Wave Propagation in Laminates Using the Nonhomogenized Dynamic Method of Cells: An Alternative to Standard Finite-Difference Hydrodynamic Approaches

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Wave propagation in laminates using the nonhomogenized dynamic method of cells: An alternative to standard finite-difference hydrodynamic approaches

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Abstract

The nonhomogenized dynamic method of cells (NHDMOC) uses a truncated expansion for the particle displacement field; the expansion parameter is the local cell position vector. In the NHDMOC, specifying the cell structure is similar to specifying the spatial grid used in a finite-difference hydrodynamic calculation. The expansion coefficients for the particle displacement field are determined by the equation of motion, any relevant constitutive relations, plus continuity of traction and displacement at all cell boundaries. We derive and numerically solve the NHDMOC equations for the first, second, and third-order expansions, appropriate for modeling a plate-impact experiment. The performance of the NHDMOC is tested, at each order, for its ability to resolve a shock-wave front as it propagates through homogeneous and laminated targets. We find for both cases that the displacement field expansion converges rapidly: given the same cell widths, the first-order theory gives only a qualitative description of the propagating stress wave; the second-order theory performs much better; and the third-order theory gives small refinements over the second-order theory. The performance of the third-order NHDMOC is then compared to that of a standard finite-difference hydrodynamic calculation. The two methods differ in that the former uses a finite-difference solution to update the time dependence of the equations, whereas the hydrodynamic calculation uses finite-difference solutions for both the temporal and spatial variables. Both theories are used to model shock-wave propagation in stainless steel arising from high-velocity planar impact. To achieve the same high-quality resolution of the stress and particle velocity profiles, the NHDMOC consistently requires less fine spatial and temporal grids, and substantially less artificial viscosity to control unphysical high-frequency oscillations in the numerical solutions. Finally, the third-order NHDMOC theory is used to calculate the particle velocity for a shock-wave experiment involving an epoxy-graphite laminate. Constitutive relations suitable for the various materials are used. This includes linear and nonlinear elasticity, and when appropriate, viscoelasticity. The results agree well with the corresponding plate-impact experiment, and are compared to the second-order theory of Clements, Johnson, and Hixson (Phys. Rev. E, 54, 6876 (1996)).
1. INTRODUCTION

In this contribution we investigate a powerful and computationally efficient theoretical method for studying mechanical wave propagation in laminated materials. Because of their recognized technological and industrial importance, laminates have already been thoroughly investigated. In these previous studies a wide range of theoretical techniques were applied. It is impossible to summarize these investigations here, thus we mention only a few references that we found helpful and that are pertinent to the present work. The book by Nayfeh [1] gives a clear presentation of the general principles of wave propagation in layered materials, and also contains a very complete listing of references on this subject. In the present work, we will focus on only two theoretical techniques, one is a finite-difference hydrodynamic calculation (FDHC) suitably formulated for wave propagation in laminates. In that context, the work of Lundergan and Drumheller [2] provides a good reference. In a FDHC, finite-differenced equations of motion, mass, and energy, plus appropriate constitutive equations, are solved, subject to the applied boundary and initial conditions. No further assumptions, regarding the displacement, or stress fields, are assumed. In contrast, there are several classes of theoretical techniques (one is used in the present contribution) which allows further analytical steps to be taken, but usually compromises some aspect of the complete theory. An example of this is encountered in approaches involving variational methods, as they are applied to wave propagation problems. The insightful paper of Nemat-Nasser [3] illustrates the point. In that work, an ansatz for a class of displacement field test functions is considered for an elastic laminated material. From the set of functions, Nemat-Nasser invoked a variational principle to determine the optimal function. While the variational technique is powerful, and possesses a large degree of analytic character (thus reducing numerical based uncertainties), an approximate displacement field is ultimately determined from the procedure. Furthermore, since the true function might lie outside of the chosen class of test functions, the success of the approach tends to hinge on the physical intuition of the investigator. As will be seen below, in this respect our present analysis bares more
resemblance to the approach taken by Nemat-Nasser, than to approaches based on a purely FDHC.

Before embarking on yet another theoretical technique, we mention our motivation for doing so. The finite-difference method is flexible in that, in principle, any microstructure and any set of constitutive equations can be considered. The price of having microstructural complexity, however, is that very fine spatial and temporal grids must be used in the finite-differencing schemes. In turn, the task of solving the problem becomes computationally expensive. Approaches similar to the variational one, tend to be less computationally consuming, but are restricted to problems where the variational principle is applicable, thus tending to limit their use to systems possessing rather simple constitutive laws. The motivation for the present work, then, is to develop a technique that alleviates the need for using excessively fine spatial and temporal grids, but at the same time allows the incorporation of complex constitutive laws. The nonhomogenized dynamic method of cells (NHDMOC) has these properties and will be investigated in the present contribution.

To avoid possible confusion, it is important to point out that there are two different versions of the method of cells: a (spatial) homogenized version, abbreviated here by MOC, and the NHDMOC version. Both are attributed to J. Aboudi [4,5], though the technique bares strong resemblance to the generalized continuum theories of Achenbach [6]. The MOC theory is a micromechanical analysis based on the assumption that a single representative volume element (RVE) can be identified for the system. The RVE repeats itself throughout the space of the entire composite. Thus the MOC is best suited for composites having nearly periodic microstructure. This implies that a RVE with sufficiently simple microstructure can be identified, for the ensuing numerical calculations to be practical. The MOC theory uses conditions of stress and displacement continuity at all interfacial boundaries associated with the RVE, plus conditions of equilibrium. When applied to dynamical situations, the dynamics enter by other means. It is thus customary to use the MOC in conjunction with either a finite-element or a finite-difference hydrodynamic calculation. In that context, the MOC theory,
through a microstructural homogenization procedure, determines an effective stiffness matrix \( B_{\text{eff}} \), and the stress \( \sigma \), given the hydrodynamic strain rates \( \dot{\varepsilon} \) at any given spatial point in the system. Fig. 1 illustrates this. Since the MOC version is frequently applied to complex three-dimensional composites, computational tractability usually forces one to make approximations that limit its utility to problems where deformational variations occur over a length scale that are long compared to the variations in the microstructure.

Figure 1. Schematic to illustrate the application of MOC in finite element simulations.

In contrast to the MOC theory, no spatial homogenization is required in the NHDMOC theory, and the dynamics enter directly by solving an equation of motion (see Sec. 2). In this contribution, we attempt to provide evidence supporting the use of the NHDMOC for investigating wave propagation in shock loaded laminates. As alluded to above, the NHDMOC formalism is based on an ansatz for the particle displacement field, which has the
form of a truncated Legendre expansion. Because the laminate has relatively simple microstructure, we are able to keep all terms in the expansion out to third-order. We note that Aboudi [5] found that, already by second-order, the NHDMOC theory gave a rather good representation for a stress wave profile for a wave propagating in a dynamically loaded bilaminate. It is informative to study the convergence rate of this expansion; we do this in the results section (Sec. 3).

We then return to another thrust of this contribution, namely, to compare the third-order NHDMOC theory with the results generated from a standard Lagrangian FDHC. Since both methods use the same constitutive relations, the difference in the results generated by the two approaches lies in the numerical solution scheme. The differences are: first, in the FDHC, no ansatz for the displacement field is needed, and second, the FDHC requires the solution of both spatial and temporal finite-difference equations, whereas the NHDMOC requires the solution of only a single temporal finite-difference equation.

Finally, we illustrate the second strength of the NHDMOC by applying the third-order theory to a system described by a complicated constitutive law. Specifically, we use the theory to model a viscoelastic epoxy-graphite bilaminate. Our results are then compared to experimental data taken from a plate impact experiment, performed on that system.

2. THEORY

In Fig. 2, a schematic is shown of a flyer and target setup; this setup is the basis for the experiments to which our analysis will be applied. The target is a bilaminate composed of periodically repeating bilayers. The impact surfaces of the flyer and the target are assumed to be parallel at impact and the lateral dimensions are sufficiently large that lateral edge-effects are negligible. Consequently, the problem that we seek to solve is longitudinal wave propagation in one dimension. The flyer impacts the bilaminate with an initial velocity denoted by $v_0$. Laser interferometry is then used to measure the time-dependence of the particle velocity (also called the material velocity) at some position in the sample. As a
function of time, the particle velocity measurements reveal the detailed history of the wave motion passing that point.

\[ N \]

Figure 2. Flyer-sample configuration used to study shock wave propagation in a bilaminate.

In the NHDMOC analysis a Lagrangian grid must be specified; one is illustrated in Fig. 2. The impact grid cell, within the flyer, is denoted \( p = p_c \). The bilaminate portion of the figure is purposely kept simple by showing the case where the grid points are made to coincide with material boundaries of the bilaminate. In general, this need not be the case, and the bilaminate may require a finer grid than the actual spacings between the bilaminate layers. We let \( p \) denote the \( p^{th} \) grid cell, then the mechanical properties of that cell require the specification of the cell length \( d_p \), its mass density \( \rho_p \), and any relevant elastic moduli. The latter might be, for example, \( E_1(p) \) related to the longitudinal sound speed by \( C_L(p) = \sqrt{E_1(p)/\rho_p} \). Further, any material parameters pertaining to quantities like the artificial viscosity, porosity, plasticity, or viscoelasticity, must be specified.

Any point within the \( p^{th} \) cell will be located by the local continuous position vector \( \vec{x}_p \). The origin of \( \vec{x}_p \) is at the center of the \( p^{th} \) cell, as shown in Fig. 2. It is defined within the region \(-d_p/2 \leq \vec{x}_p \leq +d_p/2\). The NHDMOC uses this quantity as an expansion variable for
the particle displacement field. In particular, the NHDMOC theory uses the Legendre polynomials as a complete set of basis functions; the expansion is

$$u^{(p)}(x_p,t) = \sum_{i=0}^{M} U_i^{(p)}(t) P_i\left(2x_p/d_p\right).$$

(2.1)

For the expansion to be exact, M must be infinity. The NHDMOC formalism provides an approximate method for determining the expansion coefficients $U_i^{(p)}(t)$ (here called the cells variables) for any finite truncation of this series. Here, we restrict ourselves to the third-order (and lower orders) theory. This means that the particle displacement expansion is truncated after $M = 3$, and our first approximation is to set $U_i^{(p)}(t) = 0$, $l \geq 4,1 \leq p \leq N$. To third-order, the particle displacement expansion is

$$u^{(p)}(x_p,t) = U_0^{(p)}(t) + \left(\frac{2x_p}{d_p}\right) U_1^{(p)}(t)$$

$$+ \frac{1}{2} \left[ 3\left(\frac{2x_p}{d_p}\right)^2 - 1 \right] U_2^{(p)}(t)$$

$$+ \frac{1}{2} \left[ 3\left(\frac{2x_p}{d_p}\right)^3 - 3\left(\frac{2x_p}{d_p}\right) \right] U_3^{(p)}(t),$$

(2.2)

and thus \{ $U_0^{(p)}, U_1^{(p)}, U_2^{(p)}, U_3^{(p)}$ \} for all $N$ grid cells, and all $t$ of interest, are needed for the particle displacement field to be fully determined. The particle displacement field must satisfy an equation of motion (EOM) which relates the stress gradients to the particle accelerations by

$$\frac{\partial}{\partial x_p} [\sigma^{(p)}(x_p,t) + q^{(p)}(x_p,t)] = \rho_p \ddot{u}^{(p)}(x_p,t).$$

(2.3)

The addition of an artificial viscosity term, $q^{(p)}(x_p,t)$, similar to hydrodynamic-based calculations, is useful to reduce the unphysical, high-frequency ringing observed in numerical solutions. In this work, we use

$$q^{(p)}(x_p,t) = \eta_p d_p \sqrt{E_1^{(p)} \rho_p} \dot{\epsilon}(x_p,t),$$

(2.4)

where the first order moduli, $E_1^{(p)}$, is related to the longitudinal sound speed, as described above. In Eq. (2.4), $\dot{\epsilon}$ is the strain rate, and $\eta_p$ is a nondimensional constant, on the order
of 0.01 for most applications, and is independent of \( d_p \). The relation between the strain and the particle displacement, in this one dimensional situation, is

\[
\varepsilon^{(p)}(x_p, t) = \frac{\partial u^{(p)}(x_p, t)}{\partial x_p} .
\] (2.5)

Since we will model a viscoelastic epoxy-graphite bilaminate, in Sec. 3, we use for a stress-strain relation [7]:

\[
\sigma^{(p)}(x_p, t) = T^{(p)}_1 \left\{ \varepsilon^{(p)}(x_p, t) \right\} + \frac{1}{2} T^{(p)}_2 \left\{ \left[ \varepsilon^{(p)}(x_p, t) \right]^2 \right\} ,
\] (2.6)

where \( T^{(p)}_\alpha(t) \) is a viscoelastic operator; it acts on all time-dependent quantities within the brackets. For both the first and second-order moduli, \( T^{(p)}_\alpha(t) \) has the form,

\[
T^{(p)}_\alpha(t) \equiv E^{(p)}_\alpha \delta_{\alpha', \alpha} + \left[ \frac{M^{(p)}_\alpha - E^{(p)}_\alpha}{T_p} \right] \int_0^t dt' \varepsilon^{(p)}(x_p, t') . \] (2.7)

The relaxed moduli are denoted by \( M^{(p)}_\alpha \) and the characteristic time, for going from unrelaxed to relaxed behavior, is controlled by the relaxation time \( T_p \). Further, for nonviscoelastic materials only the first term in Eq. (2.7) contributes, and the stress-strain relation simplifies to

\[
\sigma^{(p)}(x_p, t) = E^{(p)}_1 \varepsilon^{(p)}(x_p, t) + \frac{1}{2} E^{(p)}_2 \left[ \varepsilon^{(p)}(x_p, t) \right]^2 .
\] (2.8)

The details for the full NHDMOC analysis have been reviewed elsewhere [8]. Thus it suffices to simply state the results of the analysis, which follow by invoking displacement and stress-continuity between adjoining (internal) grid cells, i.e.,

\[
u^{(p-1)}(\pm \frac{d_p}{2}, t) = u^{(p)}(-\frac{d_p}{2}, t) \quad p = 2, \ldots, N ,
\]

\[
\sigma^{(p-1)}(\pm \frac{d_p}{2}, t) = \sigma^{(p)}(-\frac{d_p}{2}, t) \quad p = 2, \ldots, N ,
\] (2.9)

and integrating the moments of the EOM over a given grid cell. The result is the following set of equations:
\[\rho_{p-1}d_{p-1}\left[3\ddot{U}_0^{(p-1)} + \ddot{U}_1^{(p-1)}\right] + \rho_p d_p \left[3\ddot{U}_0^{(p)} - \ddot{U}_1^{(p)}\right] - \frac{4\eta}{d_{p-1}}\left[3\ddot{U}_2^{(p-1)} + 5\ddot{U}_3^{(p-1)}\right] - \frac{4\eta}{d_p}\left[3\ddot{U}_2^{(p)} - 5\ddot{U}_3^{(p)}\right] = \frac{12}{d_p^2}T_1^{(p)} \left\{ U_1^{(p)} + U_3^{(p)} \right\} - \frac{12}{d_{p-1}}T_1^{(p-1)} \left\{ U_1^{(p-1)} + U_3^{(p-1)} \right\}
\]
\[+ \frac{12}{d_p^2}T_2^{(p)} \left\{ \left[U_1^{(p)}\right]^2 + 3\left[U_2^{(p)}\right]^2 + 2U_1^{(p)}U_2^{(p)} + 6\left[U_3^{(p)}\right]^2 \right\}
\]
\[= \frac{12}{d_{p-1}}T_2^{(p-1)} \left\{ \left[U_1^{(p-1)}\right]^2 + 3\left[U_2^{(p-1)}\right]^2 + 2U_1^{(p-1)}U_2^{(p-1)} + 6\left[U_3^{(p-1)}\right]^2 \right\} \]
\begin{equation}
(2.10)
\end{equation}

and
\[\rho_p d_p \left[\dddot{U}_0^{(p)} - \frac{1}{5}\dddot{U}_2^{(p)}\right] - 12\eta T_2^{(p)} = \frac{12}{d_p}T_1^{(p)} \left\{ U_2^{(p)} \right\}
\]
\[+ \frac{24}{d_p^2}T_2^{(p)} \left\{ U_1^{(p)}U_2^{(p)} + 3U_2^{(p)}U_3^{(p)} \right\} , \]
\begin{equation}
(2.11)
\end{equation}

and
\[\rho_p d_p \left[\dddot{U}_1^{(p)} - \frac{3}{7}\dddot{U}_3^{(p)}\right] - 60\eta T_3^{(p)} = \frac{60}{d_p}T_1^{(p)} \left\{ U_3^{(p)} \right\}
\]
\[+ \frac{120}{d_p^2}T_2^{(p)} \left\{ \frac{3}{5}\left[U_2^{(p)}\right]^2 + U_1^{(p)}U_2^{(p)} + 12\left[U_3^{(p)}\right]^2 \right\} , \]
\begin{equation}
(2.12)
\end{equation}

and
\[\dddot{U}_0^{(p-1)} + \dddot{U}_1^{(p-1)} + \dddot{U}_2^{(p-1)} + \dddot{U}_3^{(p-1)} = \dddot{U}_0^{(p)} - \dddot{U}_1^{(p)} + \dddot{U}_2^{(p)} - \dddot{U}_3^{(p)} \]
\begin{equation}
(2.13)
\end{equation}

Similarly, zero stress at both external boundaries \((p = 1 \text{ and } N)\),
\[\sigma^{(i)}\left(-\frac{d_1}{2}, t\right) = \sigma^{(N)}\left(+\frac{d_N}{2}, t\right) = 0 , \]
\begin{equation}
(2.14)
\end{equation}
yields two more relations,
\[\dddot{U}_1^{(i)} - 3\dddot{U}_2^{(i)} + 6\dddot{U}_3^{(i)} = 0 \]
\begin{equation}
(2.15)
\end{equation}

and
\[\dddot{U}_1^{(N)} + 3\dddot{U}_2^{(N)} + 6\dddot{U}_3^{(N)} = 0 . \]
\begin{equation}
(2.16)
\end{equation}

These equations may be expressed collectively as single \(4N\) by \(4N\) matrix equation,
\[A\dddot{U}(t) + B\ddot{U}(t) = R(t) , \]
\begin{equation}
(2.17)
\end{equation}
where the zeroth-, first-, and second-derivative terms of the cell coefficients have been separated into individual matrices. Matrices \(A\) and \(B\) are \(4N\times4N\) parameter matrices
constructed from the material parameters \( \{d, \rho, \ldots\} \). The column matrices \( \bar{U}(i) \) and \( \bar{U}(i) \) are composed of first- and second-time derivatives of the cell coefficients.

Eq. (2.17) can be represented as a finite-difference in the time variable, and the result solved for \( U(t + \Delta t) \):

\[
U(t + \Delta t) = (\Delta t)^2 \left[ A + \frac{\Delta t}{2} B \right]^{-1} R(t) + 2 \left[ A + \frac{\Delta t}{2} B \right]^{-1} A U(t) - \left[ A + \frac{\Delta t}{2} B \right]^{-1} \left[ A - \frac{\Delta t}{2} B \right] U(t - \Delta t). \tag{2.18}
\]

By incrementing the time, beginning at \( t = 0 \) (impact), the cell variables at all subsequent times are determined. This requires knowing the initial conditions, and these are given elsewhere [7,8]. Eq. (2.18) is the sole finite-difference equation encountered in the NHDMOC analysis.

We now turn to a brief review of the FDHC. The EOM is

\[
\frac{\partial}{\partial x} [\sigma(x, t) + q(x, t)] = \rho(x) \dot{v}(x, t), \tag{2.19}
\]

where \( v(x, t) \) is the particle velocity. The artificial viscosity is given the same form as before:

\[
q(x, t) = \eta(x) \Delta x \sqrt{E(x) \rho(x) \dot{e}(x, t)}. \tag{2.20}
\]

Since we will apply the FDHC only to homogeneous situations, we can drop the \( x \) dependence on the \( \eta, \Delta x, \rho, \) and \( E \).

In the shock geometry, the material is in a state of uniaxial strain and thus,

\[
e_{11}(x, t) = \varepsilon(x, t),
\]

\[
e_{22}(x, t) = e_{33}(x, t) = 0.
\]

\[
\sigma_{11}(x, t) = \sigma_{33}(x, t)
\]

(2.21)

For elastically isotropic materials the Mie-Gruneisen equation of state (EOS) is widely used, and has a compressive contribution of

\[
P(x, t) = B \frac{(1 - \frac{1}{2} \gamma \varepsilon(x, t))}{(1 - s \varepsilon(x, t))^2} \varepsilon(x, t) + \rho \gamma E(x, t), \tag{2.22}
\]

while the deviatoric contribution is

\[
s_\eta(x, t) = 2 \mu e_\eta(x, t). \tag{2.23}
\]

\( P(x, t) \) is related to the stress tensor by
\[ P(x,t) = \frac{1}{3} \sum_t \sigma_{ii}(x,t). \]  (2.24)

In these equations \( B, \mu, \) and \( \gamma \) are the bulk and shear moduli, and the Gruneisen parameter. The two functions,
\[ e_{ij}(x,t) = \varepsilon_{ij}(x,t) - \frac{1}{3} \sum \varepsilon_{ii}(x,t), \]  (2.25)

and
\[ s_{ij}(x,t) = \sigma_{ij}(x,t) - P(x,t), \]  (2.26)

are the deviatoric strain and stress. The internal energy is given by
\[ \rho \quad E(x,t) = \sum \sigma_{ij}(x,t) \varepsilon_{ij}(x,t), \]  (2.27)

and the parameter \( s \) is the slope of shock \( (U_s) - \) particle \( (u_p) \) velocity relation:
\[ U_s = c_o + s \quad u_p, \]  (2.28)

where \( c_o = \sqrt{K/\rho} \) is the low-pressure bulk sound speed. By setting \( \gamma = s = 0 \) and noting that \( E = B + \frac{4}{3} \mu, \) it is easy to see that this set of equations reduces to constitutive laws used in the NHDMOC analysis. To indicate the difference between the NHDMOC and FDHC, we show the full finite-difference expressions for these equations. Eq. (2.19) become
\[ v(x,t + \Delta t) = v(x,t) + \frac{\Delta t}{\rho \Delta x} [\sigma(x + \Delta x,t) - \sigma(x,t) + q(x + \Delta x,t) - q(x,t)], \]  (2.29)

while the strain rate is
\[ \dot{\varepsilon}(x,t) = \frac{1}{\Delta x} [v(x + \Delta x,t) - v(x,t)]. \]  (2.30)

Taking the time derivative of Eqs. (2.27) and (2.28)
\[ \dot{P}(x,t) = B \quad \dot{\varepsilon}(x,t), \]  (2.31)
\[ \dot{s}_{ij}(x,t) = 2 \mu \quad \dot{e}_{ij}(x,t), \]  (2.32)

these can be inserted in to the finite-difference equations for the mean compressive and deviatoric stresses.
\[ P(x,t + \Delta t) = P(x,t) + \Delta t \quad \dot{P}(x,t), \]  (2.33)
\[ s_i(x, t + \Delta t) = s_i(x, t) + \Delta t \dot{s}_i(x, t). \] (2.34)

Finally, the stresses are updated by the equation
\[ \sigma_i(x, t + \Delta t) = P(x, t + \Delta t) + \dot{s}_i(x, t + \Delta t). \] (2.35)

3. RESULTS

In this section we present the numerical results generated by the NHDMOC equations. The first issue we address is the convergence rate of the particle displacement expansion given by Eq. (2.1). The first-order NHDMOC theory keeps only the first two terms of this expansion (M=1), which are the first two terms in Eq. (2.2). The second-order theory retains the first three terms in Eq. (2.2), and the third-order theory uses all terms in Eq. (2.2). Of course, the third-order terms must be omitted from Eqs. (2.10) - (2.16), in the case of the second-order application, etc.

In Fig. 3, stress wave profiles, for a shock-wave front, are shown that have been generated by these three approximations for the NHDMOC. For this case, the system was chosen to be a stainless steel flyer impacting, at 1 km/s, onto a stainless steel-polymethylmethacrylate (PMMA) bilaminate. Because we are interested in assessing the convergence rate of the series, we have also looked at stress wave propagation in homogeneous system rather than a bilaminate. The bilaminate shows the same trends, but obscured slightly by the more complex stress wave profile associated with the bilaminate. No artificial viscosity is used in these calculations and only the linear term in Eq. (2.8) is kept. First we comment on the homogeneous case. As a function of distance along the target sample, the stress wave profile generated by the first-order theory is composed of zero-slope linear segments that are typically characterized by a substantial stress discontinuity at cell boundaries. The second-order theory approximates the stress wave with non-zero slope linear segments. The stress is typically not continuous across cell boundaries, although, the discontinuities are smaller than in the first-order calculation. Third-order represents the stress wave by segments of a second-order polynomial. We found, numerically, that the stress discontinuity was nearly zero for this calculation. A second advantage of the third-order
theory, over the lower orders, is that the oscillations observed at the front of the shock wave are largely reduced. The final advantage is that a coarser grid-width (compared to the one used at lower orders) remains giving a suitable representation of the stress wave for the third-order theory, thus helping to reduce computational time to achieve a numerical solution. In Fig. 3, we show the stress wave profile, as a function of elapsed time. In this figure it is clear that, while first-order tends to deviate noticeably from second- and third-orders, the later two deviate only by small amounts from each other. As a result of this study we conclude that the third-order theory is suitable for most applications of stress wave propagation, and it is doubtful that much would be gained by going to yet higher orders. A further discussion of these ideas can be found in Ref. [9].

![Graph](image)

**Figure 3.** Stress wave profile for a shock wave propagating in a bilaminate. First-, second-, and third-order NHDMOC theories are compared.

The next issue concerns comparing the third-order NHDMOC theory with the FDHC
outlined in the previous section. Again we focus on the simpler problem of shock waves propagating in homogeneous stainless-steel. For simplicity, linear elasticity is assumed. In Fig. 4 (Fig. 5), the velocity profiles, obtained by the hydrodynamic calculations (third-order NHDMOC), are shown. Here, a 0.10 cm steel flyer struck a 0.30 cm steel target, moving at a speed of 1 km/s. The stress is calculated at the midpoint of the target. The NHDMOC

![Diagram](image)

**Figure 4.** FDHC calculated particle velocity profiles to illustrate the importance of artificial viscosity and grid cell spacings.

results were generated using 40 equal-spaced cells, and time step of 0.001 μs. In the FDHC, a time step of 0.0001 μs was needed, and substantially finer spatial gridding was required (the grid spacing is given in Fig. 4). With zero artificial viscosity, the third-order NHDMOC theory (Fig. 5) remains giving stable solutions, although small numerical ringing is visible. These were removed by increasing $\eta$ to 0.006. Larger values of artificial viscosity are needed in the FDHC. For the FDHC we used $\eta$ equal to 0.1. At 400 cells, in the FDHC,
and $\eta=0.1$, the two velocity profiles nearly match (the NHDMOC used 40 cells and $\eta=0.006$). No further improvement was observed by going to still finer spatial and temporal grids in the FDHC. The fact that smaller applications of artificial viscosity are required in the NHDMOC is very important when the theory is applied to complex heterogeneous structures. There, multiple reflections and transmissions at all true material boundaries will produce true physical oscillations that should not be removed, by applying artificial viscosity, from the numerical solutions.

![THIRD-ORDER NHDMOC CALCULATION](image)

**Figure 5.** NHDMOC calculated particle velocity profiles to illustrate the importance of artificial viscosity and grid cell spacings.

The final point to be made is that the NHDMOC can relatively easily be applied to problems with complex constitutive behavior [7,8]. In this example (Fig. 6) the flyer plate is composed of a front Z-cut quartz impactor, backed by a larger slab of PMMA. The initial velocity of the flyer, before striking the bilaminate target, is 500 m/s. The bilaminate portion of the target is made of 19 bilayers. The composition of each bilayer is, first, a thin section
of pure epoxy and second, a thicker section made of an epoxy-graphite mixture. The graphite, in the mixture, are fibers stacked with their axes being parallel to each other. The fibers are distributed uniformly throughout the layer, and their axes are parallel to the bilayer interface. The bilayer is backed by a PMMA window. In the experiment, the PMMA window, being transparent to the laser light, was used only to provide an acoustical impedance match with the target; the particle velocity was measured at the bilaminate-PMMA interface. Both the experimental and theoretical velocity profiles are shown in Fig. 7.

![Figure 6. Epoxy-Graphite bilaminate to which the third-order NHDMOC is applied.](image)

At 0.9 μs after impact the primary compressive wave reaches the observation point, increasing the particle velocity from zero to approximately 430 m/s. After another 0.3 μs, an abrupt drop in the particle velocity indicates the arrival of a release wave at the observation point. Its origin is from the release occurring at the left-most edge of the Z-cut quartz impactor. This is followed by a second release that occurs at about 0.5 μs after the arrival of the first release wave, caused by a wave traversing the quartz impactor twice, after impact, before finally arriving at the observation point. There are also small oscillations superimposed on the velocity profile that are caused by the laminations in the composite.
This is easily deduced from the theoretical calculation; removing the laminations removes these oscillations.

The inset in Fig. 7 shows the differences between the second- and third-order NHDMOC calculations. It was established in Ref. [9] that a coarser spatial grid, at third-order, will achieve the same resolution of a shock-wave profile, as a finer grid in second-order. Thus, for example, it is not surprising that, in the present work, only 3/4 as many grid cells were needed in the bilaminate. Of course, the disadvantage to going to higher order is that in the third-order theory one must invert and solve a $4N \times 4N$ matrix equation, but only a $3N \times 3N$ matrix equation in the second-order theory. In the second-order calculation

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.jpg}
\caption{Particle velocity profile measured experimentally [7] for the epoxy-graphite bilaminate, and calculated with the third-order NHDMOC theory. The inset shows the second-order calculation.}
\end{figure}

no artificial viscosity was used. Our numerical results infer that the combined affects of the third-order (itself tending to reduce the unphysical numerical ringing), and the mild.
application of artificial viscosity, removed the unphysical numerical ringing seen propagating just ahead of the primary compressive wave in the second-order theory. Note that the shoulder in the inset, occurring at 300 m/s, at $t = 0.9 \mu s$, is not physical, but is caused by the finite-width grids used for the spatial and time variables. We also direct the readers' attention to the fact that the artificial viscosity has slightly reduced the physical oscillations seen after the arrival of the first release wave in the third-order theory. While undesirable, the reduction is small and the benefits of applying the artificial viscosity favorably outweigh the negative aspects.

The other striking difference between the second- and third-order theories, is the overall reduction in the gap in the particle velocity profiles between the experiment and the theory, observed when going from the second-order to third-order theory. The primary reason for this is that a different value for $E_2^{(p)}$ for PMMA is used: In the present case, $E_2^{(p)}$ is taken from Ref. [5], which is smaller than the value deduced in Ref. [7]. The present value is -1.31 Mbar, while the work in Ref. [7] used -0.72 Mbar. A simple calculation shows that this change should reduce the peak compressive velocity by a factor of approximately 1.04. Indeed, this reduces the theoretical peak velocity, in the inset of Fig. 7, to essentially that of the third-order calculation. Consequently, there is no motivation to repeat the second-order calculation using the new value of $E_2^{(p)}$.

4. CONCLUSIONS

We presented evidence for the advantages of the NHDMOC method over others, such as a FDHC, when dealing with composite systems. When compared to an actual impact experiment, involving an epoxy-graphite bilaminate, the third-order NHDMOC theory, including artificial viscosity, gives very satisfactory agreement with the experimental measured particle velocity. Further, the closeness of the second- and third-order theories indicates that little would be gained by going to still higher-orders theories. The results shown here demonstrate that the NHDMOC method provides a useful technique and may
become the standard approach for wave propagation calculations in composites and other heterogeneous materials.

4. REFERENCES


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Wave propagation in laminates using the nonhomogenized dynamic method of cells: An alternative to standard finite-difference hydrodynamic approaches

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Abstract

The nonhomogenized dynamic method of cells (NHDMOC) uses a truncated expansion for the particle displacement field; the expansion parameter is the local cell position vector. In the NHDMOC, specifying the cell structure is similar to specifying the spatial grid used in a finite-difference hydrodynamic calculation. The expansion coefficients for the particle displacement field are determined by the equation of motion, any relevant constitutive relations, plus continuity of traction and displacement at all cell boundaries. We derive and numerically solve the NHDMOC equations for the first, second, and third-order expansions, appropriate for modeling a plate-impact experiment. The performance of the NHDMOC is tested, at each order, for its ability to resolve a shock-wave front as it propagates through homogeneous and laminated targets. We find for both cases that the displacement field expansion converges rapidly: given the same cell widths, the first-order theory gives only a qualitative description of the propagating stress wave; the second-order theory performs much better; and the third-order theory gives small refinements over the second-order theory. The performance of the third-order NHDMOC is then compared to that of a standard finite-difference hydrodynamic calculation. The two methods differ in that the former uses a finite-difference solution to update the time dependence of the equations, whereas the hydrodynamic calculation uses finite-difference solutions for both the temporal and spatial variables. Both theories are used to model shock-wave propagation in stainless steel arising from high-velocity planar impact. To achieve the same high-quality resolution of the stress and particle velocity profiles, the NHDMOC consistently requires less fine spatial and temporal grids, and substantially less artificial viscosity to control unphysical high-frequency oscillations in the numerical solutions. Finally, the third-order NHDMOC theory is used to calculate the particle velocity for a shock-wave experiment involving an epoxy-graphite laminate. Constitutive relations suitable for the various materials are used. This includes linear and nonlinear elasticity, and when appropriate, viscoelasticity. The results agree well with the corresponding plate-impact experiment, and are compared to the second-order theory of Clements, Johnson, and Hixson (Phys. Rev. E, 54, 6876 (1996)).
1. INTRODUCTION

In this contribution we investigate a powerful and computationally efficient theoretical method for studying mechanical wave propagation in laminated materials. Because of their recognized technological and industrial importance, laminates have already been thoroughly investigated. In these previous studies a wide range of theoretical techniques were applied. It is impossible to summarize these investigations here, thus we mention only a few references that we found helpful and that are pertinent to the present work. The book by Nayfeh [1] gives a clear presentation of the general principles of wave propagation in layered materials, and also contains a very complete listing of references on this subject. In the present work, we will focus on only two theoretical techniques, one is a finite-difference hydrodynamic calculation (FDHC) suitably formulated for wave propagation in laminates. In that context, the work of Lundergan and Drumheller [2] provides a good reference. In a FDHC, finite-differenced equations of motion, mass, and energy, plus appropriate constitutive equations, are solved, subject to the applied boundary and initial conditions. No further assumptions, regarding the displacement, or stress fields, are assumed. In contrast, there are several classes of theoretical techniques (one is used in the present contribution) which allows further analytical steps to be taken, but usually compromises some aspect of the complete theory. An example of this is encountered in approaches involving variational methods, as they are applied to wave propagation problems. The insightful paper of Nemat-Nasser [3] illustrates the point. In that work, an ansatz for a class of displacement field test functions is considered for an elastic laminated material. From the set of functions, Nemat-Nasser invoked a variational principle to determine the optimal function. While the variational technique is powerful, and possesses a large degree of analytic character (thus reducing numerical based uncertainties), an approximate displacement field is ultimately determined from the procedure. Furthermore, since the true function might lie outside of the chosen class of test functions, the success of the approach tends to hinge on the physical intuition of the investigator. As will be seen below, in this respect our present analysis bares more
resemblance to the approach taken by Nemat-Nasser, than to approaches based on a purely FDHC.

Before embarking on yet another theoretical technique, we mention our motivation for doing so. The finite-difference method is flexible in that, in principle, any microstructure and any set of constitutive equations can be considered. The price of having microstructural complexity, however, is that very fine spatial and temporal grids must be used in the finite-differencing schemes. In turn, the task of solving the problem becomes computationally expensive. Approaches similar to the variational one, tend to be less computationally consuming, but are restricted to problems where the variational principle is applicable, thus tending to limit their use to systems possessing rather simple constitutive laws. The motivation for the present work, then, is to develop a technique that alleviates the need for using excessively fine spatial and temporal grids, but at the same time allows the incorporation of complex constitutive laws. The nonhomogenized dynamic method of cells (NHDMOC) has these properties and will be investigated in the present contribution.

To avoid possible confusion, it is important to point out that there are two different versions of the method of cells: a (spatial) homogenized version, abbreviated here by MOC, and the NHDMOC version. Both are attributed to J. Aboudi [4,5], though the technique bares strong resemblance to the generalized continuum theories of Achenbach [6]. The MOC theory is a micromechanical analysis based on the assumption that a single representative volume element (RVE) can be identified for the system. The RVE repeats itself throughout the space of the entire composite. Thus the MOC is best suited for composites having nearly periodic microstructure. This implies that a RVE with sufficiently simple microstructure can be identified, for the ensuing numerical calculations to be practical. The MOC theory uses conditions of stress and displacement continuity at all interfacial boundaries associated with the RVE, plus conditions of equilibrium. When applied to dynamical situations, the dynamics enter by other means. It is thus customary to use the MOC in conjunction with either a finite-element or a finite-difference hydrodynamic calculation. In that context, the MOC theory,
through a microstructural homogenization procedure, determines an effective stiffness matrix $B_{\text{eff}}$, and the stress $\sigma$, given the hydrodynamic strain rates $\dot{\varepsilon}$ at any given spatial point in the system. Fig.1 illustrates this. Since the MOC version is frequently applied to complex three-dimensional composites, computational tractability usually forces one to make approximations that limits its utility to problems where deformational variations occur over a length scale that are long compared to the variations in the microstructure.

Figure 1. Schematic to illustrate the application of MOC in finite element simulations.

In contrast to the MOC theory, no spatial homogenization is required in the NHDMOC theory, and the dynamics enter directly by solving an equation of motion (see Sec. 2). In this contribution, we attempt to provide evidence supporting the use of the NHDMOC for investigating wave propagation in shock loaded laminates. As alluded to above, the NHDMOC formalism is based on an \textit{ansatz} for the particle displacement field, which has the
form of a truncated Legendre expansion. Because the laminate has relatively simple microstructure, we are able to keep all terms in the expansion out to third-order. We note that Aboudi [5] found that, already by second-order, the NHDMOC theory gave a rather good representation for a stress wave profile for a wave propagating in a dynamically loaded bilaminate. It is informative to study the convergence rate of this expansion; we do this in the results section (Sec. 3).

We then return to another thrust of this contribution, namely, to compare the third-order NHDMOC theory with the results generated from a standard Lagrangian FDHC. Since both methods use the same constitutive relations, the difference in the results generated by the two approaches lies in the numerical solution scheme. The differences are: first, in the FDHC, no ansatz for the displacement field is needed, and second, the FDHC requires the solution of both spatial and temporal finite-difference equations, whereas the NHDMOC requires the solution of only a single temporal finite-difference equation.

Finally, we illustrate the second strength of the NHDMOC by applying the third-order theory to a system described by a complicated constitutive law. Specifically, we use the theory to model a viscoelastic epoxy-graphite bilaminate. Our results are then compared to experimental data taken from a plate impact experiment, performed on that system.

2. THEORY

In Fig. 2, a schematic is shown of a flyer and target setup; this setup is the basis for the experiments to which our analysis will be applied. The target is a bilaminate composed of periodically repeating bilayers. The impact surfaces of the flyer and the target are assumed to be parallel at impact and the lateral dimensions are sufficiently large that lateral edge-effects are negligible. Consequently, the problem that we seek to solve is longitudinal wave propagation in one dimension. The flyer impacts the bilaminate with an initial velocity denoted by $v_i$. Laser interferometry is then used to measure the time-dependence of the particle velocity (also called the material velocity) at some position in the sample. As a
function of time, the particle velocity measurements reveal the detailed history of the wave motion passing that point.

Figure 2. Flyer-sample configuration used to study shock wave propagation in a bilaminate.

In the NHDMOC analysis a Lagrangian grid must be specified; one is illustrated in Fig. 2. The impact grid cell, within the flyer, is denoted \( p = p_c \). The bilaminate portion of the figure is purposely kept simple by showing the case where the grid points are made to coincide with material boundaries of the bilaminate. In general, this need not be the case, and the bilaminate may require a finer grid than the actual spacings between the bilaminate layers. We let \( p \) denote the \( p^{th} \) grid cell, then the mechanical properties of that cell require the specification of the cell length \( d_p \), its mass density \( \rho_p \), and any relevant elastic moduli. The latter might be, for example, \( E_l^{(p)} \) related to the longitudinal sound speed by \( c_l^{(p)} = \sqrt{E_l^{(p)}/\rho_p} \). Further, any material parameters pertaining to quantities like the artificial viscosity, porosity, plasticity, or viscoelasticity, must be specified.

Any point within the \( p^{th} \) cell will be located by the local continuous position vector \( \bar{x}_p \). The origin of \( \bar{x}_p \) is at the center of the \( p^{th} \) cell, as shown in Fig. 2. It is defined within the region \(-d_p/2 \leq \bar{x}_p \leq +d_p/2\). The NHDMOC uses this quantity as an expansion variable for
the particle displacement field. In particular, the NHDMOC theory uses the Legendre polynomials as a complete set of basis functions; the expansion is

\[ u^{(p)}(\bar{x}_p, t) = \sum_{l=0}^{M} U_l^{(p)}(t) P_l\left( 2\frac{\bar{x}_p}{d_p} \right) . \]  

(2.1)

For the expansion to be exact, M must be infinity. The NHDMOC formalism provides an approximate method for determining the expansion coefficients \( U_l^{(p)}(t) \) (here called the cells variables) for any finite truncation of this series. Here, we restrict ourselves to the third-order (and lower orders) theory. This means that the particle displacement expansion is truncated after \( M = 3 \), and our first approximation is to set \( U_l^{(p)}(t) = 0, \ l \geq 4, 1 \leq p \leq N \). To third-order, the particle displacement expansion is

\[ u^{(p)}(\bar{x}_p, t) = U_0^{(p)}(t) + \left( \frac{2\bar{x}_p}{d_p} \right) U_1^{(p)}(t) + \ldots + \frac{1}{2} \left( \frac{2\bar{x}_p}{d_p} \right)^2 \left( \frac{2\bar{x}_p}{d_p} \right) \ldots \ldots + \frac{1}{2} \left( \frac{2\bar{x}_p}{d_p} \right)^3 \ldots \ldots + \frac{1}{2} \left( \frac{2\bar{x}_p}{d_p} \right)^3 \ldots \ldots + \frac{1}{2} \left( \frac{2\bar{x}_p}{d_p} \right)^3 \ldots \ldots \right) U_2^{(p)}(t), \]

(2.2)

and thus \( \{ U_0^{(p)}, U_1^{(p)}, U_2^{(p)}, U_3^{(p)} \} \) for all \( N \) grid cells, and all \( t \)s of interest, are needed for the particle displacement field to be fully determined. The particle displacement field must satisfy an equation of motion (EOM) which relates the stress gradients to the particle accelerations by

\[ \frac{\partial}{\partial \bar{x}_p} \left[ \sigma^{(p)}(\bar{x}_p, t) + q^{(p)}(\bar{x}_p, t) \right] = \rho_p \ddot{u}^{(p)}(\bar{x}_p, t) . \]

(2.3)

The addition of an artificial viscosity term, \( q^{(p)}(\bar{x}_p, t) \), similar to hydrodynamic-based calculations, is useful to reduce the unphysical, high-frequency ringing observed in numerical solutions. In this work, we use

\[ q^{(p)}(\bar{x}_p, t) = \eta_p d_p \sqrt{E_1^{(p)} \rho_p} \dot{\varepsilon}(\bar{x}_p, t) , \]

(2.4)

where the first order moduli, \( E_1^{(p)} \), is related to the longitudinal sound speed, as described above. In Eq. (2.4), \( \dot{\varepsilon} \) is the strain rate, and \( \eta_p \) is a nondimensional constant, on the order
of 0.01 for most applications, and is independent of \(d_p\). The relation between the strain and the particle displacement, in this one dimensional situation, is
\[
\varepsilon^{(p)}(\bar{x}_p, t) = \frac{\partial u^{(p)}(\bar{x}_p, t)}{\partial \bar{x}_p} .
\]
(2.5)

Since we will model a viscoelastic epoxy-graphite bilaminate, in Sec. 3, we use for a stress-strain relation [7]:
\[
\sigma^{(p)}(\bar{x}_p, t) = T_1^{(p)} \left\{ \varepsilon^{(p)}(\bar{x}_p, t) + \frac{1}{2} T_2^{(p)} \left\{ \left[ \varepsilon^{(p)}(\bar{x}_p, t) \right]^2 \right\} \right\},
\]
(2.6)

where \(T_\alpha^{(p)}(t)\) is a viscoelastic operator; it acts on all time-dependent quantities within the brackets. For both the first and second-order moduli, \(T_\alpha^{(p)}(t)\) has the form,
\[
T_\alpha^{(p)}(t) = E_\alpha^{(p)} + \left[ \frac{M_\alpha^{(p)} - E_\alpha^{(p)}}{T_p} \right] \int_0^t d\tau' \varepsilon^{(p)}(\bar{x}_p, \tau') \quad \alpha = 1, 2 .
\]
(2.7)
The relaxed moduli are denoted by \(M_\alpha^{(p)}\) and the characteristic time, for going from unrelaxed to relaxed behavior, is controlled by the relaxation time \(T_p\). Further, for nonviscoelastic materials only the first term in Eq. (2.7) contributes, and the stress-strain relation simplifies to
\[
\sigma^{(p)}(\bar{x}_p, t) = E_1^{(p)} \varepsilon^{(p)}(\bar{x}_p, t) + \frac{1}{2} E_2^{(p)} \left\{ \varepsilon^{(p)}(\bar{x}_p, t) \right\}^2 .
\]
(2.8)
The details for the full NHDMOC analysis have been reviewed elsewhere [8]. Thus it suffices to simply state the results of the analysis, which follow by invoking displacement and stress-continuity between adjoining (internal) grid cells, i.e.,
\[
\begin{align*}
    u^{(p-1)}\left(\frac{d_{p-1}}{2}, t\right) &= u^{(p)}\left(\frac{-d_p}{2}, t\right) \quad p = 2, \ldots, N , \quad & (2.9) \\
    \sigma^{(p-1)}\left(\frac{d_{p-1}}{2}, t\right) &= \sigma^{(p)}\left(\frac{-d_p}{2}, t\right) \quad p = 2, \ldots, N ,
\end{align*}
\]
and integrating the moments of the EOM over a given grid cell. The result is the following set of equations:
\[
\rho_p d_p \left[ 3\ddot{U}_0^{(p-1)} + \ddot{U}_1^{(p-1)} \right] + \rho_p d_p \left[ 3\ddot{U}_0^{(p)} - \ddot{U}_1^{(p)} \right] \\
- \frac{4\eta_p}{d_{p-1}} \left[ 3\ddot{U}_2^{(p-1)} + 5\ddot{U}_3^{(p-1)} \right] - \frac{4\eta_p}{d_p} \left[ 3\ddot{U}_2^{(p)} - 5\ddot{U}_3^{(p)} \right] \\
= \frac{12}{d_p} T_1^{(p)} \left\{ U_1^{(p)} + U_3^{(p)} \right\} - \frac{12}{d_{p-1}} T_1^{(p-1)} \left\{ U_1^{(p-1)} + U_3^{(p-1)} \right\} \\
+ \frac{12}{d_p^2} T_2^{(p)} \left\{ \left[ U_1^{(p)} \right]^2 + 3 \left[ U_2^{(p)} \right]^2 + 2 U_1^{(p)} U_2^{(p)} + 6 \left[ U_3^{(p)} \right]^2 \right\} \\
- \frac{12}{d_{p-1}^2} T_2^{(p-1)} \left\{ \left[ U_1^{(p-1)} \right]^2 + 3 \left[ U_2^{(p-1)} \right]^2 + 2 U_1^{(p-1)} U_2^{(p-1)} + 6 \left[ U_3^{(p-1)} \right]^2 \right\} \\
\tag{2.10}
\]

and
\[
\rho_p d_p \left[ \ddot{U}_0^{(p)} - \frac{1}{5} \ddot{U}_2^{(p)} \right] - 12 \frac{\eta_p}{d_p} \ddot{U}_2^{(p)} = \frac{12}{d_p} T_1^{(p)} \left\{ U_2^{(p)} \right\} \\
+ \frac{24}{d_p^2} T_2^{(p)} \left\{ U_1^{(p)} U_2^{(p)} + 3 U_2^{(p)} U_3^{(p)} \right\} , \\
\tag{2.11}
\]

and
\[
\rho_p d_p \left[ \ddot{U}_1^{(p)} - \frac{3}{7} \ddot{U}_3^{(p)} \right] - 60 \frac{\eta_p}{d_p} \ddot{U}_3^{(p)} = \frac{60}{d_p} T_1^{(p)} \left\{ U_3^{(p)} \right\} \\
+ \frac{120}{d_p^2} T_2^{(p)} \left\{ \frac{3}{5} \left[ U_2^{(p)} \right]^2 + U_1^{(p)} U_2^{(p)} + \frac{12}{7} \left[ U_3^{(p)} \right]^2 \right\} , \\
\tag{2.12}
\]

and
\[
\ddot{U}_0^{(p-1)} + \ddot{U}_1^{(p-1)} + \ddot{U}_2^{(p-1)} + \ddot{U}_3^{(p-1)} = \ddot{U}_0^{(p)} - \ddot{U}_1^{(p)} + \ddot{U}_2^{(p)} - \ddot{U}_3^{(p)} . \\
\tag{2.13}
\]

Similarly, zero stress at both external boundaries \((p = 1 \text{ and } N)\),
\[
\sigma^{(1)} \left( -\frac{d_1}{2}, t \right) = \sigma^{(N)} \left( -\frac{d_N}{2}, t \right) = 0 , \\
\tag{2.14}
\]
yields two more relations,
\[
\dddot{U}_1^{(1)} - 3\dddot{U}_2^{(1)} + 6\dddot{U}_3^{(1)} = 0 , \\
\tag{2.15}
\]
and
\[
\dddot{U}_1^{(N)} + 3\dddot{U}_2^{(N)} + 6\dddot{U}_3^{(N)} = 0 . \\
\tag{2.16}
\]

These equations may be expressed collectively as single \(4N\) by \(4N\) matrix equation,
\[
A\dddot{U}(t) + B\dot{U}(t) = \mathbf{R}(t) , \\
\tag{2.17}
\]
where the zeroth-, first-, and second-derivative terms of the cell coefficients have been separated into individual matrices. Matrices \(A\) and \(B\) are \(4N \times 4N\) parameter matrices.
constructed from the material parameters $\{d_p, \rho_p, \ldots\}$. The column matrices $\dot{U}(t)$ and $\ddot{U}(t)$ are composed of first- and second-time derivatives of the cell coefficients.

Eq. (2.17) can be represented as a finite-difference in the time variable, and the result solved for $U(t + \Delta t)$:

$$
U(t + \Delta t) = (\Delta t)^2 \left[ A + \frac{\Delta t}{2} B \right]^{-1} R(t) + 2 \left[ A + \frac{\Delta t}{2} B \right]^{-1} A \dot{U}(t) - \left[ A + \frac{\Delta t}{2} B \right]^{-1} \left[ A - \frac{\Delta t}{2} B \right] \dot{U}(t - \Delta t).
$$

(2.18)

By incrementing the time, beginning at $t = 0$ (impact), the cell variables at all subsequent times are determined. This requires knowing the initial conditions, and these are given elsewhere [7,8]. Eq. (2.18) is the sole finite-difference equation encountered in the NHDMOC analysis.

We now turn to a brief review of the FDHC. The EOM is

$$
\frac{\partial}{\partial x} \left[ \sigma(x,t) + q(x,t) \right] = \rho(x) \dot{v}(x,t),
$$

(2.19)

where $v(x,t)$ is the particle velocity. The artificial viscosity is given the same form as before:

$$
q(x,t) = \eta(x) \Delta x(x) \sqrt{E(x)} \rho(x) \dot{\epsilon}(x,t).
$$

(2.20)

Since we will apply the FDHC only to homogeneous situations, we can drop the $x$ dependence on $\eta$, $\Delta x$, $\rho$, and $E$.

In the shock geometry, the material is in a state of uniaxial strain and thus,

$$
e_{11}(x,t) = \epsilon(x,t)
$$

$$
e_{22}(x,t) = e_{33}(x,t) = 0.
$$

(2.21)

For elastically isotropic materials the Mie-Gruneisen equation of state (EOS) is widely used, and has a compressive contribution of

$$
P(x,t) = B \left( \frac{1-\frac{1}{2} \gamma \epsilon(x,t)}{1-s \epsilon(x,t)} \right)^2 \epsilon(x,t) + \rho \gamma E(x,t),
$$

(2.22)

while the deviatoric contribution is

$$
s_{\epsilon}(x,t) = 2 \mu \epsilon_{\epsilon}(x,t).
$$

(2.23)

$P(x,t)$ is related to the stress tensor by
\[ P(x,t) = \frac{1}{3} \sum \sigma_i(x,t). \]  

(2.24)

In these equations \( B, \mu, \) and \( \gamma \) are the bulk and shear moduli, and the Gruneisen parameter.

The two functions,

\[ e_i(x,t) = e_i (x,t) - \frac{1}{3} \sum e_i (x,t), \]  

(2.25)

and

\[ s_i(x,t) = \sigma_i (x,t) - P(x,t), \]  

(2.26)

are the deviatoric strain and stress. The internal energy is given by

\[ \rho E(x,t) = \sum \sigma_i (x,t) e_i (x,t), \]  

(2.27)

and the parameter \( s \) is the slope of shock \( (U_s) \) - particle \( (u_p) \) velocity relation:

\[ U_s = c_0 + s u_p, \]  

(2.28)

where \( c_0 = \sqrt{\frac{K}{\rho}} \) is the low-pressure bulk sound speed. By setting \( \gamma = s = 0 \) and noting that \( E = B + \frac{4}{3} \mu, \) it is easy to see that this set of equations reduces to constitutive laws used in the NHDMOC analysis. To indicate the difference between the NHDMOC and FDHC, we show the full finite-difference expressions for these equations. Eq. (2.19) become

\[ v(x,t + \Delta t) = v(x,t) + \frac{\Delta t}{\rho \Delta x} [\sigma(x + \Delta x,t) - \sigma(x,t) + q(x + \Delta x,t) - q(x,t)], \]  

(2.29)

while the strain rate is

\[ \dot{\varepsilon}(x,t) = \frac{1}{\Delta x} [v(x + \Delta x,t) - v(x,t)]. \]  

(2.30)

Taking the time derivative of Eqs. (2.27) and (2.28)

\[ \dot{P}(x,t) = B \dot{e}(x,t), \]  

(2.31)

\[ \dot{s}_i(x,t) = 2 \mu \dot{e}_i(x,t), \]  

(2.32)

these can be inserted in to the finite-difference equations for the mean compressive and deviatoric stresses.

\[ P(x,t + \Delta t) = P(x,t) + \Delta t \dot{P}(x,t), \]  

(2.33)
Finally, the stresses are updated by the equation

\[ s_y(x, t + \Delta t) = s_y(x, t) + \Delta t \dot{s}_y(x, t). \]  

(2.34)

3. RESULTS

In this section we present the numerical results generated by the NHDMOC equations. The first issue we address is the convergence rate of the particle displacement expansion given by Eq. (2.1). The first-order NHDMOC theory keeps only the first two terms of this expansion (M=1), which are the first two terms in Eq. (2.2). The second-order theory retains the first three terms in Eq. (2.2), and the third-order theory uses all terms in Eq. (2.2). Of course, the third-order terms must be omitted from Eqs. (2.10) - (2.16), in the case of the second-order application, etc.

In Fig. 3, stress wave profiles, for a shock-wave front, are shown that have been generated by these three approximations for the NHDMOC. For this case, the system was chosen to be a stainless steel flyer impacting, at 1 km/s, onto a stainless steel-polymethylmethacrylate (PMMA) bilaminate. Because we are interested in assessing the convergence rate of the series, we have also looked at stress wave propagation in homogeneous system rather than a bilaminate. The bilaminate shows the same trends, but obscured slightly by the more complex stress wave profile associated with the bilaminate. No artificial viscosity is used in these calculations and only the linear term in Eq. (2.8) is kept. First we comment on the homogeneous case. As a function of distance along the target sample, the stress wave profile generated by the first-order theory is composed of zero-slope linear segments that are typically characterized by a substantial stress discontinuity at cell boundaries. The second-order theory approximates the stress wave with non-zero slope linear segments. The stress is typically not continuous across cell boundaries, although, the discontinuities are smaller than in the first-order calculation. Third-order represents the stress wave by segments of a second-order polynomial. We found, numerically, that the stress discontinuity was nearly zero for this calculation. A second advantage of the third-order
theory, over the lower orders, is that the oscillations observed at the front of the shock wave are largely reduced. The final advantage is that a coarser grid-width (compared to the one used at lower orders) remains giving a suitable representation of the stress wave for the third-order theory, thus helping to reduce computational time to achieve a numerical solution. In Fig. 3, we show the stress wave profile, as a function of elapsed time. In this figure it is clear that, while first-order tends to deviate noticeably from second- and third-orders, the later two deviate only by small amounts from each other. As a result of this study we conclude that the third-order theory is suitable for most applications of stress wave propagation, and it is doubtful that much would be gained by going to yet higher orders. A further discussion of these ideas can be found in Ref. [9].

![Stress wave profile for a shock wave propagating in a bilaminate. First-, second-, and third-order NHDMOC theories are compared.](image)

**Figure 3.** Stress wave profile for a shock wave propagating in a bilaminate. First-, second-, and third-order NHDMOC theories are compared.

The next issue concerns comparing the third-order NHDMOC theory with the FDHC
outlined in the previous section. Again we focus on the simpler problem of shock waves propagating in homogeneous stainless-steel. For simplicity, linear elasticity is assumed. In Fig. 4 (Fig. 5), the velocity profiles, obtained by the hydrodynamic calculations (third-order NHDMOC), are shown. Here, a 0.10 cm steel flyer struck a 0.30 cm steel target, moving at a speed of 1 km/s. The stress is calculated at the midpoint of the target. The NHDMOC

Figure 4. FDHC calculated particle velocity profiles to illustrate the importance of artificial viscosity and grid cell spacings. Results were generated using 40 equal-spaced cells, and time step of 0.001 \( \mu s \). In the FDHC, a time step of 0.0001 \( \mu s \) was needed, and substantially finer spatial griding was required (the grid spacing is given in Fig. 4). With zero artificial viscosity, the third-order NHDMOC theory (Fig. 5) remains giving stable solutions, although small numerical ringing is visible. These were removed by increasing \( \eta \) to 0.006. Larger values of artificial viscosity are needed in the FDHC. For the FDHC we used \( \eta \) equal to 0.1. At 400 cells, in the FDHC,
and $\eta = 0.1$, the two velocity profiles nearly match (the NHDMOC used 40 cells and $\eta = 0.006$). No further improvement was observed by going to still finer spatial and temporal grids in the FDHC. The fact that smaller applications of artificial viscosity are required in the NHDMOC is very important when the theory is applied to complex heterogeneous structures. There, multiple reflections and transmissions at all true material boundaries will produce true physical oscillations that should not be removed, by applying artificial viscosity, from the numerical solutions.

![Third-order NHDMOC calculation](image)

**Figure 5.** NHDMOC calculated particle velocity profiles to illustrate the importance of artificial viscosity and grid cell spacings.

The final point to be made is that the NHDMOC can relatively easily be applied to problems with complex constitutive behavior [7,8]. In this example (Fig. 6) the flyer plate is composed of a front Z-cut quartz impactor, backed by a larger slab of PMMA. The initial velocity of the flyer, before striking the bilaminate target, is 500 m/s. The bilamine portion of the target is made of 19 bilayers. The composition of each bilayer is, first, a thin section
of pure epoxy and second, a thicker section made of an epoxy-graphite mixture. The graphite, in the mixture, are fibers stacked with their axes being parallel to each other. The fibers are distributed uniformly throughout the layer, and their axes are parallel to the bilayer interface. The bilayer is backed by a PMMA window. In the experiment, the PMMA window, being transparent to the laser light, was used only to provide an acoustical impedance match with the target; the particle velocity was measured at the bilaminate-PMMA interface. Both the experimental and theoretical velocity profiles are shown in Fig. 7.

**Figure 6.** Epoxy-Graphite bilaminate to which the third-order NHDMOC is applied.

At 0.9 μs after impact the primary compressive wave reaches the observation point, increasing the particle velocity from zero to approximately 430 m/s. After another 0.3 μs, an abrupt drop in the particle velocity indicates the arrival of a release wave at the observation point. Its origin is from the release occurring at the left-most edge of the Z-cut quartz impactor. This is followed by a second release that occurs at about 0.5 μs after the arrival of the first release wave, caused by a wave traversing the quartz impactor twice, after impact, before finally arriving at the observation point. There are also small oscillations superimposed on the velocity profile that are caused by the laminations in the composite.
This is easily deduced from the theoretical calculation; removing the laminations removes these oscillations.

The inset in Fig. 7 shows the differences between the second- and third-order NHDMOC calculations. It was established in Ref. [9] that a coarser spatial grid, at third-order, will achieve the same resolution of a shock-wave profile, as a finer grid in second-order. Thus, for example, it is not surprising that, in the present work, only $\frac{3}{4}$ as many grid cells were needed in the bilaminate. Of course, the disadvantage to going to higher order is that in the third-order theory one must invert and solve a $4N \times 4N$ matrix equation, but only a $3N \times 3N$ matrix equation in the second-order theory. In the second-order calculation

![Graph showing particle velocity profile](image-url)

**Figure 7.** Particle velocity profile measured experimentally [7] for the epoxy-graphite bilaminate, and calculated with the third-order NHDMOC theory. The inset shows the second-order calculation.

no artificial viscosity was used. Our numerical results infer that the combined affects of the third-order (itself tending to reduce the unphysical numerical ringing), and the mild
application of artificial viscosity, removed the unphysical numerical ringing seen propagating just ahead of the primary compressive wave in the second-order theory. Note that the shoulder in the inset, occurring at 300 m/s, at $t = 0.9 \mu s$, is not physical, but is caused by the finite-width grids used for the spatial and time variables. We also direct the readers' attention to the fact that the artificial viscosity has slightly reduced the physical oscillations seen after the arrival of the first release wave in the third-order theory. While undesirable, the reduction is small and the benefits of applying the artificial viscosity favorably outweigh the negative aspects.

The other striking difference between the second- and third-order theories, is the overall reduction in the gap in the particle velocity profiles between the experiment and the theory, observed when going from the second-order to third-order theory. The primary reason for this is that a different value for $E_2^{(p)}$ for PMMA is used: In the present case, $E_2^{(p)}$ is taken from Ref. [5], which is smaller than the value deduced in Ref. [7]. The present value is -1.31 Mbar, while the work in Ref. [7] used -0.72 Mbar. A simple calculation shows that this change should reduce the peak compressive velocity by a factor of approximately 1.04. Indeed, this reduces the theoretical peak velocity, in the inset of Fig. 7, to essentially that of the third-order calculation. Consequently, there is no motivation to repeat the second-order calculation using the new value of $E_2^{(p)}$.

4. CONCLUSIONS

We presented evidence for the advantages of the NHDMOC method over others, such as a FDHC, when dealing with composite systems. When compared to an actual impact experiment, involving an epoxy-graphite bilaminate, the third-order NHDMOC theory, including artificial viscosity, gives very satisfactory agreement with the experimental measured particle velocity. Further, the closeness of the second- and third-order theories indicates that little would be gained by going to still higher-orders theories. The results shown here demonstrate that the NHDMOC method provides a useful technique and may
become the standard approach for wave propagation calculations in composites and other heterogeneous materials.

4. REFERENCES


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