To \textit{VMD}, or \textit{not} to \textit{VMD}, in the quark-gluon plasma\footnote{Talk presented At "International Workshop on Chiral Dynamics in Hadrons and Nuclei", Seoul, Korea, February 6–10, 1995.}

Robert D. Pisarski

Physics Department
Brookhaven National Laboratory
P.O. Box 5000
Upton, NY 11973-5000, USA

\textbf{DISCLAIMER}

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

\footnote{Talk presented At "International Workshop on Chiral Dynamics in Hadrons and Nuclei", Seoul, Korea, February 6–10, 1995.}

This manuscript has been authored under contract number DE-AC02-76CH00016 with the U.S. Department of Energy. Accordingly, the U.S. Government retains a non-exclusive, royalty-free license to publish or reproduce the published form of this contribution, or allow others to do so, for U.S. Government purposes.
DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.
To VMD, or not to VMD, in the quark-gluon plasma

ROBERT D. PISARSKI
Dept. of Physics, Brookhaven National Laboratory
Upton, NY, 11973, USA

Abstract
I review results on the shift of the $\rho$ meson mass at nonzero temperature in a gauged linear sigma model, under the assumption of strict vector meson dominance.

1 Toy model

In these proceedings I explain the calculations performed in [1]. The question I address is how the masses of particles, and especially vector mesons, shift in a hadronic system at nonzero temperature. (While the case of nonzero quark density can be treated by analogous methods, I am not certain that the results are qualitatively similar.) Light vector mesons — such as the $\rho$, $\omega$, and $\phi$ — are of interest because they of their coupling to low mass dileptons. For reasons which will become clear, most of my results only apply to the $\rho$ meson.

Instead of treating vector mesons directly, however, in this section I consider a toy model which turns out to exemplify much of the important physics. Consider a heavy scalar field, $\Phi$, coupled to a light scalar field, $\Phi$. The $\Phi$ field is an $O(4)$ isovector, and represents a single $J^P = 0^+$ scalar, the $\sigma$, and an isotriplet of $J^P = 0^-$ $\pi$'s: $\Phi = (\sigma, \pi)$. I assume that the $\Phi$ field is very heavy, so that only fluctuations in the $\Phi$ field matter.

Let the potential terms for the $\Phi$ and $\Phi$ fields be

$$V = \frac{m^2}{2} \Phi^2 + \frac{\kappa}{4} |\Phi|^2 \Phi^2 - \frac{1}{2} \mu^2 |\Phi|^2 + \frac{\lambda}{4} (|\Phi|^2)^2.$$  \hfill (1.1)

If the $O(4)$ symmetry is spontaneously broken at zero temperature by $\mu^2 > 0$, the $\sigma$ field acquires a vacuum expectation value, $\sigma_0^2 = \mu^2/\lambda$. After expanding $\sigma \to \sigma_0 + \sigma$, the effects on the $\Phi$ field are standard: the sigma field becomes massive, $m_\sigma^2 = 2\lambda\sigma_0^2$, there is a trilinear coupling $\sigma\pi^2$, of strength $-2\lambda\sigma_0\delta^{ab}$ (the $\delta^{ab}$ is for the isospin of
the pions) and a quartic coupling \( \sigma^2 \bar{\pi}^2 \), of strength \(-2\lambda\delta^{ab}\). (I work with euclidean conventions, so the minus signs are due to expanding \( \exp(-S) \), where \( S \) is the action, in the path integral.)

Similarly, the terms in the potential from the interaction between the two scalar fields is

\[
L = \frac{\kappa}{4} \left( \sigma_0^2 + 2\sigma_0\sigma + \sigma^2 + \bar{\pi}^2 \right) \bar{\Phi}^2 .
\]

After this shift, the mass of the \( \tilde{\Phi} \) field is

\[
m_{\tilde{\Phi}}^2 = \bar{m}^2 + \frac{\kappa}{2} \sigma_0^2 .
\]

Hence the interaction with the \( \Phi \) induces a shift in the \( \tilde{\Phi} \) mass, \( \delta m_{\tilde{\Phi}}^2(0) = \kappa\sigma_0^2/2 \). To higher order, there is also a trilinear coupling \( \sigma\bar{\pi}^2 \) of strength \(-\kappa\sigma_0 \), and two quartic couplings, \( \sigma^2 \bar{\Phi}^2 \), of strength \(-\kappa \), and \( \bar{\pi}^2 \bar{\Phi}^2 \), of strength \(-\kappa\delta^{ab} \).

Having sorted out all the proper conventions of signs and normalization, it is then straightforward to compute the thermal shifts in the effective mass of the \( \tilde{\Phi} \) field.

I start with the limit of very low temperatures. At very low temperatures, the only fluctuations are those of pions, since fluctuations in \( Q \) are suppressed by Boltzmann factors of \( e^{-\mu/T} \).

Then there are two diagrams: the quartic interaction with two pions gives

\[
-\Pi_1 = \frac{3}{2} \kappa \mathrm{tr} \frac{1}{K^2} = -\kappa \frac{T^2}{8} .
\]

Here \( K^2 = k_0^2 + k^2 \), \( k_0 = 2\pi n T \) for a bosonic integral, and \( \mathrm{tr} = T \sum_{n=-\infty}^{\infty} \int d^3 k/(2\pi)^3 \). This is a contribution to minus the self energy for the \( \Phi \) field (the minus sign comes from bringing a mass term back into the action), and so shifts the \( \Phi \) mass \( \up \). There is a factor of 3 from three types of pions, and a 1/2 for the symmetry factor of the diagram.

There is a second contribution from a tadpole diagram involving a \( \sigma \) meson propagating at zero momentum,

\[
-\Pi_2 = (-\kappa\sigma_0)(-2\lambda\sigma_0) \left( \frac{1}{m_0^2} \right) \frac{3}{2} \kappa \mathrm{tr} \frac{1}{K^2} = +\kappa \frac{T^2}{8} .
\]

Evidently, these two contributions cancel identically, \( \Pi_1 + \Pi_2 = 0 \).

Thus we have the surprising result that the mass shift in the \( \tilde{\Phi} \) field vanishes to \( \sim T^2 \). This is the simplest example I know of a general result due to Eletsky and Ioffe [2], who argued that to \( \sim T^2 \), there is no shift in the pole masses for any field. In more complicated examples, such as the vector mesons, one has to take care to compute the shift in the pole masses, and not simply the self energies at zero momentum. In the scalar example considered here, each diagram is independent of the external momentum, and so the difference doesn’t matter.
One could continue to compute the shift in the thermal $\tilde{\phi}$ mass to higher order. Including pion loops order by order in perturbation theory, such a calculation is an expansion in $\lambda(1/m_\pi^2)tr1/K^2 \sim T^2/\sigma_0^2$. The term of order $\sim T^4/\sigma_0^2$ is relatively simple to compute, and I leave it as an exercise to the reader. Notice that in this example, the only terms of $\sim T^4$ arise from diagrams at two loop order; there are no terms of $\sim T^2$ at one loop order.

Ultimately, we are not interested simply in the low temperature regime, but in what happens at the temperature of chiral symmetry restoration for the $\Phi$ field. We can easily compute at this temperature, since then $\sigma_0 = 0$, so we can work in the chirally symmetric phase. The calculation is trivial, because only quartic vertices contribute. Instead of just three $\pi$'s in the loop, we have three $\pi$'s plus one $\sigma$, so in all the self energy is

$$-\Pi(T > T_\chi) = -\frac{4}{2} \kappa \text{tr} \frac{1}{K^2} = -\kappa \frac{T^2}{6}. \quad (1.6)$$

Now I use a trick: for an $O(4)$ field, $T^2_\chi = 2\sigma_0^2$, so I can trade $T_\chi$ for $\sigma_0$. In this way, I find that the shift in the $\tilde{\phi}$ mass at $T_\chi$, $\Pi(T_\chi) = \delta m_{\tilde{\phi}}^2(T_\chi)$ is related to its shift at zero temperature, $\delta m_{\tilde{\phi}}^2(0) = \kappa \sigma_0^2/2$, as

$$\delta m_{\tilde{\phi}}^2(T_\chi) = \frac{2}{3} \delta m_{\tilde{\phi}}^2(0). \quad (1.7)$$

Amusingly enough, this relation is independent of the value of the coupling constant, $\kappa$. This is really a trick of the one loop result, and is not general: to higher loop order, further powers of $\lambda$ enter, and invalidate this simple form. Nevertheless, it is notable to find that at least in weak coupling, the shift in the thermal masses can be as large as the shifts at zero temperature. This is hardly surprising, and so is probably true outside of the weak coupling regime.

## 2 Vector meson dominance

Going onto vector fields, I begin by constructing the coupling of vector mesons to the $\Phi$ field in the standard manner, following the assumption of vector meson dominance (VMD) [3]. As I shall show, it turns out that this assumption is crucial to the results which follow. With vector meson dominance, unique predictions follow; without VMD, there is no unique prediction.

Introducing the matrices $t^0 = 1/2$ and $t^a$, $\text{tr}(t^a t^b) = \delta^{ab}/2$, the scalar field $\Phi$ is

$$\Phi = \sigma t^0 + i\vec{\pi} \cdot \vec{t}. \quad (2.8)$$
For the left and right handed vector fields I take

\[ A_{\mu}^{\pm} = (\omega^\mu \pm f_1^\mu)\xi^0 + (\vec{\rho}^\mu \pm \vec{a}_1^\mu) \cdot \vec{r}, \]  

(2.9)

where \( \omega \) and \( \vec{\rho} \) are \( J^P = 1^- \) fields, and \( f_1 \) and \( \vec{a}_1 \) are \( J^P = 1^+ \) fields. VMD [3] tells us to construct an effective lagrangian by coupling the vector fields to themselves and to \( \Phi \) exclusively through the dimensionless couplings which follow by promoting the global chiral symmetry to a local symmetry. Introducing the coupling constant \( g \) for vector meson dominance, the appropriate covariant derivative and field strengths are

\[ D^\mu \Phi = \partial^\mu \Phi - ig(A_{\mu}^\pm \Phi - \Phi A^\pm) \]

and

\[ F_{\mu\nu}^{\alpha\beta} = \partial^\mu A_{\nu}^{\alpha} - \partial^\nu A_{\mu}^{\alpha} - ig[\Phi^{\alpha\beta}, A^\pm]. \]

The effective lagrangian is then

\[ L = tr \left( |D^\mu \Phi|^2 - \mu^2 |\Phi|^2 + \lambda(|\Phi|^2)^2 - 2h t^0 \Phi \right) \]

\[ + \frac{1}{4} (F^{\mu\nu}_{\alpha\beta})^2 + \frac{1}{4} (F_{\mu\nu}^{\alpha\beta})^2 + \frac{m^2}{2} \left( (A^{\alpha\beta})^2 + (A^{\alpha\beta})^2 \right) \].  

(2.10)

Besides the parameters \( \mu^2 \) and \( \lambda \) introduced previously, there is \( g \), the coupling for VMD, a background field \( h \), to make the pions massive, and a mass term \( \sim m^2 \) for the gauge fields. The essential part of VMD is that the local gauge symmetry is only broken by an explicit mass term for the vector fields. Much of the physics of this lagrangian can be understood from the kinetic term for the scalar field,

\[ tr \left( |D^\mu \Phi|^2 \right) = \]

\[ \frac{1}{2} \left( (\partial^\mu \sigma + g \vec{a}_1 \cdot \vec{\pi})^2 + (\partial^\mu \vec{\pi} + g \vec{\rho} \times \vec{\pi} - g \vec{a}_1 \sigma)^2 + g^2 \left( \sigma^2 + \vec{\pi}^2 \right) (f_1^\mu)^2 \right) \]

(2.11)

Because it couples to the (isosinglet) current for fermion number, there are no interactions for the \( \omega^\mu \) field. There are interactions of \( \omega^\mu \) due to effects of the anomaly, but these are neglected in this work.

VMD really is a peculiar assumption. At first it sounds most reasonable: the local gauge symmetry is valid in the limit of high energies, and so controls the behavior of the dimensionless coupling constants. At low energies, the symmetry is broken by the presence of a mass term for the gauge fields. But this explanation doesn’t really make sense. One the mass term is added for the gauge fields, the ultraviolet behavior of the gauge fields is dramatically altered, so that the longitudinal terms in the gauge propagator don’t fall off as they should, like \( 1/P^2 \), but only as

\[ (\delta^{\mu\nu} - P^\mu P^\nu/m^2)/(P^2 + m^2) \sim -(P^\mu P^\nu/P^2)(1/m^2). \]

In this work and elsewhere [1] I assume that VMD works as is commonly assumed. An interesting exercise would be to test, experimentally, precisely how well VMD works. That is, parametrize the most general lagrangian consistent with the
global chiral symmetry. This will involve many more (dimensionless) coupling constants than \( VMD \), which only involves one dimensionless coupling constant. Bounds on the deviations of these coupling constants from \( VMD \) could then be obtained by comparison with the data from, say, dilepton production.

There are many terms which respect the global chiral symmetry, and yet violate \( VMD \). A simple example is

\[
\mathcal{L}_\xi = \xi \text{tr}(|\Phi|^2) \text{tr}((A_\mu^i)^2 + (A_\mu^a)^2),
\]

where \( \xi \) is a dimensionless coupling constant. This term obviously affects the masses for the vector mesons. Including it, the masses of the \( \rho \) and \( a_1 \) mesons are, in a phase with \( \sigma_0 \neq 0 \),

\[
m_\rho^2 = m^2 + \xi \sigma_0^2, \quad m_{a_1}^2 = m^2 + \xi \sigma_0^2 + g^2 \sigma_0^2. \tag{2.13}
\]

This illustrates a problem in deciphering terms which respect \( VMD \) from those which do not. At zero temperature, for both the \( \rho \) and the \( a_1 \) only the combination \( m^2 + \xi \sigma_0^2 \) enters into the masses. Thus one can’t tell, from that alone, if the mass of the \( \rho \) arises from an explicit mass term, \( \sim m \), or dynamically, from a coupling such as \( \sim \xi \sigma_0^2 \). Notice that one still needs the \( VMD \) coupling \( \sim g^2 \) to split the \( a_1 \) from the \( \rho \).

In principle, the effects of the term \( \sim \xi \) could be determined indirectly. Including the effects of the \( L_\xi \), the current which couples to the photon is

\[
j^\mu = \frac{m^2 + \xi |\Phi|^2}{g^2} \rho_3^\mu. \tag{2.14}
\]

The term \( \sim m^2 \) is the standard term of \( VMD \); the second, \( \sim \xi \), is new. Now \( |\Phi|^2 = \sigma_0^2 + \ldots \), so to lowest order its like the masses: only the combination \( m^2 + \xi \sigma_0^2 \) enters, and only can’t tell \( m^2 \) from \( \xi \sigma_0^2 \). But there is also a term in the current \( \sim \xi \bar{\pi}^2 \rho^\mu \), which opens up above the \( \rho \pi \pi \) threshold. A bound could be placed on \( \xi \) from the absence of such an increase in the dilepton rate above this threshold. As a multiparticle state this is a smooth threshold, but it should be measurable (or not) nevertheless.

### 3 \( VMD \) at high temperature

The analysis of gauged linear sigma models at nonzero temperature is elementary, at least in perturbation theory; for earlier works, see [4,5]. Of course one wishes to use the model in a regime of strong coupling, but perhaps the qualitative features of a weak coupling analysis are correct.

First I deal with the analysis of the mass shift at the point of phase transition. This case is especially simple to consider, because when \( \sigma_0 = 0 \), many trilinear vertices
proportional to \( \sigma_0 \) vanish. Also, mixing between \( \partial^\mu \pi \) and \( \sigma_0 \alpha_1^\mu \), which is implicit from Eq. (2.11), vanishes.

A technical comment about the self energies of massive vector bosons is in order. For a massive vector boson of mass \( m \) and momentum \( P \), at tree level the inverse propagator is

\[
\Delta^{-1}_{\mu\nu} = \delta^{\mu\nu} \left( P^2 + m^2 \right) - P^\mu P^\nu .
\]  

(3.15)

For a gauge vector boson, \( m = 0 \), the propagator has a zero mode, \( P^\mu \Delta^{-1}_{\mu\nu} = 0 \), which is eliminated by gauge fixing. For a massive vector boson, when \( m \neq 0 \) there is no zero mode in \( \Delta^{-1}_{\mu\nu} \), and so the propagator is trivially obtained simply by inverting the inverse propagator. At tree level,

\[
\Delta^{\mu\nu} = \left( \delta^{\mu\nu} - \frac{P^\mu P^\nu}{m^2} \right) \frac{1}{P^2 + m^2} .
\]  

(3.16)

Going on to include loop effects, at zero temperature the most general form of the self energy involves two functions, proportional to \( \delta^{\mu\nu} \) and \( P^\mu P^\nu \). The physical modes of the propagator can be shown to be uniquely determined by the coefficient of the inverse propagator proportional to \( \delta^{\mu\nu} \).

At nonzero temperature, in general things are more complicated. The self energy involves four independent functions, \( \delta^{\mu\nu} \), \( P^\mu P^\nu \), \( n^\mu n^\nu \), and \( P^\mu n^\nu + n^\mu P^\nu \), where \( n^\mu = (1, 0, 0, 0) \). Since without gauge invariance the vector propagator is immediately invertible, this causes no complication. One can compute the general form of the propagator as a function of these four functions in the self energy. Of course the coefficient of the propagator \( \sim \delta^{\mu\nu} \) is easiest to compute, since it is just the inverse of the coefficient of \( \sim \delta^{\mu\nu} \) in the inverse propagator. The full calculation shows that, assuming no untoward cancellations, a pole in the term in the propagator \( \sim \delta^{\mu\nu} \) is a pole in all of its coefficients. Consequently, for simplicity sake I limit myself to this term.

There is a second complication of more significance. In general, the vector boson self energy, \( \Pi(P) \), is a function of the momentum \( P \). While the limit at zero momentum, \( \Pi(0) \), is of interest for determining the correlation functions at large distances, the effects upon couplings to dileptons are governed by the behavior of the propagator on the mass shell. To lowest order, this is given by \( \Pi(-M^2) \), where \( M \) is the mass at tree level.

With these preliminaries aside, in the symmetric phase, \( T \geq T_x \), the self energies of the vector bosons are:

\[
\Pi^\mu_{\rho}(P) = \Pi_{\alpha_1}^{\mu\nu}(P) = \left( \frac{g^2 + \xi}{6} \right) T^2 \delta^{\mu\nu} + \ldots ,
\]  

(3.17)
\[ \Pi^{\mu\nu}(P) = \left(\frac{\xi}{6}\right) T^2 \delta^{\mu\nu} + \ldots, \]  
\[ \Pi^{\mu\nu}_{f_1}(P) = \left(\frac{2g^2 + \xi}{6}\right) T^2 \delta^{\mu\nu} + \ldots, \]

where the terms neglected are proportional to \( P^\mu P^\nu \).

In general, all masses shift with temperature. There is a single exception, which may be of experimental significance. VMD implies that \( g \neq 0, \xi = 0 \). In this case, from (3.18) we see that the \( \omega \) meson mass does \textit{not} shift with temperature. This is due to the form of the covariant coupling for the \( \omega \) meson in (2.11), where the \( \omega \) meson decouples from the kinetic term for the scalar field.

Indeed, one can turn this observation around. \textit{If the \( \omega \) meson mass shifts with temperature, then it must be due to terms which violate vector meson dominance.}

Now in this paper I assume the VMD holds. But since the \( \omega \) meson, being a narrow resonance, is relatively simple to pick out of the high multiplicity environment of ultrarelativistic heavy ion collisions, \textit{any} shift in its mass would be of enormous interest.

A priori, if one is willing to abandon VMD, one cannot predict whether the mass of the \( \omega \) goes up, or down, with temperature. In the limit that \( m^2 = 0 \), then the \( \rho \) and the \( \omega \), and the \( a_1 \) and the \( f_1 \), move together uniformly with temperature. At the point of phase transition, all four fields have the same effective mass:

\[ m^2_p(T_x) = m^2_{a_1}(T_x) = m^2_{\omega}(T_x) = m^2_{f_1}(T_x)|_{m^2=0} = \frac{2}{3} m^2_\rho(0) = (629 \text{ MeV})^2. \]  

This relation is analogous to (1.7), where the coupling constant \( \xi \) has been eliminated.

On the other hand, if \( m^2 \neq 0 \) and \( \xi \neq 0 \), then the \( \omega \) mass could be greater at \( T_x \) than at zero temperature. In this case, there is no simple relation between \( m^2_{\omega}(T_x) \) and \( m^2_{\rho}(0) \).

If VMD holds, so \( m^2 \neq 0 \) and \( \xi = 0 \), then

\[ m^2_p(T_x)|_{\xi=0} = m^2_{a_1}(T_x)|_{\xi=0} = \frac{1}{3} \left( 2m^2_\rho(0) + m^2_{a_1}(0) \right) = (962 \text{ MeV})^2, \]
\[ m^2(\omega)(T_x)|_{\xi=0} = m^2_{\omega}(0) = (782 \text{ MeV})^2, \]
\[ m^2_{f_1}(T_x)|_{\xi=0} = \frac{1}{3} \left( m^2_\rho(0) + 2m^2_{a_1}(0) \right) = (1120 \text{ MeV})^2. \]  

The dependence of the mass shifts on the choice of lagrangian helps to relate these results with those of other authors, especially those of Brown and Rho [6]. Following Georgi [7], Brown and Rho work in a nonlinear sigma model, so the lagrangian includes \( f_\pi^2 |D_\mu U|^2 \) for the nonlinear pion field \( U \) and a mass term for the vector fields...
as in (2.10). Georgi takes \( m^2 = \beta f^2 \), where \( \beta \) is a dimensionless coupling constant. Thus when \( f_\pi \) vanishes, as it does for \( T \geq T_\chi \), the masses for the \( \rho, \omega, a_1 \), and \( f_1 \) do as well, at least at tree level. My difficulty is that I do not understand why in this effective theory, the dimensional coupling \( m^2 \) is directly proportional to \( f^2 \); I would expect the theory to have \textit{two} dimensional coupling constants which vary independently. Hence I would expect that at \( T_\chi \), while \( f_\pi = 0, m^2 \neq 0 \).

Studies using sum rules typically find that the mass of the \( \rho \) meson decreases by the time of \( T_\chi \) [8] (see, however, [9]). This can be explained by assuming that \( VMD \) is violated by terms such as \( \xi \neq 0 \). From the measured phase shifts, Shuryak and Shuryak and Thorsson [10] also find that the \( \rho \) meson mass has decreased by the time of \( T_\chi \).

I have adopted \( VMD \) [1] because it appears to me to be such a beautiful and mysterious principle. Beauty can be deceptively seductive, however, and so one should be aware that violations of \( VMD \) would lead to phenomenon — such as decreasing masses, and shifts in the \( \omega \) mass — that are forbidden under \( VMD \). Ultimately, experiment will decide.

In that vein, determining whether or not these shifts in the vector meson peaks are observable is an involved question. On the theoretical side, it is necessary to do a complete analysis, including especially the effects of thermal broadening of the widths for the \( \pi \)'s, \( \sigma \)'s, and all other fields. While an analysis in weak coupling has its obvious limitations, it would be well worth checking that at least the \( \pi \)'s have a relatively narrow thermal width. A necessary condition for any quasiparticle picture to make sense is that the quasiparticle width is less than the effective mass of the field. Secondly, and more importantly, is to establish that the bulk of the dileptons are in fact to a phase with temperatures \( \sim T_\chi \), so that one can at least hope to pick out thermal dileptons from the usual zero temperature backgrounds.

### 4 VMD at low temperature

In this section I conclude by analyzing the shifts in the thermal masses in the limit of low temperature. This analysis is mainly of theoretical interest, and is rather more involved than that at the point of phase transition. At the outset, I work in the chiral limit, \( h = 0 \).

There are three diagrams which contribute to self energy of the \( \rho \) meson at one loop order. There is a tadpole diagram from the \( \rho_\mu^2 \pi^2 \) coupling,

\[
\delta_1 \Pi_\rho^{\mu\nu}(P) = +2g^2 \left( \frac{m_{a_1}}{m_\rho} \right)^2 \delta^{\mu\nu} tr \frac{1}{K^2},
\]

(4.22)
a diagram from $\rho \to \pi \pi \to \rho$,
\begin{equation}
\delta_2 \Pi^\mu_\nu(r)(P) = -g^2 \, tr \left( \frac{(2K + P)^\mu(2K + P)^\nu}{K^2(K + P)^2} \right),
\end{equation}
and, lastly, a diagram from $\rho \to a_1 \pi \to \rho$,
\begin{equation}
\delta_3 \Pi^\mu_\nu(r)(P) = -2g^2 \left( m_{a_1}^2 - m_r^2 \right) \left( \frac{m_{a_1}}{m_r} \right)^2 \times \nonumber \\
tr \left( \delta^\mu_\nu - \frac{(P + K)^\mu(P + K)^\nu}{m_{a_1}^2} \right) \frac{1}{K^2((P + K)^2 + m_{a_1}^2)}.
\end{equation}

The analysis is greatly simplified by using two tricks. The first is to concentrate on the transverse modes, by extracting only the term proportional to $\sim \delta^\mu_\nu$. The second is to recognize that we need the value of the self energies on their mass shell. At low temperatures, only fluctuations from pions matters, so the loop momenta $K \sim T$; hence, since $P \sim m_r, m_{a_1} \gg T \sim K$, in the loop integrals we can expand in powers of $K$, treating $K$ as small relative to $P$.

In this manner, we see that we can neglect $\delta_2 \Pi$, since it cannot produce a term $\sim \delta^\mu_\nu$ to $\sim T^2$. In $\delta_3 \Pi$, we can keep just the term $\sim \delta^\mu_\nu$, and approximate
\begin{equation}
\frac{1}{(P + K)^2 + m_{a_1}^2} \sim \frac{1}{P^2 + m_{a_1}^2}.
\end{equation}

Thus, to order $\sim T^2$ the term in the $\rho$ self energy proportional to $\delta^\mu_\nu$ is
\begin{equation}
\Pi^\mu_\nu \sim 2g^2 \left( \frac{m_{a_1}}{m_r} \right)^2 \left( 1 - \frac{m_{a_1}^2 - m_r^2}{P^2 + m_{a_1}^2} \right) \delta^\mu_\nu \tr \frac{1}{K^2}.
\end{equation}

This form manifestly vanishes on the $\rho$ mass shell, $\Pi_\rho(P^2) \sim 0$ at $P^2 = -m_r^2$, as is should by the analysis of Eletsky and Ioffe [2]. It also shows that the self energy does not vanish off of the mass shell. This explicit analysis was verified in the linear model by Lee, Song, and Yabu [5] and, independently, by myself [1]. Song [5] also demonstrated that it holds in the nonlinear model.

For the $a_1$ meson, there are three diagrams which contribute at one loop order. There is a tadpole diagram from the $(a_1^+)^2\pi^2$ coupling,
\begin{equation}
\delta_1 \Pi^\mu_\nu(r)(P) = +g^2 \left( \frac{m_{a_1}}{m_r} \right)^2 \delta^\mu_\nu \tr \frac{1}{K^2},
\end{equation}

a tadpole diagram, involving the cubic couplings between $a_1^+\sigma$ and $\sigma\pi^2$, and the $\sigma$ propagator at zero momentum,
\begin{equation}
\delta_2 \Pi^\mu_\nu(r)(P) = -3g^2 \left( \frac{m_{a_1}}{m_r} \right)^2 \frac{2\lambda\sigma_0^2}{m_\sigma^2} \delta^\mu_\nu \tr \frac{1}{K^2},
\end{equation}

This form manifestly vanishes on the $\rho$ mass shell, $\Pi_\rho(P^2) \sim 0$ at $P^2 = -m_r^2$, as is should by the analysis of Eletsky and Ioffe [2]. It also shows that the self energy does not vanish off of the mass shell. This explicit analysis was verified in the linear model by Lee, Song, and Yabu [5] and, independently, by myself [1]. Song [5] also demonstrated that it holds in the nonlinear model.

For the $a_1$ meson, there are three diagrams which contribute at one loop order. There is a tadpole diagram from the $(a_1^+)^2\pi^2$ coupling,
\begin{equation}
\delta_1 \Pi^\mu_\nu(r)(P) = +g^2 \left( \frac{m_{a_1}}{m_r} \right)^2 \delta^\mu_\nu \tr \frac{1}{K^2},
\end{equation}

a tadpole diagram, involving the cubic couplings between $a_1^+\sigma$ and $\sigma\pi^2$, and the $\sigma$ propagator at zero momentum,
a diagram from $a_1 \to \rho \pi \to a_1$, 

$$
\delta_3 \Pi_{a_1}^{\mu \nu} (P) = -2g^2 \left( m_{a_1}^2 - m_\rho^2 \right) \left( \frac{m_{a_1}}{m_\rho} \right)^2 \times 
$$

$$
tr \left( \delta^{\mu \nu} - \frac{((P + K)^\mu (P + K)^\nu)}{m_\rho^2} \right) \frac{1}{K^2((P + K)^2 + m_\rho^2)} ,
$$

(4.29)

and, lastly, a diagram from $a_1 \to \sigma \pi \to a_1$, 

$$
\delta_4 \Pi_{a_1}^{\mu \nu} (P) = -g^2 \left( m_{a_1}^2 - m_\rho^2 \right) \left( \frac{m_{a_1}}{m_\rho} \right)^2 \times 
$$

$$
tr \left( P^\mu + 2 \left( \frac{m_\rho}{m_{a_1}} \right)^2 K^\mu \right) \left( P^\nu + 2 \left( \frac{m_\rho}{m_{a_1}} \right)^2 K^\nu \right) \frac{1}{K^2((P + K)^2 + m_\rho^2)} ,
$$

(4.30)

The calculation of the behavior of these diagrams to leading order in $\sim T^2$ is as easy as for the $\rho$. The first two contributions are independent of momentum; the third reduces in the limit of large $P$ to a simple form, while the fourth has no term $\sim \delta^{\mu \nu}$ to this order. The sum is

$$
\Pi_{a_1}^{\mu \nu} (P) \sim -2g^2 \left( \frac{m_{a_1}}{m_\rho} \right)^2 \left( 1 + \frac{m_{a_1}^2 - m_\rho^2}{P^2 + m_\rho^2} \right) \delta^{\mu \nu} tr \frac{1}{K^2} .
$$

(4.31)

Again, we see that $\Pi_{a_1} (P) \sim 0$ on the $a_1$ mass shell, $P^2 = -m_{a_1}^2$.

So far I have worked in the chiral limit, $h = 0$. It is trivial extending the analysis away from the chiral limit. The only difference for $h \neq 0$ is that the mass of the $\sigma$ depends upon $h$ as $2 \sigma_0 + h/\sigma_0$. Thus there is no change in the result for the $\rho$ meson: its mass does not move to $\sim T^2$. For the $a_1$ self energy, the previous cancellation is now invalid; $\delta_2 \Pi_{a_1}$ changes, with the total result

$$
m_{a_1}^2 (T) \sim m_{a_1}^2 + \frac{g^2 m_\pi^2 T^2}{4m_\rho^2} + \ldots .
$$

(4.32)

That is, the mass of the $a_1$ increases with temperature, to leading order $\sim T^2$ at low temperature. Of course this result is special to the linear sigma model: it would not be seen in the nonlinear model, where $m_\sigma \to \infty$.

The corrections at next to leading order, $\sim T^4$, are relatively simple to compute. Remember that in the toy model, corrections of $\sim T^4$ only arose from diagrams at two loop order. Such corrections are of course present in a gauged linear sigma model. But there are also corrections at one loop order. After deriving these corrections, I show in which regime the terms $\sim T^4$ at one loop order dominate those to two loop order.
To derive these corrections of $\sim T^4$, terms $\sim K/P$ are retained. For example,

$$
\frac{1}{(P + K)^2 + m_{a_1}^2} \sim \frac{1}{P^2 + m_{a_1}^2} \left(1 - \frac{2K \cdot P}{P^2 + m_{a_1}^2} - \frac{K^2}{P^2 + m_{a_1}^2} + 4 \frac{(K \cdot P)^2}{(P^2 + m_{a_1}^2)^2} + \ldots \right).
$$

(4.33)

The term $\sim 1$ on the right hand side gives rise to a contribution of order $T^2$. Corrections $\sim K \cdot P$ typically vanish after integrating over $K$. Terms $\sim K^2$ contribute to the integral as $tr K^2/K^2 = tr 1$, which doesn't have a term $\sim T^4$. The remaining term, $\sim (K \cdot P)^2$, produces an integral which does contribute a term $\sim T^4$:

$$
tr \frac{K^\mu K^\nu}{K^2} = \frac{\pi^2 T^4}{90} (g^{\mu\nu} - 4n^\mu n^\nu).
$$

(4.34)

Calculating in this manner, it is direct to compute the shift in the $\rho$ mass:

$$
m^2_\rho(T) \sim m^2_\rho - \frac{g^2 \pi^2 T^4}{45 m^2_\rho} \left(\frac{4 m^4_{a_1} (3 m^2_\rho + 4 p^2)}{(m^2_{a_1} - m^2_\rho)^2} - 3\right) + \ldots.
$$

(4.35)

In Eqs. (4.35) and (4.36), all masses on the right hand side refer to values at zero temperature. The first term on the right hand side is due to $\rho \rightarrow a_1 \pi \rightarrow \rho$, from the piece of $\delta_3 \Pi_\rho$ proportional to $\delta^{\mu\nu}$; notice that it is proportional to $1/(m^2_{a_1} - m^2_\rho)^2$. The second term on the right hand side is due to $\delta_2 \Pi_\rho$ and $\delta_3 \Pi_\rho$, from the terms proportional to $K^\mu K^\nu$.

For the $a_1$, the shift in its thermal mass is found to be:

$$
m^2_{a_1}(T) \sim m^2_{a_1} + \frac{g^2 \pi^2 T^4}{45 m^2_\rho} \left(\frac{4 m^2_{a_1} (3 m^2_{a_1} + 4 p^2)}{(m^2_{a_1} - m^2_\rho)^2} + \frac{2 m^4_\rho}{m^2_{a_1} (m^2_{a_1} - m^2_\rho)} - \frac{m^2_{a_1}}{m^2_\rho}\right) + \ldots.
$$

(4.36)

Similar to the $\rho$, the first term on the right hand side is due to $a_1 \rightarrow \rho \pi \rightarrow a_1$, from the term in $\delta_3 \Pi_{a_1}$ proportional to $\delta^{\mu\nu}$, and is also proportional to $1/(m^2_{a_1} - m^2_\rho)^2$. The second term arises from $a_1 \rightarrow \sigma \pi \rightarrow a_1$, $\delta_4 \Pi_{a_1}$, and so involves the mass of the $\sigma$ meson. Lastly, the third term on the right hand side arises from the piece of $\delta_3 \Pi_{a_1}$ proportional to $K^\mu K^\nu$.

The fascinating feature of these results is that the while the $\rho$ mass goes up, and the $a_1$ mass down by $T^4$, they start out in the opposite direction: the $\rho$ mass goes down to $\sim T^4$, and the $a_1$ mass, up! Technically, this happens because of the leading term in each mass, proportional to $1/(m^2_{a_1} - m^2_\rho)^2$. For the $\rho$, this factor arises from a factor of $g^2 \sigma_0^2 = (m^2_{a_1} - m^2_\rho)$ from the vertices, times the expansion of $1/((P + K)^2 + m^2_{a_1})$ in the denominator. From (4.33), the expansion of this denominator to the relevant order gives a factor of $(m^2_{a_1} - m^2_\rho)/(P^2 + m^2_{a_1})^3$, so overall, at $P^2 = -m^2_\rho$ the factor is $-1/(m^2_{a_1} - m^2_\rho)^2$. For the $a_1$, there is again a factor of $g^2 \sigma_0^2 = (m^2_{a_1} - m^2_\rho)$ from the vertices, times the expansion of $1/((P + K)^2 + m^2_{a_1})$ in the denominator. Expanding
to the relevant order gives of \((m_{a_1}^2 - m_\rho^2)/(P^2 - m_\rho^2)^3\), so at \(P^2 = -m_{a_1}^2\) the overall factor is \(+1/(m_{a_1}^2 - m_\rho^2)^2\), which has the opposite sign as the analogous term for the \(\rho\).

As of yet, I do not know a deeper explanation for the most peculiar behavior of the vector meson masses to leading order in low temperature. I can argue, however, that these results are dominant in the limit of low temperature, at least in certain regimes. To show this, remember \(m_{a_1}^2 - m_\rho^2 = g^2\sigma_0^2 = g^2\mu^2/\lambda\); thus the shift in the vector meson masses is, to one loop order, of order

\[
\delta m^2_{\text{one loop}} \approx g^2 \frac{m_\rho^2}{(m_{a_1}^2 - m_\rho^2)^2} T^4 \approx \frac{\lambda^2}{g^2} \frac{m_\rho^2}{\mu^2} T^4 .
\]

I must confess that the appearance of a factor of \(1/g^2\) is disturbing. However, taking \(\lambda \sim g^2\), overall this term is \(\sim g^2\), which is small; if \(\lambda \sim g^4\), \(\lambda^2/g^2 \sim g^6\) is even smaller. This is assuming that the mass scales \(m\) and \(\mu\) are held fixed. Since in nature \(m_\rho \sim m_\sigma \sim \mu\), this is reasonable.

Now of course the vector mesons masses will shift not just from diagrams at one loop order, but from diagrams at two loop order and beyond. These diagrams involve either higher powers of \(\lambda\), or higher powers of \(g^2\). Consider first diagrams with higher powers of \(\lambda\): one example is a one loop correction to the \(\sigma\) propagator on a one loop tadpole graph. Such a graph is of order of magnitude

\[
\delta m^2_{\text{two loop}} \approx (g^2\sigma_0)(\lambda\sigma_0) \frac{1}{(m_\sigma^2)^2} (\lambda T^2) T^2 \approx \lambda g^2 \frac{T^4}{\mu^2} .
\]

An example of a two loop diagram involving higher powers of \(g^2\) comes from, say, \(\rho \rightarrow a_1\pi \rightarrow a_1\pi \rightarrow \rho\). This diagram is large because it involves two \(a_1\) propagators. In magnitude it is

\[
\delta m^2_{\text{two loop}} \approx (g^2\sigma_0)^2 \frac{g^2}{(m_{a_1}^2 - m_\rho^2)^2} T^4 \approx \lambda g^2 \frac{T^4}{\mu^2} .
\]

Thus the ratio of these two loop diagrams to one loop diagrams is

\[
\frac{\delta m^2_{\text{two loop}}}{\delta m^2_{\text{one loop}}} \sim \frac{g^4}{\lambda} \frac{\mu^2}{m_\rho^2} .
\]

Thus for \(\lambda \sim g^2\), the one loop term dominates. It is certainly possible to choose \(\lambda\) in this range. If \(\lambda \sim g^4\) and \(\mu \sim m_\rho\), the terms at one and two loop order are comparable. If \(m_\rho \gg m_\sigma \sim \mu\), then the one loop term dominates without restriction on the ratio of the coupling constants.

What, then, of the opposite limit, when \(m_\sigma \gg m_\rho\)? This is the limit which is probed by the nonlinear \(\sigma\) model. In this case the power counting is slightly different,
because then one wants to keep the vacuum expectation value, \( \sigma_0^2 = \mu^2/\lambda \), fixed. In this limit, the one loop term is

\[
\delta m^2_{\text{one loop}} \sim \frac{1}{g^2} \frac{m_p^2}{\sigma_0^2} \frac{T^4}{\sigma_0^2},
\]  

(4.41)

while the two loop term is

\[
\delta m^2_{\text{two loop}} \sim g^2 \frac{T^4}{\sigma_0^2},
\]  

(4.42)

Thus

\[
\frac{\delta m^2_{\text{two loop}}}{\delta m^2_{\text{one loop}}} \sim g^4 \frac{\sigma_0^2}{m_p^2},
\]  

(4.43)

and for small \( g \) the two loop term is much smaller than that at one loop order. According to this analysis, the terms \( \sim T^4 \), Eqs. (4.35) and (4.36), would be the same if computed in a gauged nonlinear sigma model with strict VMD, taking of course the limit \( m_p \to \infty \).

The corrections of \( \sim T^4 \) have also been analyzed by Eletsky and Ioffe [11], who find that both masses decrease with temperature. This can happen in a nonlinear model in which \( \xi \neq 0 \), following the discussion in the previous section. In this conference, S.-H. Lee [12] argued that the terms \( \sim T^4 \) are not uniquely defined. We agree: there is a unique prediction only under the assumption of strict VMD.

5 References


