THE ELEMENTARY TRANSCENDENTAL FUNCTIONS OF A COMPLEX VARIABLE AS DEFINED BY INTEGRATION

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THESIS

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CHAPTER I

INTRODUCTION

The object of this paper is to define the elementary transcendental functions of a complex variable by means of integrals, and to discuss their properties. We shall assume that the reader has a knowledge of the theory of functions of a complex variable.

We give here a list of definitions and theorems to which references will be made in the later chapters.

Let $f(z)$ be defined at each point of a rectifiable arc $z(t) = x(t) + iy(t)$, $t_0 \leq t \leq T$, in the z-plane. We subdivide the arc into smaller arcs by the points $z_0 = z(t_0)$, $z_1 = z(t_1)$, $z_2 = z(t_2)$, ..., $z_n = z(t_n) = Z$, $t_0 < t_1 < t_2 < \dots < t_n = T$. Next we form the sum $\sum_{r=1}^{n} f(\zeta_r)(z_r - z_{r-1})$, where ζ_r is a point $\zeta_r = z(\eta_r)$ such that $t_{r-1} < \eta_r < t_r$. If this sum tends to a limit J, for all possible subdivisions of the above type and all such choices of η_r, as max $|z_r - z_{r-1}|$ tends to zero, we say that $f(z)$ is "integrable Riemann" along C, and write

$$J = \int_C f(z)\, dz.$$

<u>Theorem 1.1. If $f(z)$ is continuous on a regular arc C whose equation is $z(t) = x(t) + iy(t)$</u> where <u>$t_0 < t \leq T$, then $f(z)$ is integrable Riemann along C and</u>

$$\int_c f(z)\,dz = \int_{t_0}^{T} F(t)\left\{\frac{dx}{dt} + i\frac{dy}{dt}\right\}dt.$$

where $F(t)$ denotes the value of $f(z)$ on c, i.e. $f\big(x(t)+iy(t)\big) = F(t)$.

Theorem 1.2. If $f(z)$ is regular in a simply connected region R, then $\int_{z_0}^{z} f(z)\,dz$ is independent of the path provided z_0 and z lie in R, and the paths considered lie entirely in R.

Theorem 1.3. If $F(z) = \int_{z_0}^{z} f(z)\,dz$ is independent of the path in a region R, and if $f(z)$ is continuous in R, then $F'(z)$ exists and is equal to $f(z)$ in R.

Theorem 1.4. Let $f(z)$ be an analytic function, regular in a neighborhood of the point z_0, at which it takes the value w_0. The necessary and sufficient condition that the function $f(z) = w$ should have a unique inverse $z = F(w)$, regular in a neighborhood of w_0, is that $f'(z)$ should not vanish.[1]

Theorem 1.5. Suppose that $f(z)$ is regular on and inside a simple close contour C, and let a be a point inside C. Then

$$(1)\quad f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)(z-a)^2}{2!} + \cdots\cdots$$

The series is convergent if $|z - a| < d$, where d is the smallest distance of a point t on C from a. (1) is known as Taylor's series. If $a = 0$ it is called MacLaurin's series.

If $f(z)$ is regular in some region containing $z = a$, and if $f(a) = 0$, then a is said to be a zero of $f(z)$.

By Taylor's series $f(z)$ can be expanded in powers of $(z - a)$ in the form

$$f(z) = \sum_{k=0}^{\infty} \alpha_k (z-a)^k.$$

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1 E. T. Copson, Theory of Function of a Complex Variable, p.121

If α_m is the first non-vanishing coefficient in Taylor's series of *f*(*z*) in powers of (*z* - *a*), then *a* is said to be a *zero* of *order m*.

If *f*(*z*) has a zero of order *m* at *a*, 1/*f*(*z*) has a *pole* of *order m* there.

Theorem 1.6. The modulus of the product of two complex numbers (and hence, by induction, of any number of complex numbers) *is equal to the product of their moduli.*[1]

Theorem 1.7. Let the region R(z) in which the path of integration C_z lies be mapped upon the region B_w by means of a single valued analytic function whose inverse is single valued. If C_z is mapped upon C_w in the B_w region then[2]

$$\int_{c_3}^{1} f(z)\,dz = \int f[\phi(w)]\,\phi'(w)\,dw .$$

If *a* - *b* = *m*α, where *m* is an integer, we say that

$$a \equiv b \pmod{\alpha} .$$

It is obvious

a). That the relationship is reciprocal, i.e. if *a* $\equiv$ *b*(mod α), then *b* $\equiv$ *a*(mod α).

b). The relationship is transitive, i.e. if *a* $\equiv$ *b*(mod α) and *b* $\equiv$ *c*(mod α), then *a* $\equiv$ *c*(mod α).

c). If *a* $\equiv$ *b*(mod α) and *c* $\equiv$ *d*(mod α), then *a* + *c* $\equiv$ *b* + *d*(mod α).

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1 E. T. Copson, *Theory of Function of a Complex Variable*, p. 7.

2 Ludwig Bieberbach, *Lehrbuch Der Funktionentheorie*, p. 112.

All numbers which are congruent to a given number $\underline{a}$ (mod α) constitute the residue class $[\underline{a}]$ (mod α). A number $\underline{b}$ is in $[\underline{a}]$ (mod α), if $\underline{b} = \underline{a} + \underline{k}\alpha$, where $\underline{k}$ is an integer. Any number of $[\underline{a}]$ (mod α) is said to represent it.

We shall define

$$[\underline{a}] + [\underline{b}] = [\underline{a} + \underline{b}] \pmod{\alpha} .$$

A <u>complete set of residues</u> (mod α) is a set $\{\underline{z}\}$ such that every number can be written as $\underline{z} + \underline{k}\alpha$, where $\underline{k}$ is an integer.

The <u>complete set of residues</u> (mod α), α imaginary, such that $\underline{z} = \underline{x} + \underline{t}\alpha$, where $\underline{x}$ is any real number and $-\frac{1}{2} < \underline{t} \leq \frac{1}{2}$ is called a <u>principal set of residues</u>.

If α is real, the set $\underline{z} = \underline{iy} + \underline{t}\alpha$, where $\underline{y}$ is real and $-\frac{1}{2} < \underline{t} \leq \frac{1}{2}$ is called a <u>principal set of residues</u>.

CHAPTER II

THE LOGARITHM FUNCTION

For complex values of $z \neq 0$ we now define

$$l(z) = \int_1^z \frac{du}{u} .$$

Since the integrand is regular with the exception of the point $u = 0$, we may integrate $\int_1^z \frac{du}{u}$ along any path that does not include the point $u = 0$. If it is included we get different values for $l(z)$.

Evaluating $l(z)$, we find

$$l(z) \cong (\log|z| + i\theta) (\text{mod } 2\pi i)$$

where θ is the amplitude of z.

Proof:

As the integrand is regular with the exception of the point $u = 0$, we choose the path, (1) $x = t$, $y = 0$ (if $|z| > 1$ then $1 \leq t \leq |z|$ or if $|z| < 1$ then $|z| \leq t \leq 1$), calling this c_1, and (2) $x = |z| \cos t$, $y = |z| \sin t$ (if $\theta + 2k\pi > 0$ then $0 \leq t \leq \theta + 2k\pi$ or if $\theta + 2k\pi < 0$ then $\theta\ 2k\pi \leq t \leq 0$) (where θ is the amplitude of z and $-\pi < \theta \leq \pi$), calling this c_2. Hence

$$\int_{1c}^{z} \frac{du}{u} = \int_{1c_1}^{|z|} \frac{du}{u} + \int_{|z|c_2}^{z} \frac{du}{u} .$$

If we consider the first of these integrals, we have

$$\int_{1c_1}^{|z|} \frac{du}{u} = \int_{1c_1}^{|z|} \frac{1}{t}\left(\frac{dx}{dt} + i\frac{dy}{dt}\right) dt,$$

but $dx/dt = 1$ and $dy/dt = 0$, therefore

$$\int_{1c_1}^{|z|} \frac{du}{u} = \int_{1c_1}^{|z|} \frac{dt}{t} = \log |z|.$$

Considering the second of the integrals

$$\int_{|z|c_2}^{z} \frac{du}{u} = \int_{0c_2}^{\theta + 2k\pi} \frac{1}{|z|(\cos t + i \sin t)} \left(\frac{dx}{dt} + i\frac{dy}{dt}\right) dt,$$

but $dx/dt = -|z|\sin t$ and $dy/dt = |z|\cos t$, therefore

$$\int_{|z|c_2}^{z} \frac{du}{u} = \int_{0}^{\theta + 2k\pi} \frac{|z|(-\sin t + i\cos t)}{|z|(\cos t + i \sin t)} dt = i\int_{0}^{\theta + 2k\pi} \frac{\cos t + i \sin t}{\cos t + i \sin t} dt$$

$$= i\int_{0}^{\theta + 2k\pi} dt = i\theta + 2k\pi i,$$

where $k = 0, \pm 1, \pm 2, \dots$.

We see that $l(z) = \log |z| + i\theta + 2k\pi i$, therefore we set

$$l(z) \equiv [\log |z| + i\theta] \pmod{2\pi i} .$$

Making a cut along the negative real axis we find that in this region $w = l(z) = \int_1^z \frac{du}{u}$ is independent of the path of integration with the exception of the point $z = 0$, and $1/z$ is continuous in that same region, hence by Theorem 1.3, dw/dz exists and is equal to $1/z$ in that region.

Next we have the addition theorem

$$l(z_1) + l(z_2) = l(z_1 z_2) .$$

Proof:

$$l(z_1) = [\log |z_1| + i\theta_1] \pmod{2\pi i},$$

$$l(z_2) = [\log |z_2| + i\theta_2] \pmod{2\pi i},$$

$$\underline{l}(\underline{z}_1) + \underline{l}(\underline{z}_2) = [\log|\underline{z}_1| + \log|\underline{z}_2| \ i(\underline{\theta}_1 + \underline{\theta}_2)] \pmod{2\pi \underline{i}}$$
$$= [\log|\underline{z}_1||\underline{z}_2| + i(\underline{\theta}_1 + \underline{\theta}_2)] \pmod{2\pi \underline{i}}$$
$$= [\log|\underline{z}_1 \underline{z}_2| + i(\underline{\theta}_1 + \underline{\theta}_2)] \pmod{2\pi \underline{i}}$$
$$= \underline{l}(\underline{z}_1 \underline{z}_2).$$

The function $\log|\underline{z}| + i\underline{\theta}$, $-\pi < \underline{\theta} \leq \pi$, is said to represent the <u>principal branch</u> of $\underline{l}(\underline{z})$. We can now state the theorem;

<u>The principal branch of $\underline{l}(\underline{z})$ takes on every value of the principal set of residues mod $2\pi i$.</u>

<u>Proof</u>:

Let $\underline{w}_1 = \underline{u}_1 + i\underline{v}_1$, be any number from the principal set of residues mod $2\pi i$, that is $-\pi < \underline{v}_1 \leq \pi$. Then setting

$$\log|\underline{z}| + \underline{i\theta} = \underline{u}_1 + \underline{iv}_1,$$

we find that $|\underline{z}| = \underline{e}^{\underline{u}_1}$, and $\underline{\theta} = \underline{v}_1$.

Hence $\log|\underline{z}| + i\underline{\theta} = \underline{u}_1 + i\underline{v}_1$.

It follows at once that $\underline{l}(\underline{z})$ takes on any complex value.

We shall designate the principal branch of $\underline{l}(\underline{z})$ by $\underline{L}(\underline{z})$.

We make a further investigation of $\underline{L}(\underline{z})$ by setting

$$\underline{L}(\underline{z}) = 1 .$$

Then
$$\log|\underline{z}| + i\underline{\theta} = 1,$$
$$\log|\underline{z}| = 1, \text{ and } \underline{\theta} = 0,$$
$$|\underline{z}| = \underline{e}.$$

Hence we find that for $\underline{L}(\underline{z}) = 1$, $\underline{z} = \underline{e}$. Also if $\underline{L}(\underline{z}) = \underline{x}$(real) we find
$$\log|\underline{z}| + i\underline{\theta} = \underline{x},$$
$$\log|\underline{z}| = \underline{x}, \text{ or } \underline{z} = \underline{e}^{\underline{x}}, \text{ and } \underline{\theta} = 0.$$

Hence $\underline{L}(\underline{e}^{\underline{x}}) = \underline{x}$.

Since $\underline{L}(\underline{z})$ reduces for real $\underline{z} = \underline{x}$ to $\log \underline{x}$, we designate

$\underline{L}(\underline{z})$ by Log $\underline{z}$ and call it the principal logarithm of $\underline{z}$. The residue class $[\log|\underline{z}| + i\theta]$ (mod $2\pi\underline{i}$) will be denoted by log $\underline{z}$ and called the general logarithm of $\underline{z}$.

We define

$$l_c(z) = \frac{\log|z| + i\theta}{\log|c| + i\phi} \left(\mathrm{mod}\ \frac{2\pi(\phi + i\log|c|)}{\log^2|c| + \phi^2}\right)$$

where ϕ is the amplitude of $\underline{c}$, and $-\pi < \phi \le \pi$. $\underline{L}_c(\underline{z})$ is defined to be the values of $\underline{l}_c(\underline{z})$ which lie in the principal set of residues $\left(\mathrm{mod}\ \frac{2\pi(\phi + i \log|c|)}{\log^2|c| + \phi^2}\right)$. Hence $\underline{L}_c(\underline{z})$ will lie in the strip containing $\underline{w} = \underline{u} + \underline{iv}$ such that

$$\frac{-\pi \log|c|}{\log^2|c| + \phi^2} < v \le \frac{\pi \log|c|}{\log^2|c| + \phi^2}.$$

If $|c| = 1$, say $\underline{c} = \cos\phi + \underline{i}\sin\phi$ $(-\pi < \phi \le \pi)$, then

$$\underline{l}_1(\underline{z}) = \left[\frac{\log|z| + i\theta}{i\phi}\right]\left(\mathrm{mod}\ \frac{2\pi}{\phi}\right).$$

Hence $\underline{L}_c(\underline{z})$, which is defined to be the principal set of residues (mod $\frac{2\pi}{\phi}$), will lie in the strip containing all $\underline{w} = \underline{u} + \underline{iv}$ such that $\frac{-\pi}{\phi} < \underline{u} \le \frac{\pi}{\phi}$.

If $\underline{c} = \underline{e}$, we see that $\underline{l}_c(\underline{z}) = \log \underline{z}$. Hence $\underline{l}_c(\underline{z})$ will be called the general logarithm of $\underline{z}$ to the base $\underline{c}$, written $\log_c \underline{z}$. The principal logarithm shall be $\underline{L}_c(\underline{z})$ and be denoted by $\mathrm{Log}_c\ \underline{z}$.

We see that the whole $\underline{z}$-plane is mapped bi-uniquely by the function $\underline{w} = \underline{u} + \underline{iv} = \log \underline{z}$ upon any strip $(2\underline{k} - 1)\pi < \underline{v} \le (2\underline{k} + 1)\pi$, $(\underline{k} = 0, \pm1, \pm2, \ldots)$. All values of log $\underline{z}$ lying in any one such strip will be called a branch of log $\underline{z}$. Each branch is a regular function in the cut $\underline{z}$-plane, whose derivative is $1/\underline{z}$.

Similarly all values of $w = u + iv = \log_c z$, $c \neq 0$, $|c| \neq 1$, which lie in a strip $\frac{(2k - 1)\log|c|}{\log^2|c| + \phi^2} < v \leq \frac{(2k + 1)\log|c|}{\log^2|c| + \phi^2}$ ($k = 0, \pm 1, \pm 2, \ldots.$) will be called a branch of $\log_c z$. If $|c| = 1$, $\phi \neq 0$, then all the values of $w = u + iv = \log_c z$ which lie in a strip $(2k - 1)\pi/\phi < u \leq (2k + 1)\pi/\phi$ (where $k = 0, \pm 1, \pm 2, \ldots.$) will be called a branch of $\log_c z$. Each branch is a regular function of z in the cut z-plane whose derivative is $1/z\,\mathrm{Log}\, c$.

CHAPTER III

THE EXPONENTIAL FUNCTION

If $w = \log z$, and w is given, then it follows from our previous discussion that z is uniquely determined. Hence the inverse function of $w = \log z$ is a single valued function, say $E(w)$, which never vanishes.

Since the points $w + 2k\pi i$ ($k = 0, \pm 1, \pm 2, \ldots$) all correspond to the same value of z, $E(w + 2k\pi i) = E(w)$. Hence $E(w)$ is a periodic function of period $2\pi i$. Also by the addition theorem for $\log z$

$$E(w_1 + w_2) = E(\log z_1 + \log z_2) = E(\log z_1 z_2) =$$
$$z_1 z_2 = E(w_1) \cdot E(w_2).$$

If $w = \operatorname{Log} z$, then $z = E(w) = E(u + iv)$ where $-\pi < v \leq \pi$. Here w is a regular function, and hence $dw/dz = 1/z$, $z \neq 0$, and $dz/dw = z = E(w)$. Therefore $E(w)$ is a regular function in the strip $-\pi < v \leq \pi$. But since $E(w)$ is periodic, $E(w)$ is regular in the whole plane and $d(E(w))/dw = E(w)$. We found for real values of z that $\log 1 = 0 \pmod{2\pi i}$, hence $E(2k\pi i) = 1$, ($k = 0, \pm 1, \pm 2, \ldots$); in particular $E(0) = 1$, $\operatorname{Log} x$ (x real) is the ordinary logarithm of x whose inverse is e^x, hence $E(x) = e^x$. Therefore we shall designate $E(w)$ by e^w, where w can be any complex number. Hence $d(e^w)/dw = e^w$. Since the function is regular in the whole w-plane and $e^0 = 1$, we may write it in terms of the MacLaurin's expansion

$$e^w = 1 + w + w^2/2! + w^3/3! + \dots + w^n/n! + \dots .$$

Since this expansion converges for all values of w, w can take on any value. Suppose $w = iy$, then

$$e^{iy} = 1 + iy - y^2/2! - iy^3/3! + y^4/4! + iy^5/5! - \dots .$$

Now we know that the expansion of sin y, and cos y (y real) is

$$\sin y = y - y^3/3! + y^5/5! - y^7/7! + \dots ,$$

$$\cos y = 1 - y^2/2! + y^4/4! - y^6/6! + \dots .$$

By observation we see that $e^{iy} = \cos y + i \sin y$, or we may write $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$. Since, for y real,

$$e^{iy} = \cos y + i \sin y ,$$

$$e^{-iy} = \cos y - i \sin y .$$

Adding the two, we get

$$e^{iy} + e^{-iy} = 2 \cos y ,$$

$$\cos y = \frac{e^{iy} + e^{-iy}}{2} .$$

Now subtracting the two, we have

$$e^{iy} - e^{-iy} = 2i \sin y ,$$

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i} .$$

Next we shall find the inverse of $\log_c z$. For the same reasons as those given for log z, $\log_c z$ has an inverse. Call this inverse $E_c(z)$. To find the value of the inverse we set

$$w = \log_c z = \left[\frac{\log|z| + i\theta}{\log|c| + i\phi}\right] \left(\bmod \frac{2\pi(\phi + i \log|c|)}{\log^2|c| + \phi^2}\right).$$

Hence $\quad w(\log|c| + i\phi) = [\log|z| + i\theta] \pmod{2\pi i}$,

$$w \operatorname{Log} c = \log z ,$$

$$z = e^{w \operatorname{Log} c} = E_c(w) .$$

Since for $\underline{x}$ (real), $\underline{c} > 0$, $\underline{E}_c(\underline{w}) = \underline{E}_c(\underline{x}) = \underline{e}^{x \text{ Log } c} = \underline{c}^x$, we will designate $\underline{E}_c(\underline{w})$ by $\underline{c}^w = \underline{e}^{w \text{ Log } c}$.

If $\underline{c}^w = \underline{e}^{w \text{ Log } c}$, then taking the logarithm of both sides we have $\text{Log } \underline{c}^w = \underline{w} \text{ Log } \underline{c}$.

From the definition $\underline{c} = \underline{e}^{\log c}$, therefore $\underline{c}^w = (\underline{e}^{\text{Log } c})^w$. But since $\underline{c}^w = \underline{e}^{w \text{ Log } c}$, $(\underline{e}^{\log c})^w = \underline{e}^{w \text{ Log } c}$.

Next, $(\underline{c}^{w_1})^{w_2} = (\underline{e}^{w_1 \text{ Log } c})^{w_2} = \underline{e}^{w_1 w_2 \text{ Log } c} = \underline{c}^{w_1 w_2}$.

From the last chapter we found $\underline{d}(\log_c \underline{z})/\underline{dz} = 1/\underline{z} \cdot \text{Log } \underline{c}$, hence if $\underline{z} = \underline{c}^w$, $\underline{dz}/\underline{dw} = \underline{c}^w \cdot \text{Log } \underline{c}$.

CHAPTER IV

THE TRIGONOMETRIC FUNCTIONS

Arc tan z. We shall define the arc tan z, $z \neq i$, $z \neq -i$, as follows:

$$\text{arc tan } z = \int_0^z \frac{du}{1+u^2}.$$

The integrand has the poles i and $-i$, therefore the integral is independent of the path if we do not go around these points. Since we know the integral of $\int_1^t \frac{dt}{t}$, we shall make a transformation which will get the arc tan z in this form. We shall attempt to use a linear fractional transformation of the form $\frac{au+b}{cu+d}$ which will put one of the poles at zero, the other at infinity, and zero will go into one. Hence if $t = \frac{au+b}{cu+d}$, then for $u = i$, $t = 0$, or $0 = ia + b$ and $a = ib$. If $u = -i$, $t = \infty$ hence $-ci + d = 0$, or $c = -di$. Making these substitutions we have $t = \frac{b(iu+1)}{d(-iu+1)}$. Furthermore if $u = 0$, $t = 1$ or $1 = b/d$. Therefore the transformation we shall use is

$$t = \frac{1+iu}{1-iu}, \text{ or } \quad u = \frac{1}{i}\frac{t-1}{t+1}.$$

By this transformation we have transformed all the points above the axis of reals in the u-plane into points within the unit circle in the t-plane, and all the points below the axis of reals in the u-plane go into points outside the unit circle in the t-plane. The axis of reals in the u-plane is the unit circle in the t-plane, and the line segment from i to $-i$ in the

u-plane is the ray along the positive half of the real axis.

To integrate $\int_1^t \frac{dt}{t}$, we go along the axis of reals from 1 to $|t|$ and then along a circle $|t| = c$. In the u-plane this is the same as going from 0 to $-i\,\frac{|t| - 1}{|t| + 1}$ and then along the path found thus, if $u = x + iy$

$$c^2 = |t|^2 = \left|\frac{1 + iu}{1 - iu}\right|^2 = \frac{(1 - y)^2 + x^2}{(1 + y)^2 + x^2},$$

$$c^2 + 2c^2y + c^2y^2 + c^2x^2 = 1 - 2y + y^2 + x^2,$$

$$(c^2 - 1)x^2 + (c^2 - 1)y^2 + 2(c^2 + 1)y = 1 - c^2,$$

$$x^2 + y^2 + 2\left(\frac{c^2 + 1}{c^2 - 1}\right)y = -1,$$

$$x^2 + \left(y + \frac{c^2 + 1}{c^2 - 1}\right)^2 = -1 + \left(\frac{c^2 + 1}{c^2 - 1}\right)^2,$$

$$x^2 + \left(y + \frac{c^2 + 1}{c^2 - 1}\right)^2 = \frac{4c^2}{(c^2 - 1)^2}.$$

Hence we see that this path is a circle of radius $2c/(c^2 - 1)$ and center at $-i(c^2 + 1)/(c^2 - 1)$ which passes through the points $-i(|t| - 1)/(|t| + 1)$ and $u = z$. If $|t| > 1$, then $-i(|t| - 1)/(|t| + 1)$ lies on the line segment between 0 and $-i$. If $|t| < 1$, then $-i(|t| - 1)/(|t| + 1)$ lies on the line segment between 0 and i, also $-i(c^2 + 1)/(c^2 - 1)$ lies on the imaginary axis above i.

Making the transformation $z = (t - 1)/i(t + 1)$ in the integral we find that

$$\frac{du}{dt} = \frac{2}{i(t^2+1)^2}.$$

Hence

$$\int_0^z \frac{du}{1+u^2} = \int_1^t \frac{\frac{2}{i(t^2+1)^2}}{1+\left[\frac{t-1}{i(t+1)}\right]^2}\,dt = \frac{1}{2i}\int_1^t \frac{dt}{t},$$

$$\frac{1}{2i}\int_1^t \frac{dt}{t} = \frac{1}{2i}\log t = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right).$$

Hence $w = \arc\tan z = \left[\frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)\right] (\mathrm{mod}\ \pi)$.
The principal set of residues $\left[\frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)\right](\mathrm{mod}\ \pi)$ are all values of $w = u + iv$ such $-\pi/2 \leq u \leq \pi/2$. These values of arc tan z will be denoted by Arc tan z and called the principal arc tan z.

To find z we set

$$w = u + iv = \mathrm{Arc}\tan z = \frac{1}{2i}\mathrm{Log}\left(\frac{1+iz}{1-iz}\right),$$

$$2iw = \mathrm{Log}\left(\frac{1+iz}{1-iz}\right),$$

$$e^{2iw} - ize^{2iw} = 1 + iz,$$

$$z(ie^{2iw} + i) = e^{2iw} - 1,$$

$$\tan w = z = \frac{1}{i}\,\frac{e^{2iw}-1}{e^{2iw}+1}.$$

To prove the addition formula for the tangent of two complex numbers, we set

$$\tan(w_1 + w_2) = \frac{1}{i}\,\frac{e^{2i(w_1+w_2)}-1}{e^{2i(w_1+w_2)}+1}.$$

If we multiply numerator and denominator by 2 then add and subtract e^{2iw_1} and e^{2iw_2}, we get

$$\frac{1}{i}\left[\frac{2e^{2i(w_1+w_2)} - e^{2iw_1} + e^{2iw_1} - e^{2iw_2} + e^{2iw_2} - 2}{2e^{2i(w_1+w_2)} - e^{2iw_1} + e^{2iw_2} - e^{2iw_2} + e^{2iw_2} + 2}\right] =$$

$$\frac{1}{i}\,\frac{(e^{2iw_1}-1)(e^{2iw_2}+1) + (e^{2iw_1}+1)(e^{2iw_2}-1)}{(e^{2iw_1}+1)(e^{2iw_2}+1) + (e^{2iw_1}-1)(e^{2iw_2}-1)},$$

dividing numerator and denominator by $(e^{2iw_1} + 1)(e^{2iw_2}+1)$

$$\frac{\frac{1}{i}\left(\frac{e^{2iw_1}-1}{e^{2iw_1}+1}\right)+\frac{1}{i}\left(\frac{e^{2iw_2}-1}{e^{2iw_2}+1}\right)}{1-\frac{1}{i}\,\frac{e^{2iw_1}-1}{e^{2iw_1}+1}\cdot\frac{1}{i}\,\frac{e^{2iw_2}-1}{e^{2iw_2}+1}} =$$

$$\frac{z_1+z_2}{1-z_1z_2} = \frac{\tan w_1+\tan w_2}{1-\tan w_1\cdot\tan w_2}.$$

From the definition of $w = \text{arc}\tan z$ we see that $\tan w$ is a periodic function of period π. If we make a cut along iy, $y > 1$, and iy, $y < -1$ (which corresponds to a cut in the t-plane along the negative real axis), $w = u + iv = \text{Arc}\tan z = \int_0^z \frac{du}{1+u^2}$ is independent of the path of integration. In this cut plane w is a regular function, and hence $dw/dz = 1/(1+z^2)$, $z \neq i$, $z \neq -i$, hence $dz/dw = 1 + z^2 = 1 + \tan^2 w$, in the strip $-\pi/2 < u \leq \pi/2$ of the w-plane except for the pole $w = \pi/2$. But since $\tan w$ is a periodic function it will be analytic in the whole plane except for the poles $w = (2k+1)\pi/2$.[2] These points are simple poles since $1/\tan w = 0$ and $\frac{d}{dw}\left(\frac{1}{\tan w}\right) \neq 0$.

Arc sin z. We next define the arc sin z for $z \neq 1$ and $z \neq -1$.

$$\text{arc}\sin z = \int_0^z \frac{du}{\sqrt{1-u^2}}.$$

We shall attempt to get this integral in the form $\int_1^t \frac{dt}{t}$ by the transformation $t = iu + \sqrt{1-u^2}$[1]. By this transformation we have

$$u = (t^2-1)/2it \quad \text{and} \quad \sqrt{1-u^2} = (t^2+1)/2t.$$

Therefore $\displaystyle\int_0^z \frac{du}{\sqrt{1-u^2}} = \int_1^t \frac{\frac{1}{i}\,\frac{t^2+1}{2t^2}}{\frac{t^2+1}{2t}}\,dt = \frac{1}{i}\int_1^t \frac{dt}{t} =$

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[1] Ludwig Bieberbach, Lehrbuch Der Funktionentheorie, p. 113.

[2] Here k is any integer.

$$\frac{1}{i}\int_1^t \frac{dt}{t} = \frac{1}{i}\log t = \frac{1}{i}\log\left(iz + \sqrt{1-z^2}\right).$$

Now $\sqrt{1 - z^2}$ has two values; hence $\{\log(iz + \sqrt{1 - z^2})\}/i$ has two corresponding values. If $s = iz + \sqrt{1 - z^2}$, then $-1/s = iz - \sqrt{1 - z^2}$. The $\log(-1/s) = (2k + 1)\pi i - \log s$. Hence we see that the sum of the two evaluations differ by an odd multiply of π. Hence if one value is w_1, the other value will be $w_2 = (2k + 1)\pi - w_1$. Therefore we find that $w = \text{arc} \sin z = [\{\log(iz + \sqrt{1 - z^2})\}/i]$ (mod 2π) and also $[\pi - \{\log(iz + \sqrt{1 - z^2})\}/i]$ (mod 2π). If $w = u + iv$, then the principal value of arc sin z = Arc sin z and will be the principal set of residues mod 2π such that $-\pi < u \leq \pi$. Since for any given value of w, $[\frac{1}{i}\log(iz + \sqrt{1 - z^2})]$ (mod 2π) determines one value of z, therefore we may find z. We see that if

$$w = \{\text{Log}(iz + \sqrt{1 - z^2})\}/i,$$

then

$$iw = \text{Log}(iz + \sqrt{1 - z^2}),$$

$$e^{iw} = iz + \sqrt{1 - z^2},$$

$$e^{iw} - iz = \sqrt{1 - z^2},$$

$$e^{2iw} - 2ize^{iw} - z^2 = 1 - z^2,$$

$$2ize^{iw} = e^{2iw} - 1,$$

$$\sin w = z = \frac{e^{2iw} - 1}{2i} = \frac{e^{iw} - e^{iw}}{2i}.$$

The sin w is regular in the entire w-plane, and is zero for $w = k\pi$ (where $k = 0, \pm 1, \pm 2, \ldots$). By the definition of arc sin z, sin w is a periodic function of period 2π.

We define

$$\cos w = \sin(w + \pi/2).$$

Hence we find that

$$\cos w = \frac{e^{i(w+\frac{\pi}{2})} - e^{-i(w+\frac{\pi}{2})}}{2i} = \frac{e^{iw} + e^{-iw}}{2} .$$

We see that the derivative of sin $\underline{w}$ is the cos $\underline{w}$, and the derivative of the cos $\underline{w}$ is minus the sin $\underline{w}$.

If we divide sin $\underline{w}$ by cos $\underline{w}$, we have tan $\underline{w}$, for

$$\frac{\sin w}{\cos w} = \frac{e^{iw} - e^{-iw}}{2i} \cdot \frac{2}{e^{iw} + e^{-iw}} =$$

$$\frac{1}{i} \; \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} = \tan w .$$

Next we find that $\sin^2 \underline{w} + \cos^2 \underline{w} = 1$, since

$$\sin^2 w = \frac{e^{2iw} - 2 + e^{-2iw}}{-4} ,$$

$$\cos^2 w = \frac{e^{2iw} + 2 + e^{-2iw}}{4} .$$

Adding the two, we get

$$\frac{-e^{2iw} + 2 - e^{-2iw} + e^{2iw} + 2 + e^{-2iw}}{4} = 1 .$$

Hence $\sin^2 w + \cos^2 \underline{w} = 1$.

To examine the sine of the sum of two angles we set

$$\sin(x+y) = \frac{e^{i(x+y)} - e^{-i(x+y)}}{2i} ,$$

but

$$\sin x \cos y = \frac{e^{ix} - e^{-ix}}{2i} \cdot \frac{e^{iy} + e^{-iy}}{2} =$$

$$\frac{e^{i(x-y)} - e^{i(y-x)} + e^{i(x+y)} - e^{-i(x+y)}}{4i}$$

and

$$\cos x \sin y = \frac{e^{ix} + e^{-ix}}{2} \cdot \frac{e^{iy} - e^{-iy}}{2i} =$$

$$\frac{e^{i(x+y)} - e^{i(x-y)} + e^{i(y-x)} - e^{-i(x+y)}}{4i} .$$

If we add sin **x** cos **y** and cos **x** sin **y**, we get

$$\underline{e^{i(x+y)}+e^{i(x-y)}-e^{i(y-x)}-e^{-i(x+y)}+e^{i(x+y)}-e^{i(x-y)}+e^{i(-y-x)}-e^{-i(x+y)}} =$$

$$\frac{2e^{i(x+y)}-2e^{-i(x+y)}}{4i} = \frac{e^{i(x+y)}-e^{i(x+y)}}{2i}.$$

Hence $\sin(x+y) = \cos x \sin y + \sin x \cos y$.

Now we know that

$$\cos(x+y) = \frac{e^{i(x+y)}+e^{-i(x+y)}}{2}.$$

Also $\cos x \cos y = \frac{e^{ix}+e^{-ix}}{2} \cdot \frac{e^{iy}+e^{-iy}}{2} =$

$$\frac{e^{i(x+y)}+e^{i(x-y)}+e^{i(y-x)}+e^{-i(x+y)}}{4},$$

and $\sin x \sin y = \frac{e^{ix}-e^{-ix}}{2i} \cdot \frac{e^{iy}-e^{-iy}}{2i} =$

$$\frac{e^{i(x+y)}-e^{i(x-y)}-e^{i(y-x)}+e^{-i(x+y)}}{-4}.$$

By subtracting sin **x** sin **y** from cos **x** cos **y**, we get

$$\frac{e^{i(x+y)}+e^{i(x-y)}+e^{i(y-x)}+e^{-i(x+y)}+e^{i(x+y)}-e^{i(x-y)}-e^{i(y-x)}+e^{i(x+y)}}{4} =$$

$$\frac{2e^{i(x+y)}+2e^{-i(x+y)}}{4} \; . \; \frac{e^{i(x+y)}+e^{-i(x+y)}}{2} = \cos(x+y).$$

In brief we have

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

CHAPTER V

THE HYPERBOLIC FUNCTIONS

__Arc__ __tanh__ __z__. For $z \neq 1$ and $z \neq -1$, we define

$$\text{arc tanh } z \int_0^z \frac{du}{1-u^2}.$$

By means of the transformation $u = it$ we have

$$\int_0^z \frac{du}{1-u^2} = i\int_0^t \frac{dt}{1+t^2} = \frac{1}{2}\log\left(\frac{1+it}{1-it}\right).$$

Hence $w = \text{arc tanh } z = \left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right] (\text{mod } \pi i)$ or arc tanh $z = i$ arc $\tan(z/i)$.

The principal set of residues of $\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right](\text{mod } \pi i)$ are those values of $w = u + iv = \text{arc tanh } z$ such that $-\pi/2 < v \leq \pi/2$.

If $w = u + iv = \text{arc tanh } z = i \text{ arc } \tan(z/i)$, then

$$w/i = \text{arc } \tan(z/i).$$

Hence

$$z/i = \tan(w/i) = \frac{1}{i}\,\frac{e^w - e^{-w}}{e^w + e^{-w}},$$

$$z = \tanh w = \frac{e^w - e^{-w}}{e^w + e^{-w}}.$$

From the definition of arc tanh z, tanh z is a periodic function of period πi. Since $w = \text{arc tanh } z = \int_0^z \frac{du}{1+u^2}$, $dw/dz = 1/(1 - z^2)$ and $dz/dw = 1 - z^2 = 1 - \tanh^2 z$. If $w = (2k + 1)\pi i$, then $1/\tanh w = 0$, but $\frac{d}{dw}\left(\frac{1}{\tanh w}\right) \neq 0$. Hence at these points tanh w has simple poles.

Arc sinh z. We define for $z \neq i$ and $z \neq -i$

$$\operatorname{arc\,sinh} z = \int_0^z \frac{du}{\sqrt{1+u^2}} .$$

Making the substitution $u = it$ we get

$$\operatorname{arc\,sinh} z = \int_0^z \frac{du}{\sqrt{1+u^2}} = i\int_0^t \frac{dt}{\sqrt{1-t^2}} =$$

$$\log(it + \sqrt{1-t^2}) = \log(z + \sqrt{1+z^2}) .$$

Therefore $w = \operatorname{arc\,sinh} z = \left[\log(z + \sqrt{1+z^2})\right] (\mathrm{mod}\ 2\pi i)$

$\left[\pi i - \log(z + \sqrt{1+z^2})\right] (\mathrm{mod}\ 2\pi i)$.

If $w = u + iv$, the the principal set of residues (mod $2\pi i$) such that $-\pi < v \leq \pi$, is the principal value of the arc sinh z, and is denoted by Arc sinh z.

Since $w = \operatorname{arc\,sinh} z = i \operatorname{arc\,sin}(z/i)$, $w/i = \operatorname{arc\,sin}(z/i)$. Therefore $\sin(w/i) = z/i$, and $z = \sinh w = i \sin(w/i)$. It is easily seen that $\sinh w$ is a periodic function of period $2\pi i$. To find the derivative of sinh w, we set $w = \operatorname{arc\,sinh} z = \int_0^z \frac{du}{\sqrt{1+u^2}}$ hence $dw/dz = 1/\sqrt{1+z^2}$ and $dz/dw = \sqrt{1+z^2} = \sqrt{1+\sinh^2 w}$.

Since for real x $\cos(x/i) = \cosh x$, we define

$$\cos(w/i) = \cosh w .$$

From this we see that $\sinh(z + \pi i/2) = i \cosh z$.

We have the following formulas for hyperbolic functions.

(1) $\cosh^2 x - \sinh^2 x = 1$

For $\cosh^2 x = \dfrac{e^x + e^{-x}}{2} \cdot \dfrac{e^x + e^{-x}}{2} = \dfrac{e^{2x} + e^{-2x} + 2}{4}$,

and $\sinh^2 x = \dfrac{e^x - e^{-x}}{2} \cdot \dfrac{e^x - e^{-x}}{2} = \dfrac{e^{2x} + e^{-2x} - 2}{4}$,

hence $\cosh^2 x - \sinh^2 x = 1$.

(2) $\sinh 2x = 2 \sinh x \cos x$.

For $\sinh 2x = \dfrac{e^{2x}-e^{-2x}}{2} = 2\left(\dfrac{e^{x}-e^{-x}}{2}\right)\left(\dfrac{e^{x}+e^{-x}}{2}\right) = 2 \sinh x \cos x$.

(3) $\cosh 2x = \cosh^2 x + \sinh^2 x$.

For by adding the two in (1) we get

$$\cosh^2 x + \sinh^2 x = \frac{e^{2x}+e^{-2x}+2+e^{2x}+e^{-2x}-2}{4} =$$

$$\frac{e^{2x}+e^{-2x}}{2} = \cosh 2x .$$

(4) $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$.

Since $\sin(x + iy) = \sin x \cos iy + \cos x \sin iy$,

but $\cos iy = \dfrac{e^{-y}+e^{y}}{2} = \cosh y$,

$\sin iy = \dfrac{e^{-y}-e^{y}}{2i} = i \sinh y$,

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y.$$

(5) $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$.

Since $\cos(x + iy) = \cos x \cos iy - \sin x \sin iy$,

but $\cos iy = \cosh y$ and $\sin iy = i \sinh y$,

$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y.$$

(6) $\tan(x + iy) = \dfrac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}$.

For $\tan(x + iy) = \dfrac{\sin(x+iy)}{\cos(x+iy)} =$

$$\frac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y} =$$

$$\frac{\sin x \cos x \cosh^2 y - \cos x \sin x \sinh^2 y + i(\sin^2 x \cosh y \sinh y + \cos^2 x \sinh y \cosh y)}{\cos^2 x \cosh^2 y + \sinh^2 y \sin^2 x}$$

$$\frac{2 \sin x \cos x (\cosh^2 y - \sinh^2 y) + i\, 2 \sinh y \cosh y (\sin^2 x + \cos^2 x)}{(\cos^2 x - \sin^2 x)(\cosh^2 y - \sinh^2 y) + (\cos^2 x + \sin^2 x)(\cosh^2 y + \sinh^2 y)} =$$

$$\frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y} .$$

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