THE CANTOR TERNARY SET AND CERTAIN OF ITS GENERALIZATIONS AND APPLICATIONS

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CHAPTER I

INTRODUCTION

Preceding the principal discussions in this paper, we shall give the following necessary definitions, notations and related theorems.

A <u>point set</u> will be regarded as a set of distinct real numbers.[1] We shall use the term point and real number interchangeably.

The points x such that $a \le x \le b$ will be denoted by $[a,b]$ and will be called a <u>closed interval</u>. Similarly, the points x such that $a < x < b$ will be denoted by (a,b) and will be called an <u>open interval</u>. We shall say that (c,d) is a sub-interval of $[a,b]$ or (a,b) if $a < c$ and $d < b$, if $a \le c$ and $d < b$, or if $a < c$ and $d \le b$.

A <u>closed set</u>, S, is a set which contains all of its limit points. If x is a <u>limit point</u> of S, every neighborhood of x (every open interval containing x) contains points of S. If S is an <u>open set</u>, every point of S is an <u>interior point</u> of S, that is, if x is a point of S, there exists a neighborhood of x such that every point of that neighborhood is a point of S. The set S_1 is said to be a subset of the

[1] G. Cantor, Math. Annalen, Vol. XLVI (1895), p. 481.

set S if every element of S_1 is an element of S. The complement of S, the set of all points not in S, will be denoted by $\bar{S}$. We shall assume that the complement of an open set is closed, and conversely, that the complement of a closed set is open.[2]

When we say that there exists a biunique correspondence between the elements of S_1 and S_2, we mean that there exists a method of mating the elements of S_1 with the elements of S_2 in such a way that every element of S_1 will be mated to one and only one element of S_2 and that every element of S_2 will be mated to one and only one element of S_1.

If there exists a biunique correspondence between the elements of S_1 and S_2, then S_1 and S_2 are said to have the same cardinal number. A set which can be put into biunique correspondence with the set of all positive integers is said to be __denumerable__ and to have the cardinal number $\aleph_0$.[3] It is evident that the elements of a denumerable set can be enumerated as in infinite sequence, $p_1, p_2, p_3, \ldots$. A set which has the same cardinal number as the continuum (set of all real numbers) will be said to have the cardinal

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[2] E. W. Hobson, __The Theory of Functions of a Real Variable and the Theory of Fourier's Series__, Vol. I, third edition, p. 79.

[3] G. Cantor, __Journal für die reine und angewandte mathematik__, Vol. LXXXIV (1877), p. 24.

number c.[4] If S_1 can be mated biuniquely with a subset of S_2, and S_2 can be mated biuniquely with a subset of S_1, then S_1 has the same cardinal number as S_2.[5]

A set is dense in the interval (a,b) if every point x such that $a < x < b$ is a limit point of S. A set S is non-dense if every interval contains a sub-interval which contains no point of S. If every point of S is a limit point of S, then we say that S is dense-in-itself. The sum of at most $\aleph_0$ non-dense sets is exhaustible or first category of Baire. The complement of an exhaustible set is said to be residual or second category of Baire.[6] A set that is both closed and dense-in-itself is called a perfect set.[7]

If all the points of S lie in an interval, then S is bounded and has a least upper bound, or upper boundary, and a greatest lower bound, or lower boundary.[8]

Let Γ be any denumerable set of open intervals covering the set S lying in $[a,b]$, and denote by l_Γ the length sum of the intervals of Γ. Let Me(S) be the lower boundary

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[4] Ibid., p. 258.

[5] E. Borel, Leçons sur la théorie des fonctions, p. 103.

[6] R. Baire, Leçons sur les fonctions discontinues, p. 78.

[7] Cantor, Math. Annalen, Vol. XXI (1883), p. 575.

[8] Hobson, op. cit., p. 62.

of l_r for all such Γ. Me(S) is called the <u>exterior Lebesgue measure</u> of S. Let $\bar{S}$ be the complement of S with respect to $[a,b]$. $Mi(S) = b - a - Me(\bar{S})$ is called the <u>interior Lebesgue measure</u> of S. If $Me(S) = Mi(S)$, the set S is said to be <u>measurable Lebesgue</u> and the common value is called the <u>Lebesgue measure</u> of S and is denoted by M(S).[9] The Lebesgue measure of the sum of two sets is less than or equal to the sum of the measures of the two sets, or in notation: $M(S_1 + S_2) \leq M(S_1) + M(S_2)$.[10] The Lebesgue measure of a subset of S is less than or equal to the Lebesgue measure of S. Hereafter we shall use the word measure to denote the Lebesgue measure of a set.

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9) H. Lebesgue, <u>Journal für die reine und angewandte mathematik</u>, series 6, Vol. I (1905), p. 165. For the original paper by Lebesgue see also Lebesgue, <u>Lecons sur l'Integration</u>, first edition (1904).

10) Hobson, <u>op</u>. <u>cit</u>., p. 166.

CHAPTER II

THE CANTOR TERNARY SET

The Cantor Ternary Set[1] is defined as the set of all numbers in $[0, 1]$ which can be represented in ternary notation (notation in the number system with base 3) without the use of the digit 1.

Theorem I. The Cantor Ternary Set has cardinal number c.[2]

Consider the numbers between zero and one which in ternary notation end with the digit 1 but contain no other digit 1. These numbers belong to the Cantor Ternary Set, for they can be represented by equivalent numbers where the digit 1 is replaced by 022222... . Write these numbers in a sequence

$$\{p_\nu\} = p_1, p_2, \ldots, p_n, \ldots .$$

Write the rational numbers in a sequence

$$\{r_\nu\} = r_1, r_2, \ldots, r_n, \ldots .$$

We shall think of the points of $\{p_\nu\}$ as measured on a line and the points of $\{r_\nu\}$ as measured on another line. Mate

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[1]A. M. Harding and G. W. Mullins, College Algebra, p. 325.

[2]G. Cantor, "Über unendliche lineare Punktmannigfaltigkeiten," Math. Annalen, Vol. XXI (1892), p. 43.

p_1 with r_1, which we will now call q_1. Mate p_2 with the first r_ν which is to the right or left of r_1 according as p_2 is to the right of left of p_1. Call this r_ν, q_2. In like manner, mate p_3 with the first r_ν which is to the right of q_1 and q_2, between q_1 and q_2, or to the left of q_1 and q_2 according as p_3 is to the right of p_1 and p_2, between p_1 and p_2, or to the left of p_1 and p_2. And in general, mate p_n with the first r_ν which has the same position in regard to q_1, q_2, q_3, ..., q_{n-1} as p_n has to p_1, p_2, p_3, ..., p_{n-1}.

Clearly every p_ν will have a unique mate in $\{r_\nu\}$. But suppose there exist some points of $\{r_\nu\}$ which do not have mates in $\{p_\nu\}$. In that case, there must exist some first point of that kind. Call it r_α. Evidently r_α must be to the left of r_1, r_2, r_3, ..., $r_{\alpha-1}$, to the right of r_1, r_2, r_3, ..., $r_{\alpha-1}$, or between some two of those points. But, in either case, r_α must be taken as a mate of some p_ν within a finite number of steps. For suppose $r_\beta = q_m$ is the mate of p_m where p_m is the last p_ν used as mate of r_1, r_2, ..., or $r_{\alpha-1}$. Then there exists an n such that

$$1+2+4+\ldots+2^{n-1} \leq m \leq 1+2+4+\ldots+2^{n-1}+2^n.$$

Evidently r_α will be the mate of some p_k where

$$1+2+4+\ldots+2^{n-1} \leq k \leq 1+2+4+\ldots+2^n+2^{n+1}.$$

Let x be any non-terminating ternary point of the Cantor Set. Consider the two sets of left end-points, A and B, where p_ν belongs to A if p_ν is less than x, and where p_ν belongs to B if p_ν is greater than x. It is evident that p_ν is not equal to x. Consider also the two sets of rational numbers, C and D, where r belongs to C if r is a mate of a p_ν in A, and r is in D if r is a mate of a p_ν in B. The set A is not empty, for if x is a non-terminating ternary point

$$0 < x < 1.$$

But any interval (0,x) contains left end-points (p_ν's of the Cantor Ternary Set). Since x is not a left end-point, there must be in the interval (0,x) a point $p_\nu < x$. And p has a mate r_ν in C. Thus C is not empty. Similarly we can show that D is not empty by considering the interval (x,1).

Since C is not empty, D has as a lower bound any point in C. Since D has a lower bound, D also has a lower boundary. Call this lower boundary y.

Mate x with y. Clearly though y is the lower boundary of D, y is not in D, for there can be no least left end-point in B and hence no least rational number in D. Similarly, y is not in C. Thus y is not a rational number.

It is evident that we will thus find unique, irrational mates for every non-terminating ternary point x. But there may possibly be some irrational number w which in this mating receives no mate. But w is the lower boundary of some set of rational numbers which by our first mating were mated to a corresponding set of left end-points which are bounded and which must then have a lower boundary z. But z is a point of the Cantor Ternary Set for the Cantor Ternary Set is closed, that is, it contains all of its limit points. (See Theorem II.) Thus z would have been mated to w.

We have shown that a subset of the Cantor Ternary Set can be mated biuniquely with the whole continuum. It is obvious that a subset of the continuum, the Cantor Ternary Set itself, can be mated biuniquely with the Cantor Ternary Set. By Bernstein's Theorem,[3)] we can say that the Cantor Ternary Set has the cardinal number c.

<u>Theorem II. The Cantor Ternary Set is perfect.</u>

The Cantor Ternary Set is the complement of at most $\aleph_0$ non-overlapping, non-abutting open intervals, which is exactly the structure of a perfect set.

<u>Theorem III. The Cantor Ternary Set is non-dense.</u>

Every interval contains a sub-interval which contains no points of the Cantor Ternary Set.

<u>Theorem IV. The Cantor Ternary Set is exhaustible.</u>

Any non-dense set is exhaustible.

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[3]Borel, op. cit., p. 103.

Theorem V. The Cantor Ternary Set is of measure zero.

Denote by S' all the points of the Cantor Ternary Set, except the end-points of the open intervals left out.

$$Me(S') \leq \frac{2}{3}.$$
$$Me(S') \leq \left(\frac{2}{3}\right)^2.$$
$$\vdots$$
$$Me(S') \leq \left(\frac{2}{3}\right)^n$$
$$\vdots$$

But $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$. Thus, $Me(S') = 0$.

The remaining points of S can be covered in the following manner. Choose $\epsilon > 0$. Write the end-points in a sequence. Cover the first with an interval of length less than $\frac{\epsilon}{2}$, the second with an interval of length less than $\frac{\epsilon}{4}$, the third with an interval of length less than $\frac{\epsilon}{8}$, ..., the nth with an interval of length less than $\frac{\epsilon}{2^n}$, ... The length sum of the intervals

$$L_1 < \frac{\frac{\epsilon}{2}}{1 - \frac{1}{2}} = \epsilon.$$

Thus $Me(S - S') = 0$ and hence $Me(S) = 0$.

Consider $\overline{S}$. $\overline{S}$ can be covered with the intervals of $\overline{S}$.

$$Me(\overline{S}) \leq \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \dots, + \left(\frac{2}{3}\right)^n + \dots = 1.$$
$$Me(\overline{S}) \leq 1.$$

But suppose $Me(\overline{S}) < 1$. $Me\,[0,1] \doteq Me(\overline{S}) + Me(S) < 1$, but this is obviously a contradiction. Thus $Me(\overline{S}) = 1$. And $M(S) = 0$.

CHAPTER III

GENERALIZATIONS OF THE CANTOR SET

An obvious generalization of the Cantor Ternary Set is the set of all numbers in $[0,1]$ which can be represented in n-ary notation with only the digits 0 and m, where $m = n - 1$. This set is perfect, non-dense and hence exhaustible, of measure zero, and of cardinal number c, the proofs being essentially the same as for the Cantor Ternary Set.

A more interesting generalization is the following. Given any closed interval $[a,b]$, take out any open sub-interval (a_{11}, b_{11}) such that $a_{11} > a$ and $b_{11} < b$. Out of each of the two remaining intervals, take an open sub-interval which does not abut (a_{11}, b_{11}) and such that its end-points are not a or b. Call these sub-intervals (a_{21}, b_{21}) and (a_{22}, b_{22}). And in general, out of each of the remaining 2^n intervals take an open sub-interval which does not abut any interval previously removed and such that its end-points are not a or b. Call these intervals (a_{n1}, b_{n1}), (a_{n2}, b_{n2}), ..., (a_{nn}, b_{nn}). S is defined as the set of all points in $[a,b]$ which are in none of the open intervals (a_{11}, b_{11}), (a_{21}, b_{21}), ..., (a_{nn}, b_{nn}), ...

<u>Theorem VI</u>. <u>S is perfect</u>.

S is the complement of at most $\aleph_0$ non-overlapping, non-abutting open intervals.

The set S may be non-dense, the Cantor Ternary Set being a case in point. However, S may fail to have this property as, for example, when all the intervals (a_{11}, b_{11}), (a_{21}, b_{21}), ... are taken from the middle third of $[a, b]$.

Theorem VII. S has cardinal number c.

If S is non-dense, the proof is essentially the same as the proof of the corresponding theorem concerning the Cantor Ternary Set. If S is dense in some interval, say $[c, d]$, then (c, d), a subset of S, can be put into biunique correspondence with the points of the whole continuum. And S, a subset of the continuum, can be put into biunique correspondence with S itself. By Bernstein's Theorem, the set S has the same cardinal number as the continuum, that is, the cardinal number of S is c.

Theorem VIII. The measure of the generalized set S is equal to the length of the interval $[a, b]$ less the sum of the lengths of the sub-intervals removed.

$$Me(S) \leq (b - a) - [(b_{11} - a_{11}) + (b_{21} - a_{21}) + \dots + (b_{nn} - a_{nn}) + \dots].$$

$$Me(\overline{S}) \leq (b_{11} - a_{11}) + (b_{21} - a_{21}) + \dots + (b_{nn} - a_{nn}) + \dots .$$

But ME(S) cannot be less than $(b - a) - [(b_{11} - a_{11}) + (b_{21} - a_{21}) + \dots + (b_{nn} - a_{nn}) + \dots$ and $Me(\overline{S})$ cannot be less than $(b_{11} - a_{11}) + (b_{21} - a_{21}) + \dots + (b_{nn} - a_{nn}) + \dots$, for suppose either or both is true. Then $Me(S) + Me(\bar{S}) < (b - a)$. But

$$Me(S) + Me(\bar{S}) \geq Me(S + \bar{S}) = b - a.$$

Therefore

$$Me(S) = (b - a) - [(b_{11} - a_{11}) + (b_{21} - a_{21}) + \dots + (b_{nn} - a_{nn}) + \dots].$$

And

$$Me(\bar{S}) = (b_{11} - a_{11}) + (b_{21} - a_{21}) + \dots + (b_{nn} - a_{nn}) + \dots .$$

$$Mi(S) = (b - a) - [(b_{11} - a_{11}) + (b_{21} - a_{21}) + \dots + (b_{nn} - a_{nn}) + \dots ,$$

and since $Me(S) = Mi(S)$, the $M(S)$ exists and

$$M(S) = (b - a) - [(b_{11} - a_{11}) + \dots + (b_{nn} - a_{nn}) + \dots].$$

<u>Example</u>: It is possible to construct in the interval $[0,1]$, a non-dense set of measure $\frac{q}{p}$, where q and p are any positive numbers such that $0 < q < p$.

Consider the set S constructed in the following manner. In the interval $[0,1]$, take out the middle $\frac{r}{2p}$th part of $[0,1]$, where $r = p - q$. Then out of the middle of each of the two remaining intervals, take an open interval of length $\frac{r}{2^3 p}$. Continue in this manner, taking out of the middle of each of the 2^{n-1} remaining intervals, an open interval of length $\frac{r}{2^{2n-1} p}$. Let S be the set of points never taken in any of the open intervals. From the preceding theorem we have

$$M(S) = 1 - \left[\frac{r}{2p} + \frac{r}{2^2 p} + \frac{r}{2^3 p} + \dots\right].$$

$$M(S) = 1 - \frac{r}{p} = \frac{q}{p}.$$

CHAPTER IV

DENUMERABILITY, EXHAUSTIBILITY AND ZERO MEASURE

We shall now give a complete existential theory for the three set properties: denumerability, exhaustibility and zero measure.

Theorem IX. All denumerable sets are of measure zero, but not conversely.

Let S be any denumerable set. Write the points of S in a sequence

$$S = p_1, p_2, p_3, \ldots, p_n, \ldots \quad .$$

Choose any $\epsilon > 0$. Cover p_1 with an interval of length less than $\frac{\epsilon}{2}$. Cover p_2 with an interval of length less than $\frac{\epsilon}{4}$. In general, cover p_n with an interval of length less than $\frac{\epsilon}{2^n}$. Consider the length sum of these intervals covering S.

$$L_1 < \frac{\frac{\epsilon}{2}}{1 - \frac{1}{2}} = \epsilon.$$

But this can be done for any $\epsilon > 0$. Thus the set is of measure zero.

The converse is not true. The Cantor Ternary Set, for example, is of measure zero, but not denumerable.

Theorem X. All denumerable sets are exhaustible, but

<u>not conversely</u>.

Let S be any denumerable set. Write the points of S in a sequence

$$S = p_1, p_2, p_3, \ldots, p_n, \ldots .$$

Consider the non-dense sets $S_1 = p_1$, $S_2 = p_2$, $S_3 = p_3$, ..., $S_n = p_n$, ... Thus S is the sum of at most $\aleph_0$ non-dense sets and is therefore exhaustible.

The converse is not true. For example, consider the Cantor Ternary Set which is exhaustible but not denumerable.

<u>Examples</u>: (a) Let us now consider an example of an exhaustible set which is not only of measure zero but is, in fact, of relative measure 1. In the interval $[0,1]$, construct a non-dense perfect set, S_1, of measure $\frac{1}{2}$. In the same interval, construct a non-dense perfect set, S_2, of measure $\frac{3}{4}$. Continue in this manner, in general, constructing in the same interval, a non-dense perfect set, S_n, of measure $\frac{2^n - 1}{2^n}$. Let $S = S_1 + S_2 + S_3 + \ldots + S_n + \ldots$. Thus S is the sum of at most $\aleph_0$ non-dense sets and is therefore exhaustible. But $M(S) \geq \frac{2^n - 1}{2^n}$ and $\lim_{n\to\infty} \frac{2^n - 1}{2^n} = 1$. Therefore $M(S) \geq 1$. But M(S) obviously cannot be greater than the measure of the interval $[0,1]$. Thus $M(S) = 1$.

(b) The set $\bar{S}$, the complement of set S in example (a), has zero measure but is not exhaustible. For suppose $\bar{S}$ had any positive measure p. Then

$$M(\bar{S}) + M(S) = 1 + p.$$

But $M(\overline{S}) + M(S) = 1$. Thus $M(\overline{S}) = 0$.

(c) An example of a set that is non-denumerable, non-exhaustible, and not of measure zero is the set of points in the interval (0,1).

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