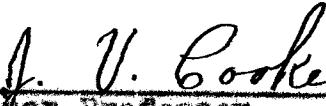


SOME PROPERTIES OF CERTAIN GENERALIZATIONS
OF THE SUM OF AN INFINITE SERIES

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SOME APPROPRIATIONS OF CERTAIN GENERALIZATIONS
OF THE SUM OF AN INFINITE SERIES

THESIS

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By

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CHAPTER I

INTRODUCTION

We shall define the series $a_1 + a_2 + a_3 + \dots$ to be convergent whenever $\lim_{n \rightarrow \infty} s_n$ exists, where $s_n = a_1 + a_2 + \dots + a_n$. This limit will be called the sum of the series. If $\lim_{n \rightarrow \infty} s_n$ does not exist, we shall say that the sequence (s_n) is divergent and that the series is divergent or not summable. To further emphasize our definition of divergence it is pointed out that if the sequence (s_n) neither converges to a limit nor diverges to $+\infty$ or $-\infty$, the sequence is divergent.

Since it is necessary that $\lim_{n \rightarrow \infty} a_n = 0$ in order for $\lim_{n \rightarrow \infty} s_n$ to exist, only a relatively small class of series is convergent. It is our purpose to include other series by means of more general definitions, namely, those of Hölder and Cesáro, which satisfy the generally accepted fundamental requirements of any generalized definition of summability of a series, namely:¹

- (1) The generalized sum must exist, whenever the series converges.
- (2) The generalized sum must be equal to the ordinary sum, whenever the series converges.

¹I. L. Silverman, Sum of a Divergent Series, p. 2.

(3) Each of the series

$$a_0 + a_1 + a_2 + \dots + a_n + \dots$$

$$a_1 + a_2 + \dots + a_n + \dots$$

has a generalized sum, whenever the other has, and

$t = s + a_0$ if t and s are their respective sums.

(4) If the series,

$$a_0 + a_1 + a_2 + \dots + a_n + \dots$$

$$b_0 + b_1 + b_2 + \dots + b_n + \dots,$$

have the generalized sums s and t , respectively,

then the series $(a_0 + b_0) + (a_1 + b_1) + \dots + (a_n + b_n)$
 $+ \dots$ has the generalized sum $s + t$.

(5) If the series $a_0 + a_1 + a_2 + \dots + a_n + \dots$ has s for its generalized sum, then $ka_0 + ka_1 + \dots + ka_n + \dots$ has a generalized sum which is ks .

An attempt will be made to establish properties of Hölder and Cesáro summable series analogous to those of ordinary convergent series and also to establish properties that are possibly different from those of convergent series.

For the purpose of simplifying our notation, we shall use $\sum a_n$ and $\lim a_n$ for $\sum_{n=0}^{\infty} a_n$ and $\lim_{n \rightarrow \infty} a_n$.

CHAPTER II

CESÁRO SUM OF ORDER 1

First, we shall define the sum of order 1 of a series and the convergence of order 1 of a sequence.

Definition: If the sequence x_0, x_1, x_2, \dots is such that $\lim (x_0 + x_1 + \dots + x_n)/(n + 1) = s$, we shall say the sequence is Cesáro convergent of order 1, or more briefly C_1 convergent, to s .

Definition: If the series $\sum a_n$ is such that the sequence of partial sums, s_0, s_1, s_2, \dots , is C_1 convergent, then we shall say that the series has the Cesáro sum s of order 1, or briefly the C_1 sum s .

We shall proceed to show in the succeeding theorems that the class of series which are C_1 summable fulfills the fundamental requirements for any generalized definition of summability of series.

Theorem 2.1: If a series converges to s , then the C_1 sum is equal to s .

Let $s_0, s_1, s_2, \dots, s_n, \dots$ be the sequence of partial sums of the series. We now choose a positive number ϵ and let $h_n = (s_0 + s_1 + s_2 + \dots + s_n)/(n + 1)$. If we let $s_n - s = k_n$, then there is a positive integer N such that for every $n > N$ we have $|k_n| < \epsilon/2$. But $s_n = s + k_n$; and

thus by substituting for s_i , $i = 1, 2, \dots, n$, in

$$h_n = \frac{s_0 + s_1 + s_2 + \dots + s_N}{n+1} + \frac{s_{N+1} + s_{N+2} + \dots + s_n}{n+1},$$

we have $h_n = s + \sum_{i=0}^N k_i/(n+1) + \sum_{i=N+1}^n k_i/(n+1)$. We shall denote $\sum_{i=0}^N k_i$ by K . Now $|\sum_{i=N+1}^n k_i|/(n+1) \leq \sum_{i=N+1}^n |k_i|/(n+1) < (n-N)/(n+1) \cdot \epsilon/2 < \epsilon/2$. Thus for every $n > N$, we have $|h_n - s| < |K|/(n+1) + \epsilon/2$. Hence, for every $n > \max [N, 2K/\epsilon - 1]$, we have $|h_n - s| < \epsilon$. Therefore,

$$\lim (s_0 + s_1 + s_2 + \dots + s_n)/(n+1) = s.$$

Lemma 2.1.1: If the sequence $s_0, s_1, s_2, \dots, s_n, \dots$ is c_1 convergent to s , then the sequence $s_1, s_2, \dots, s_n, \dots$ is c_1 convergent to s and conversely.

For,

$$\begin{aligned} \lim \frac{s_1 + s_2 + \dots + s_n}{n} &= \lim \left[\frac{s_0 + s_1 + s_2 + \dots + s_n}{n+1} \cdot \frac{n+1}{n} \right. \\ &\quad \left. - \frac{s_0}{n+1} \right] \\ &= \lim \frac{s_0 + s_1 + s_2 + \dots + s_n}{n+1} \\ &= s. \end{aligned}$$

Therefore the sequence $s_1, s_2, \dots, s_n, \dots$ is c_1 convergent to s . In a similar manner the converse statement may be shown to be true.

Theorem 2.2: Each of the series

$$s_0 + s_1 + s_2 + \dots + s_n + \dots$$

$$a_1 + a_2 + \dots + a_n + \dots$$

is C₁ summable when the other is; and s and t, their respective sums, are connected by the relation s = t + a₀.

Let $s_n = a_0 + a_1 + a_2 + \dots + a_n$. Now by making use of the preceding lemma the sequence, $s_1, s_2, \dots, s_n, \dots$, is C₁ convergent to s when $s_0, s_1, s_2, \dots, s_n, \dots$ is C₁ convergent to s. But $(s_n - a_0)$ is the nth partial sum of $a_1 + a_2 + \dots + a_n + \dots$. Thus,

$$\begin{aligned} t &= \lim_{n \rightarrow \infty} \frac{(s_1 - a_0) + (s_2 - a_0) + \dots + (s_n - a_0)}{n} \\ &= \lim_{n \rightarrow \infty} \frac{s_1 + s_2 + \dots + s_n}{n} - a_0 = s - a_0. \end{aligned}$$

In a similar manner it follows that the series, $a_0 + a_1 + a_2 + \dots + a_n + \dots$, is C₁ summable to s if the series, $a_1 + a_2 + \dots + a_n + \dots$, is C₁ summable to $s + a_0$.

Theorem 2.3: If $\sum a_n$ is C₁ summable to s and $\sum b_n$ is C₁ summable to t, then the series $\sum (a_n + b_n)$ is C₁ summable to s + t.

Let $s_n = a_0 + a_1 + \dots + a_n$ and $t_n = b_0 + b_1 + \dots + b_n$; then $(a_0 + b_0) + (a_1 + b_1) + \dots + (a_n + b_n) = (s_n + t_n)$ is the nth partial sum of $\sum (a_n + b_n)$. Thus,

$$\lim_{n \rightarrow \infty} \frac{(s_0 + t_0) + (s_1 + t_1) + \dots + (s_n + t_n)}{n+1}$$

$$= \lim \frac{s_0 + s_1 + \dots + s_n}{n+1} + \lim \frac{t_0 + t_1 + \dots + t_n}{n+1}$$

$$= s + t.$$

Theorem 2.4: If $\sum a_n$ is C_1 summable to s , then the series ka_n is C_1 summable to ks .

For $s_n = a_0 + a_1 + \dots + a_n$ is the n^{th} partial sum for the first series. In the last series mentioned, the partial sum is $ka_0 + ka_1 + ka_2 + \dots + ka_n = ks_n$. But

$$\lim \frac{ks_0 + ks_1 + \dots + ks_n}{n+1} = k \lim \frac{s_0 + s_1 + s_2 + \dots + s_n}{n+1}$$

$$= ks.$$

We may now study further properties of the Cesáro sum. First, we shall consider the class of series for which the Cesáro sum exists. The following theorems will give some notion of the range of this class.

Theorem 2.5: A necessary condition for $\sum a_n$ to be C_1 summable is that $\lim \frac{a_n}{n} = 0$.

Now,

$$\lim \frac{a_n}{n} = \lim \frac{s_n - s_{n-1}}{n}$$

$$\lim \left[\left(\frac{s_0 + \dots + s_n}{n} - \frac{s_0 + \dots + s_{n-1}}{n} \right) \right. \\ \left. + \left(\frac{s_0 + \dots + s_{n-1}}{n} - \frac{s_0 + \dots + s_{n-2}}{n} \right) \right]$$

$$\begin{aligned}
 &= \lim \left[\left(\frac{s_0 + \dots + s_n}{n} \cdot \frac{n}{n+1} - \frac{s_0 + \dots + s_n - 1}{n} \right) \right. \\
 &\quad \left. + \left(\frac{s_0 + \dots + s_{n-1}}{n} - \frac{s_0 + \dots + s_{n-2}}{n} \cdot \frac{n}{n-1} \right) \right] \\
 &= \lim \left[\left(\frac{s_0 + \dots + s_n}{n+1} - \frac{s_0 + \dots + s_{n-1}}{n} \right) \right. \\
 &\quad \left. + \left(\frac{s_0 + \dots + s_{n-1}}{n} - \frac{s_0 + \dots + s_{n-2}}{n-1} \right) \right] \\
 &= 0.
 \end{aligned}$$

CHAPTER III

CESÁRO AND HÖLDER SUMMABILITY OF ANY NON-NEGATIVE INTEGRAL ORDER

We can not evaluate the series,

$$1 - r + \frac{r(r+1)}{2!} - \frac{r(r+1)(r+2)}{3!} + \dots, \text{ for } r > 1,$$

although we obtain by Euler's "scheme", if $x = -1$,

$$\frac{1}{(1-x)^r} = \frac{1}{2^r} = 1 - r + \frac{r(r+1)}{2!} - \frac{r(r+1)(r+2)}{3!} + \dots$$

It is necessary that we now extend the notion of Cesáro sum by a more general definition.

Definition: The series $\sum a_n$ is Cesáro summable of order r to s , if r is a positive integer such that

$$\lim_{n \rightarrow \infty} \frac{s_0 \binom{r+n-1}{n} + s_1 \binom{r+n-2}{n-1} + \dots + \binom{r}{1} s_{n-1} + s_n}{\binom{r+n}{n}} = s,$$

where $\binom{a}{b} = \frac{a(a-1)\dots(a-b+1)}{b!}$. If $r = 0$, the sum shall

be the same as that for the ordinary sum.

We may briefly say that the series is C_r summable to s . This general definition of summability of order r evidently reduces to the definition already given for $r = 1$.

An equivalent definition can be given directly using the terms of the series. We may substitute for the above limit,

$$\lim_{n \rightarrow \infty} \frac{a_0 \binom{r+n}{n} + a_1 \binom{r+n-1}{n-1} + \dots + \binom{r+1}{1} a_{n-1} + a_n}{\binom{r+n}{n}} = s.$$

For,

$$\frac{1}{(1-x)^{r+1}} = (1+x+\dots+x^n+\dots) \frac{1}{(1-x)^r}, \quad |x| < 1,$$

and if each side be expanded in a power series, we have an identity. Now, if we write the coefficients of x^n on each side of the equation, we have

$${r+n \choose n} = {r+n-1 \choose n} + {r+n-2 \choose n-1} + \dots + {r \choose 1} + 1.$$

The sum of the coefficients of the numerator is equal to the denominator; hence, we have a weighted arithmetic mean.

Definition: Let $s_n = a_0 + a_1 + \dots + a_n$, and

$$h_n^{(0)} = s_n,$$

$$h_n^{(1)} = \frac{h_0^{(0)} + h_1^{(0)} + h_2^{(0)} + \dots + h_n^{(0)}}{n+1}$$

$$h_n^{(r)} = \frac{h_0^{(r-1)} + h_1^{(r-1)} + \dots + h_n^{(r-1)}}{n+1};$$

then any integer r for which $\lim h_n^{(r)} = s$ the series will be Hölder summable of order r to s . We may briefly say that it is H_r summable to s .

It has been proved that if a series has a C_r sum, it has the same H_r sum and conversely.¹ Since this is true, it will

¹ E.W. Hobson, Theory of Functions of a Real Variable, Vol. II, p. 85, or Knopp, op. cit., p. 481.

only be necessary to deal with one in order to get the properties in general of both.

The Cesáro definition satisfies the fundamental requirements for a generalized sum. This fact shall be exemplified in the four succeeding theorems.

Lemma 3.0.1: If

$$s_n^{(1)} = s_0 + s_1 + \dots + s_n,$$

$$s_n^{(2)} = s_0^{(1)} + s_1^{(1)} + \dots + s_n^{(1)},$$

⋮

$$s_n^{(r)} = s_0^{(r-1)} + s_1^{(r-1)} + \dots + s_n^{(r-1)},$$

⋮

⋮

then

$$s_n^{(r)} = s_0 \binom{r+n-1}{n} + s_1 \binom{r+n-2}{n-1} + \dots + s_n.$$

For, by using our formula for a combination of n things taken $r-1$ at a time, we see that in the sum represented by $s_n^{(r)}$, we have $\binom{r+n-1}{n}$ of the s_0 terms. Likewise for the s_1 terms we have, by using the formula for a combination of $n-1$ things taken $r-1$ at a time, $\binom{r+n-2}{n-1}$ of the s_1 terms. Now by continuing this process until we arrive at the n th term, we have only one s_n term. Thus our lemma follows.

Theorem 3.1: If the C_r sum is s , then the $C_{(r+k)}$ sum is s , where k is also a positive integer.

A well known theorem states: "If (B_n) denotes a monotone increasing sequence of positive numbers, such that B_n increases indefinitely with n , and if (a_n) be any sequence of numbers then ...; and in particular if $\lim \frac{a_n + 1 - a_n}{B_n + 1 - B_n}$ has a definite value, then $\lim \frac{a_n}{B_n}$ has the same definite value."²

Now, if we use the notation of our lemma and let

$a_n = \frac{s_n(r+1)}{s_n(r)}$, $B_n = \frac{(r+n)}{(r+n-1)}$, we have $a_n - a_{n-1} = \frac{s_n(r)}{s_n(r)}$ and
 $B_n - B_{n-1} = \frac{(r+n-1)}{(r+n-2)}$. Thus if $\lim \frac{s_n(r)}{(r+n-1)} = s$, then

$\lim \frac{s_n(r+1)}{(r+n)} = s$. But $\lim \frac{s_n(r)}{(r+n-1)} = \lim \frac{s_n(r)}{(r+n)}$; hence,

if the C_r sum is s , then the C_{r+1} is s .

By continuing this process we see that if the C_r sum is equal to s , then the $C_{(r+k)}$ sum is equal to s .

It is evident if we let $r = 0$, that requirements (1) and (2) for a generalized sum are satisfied.

Theorem 3.2: If the series $\sum_{n=0}^{\infty} a_n$ is C_r summable to s , then $\sum_{n=1}^{\infty} a_n$ is C_r summable to $s + s_0$ and conversely.

²Hobson, E. W., Theory of Functions of a Real Variable, Vol. II, p. 7.

If $s_0, s_1, s_2, \dots, s_n, \dots$ are the partial sums of the first series, the partial sums of the second series are $(s_1 - s_0), (s_2 - s_0), (s_3 - s_0), \dots, (s_n - s_0), \dots$.

Substituting in our definition of a C_r sum, we have

$$\begin{aligned} & \lim \frac{(s_1 - s_0) \left(\frac{r+n-2}{n-1} \right) + (s_2 - s_0) \left(\frac{r+n-3}{n-2} \right) + \dots + (s_n - s_0) \left(\frac{r+n-1}{n-1} \right)}{\left(\frac{r+n-1}{n-1} \right)} \\ &= \lim \frac{s_1 \left(\frac{r+n-2}{n-1} \right) + s_2 \left(\frac{r+n-3}{n-2} \right) + \dots + s_n \left(\frac{r+n-1}{n-1} \right)}{\left(\frac{r+n-1}{n-1} \right)} - s_0, \end{aligned}$$

since $\left(\frac{r+n-2}{n-1} \right) + \left(\frac{r+n-3}{n-2} \right) + \dots + 1 = \left(\frac{r+n-1}{n-1} \right)$ as we have previously shown in this chapter. Now it is necessary that we establish the following lemma:

Lemma 3.2.1: If the sequence $s_0, s_1, s_2, \dots, s_n, \dots$ is C_r convergent, then the sequence $s_1, s_2, s_3, \dots, s_n, \dots$ is C_r convergent to the same number and conversely.

For,

$$\begin{aligned} & \lim \frac{s_1 \left(\frac{r+n-2}{n-1} \right) + s_2 \left(\frac{r+n-3}{n-2} \right) + \dots + s_n \left(\frac{r+n-1}{n-1} \right)}{\left(\frac{r+n-1}{n-1} \right)} \\ &= \lim \frac{s_1 \left(\frac{r+n-2}{n-1} \right) + s_2 \left(\frac{r+n-3}{n-2} \right) + \dots + s_n \left(\frac{r+n-1}{n-1} \right)}{\left(\frac{r+n-1}{n-1} \right)} + \lim \frac{rs_0}{n} \\ &= \lim \frac{s_0 \left(\frac{r+n-1}{n} \right) + s_1 \left(\frac{r+n-2}{n-1} \right) + \dots + s_n \left(\frac{r+n-1}{n-1} \right)}{\left(\frac{r+n-1}{n-1} \right)} \cdot \frac{n}{r+n} \end{aligned}$$

$$\lim \frac{s_0 \left(\frac{r+n-1}{n} \right) + s_1 \left(\frac{r+n-2}{n-1} \right) + s_2 \left(\frac{r+n-3}{n-2} \right) + \dots + s_n}{\left(\frac{r+n}{n} \right)}$$

This last limit is, by definition, the number to which the sequence is C_r convergent if this limit exists. Hence, both sequences are C_r convergent to same number.

The converse follows in a similar manner. Now the first term in the limit preceding this lemma approaches s as $n \rightarrow \infty$, since it is equal to the limit in our definition and since by hypothesis $\sum_{n=0}^{\infty} s_n$ is C_r summable. Hence, we have $\sum_{n=1}^{\infty} s_n$ to be C_r summable to $s - s_0$.

The converse follows in a similar manner using the converse of the lemma.

Theorem 3.3: If $\sum a_n$ and $\sum b_n$ are C_r summable to s and t respectively, then $\sum (a_n + b_n)$ is C_r summable to $s + t$.

Let $s_0, s_1, s_2, \dots, s_n, \dots$ and $t_0, t_1, t_2, \dots, t_n, \dots$ be the sequences of partial sums of a_n and b_n respectively; then $(s_0 + t_0), (s_1 + t_1), (s_2 + t_2), (s_3 + t_3), \dots, (s_n + t_n), \dots$ is the sequence of partial sums of $\sum (a_n + b_n)$. Thus, we have

$$\lim \frac{(s_0 + t_0) \left(\frac{r+n-1}{n} \right) + (s_1 + t_1) \left(\frac{r+n-2}{n-1} \right) + \dots + (s_n + t_n)}{\left(\frac{r+n}{n} \right)}$$

$$= \lim \frac{s_0 \left(\frac{r+n-1}{n} \right) + s_1 \left(\frac{r+n-2}{n-1} \right) + \dots + s_n}{\left(\frac{r+n}{n} \right)}$$

$$+ \lim \frac{t_0 \left(\frac{r+n-1}{n} \right) + t_1 \left(\frac{r+n-2}{n-1} \right) + \dots + t_n}{\left(\frac{r+n}{n} \right)} = s + t,$$

since the first and second terms are respectively s and t by the definition of a C_p sum.

Theorem 3.4: If $\sum a_n$ is C_p summable to s , then $\sum ka_n$ is C_p summable to ks .

For, if we let $s_0, s_1, s_2, \dots, s_n, \dots$ be the sequence of partial sums of $\sum a_n$, then the sequence of partial sums of $\sum ka_n$ is $ks_0, ks_1, ks_2, \dots, ks_n, \dots$. Now, if we substitute in our definition of a C_p sum and factor out k , we have

$$\lim k \cdot \frac{s_0 \left(\frac{r+n-1}{n} \right) + s_1 \left(\frac{r+n-2}{n-1} \right) + \dots + s_n}{\left(\frac{r+n}{n} \right)} = ks$$

since the second factor of the limit is by definition the C_p sum of $\sum a_n$.

Theorem 3.5: A necessary condition for C_p summability of the series $\sum a_n$ is that $C_p\text{-}\lim a_n = 0$.

For,

$$\lim \frac{\left(\frac{r+n-1}{n} \right) a_0 + \left(\frac{r+n-2}{n-1} \right) a_1 + \dots + a_n}{\left(\frac{r+n}{n} \right)}$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \frac{\left(\frac{r+n-1}{n}\right)s_0 + \left(\frac{r+n-2}{n-1}\right)(s_1 - s_0) + \dots + (s_n - s_{n-1})}{\left(\frac{r+n}{n}\right)} \\
 &= \lim_{n \rightarrow \infty} \left[\frac{\left(\frac{r+n-1}{n}\right)s_0 + \left(\frac{r+n-2}{n-1}\right)s_1 + \dots + s_n}{\left(\frac{r+n}{n}\right)} \right. \\
 &\quad \left. - \frac{\left(\frac{r+n-1}{n}\right)s_0 + \left(\frac{r+n-2}{n-1}\right)s_1 + \dots + s_{n-1} \cdot \frac{r+n}{n}}{\left(\frac{r+n}{n}\right)} \right] \\
 &= 0.
 \end{aligned}$$

Theorem 3.6: A necessary condition for C_r summability is that $\lim_{n \rightarrow \infty} \frac{s_n}{n^r} = 0$.

Using the notation of Lemma 3.0.1., we have

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \frac{s_n^{(r)}}{\left(\frac{r+n}{n}\right)} &= \lim_{n \rightarrow \infty} \frac{s_0^{(r-1)} + s_1^{(r-1)} + \dots + s_{n-1}^{(r-1)}}{\left(\frac{r+n}{n}\right)} \\
 &= \lim_{n \rightarrow \infty} \frac{s_0^{(r-1)} + s_1^{(r-1)} + \dots + s_{n-1}^{(r-1)}}{\left(\frac{r+n-1}{n-1}\right)} \\
 &= \lim_{n \rightarrow \infty} \frac{s_0^{(r-1)} + s_1^{(r-1)} + \dots + s_{n-1}^{(r-1)}}{\left(\frac{r+n}{n}\right)},
 \end{aligned}$$

since $\lim_{n \rightarrow \infty} \left(\frac{r+n-1}{n-1}\right) = \lim_{n \rightarrow \infty} \left(\frac{r+n}{n}\right)$. Now if we take the difference between the last limit and the second limit in the preceding expression, we have

$$\lim \frac{s_n^{(r+1)}}{(r+n)} = 0;$$

thus

$$\begin{aligned} \lim \frac{s_n^{(r+1)}}{(r+n)} &= \lim \frac{r! s_n^{(r+1)}}{(n+1)(n+2) \dots (n+r)} \\ &= \lim \frac{s_n^{(r+1)}}{n^r} = 0. \end{aligned}$$

In a similar manner we have

$$\lim \frac{\frac{s_n^{(r+1)}}{n^r} - 1}{n^r} = \lim \frac{s_n^{(r+1)} - 1}{(n+1)^r} = 0.$$

But

$$\lim \left[\frac{s_n^{(r+1)}}{n^r} - \frac{s_n^{(r+1)}}{n^r} \right] = \lim \frac{s_n^{(r+2)}}{n^r} = 0,$$

and in like manner we get

$$\lim \frac{s_n^{(r+3)}}{n^r} = 0, \dots, \lim \frac{s_n^{(1)}}{n^r} = 0, \lim \frac{s_n}{n^r} = 0,$$

and

$$\lim \frac{s_n}{n^r} = 0.$$

Theorem 3.7: If $\sum a_n$ is C_r summable to s , and $\sum b_n$ is C_k summable to t , then the Cauchy product, $\sum p_n = \sum (a_0 b_n + a_1 b_{n-1} + \dots + a_{n-1} b_1 + a_n b_0)$, is C_r summable to st , where $r+k+1$.

Let us denote $a_0\left(\frac{r+n}{n}\right) + a_1\left(\frac{r+n-1}{n-1}\right) + \dots + a_n$ by

$s_n^{(r)}$, $b_0\left(\frac{k+n}{n}\right) + b_1\left(\frac{k+n-1}{n-1}\right) + \dots + b_n$ by $t_n^{(k)}$, and

$p_0\left(\frac{s+n}{n}\right) + p_1\left(\frac{s+n-1}{n-1}\right) + \dots + p_n$ by $p_n^{(s)}$. If $|x| < 1$, then

$$(1) \frac{1}{(1-x)^{r+1}} \sum a_n x^n = \frac{1}{(1-x)^{k+1}} \sum b_n x^n = \frac{1}{(1-x)^{s+1}} \sum p_n x^n.$$

$\sum p_n x^n$, since each power series is convergent. For, if we choose a positive ϵ , there is an N such that for every positive $n > M$, $\left|\frac{a_n}{n^r}\right| < \epsilon$ by Theorem 3.6. Now $\sum a_n x^n$, $|x| < 1$, may be written in the form $\sum \frac{a_n}{k^n}$, $|k| > 1$, but also there is an M such that for all $n > M$, $\left|\frac{a_n}{k^n}\right| < \left|\frac{a_n}{n^{r+s}}\right|$. Thus, for $n > \max[N, M]$,

$$\left|\frac{a_n}{k^n}\right| < \frac{\frac{a_n}{n^r}}{n^{s+r}} < \frac{1}{n^s}.$$

Since $\sum \frac{1}{n^s}$ is convergent, we know that $\sum a_n x^n$, $|x| < 1$, is convergent. In like manner $\sum b_n x^n$, $|x| < 1$, may be shown to be convergent.

Lemma 3.7.1:

$$\frac{1}{(1-x)^{r+1}} \sum a_n x^n = \sum s_n^{(r)} x^n, |x| < 1.$$

For,

$$\frac{1}{1-x} \sum a_n x^n = \sum a_n x^n \cdot \sum x^n = \sum (a_0 + a_1 + \dots + a_n) x^n, |x| < 1.$$

By repeated differentiation of $\frac{1}{1-x}$ and $\sum x^n$ r times and dividing each one by $(r-1)!$, we have

$$\begin{aligned}
 \frac{1}{(1-x)^{r+n}} \sum a_n x^n &= \sum \binom{r+n}{n} x^n \cdot \sum a_n x^n \\
 &= \sum \left[\binom{r+n}{n} a_0 + \binom{r+n-1}{n-1} a_1 + \dots + a_n \right] x^n \\
 &= \sum s_n \binom{r+n}{n} x^n, \quad |x| < 1.
 \end{aligned}$$

Statement (1) now results in

$$\begin{aligned}
 \sum p_n^{(s)} x^n &= \sum s_n^{(r)} x^n \cdot \sum t_n^{(k)} x^n \\
 &= \sum \left(s_0^{(r)} t_n^{(k)} + s_1^{(r)} t_{n-1}^{(k)} + \dots + s_n^{(r)} t_0^{(k)} \right) x^n, \quad |x| < 1.
 \end{aligned}$$

Since coefficients of power series have to be equal in order for the power series to be equal,

$$p_n^{(s)} = s_0^{(r)} t_n^{(k)} + s_1^{(r)} t_{n-1}^{(k)} + \dots + s_n^{(r)} t_0^{(k)}.$$

For, if we place $s_n^{(r)} = (s - \delta_n) \binom{r+n}{n}$, $t_n^{(k)} = (t + e_n) \binom{k+n}{n}$,

$\binom{k+n}{n}$, where $\delta_n \rightarrow 0$ and $e_n \rightarrow 0$ as $n \rightarrow \infty$ by hypothesis. Since

it can be shown that

$$\binom{r+n}{n} \binom{k}{k} + \binom{r+n+1}{n+1} \binom{k+1}{1} + \dots + \binom{r}{r} \binom{k+n}{n} = \binom{r+n}{n},$$

using Lemma 3.7.1. by substituting, in (1), $\sum x^n$ for each $\sum a_n x^n$ and $\sum p_n x^n$ and omitting the factor $\sum b_n x^n$, we need only to show that

$$\left[s_0 \theta_n \binom{r+n}{n} \binom{k}{k} + s_1 \theta_{n-1} \binom{r+n-1}{n-1} \binom{k+1}{1} + \dots + s_n \theta_0 \binom{r}{r} \binom{k+n}{n} \right]$$

divided by $\binom{r+n}{n}$ approaches 0 as $n \rightarrow \infty$. Since $\delta_n \rightarrow 0$ and

$\theta_n \rightarrow 0$, we may choose K so that, for every n , $|s_n| < K$ and $\theta_n < K$.

Now, if an arbitrary positive ϵ be given and if N' be chosen so that, for every $n > \frac{1}{2}N'$, $|s_n|$ and $|\theta_n|$ are less than ϵ/k then the absolute value of the preceding expression is for every $n > N'$,

$$\leq k \cdot \frac{\epsilon}{k} \left[\binom{r+n}{n} \binom{k}{k} + \dots + \binom{r}{n} \binom{k+n}{n} \right] = \epsilon \binom{r+n}{n}.$$

Hence, the limit of this expression divided by $\binom{r+n}{n}$ is 0.

Therefore, by this fact,

$$\lim \frac{\binom{r}{n}}{\binom{r+n}{n}} = st.$$

Theorem 3.8: If $\sum a_n$ and $\sum b_n$ have the sums s and t respectively, then the Cauchy product

$$\sum (a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0)$$

has the C₁ sum, st.

An analogy may be noted with the well-known theorem³ that if two series are absolutely convergent, then the Cauchy product is absolutely convergent to the product of their sums. In fact the Cauchy product⁴ of a C_p summable series with an absolutely convergent series is itself C_p summable to the product of the sums of the two series.

³Knopp, Theory and Application of Infinite Series, p. 146.

⁴L. L. Silverman, Sum of a Divergent Series, p. 27.

Another generalization⁵ may be made using the idea of a Cesaro form in a weighted arithmetic mean as illustrated by the following form:

$$s = \lim \frac{b_0 s_0 + b_1 s_1 + \dots + b_n s_n}{b_0 + b_1 + \dots + b_n},$$

where the b_i values depend upon n and r , r being a fixed number.

We shall make the following definition, substituting for b_n the function $\frac{r^n}{n!}$.

Definition:

$$s = \lim_{r \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{s_0 + s_1 \frac{r}{1!} + s_2 \frac{r^2}{2!} + \dots + s_n \frac{r^n}{n!}}{1 + \frac{r}{1!} + \frac{r^2}{2!} + \dots + \frac{r^n}{n!}}.$$

Since $e^r = 1 + \frac{r}{1!} + \frac{r^2}{2!} + \dots + \frac{r^n}{n!} + \dots$,

$$s = \lim_{r \rightarrow \infty} \lim_{n \rightarrow \infty} e^{-r} \left[s_0 + s_1 \frac{r}{1!} + s_2 \frac{r^2}{2!} + \dots + s_n \frac{r^n}{n!} \right].$$

This limit exists if $\lim s_n$ exist, for we only have to apply the main rearrangement theorem.⁶ We shall let $s(r)$ represent the function of n and r in the brackets, where $n \rightarrow \infty$. Then

$$s'(r) = s_1 + s_2 r + s_3 \frac{r^2}{2!} + \dots$$

Let

$$s_1(r) = s'(r) - s(r) = s_1 + s_2 r + s_3 \frac{r^2}{2!} + \dots$$

⁵Ibid., p. 11.

⁶One proof given by Knopp, Op. cit., p. 145.

Since

$$\begin{aligned}\frac{d}{dr} e^{-r} [s(r)] &= e^{-r} [s'(r)] - e^{-r} [s(r)] \\ &= e^{-r} [a_1(r)]\end{aligned}$$

we have

$$e^{-r} [s(r)] = a_0 + \int_0^r e^{-r} [s'(r) - s(r)] ;$$

$$\text{and applying the limit, } s - a_0 = \int_0^\infty e^{-r} [a_1(r)] .$$

If we integrate by parts, it is seen that

$$s - a_0 = \left[e^{-r} \int_0^r a_1(r) \right]_0^\infty + \int_0^\infty e^{-r} \left[\int_0^r a_1(r) \right] .$$

letting

$$e_0(r) = a_0 + a_1 r + a_2 \frac{r^2}{2!} + \dots + \frac{a_n r^n}{n!} + \dots = a_0 + \int_0^r a_1(r) ,$$

the above equation results in

$$\begin{aligned}s - a_0 &= \left[e^{-r} [e_0(r)] - a_0 \right]_0^\infty + \int_0^\infty e^{-r} [e_0(r) - a_0] \\ &= \left[e^{-r} [e_0(r)] \right]_0^\infty - \left[\frac{a_0}{e^r} \right]_0^\infty + \int_0^\infty e^{-r} [e_0(r)] - \left[\frac{a_0}{e^r} \right] \\ &= \left[e^{-r} [e_0(r)] \right]_0^\infty + \int_0^\infty e^{-r} [e_0(r)] \\ &= \lim_{r \rightarrow \infty} e^{-r} [e_0(r)] - a_0 + \int_0^\infty e^{-r} [e_0(r)] ,\end{aligned}$$

and finally,

$$s = \lim_{r \rightarrow \infty} e^{-r} [e_0(r)] + \int_0^\infty e^{-r} [e_0(r)] .$$

If the integral in the last statement is convergent, then for the statement to be true, the $\lim_{r \rightarrow \infty} e^{-r} [e_0(r)]$ must exist. But in order for our integral to converge, it is

necessary that this limit be zero. Hence,

$$s = \int_0^\infty e^{-r} [a_0(r)],$$

where

$$a_0(r) = a_0 + a_1 r + a_2 \frac{r^2}{2!} + \dots + a_n \frac{r^n}{n!} + \dots,$$

when the integral converges.

It has been shown⁷ that this Borel Integral generalization satisfied the five fundamental requirements with the exception of requirement (3). To avoid this difficulty, Borel has stated this definition⁸ in the following manner:

Definition: The series shall be absolutely summable if both the integrals, $\int_0^\infty e^{-r} |a_0(r)|$ and $\int_0^\infty e^{-\frac{r^D}{D}} |a_0(r)|$, converge, where D denotes the order of any derivative.

We have omitted proofs of the fulfillment of these fundamental requirements, since it is our purpose to confine ourselves chiefly to the Cesáro and Hölder definitions of summability.

Examples of Cesáro Summability

We shall here confine our examples to those series with unbounded partial sums.

⁷A discussion of or references to these properties is given by Silverman, Sum of a Divergent Series, p. 14.

⁸Ibid.

Writing the series,

$$\sum (-1)^n (n+1) = 1 - 2 + 3 - 4 + 5 - 6 + \dots$$

and the sequence of partial sums,

$$1, -1, +2, -2, +3, -3, \dots, +\frac{n+1}{2} - \frac{n+1}{2}, \dots$$

it follows that the sequence of arithmetic means of the partial sums is

$$1, 0, 2/3, 0, 2/5, 0, 4/7, \dots$$

It can be readily seen that this sequence does not converge, and consequently the series is not C_1 summable.

If we substitute these partial sums in our definition of C_2 summability, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{(n+1) - n + 2(n-1) - \dots + (-1)^{n+1}(2) \cdot n/2 + (-1)^n (n/2)}{(n+1)(n+2)/2} \\ = \lim_{n \rightarrow \infty} \frac{1 + 3 + 5 + 4 + 5 + \dots + (n - n/2)}{(n+1)(n+2)/2} \\ = \lim_{n \rightarrow \infty} \frac{(n/2 + 1)}{(n+1)} \frac{(n/2 + 2)}{(n+2)} = \frac{1}{4}. \end{aligned}$$

We notice that our series has its C_2 sum equal to $1/(1-x)^2$ at $x = -1$; our series may be derived by expanding $1/(1-x)^2$ into a power series and substituting $x = -1$.

If we consider the series,

$$\begin{aligned} 1 - 1 + 2 - 2 + 1/2 + 2 - 3 + 1/4 + 1/8 + 4 - 4 \\ + 1/16 + 1/32 + 1/64 + \dots, \end{aligned}$$

and write the sequence of partial sums, we have

$$1, 0, 2, 0, 1/2, 3, 0, 1/4, 1/2, 4, 0, 1/16, 1/32, 1/64, \dots$$

By substituting in our definition for the G_1 sum and choosing n so that the \underline{n} th term is the integer k , we have

$$\frac{1 + 0 + 2 + 0 + 1/2 + 3 + 0 + 1/4 + 1/8 + 4 + 0 + 1/16 + \dots + k}{n}$$

The required limit can be found readily. For there is no loss in generality in letting k be the last term, since the sum of all the terms to the next integral term is less than 1 because the geometric series, $1/2 + 1/4 + 1/8 + \dots$ converges to 1.

Now the sum of the integers is $k(k + 1)/2$, and hence there are $k(k - 1)/2$ terms in sequence of partial sums out to and including k . It thus appears that $n = k(k + 1)/2$. Hence, our limit is $\lim (n + \lambda)/n$, where $0 < \lambda < 1$. Hence the series is G_1 summable to 1.

CHAPTER IV.

CESÁRO SUMMABILITY OF ORDER r ,

r BEING REAL AND $r > -1$

Definition: let r be any real number > -1 . If

$$\lim \frac{\left(\frac{r+n-1}{n}\right)s_0 + \left(\frac{r+n-2}{n-1}\right)s_1 + \dots + s_n}{\left(\frac{r+n}{n}\right)} = s,$$

where s_0, s_1, s_2, \dots , are the partial sums of $\sum a_n$ and

$\binom{r+k}{k} = \frac{(r+1)(r+2)\dots(r+k)}{k!}$, s will be called the

Cesáro sum of order r of $\sum a_n$. We may briefly say that the series has the C_r sum s .

If we again denote

$$\left(\frac{r+n-1}{n}\right)s_0 + \left(\frac{r+n-2}{n-1}\right)s_1 + \dots + \left(\frac{r}{1}\right)s_{n-1} + s_n$$

by $S_n^{(r)}$, then the limit in our definition of the C_r sum is

$$\lim \frac{S_n^{(r)}}{\left(\frac{r+n}{n}\right)}.$$

Since¹ $\left(\frac{r+n}{n}\right) = \frac{\Gamma(n+r+1)}{\Gamma(r+1)\Gamma(n+1)}$ and by making use of
Sterling's² asymptotic formula,

¹ For a discussion of $\Gamma(x)$, where x is any real number, see R. L. Nolen, Some Fundamental Properties of Gamma and Beta Functions, pp. 6-12.

² A proof of Sterling's formula is given by Courant, Differential Calculus, Vol. II, p. 461.

$$\Gamma(x+1) = e^x x^x (2\pi x)^{\frac{1}{2}} (1 + \epsilon_x),$$

where $\epsilon_x \rightarrow 0$ as $x \rightarrow \infty$, we get

$$\frac{\Gamma(n+r+1)}{\Gamma(r+1)\Gamma(n+1)}$$

$$= \frac{1}{\Gamma(r+1)} \cdot \frac{e^{-n}(r+n)}{e^{-n}} \frac{(r+n)^{(r+n)}}{n^n} \frac{(r+n)^{\frac{1}{2}}(1+\delta_n)}{n^{\frac{1}{2}}},$$

where $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. Also, since $\lim \frac{(r+n)^{\frac{1}{2}}}{n^{\frac{1}{2}}} = 1$ and

$$\lim \frac{(r+n)^{(r+n)}}{n^n} = \lim \frac{n^{(r+n)}}{n^n} = \lim n^r, \text{ we have}$$

$$(r+n)^{\frac{1}{n}} = \frac{1}{\Gamma(r+1)} \cdot \frac{\Gamma(r+n+1)}{\Gamma(n+1)} = \frac{n^r}{\Gamma(r+1)} (1 + \delta_n).$$

Therefore,

$$\lim \frac{s_n(r)}{(r+n)^{\frac{1}{n}}} = \lim \Gamma(r+1) \frac{s_n(r)}{n^r} \cdot 1$$

Theorem 4.1: If $\sum a_n$ and $\sum b_n$ are C_r and C_k summable respectively, where $r > -1$ and $k > -1$, then the series $\sum p_n$, where

$$p_n = a_n b_0 + a_{n-1} b_1 + \dots + a_0 b_n,$$

is $C(r+k+1)$ summable to the product of the two sums.

This theorem follows from Theorem 3.7 which holds true for real numbers.

As a consequence of Theorem 4.1 we have the following interesting theorem due to Cesaro.

³Hobson, Theory of Functions of a Real Variable, Vol. II, p. 70.

Theorem 4.2: If (A_n) and (B_n) are sequences of numbers such that $\frac{A_n}{n^r}$ and $\frac{B_n}{n^k}$ converge respectively to definite limits A and B as $n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} \frac{A_0 B_n + A_1 B_{n-1} + \dots + A_n B_0}{n^{r+k+1}} = \frac{\Gamma(r+1) \Gamma(k+1)}{\Gamma(r+k+2)} \cdot AB,$$

provided r and k are both > -1 .

The following proof is an immediate consequence of

Theorem 4.1. By substituting $\lim \Gamma(r+1) \frac{s_n(r)}{n^r}$,

$\Gamma(k+1) \frac{s_n(k)}{n^k}$, and $\Gamma(r+k+2) \frac{s_n(r+k+1)}{n^{r+k+1}}$ for the

c_r , c_k , and c_{r+k+1} sums respectively and letting $A_n = s_n(r)$, $B_n = s_n(k)$, the truth of the theorem follows since these limits exist when these Cesáro sums exist and conversely.

A series that is summable of order 0 may be summable of a negative order $r > -1$. At -1 the limit of our definition ceases to be defined. A consideration of this negative order would tell something of the nature of the convergence of the series.

According to Lemma 3.6.1 the Cauchy product of the finite series $s_0 + s_1 x + s_2 x^2 + \dots + s_n x^n$ with

$$1 + (\frac{r}{1})x + (\frac{r+1}{2})x^2 + \dots + (\frac{r+n-1}{n})x^n + \dots, |x| < 1,$$

gives the first series whose first $n+1$ terms are

$$s_0^{(r)} + s_1^{(r)}x + s_2^{(r)}x^2 + \dots + s_n^{(r)}x^n.$$

Thus, the sum of this product series is

$$(s_0 + s_1x + \dots + s_nx^n) (1 - x)^{-r}.$$

In this operation we have not made any assumption concerning convergence of $\sum s_n^{(r)} x^n$.

Since $(s_0 + s_1x + \dots + s_nx^n)$ is the product of $(1 - x)^{-r}$ with the sum of the series for which we have stated the first $n + 1$ terms, we have

$$s_n = s_n^{(r)} - \binom{r}{1}s_{n-1}^{(r)} + \binom{r}{2}s_{n-2}^{(r)} - \dots + (-1)^n \binom{r}{n}s_0^{(r)}$$

which holds for any value of n . If r is a positive integer, the series has only $n + 1$ terms for $n > r$.

In like manner, multiplying $s_0 + s_1x + \dots + s_nx^n$ by the series for $(1 - x)^{-(r+1)}$, we obtain

$$s_n = s_n^{(r)} - \binom{r+1}{1}s_{n-1}^{(r)} + \binom{r+1}{2}s_{n-2}^{(r)} - \dots + (-1)^n \binom{r+1}{n}s_0^{(r)}.$$

Now

$$\begin{aligned} & (1 - x)^{-r}(s_0 + s_1x + \dots + s_nx^n) \\ &= (1 - x)^{-(r - r')} [(1 - x)^{-r}(s_0 + s_1x + \dots + s_nx^n)], \end{aligned}$$

for $|x| < 1$; thus the product of $(1 - x)^{-(r - r')}$ with the sum of the convergent series of which the first $n + 1$ terms are

$$s_0^{(r')} + s_1^{(r')}x + \dots + s_n^{(r')}x^n$$

is equal to the right hand member of the above equation.

The Cauchy product of the series for $(1 - x)^{-(r - r')}$ and the series for which we last gave the sum of the first

$n + 1$ terms is a series of which the sum of the first $n + 1$ terms are

$$s_0^{(r)} + s_1^{(r)}x + \dots + s_n^{(r)}x^n.$$

Thus, it follows⁴ that $s_n^{(r)}$ is equivalent to

$$s_n^{(r')} + \binom{r - r'}{1}s_{n-1}^{(r)} + \binom{r - r' - 1}{2}s_{n-2}^{(r')} + \dots + \binom{r - r' - n + 1}{n}s_0^{(r')}$$

for there can not be two different power series which converge for $|x| < 1$.

Theorem 4.3: If a series is C_r summable to s , then it is $C_{r'}$ summable to s , where r and r' are any real numbers such that $r' > r > -1$.

Now $\frac{s_n^{(r)}}{n^r}$ converges to the limit $\frac{s}{\Gamma(r+1)}$ since

$$\lim \frac{s_n^{(r)}}{\left(\frac{r+n}{n}\right)} = s \text{ and also } \lim \Gamma(r+1) \frac{s_n^{(r)}}{n^r} = s.$$

If we let $B_n = \left(\frac{r' - r + n - 1}{n}\right)$ where $\frac{B_n}{r' - r - 1}$ converges to $\frac{1}{\Gamma(r' - r)}$, where $r' - r - 1 = k$ and $r' > r$, then by making use of the relation immediately preceding this theorem, we have

$$s_n^{(r')} = s_n^{(r)} + B_1 s_{n-1}^{(r)} + B_2 s_{n-2}^{(r)} + \dots + B_n s_0^{(r)}.$$

⁴Hobson, op. cit. pp. 71-72

It follows from this equation that $\frac{s(r^*)}{n^r}$ converges to $\frac{s}{f(r^* - 1)}$.

Hence by Theorem 4.1

$$\lim \frac{\frac{s(r^*)}{n^r}}{\frac{n}{r^* + n}} = s.$$

Theorem 4.4: If $\lim \frac{s_n(r)}{\left(\frac{n+r}{n}\right)} = +\infty$ or $-\infty$, then

$$\lim \frac{s_n(r+1)}{\left(\frac{n+r+1}{n}\right)} = +\infty \text{ or } -\infty.$$

Since $s_n(r) = s_n(r+1) - s_{n-1}(r+1)$ and $\left(\frac{n+r}{n}\right) = \left(\frac{n+r+1}{n}\right) - \left(\frac{n+r-1}{n-1}\right)$

$$-\left(\frac{n+r}{n-1}\right), \quad \lim \frac{s_n(r+1) - s_{n-1}(r+1)}{\left(\frac{n+r+1}{n}\right) - \left(\frac{n+r-1}{n-1}\right)} = +\infty \text{ or } -\infty.$$

Furthermore, since $\left(\frac{n+r+1}{n}\right)$, for $n = 1, 2, \dots$, is a monotone increasing sequence of positive numbers, then by a well-known theorem⁵ of sequences as a result of the following limit, we have

$$\lim \frac{s_n(r+1)}{\left(\frac{n+r+1}{n}\right)} = +\infty \text{ or } -\infty.$$

From this preceding theorem and theorem 4.2, we are led to make the following conclusion:

⁵A proof is given by Hobson, op. cit., II, p. 7.

If a series is C_r summable and divergent, then the sequence of partial sums oscillate between finite or infinite limits.

From Theorem 4.3 it is evident that the fundamental requirements (1) and (2) are satisfied where r is any real number. Since Theorems 3.2, 3.3., and 3.4. hold equally as well, where r is any real number, we also have the last three requirements fulfilled.

Theorem 4.5: A necessary condition for C_r summability of the series $\sum a_n$ is that

$$\lim_{n \rightarrow \infty} \frac{a_n}{n^r} = 0.$$

Proof follows similarly as for theorem 3.5. By induction, we get from $\lim \frac{s_n(r)}{(r+n)} = 0$, that

$$\lim \frac{s_n(r-1)}{(r+n)} = 0, \lim \frac{s_n(r-2)}{(r+n)} = 0, \dots, \lim \frac{s_n}{(r+n)} = 0,$$

and finally

$$\lim_{r \rightarrow \infty} \frac{a_n}{(r+n)^{\frac{1}{r}}} = 0.$$

But this limit is equal to

$$\begin{aligned} \lim \frac{a_n}{\frac{1}{\Gamma(r+1)} \frac{\Gamma(r+n+1)}{\Gamma(n+1)}} &= \lim \frac{\Gamma(r+1) a_n}{n^r} \\ &= \lim \frac{a_n}{n^r} = 0, \end{aligned}$$

since we have demonstrated that

$$\lim \frac{\Gamma(r+n+1)}{\Gamma(r+1)\Gamma(n+1)} = \lim \frac{n^r}{\Gamma(r+1)}.$$

We have now by the more general definitions of Hölder and Cesáro extended the class of series which are summable. It has been shown that these definitions satisfy the five generally accepted fundamental requirements of any generalization of summability, and have as well, other properties analogous to those of ordinary convergent series.

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