omega[(2^n\ n)^{1/2}]\}$. For classification problems, one of the first results was that a neural network with only one hidden layer having $m-1$ nodes could compute an arbitrary dichotomy, showing that for binary inputs the size grows exponentially as $m \leq 2^n$. A different approach for classification problems has been presented in [1, 4, 8], and is based on computing the entropy (i.e. number-of-bits) of the given data-set. Establishing bounds on the needed number-of-bits for solving a dichotomy is important for engineering applications. Knowledge of the bounds can improve certain constructive neural learning algorithms [2]. Moreover it can result in reducing the area of future VLSI implementations of neural networks [3, 6].

The paper will present an effective method for the exact calculation of the volume of any n-dimensional complex. The method uses a divide-and-conquer approach consisting of: (i) partitioning a complex into simplices; and (ii) computing the volumes of these simplices. It will be shown that this optimal choice is related to the symmetries of the complex, and can significantly reduce the computations involved. We shall use these results in conjunction with previous ones pertaining to lower entropy bounds for classification problems [1, 4, 5, 8]. They lead to an improvement over the best known lower entropy bound [5] for the case of neural networks with integer weights and thresholds in the range $[-p, p]$ (i.e. limited weights and thresholds).

II. Volume of nD Complexes

A. Complexes, Simplices and their Volumes

Any n-dimensional body bounded by $(n-1)$- dimensional hyperplanes is a complex. An n-dimensional complex with minimal possible number of vertices, i.e. $(n+1)$ vertices, is called a simplex. The general formula for the calculation of the volume of a simplex is

$$V_n = \frac{1}{n} h_n V_{n-1} = \ldots = \frac{1}{n!} \prod_{i=1}^{n} h_i \qquad (1)$$

where h_i is the height of an i-dimensional simplex with its spot on $i-1$ dimensional simplex with volume V_{i-1} (its basis). Any complex can be divided into a sum of simplices [7].

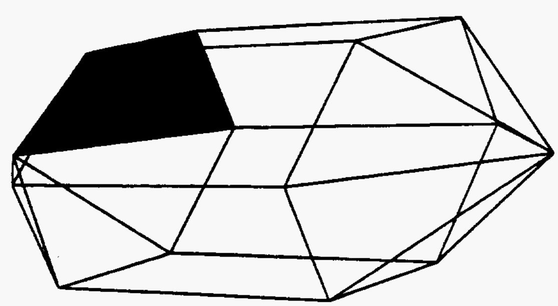

Fig. 1. 3D complex with a distinguished simplex.

The partition is not unique. This non-uniqueness gives us the freedom to choose that specific partitioning which is convenient for a particular case. The two possible approaches to this problem are as follows:

- General algorithm which works in all cases, but is a rather long, tedious method – symbolic computer calculations are recommended.
- Simplification of the problem by taking advantage of the symmetries of the particular complex – this method can be applied only in case of a highly symmetric complex.

B. General algorithm of the partition a complex into simplices

A general algorithm for finding the volume of any n-dimensional complex possessing k vertices ($k > n$) is based on partitioning the complex into a sum of simplices, and works as follows:

- If $k = n + 1$ it is already a simplex.
- If $k > n + 1$ choose one vertex, v_1.
- Consider the set consisting of all remaining vertices (i.e. $k - 1$ elements.)
- Take all the possible subsets of this set, containing $n - 1$ elements each. The vertex v_1 and any such subset define uniquely an $n-1$ dimensional hyperplane.
- Take all the hyperplanes obtained above. They define the faces of the simplices onto which the complex is partitioned.
- Calculate and add the volumes of the simplices.

The algorithm allows for the automation of the whole procedure, including the calculation of the volumes of the simplices. Hence, the exact calculation of the volume of any complex becomes possible. The partitioning into simplices obtained by using a direct computer program might not always be the optimal one in terms of ease of calculations, but it always leads to an exact solution.

C. Application of the symmetries of the complex

In particular cases the symmetries of the complex may significantly simplify calculations. Fortunately, the n-dimensional complexes of practical application for entropy calculations usually are highly symmetric. Let us consider, as an example, a complex used by Beiu & Draghici [5], for bounding the number-of-bits. The n-dimensional complex considered there consists of two hyperprisms, which have as common basis an $n-1$-dimensional complex.

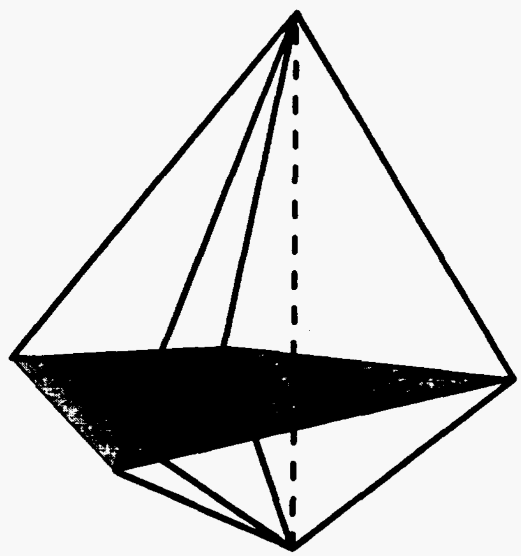

Fig. 2. The highly symmetric complex from [5] in 3D with 2D basis. The highs h in all three dimensions are dashed.

The sum of the heights of these hyperprisms is the same in every dimension and equal to h:

$$h = h_1 + h_2 = 1/p = d \tag{2}$$

were d is the smallest Euclidean distance between examples from opposite classes.

This means that every simplex with a height $h_1 = \frac{1}{p+1}$, has its counterpart, a simplex with the same basis b and height $h - h_1 = \frac{1}{p(p+1)}$. The sum of their volumes is equal to a volume of a simplex with height h. The sum of the volumes of these bases for all such pairs is also known: this is the $(n-1)$-dimensional complex described above. To find the volume of this complex one has to continue to repeat the same procedure in $(n-1)$-dimension, $(n-2)$..., down to 1-dimension. In this particular case it leads us to a simple formula:

$$V(h, n) = h^n/n! \ . \tag{3}$$

Taking into account that $h = d$, $V(d, n) = \frac{1}{d^n \, n!}$ and gives us the following lover entropy bound:

$$\begin{aligned} & \# \text{ bits }_{example} && (4)\\ = & \lceil \log \{V(D,n)/V(d,n)\} \rceil \\ = & \lceil \log (2\alpha(n-1)\, D^n\, a(n)\, p^n\, n!) \rceil \\ = & \lceil 1 + \log \alpha(n-1) + n \log D + \log a(n) + n \log p + \log n! \rceil \end{aligned}$$

where $V(D, n)$ is volume of the intersection of two spheres in n-dimensions, having the same radius D

and placed such that the center of each one is on the boundary of the other one. D is the largest Euclidean distance between examples from opposite classes.

Becouse $d = 1/p$, and knowing that $\log a(n) \geq -1.4667n + 0.0665$ (see [5]), and using Stirling formula we obtain:

$$\# \text{ bits }_{example} = \left\lceil n \log(D/d) - 1.6880n + \frac{\log n}{2} + 1.5667 \right\rceil \quad (5)$$

which slightly improves over the best previous known lower entropy boun from [5]:

$$\# \text{ bits }_{example} = \lceil n \log(D/d) - 0.4667n + \log n + 0.0665 \rceil \quad (6)$$

III. Volume of Solids Bounded by Curved Hypersurfaces

Computing the volumes of multidimensional bodies bounded by curved hypersurfaces is in general a complicated task. Fortunately, the solids under consideration in the calculation of the entropy bounds from [5, 8] are highly symmetric. These symmetries make it possible to exactly compute these volumes.

The preliminary step for computing the volume of a solid bounded by curved hypersurfaces is to look for symmetries of these surfaces. In case when the surfaces are parts of spheres, paraboloids, hyperboloids, cylinders, cones, the problem can be significantly simplified by a proper choice of a curvilinear orthogonal coordinate system. The choice is not always obvious — even in low dimensions — but finding such a coordinate system is highly rewarding: it tremendously simplifies calculations. Examples of such coordinate systems are: multidimensional spherical system for or cylindrical systems, system of orthogonal curvilinear coordinates built of families of ellipsoids and one sheet hyperboloids two sheets hyperboloids. These families are orthogonal to each other.

Unfortunately, not every curved hypersurface is a surface of a fixed coordinate in a reasonably simple orthogonal coordinate system.

IV. Slicing: The Universal Method

For n-dimensional solids bounded by curved hypersurfaces the essence of an effective method for the calculation of the volume is also based on recursive slicing the n-dimensional solid into $(n-1)$-dimensional solids. After k steps we obtain $(n-k)$-dimensional solids of known volumes. In order to obtain the volume of the n-dimensional solid one needs to integrate backward k-times. There is no guarantee that these integrations can always be done exactly. Even if the analytical integration is not possible, numerical results can always be obtain.

As an example let us consider volume of a part of an $(n+1)$-dimensional sphere.

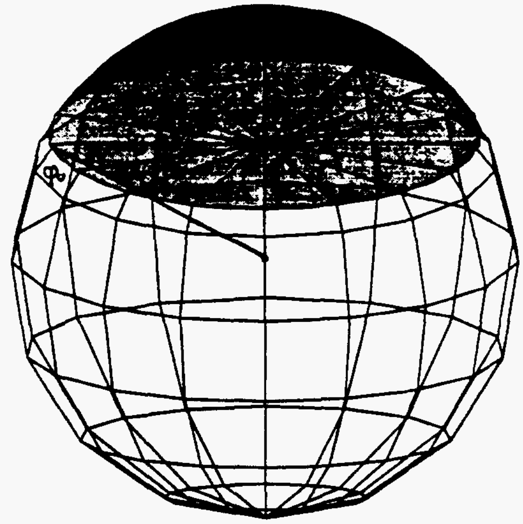

Fig. 3. Volume of a part of 3*-dimensional sphere.*

To find the volume we have to integrate over the angle ϕ in the range $(0, \phi_o)$ in the $(n+1)$-th dimension. This is an integral of the function which describes the volume of n-dimensional spheres constituting the $(n+1)$-dimensional one.

The first step is to find the volumes of the slices. We follow Maurin [10] for this calculation. The n-dimensional sphere can be sliced by the planes $x_n = const$. The slices are $(n-1)$-dimensional spheres which are reduced $\sqrt{1 - \frac{x_n^2}{r^2}}$ times with respect to the n-dimensional unit one. Therefore, we have the following expression for volume of the n-dimensional sphere:

$$|K^n| = \int_{-r}^{r} \left(\sqrt{1 - \frac{x_n^2}{r^2}}\right)^{n-1} |K^{n-1}| dx_n \quad (7)$$

where $|K^{n-1}|$ is the volume of $(n-1)$-dimensional sphere in our slicing, and r is the radius of the n-dimensional sphere. We substitute $x_n = r \sin t$, then $\sqrt{1 - \frac{x_n^2}{r^2}} = \cos t$, and:

$$\begin{aligned} |K^n| &= 2\int_0^{\frac{\pi}{2}} r|K^{n-1}| \cos^n t\, dt = 2|K^{n-1}| r \int_0^{\frac{\pi}{2}} \cos^n t\, dt \\ &= 2^n r^n \prod_{i=1}^{n} \int_0^{\frac{\pi}{2}} r \cos^i t\, dt \end{aligned} \quad (8)$$

There are only two possible cases.

For $n = 2k$ the product of the integrals gives:

$$|K^{2k}| = \frac{\pi^k}{k!} r^{2k} \quad , \quad (9)$$

while for $n = 2k+1$ it gives:

$$|K^{2k+1}| = 2^{k+1} \frac{\pi^k}{(2k+1)!!} r^{2k+1} \quad . \quad (10)$$

The final integral in $(n+1)$-dimension gives the volume we are looking for:

$$
\begin{aligned}
& V_{n+1} = \\
= \quad & \int_0^{\phi_o} \quad r|K^n| \sin^{n+1} \phi \, d\phi = r|K^n| \int_0^{\phi_o} \sin^{n+1} \phi \, d\phi \\
= \quad & r|K^n| \begin{cases} \frac{(-1)^{n/2}}{2^n} \Sigma_{k=0}^{\frac{n}{2}} (-1)^k \binom{n+1}{k} \frac{\cos((n+1-2k)\phi_o)}{n+1-2k} \\ \text{for } n{+}1 \text{ odd} \\ \frac{(-1)^{(n+1)/2}}{2^n} \Sigma_{k=0}^{(n-1)/2} (-1)^k \binom{n+1}{k} \frac{\sin((n+1-2k)\phi_o)}{n+1-2k} \\ + \frac{1}{2^{n+1}} \binom{n+1}{(n+1)/2} \phi_o \\ \text{for} n+1 \text{even} \end{cases}
\end{aligned}
$$

A bound on V_{n+1} was detailed in [4].

V. Conclusions

The geometrical results for computing volumes of multidimensional solids can be used to improve on the best known lower bound on the number-of-bits [5].

- The exact calculation of the volume of n-dimensional complexes is always possible.
- The exact calculation of the volume of n-dimensional solids bounded by curvilinear, but highly symmetric hypersurfaces (segments of spheres, tori, cones, cylinders, ellipsoids, hyperboloids, etc.), is usually possible, especially when one is able to introduce a proper system of curvilinear orthogonal coordinates.
- The calculation of the volume of an $n < D$ dimensional solid bounded by hypersurfaces without high symmetries is in general also possible by applying the slicing method, but usually one can get only numerical results.
- Using the methods presented in this paper, the exact calculation of a quotient of two volumes of multidimensional solids becomes feasible. It leads to a tighter lower bound on the number-of-bits (entropy) for solving a dichotomy problem. It has applications to constructive neural learning and VLSI efficient implementations of neural networks.

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