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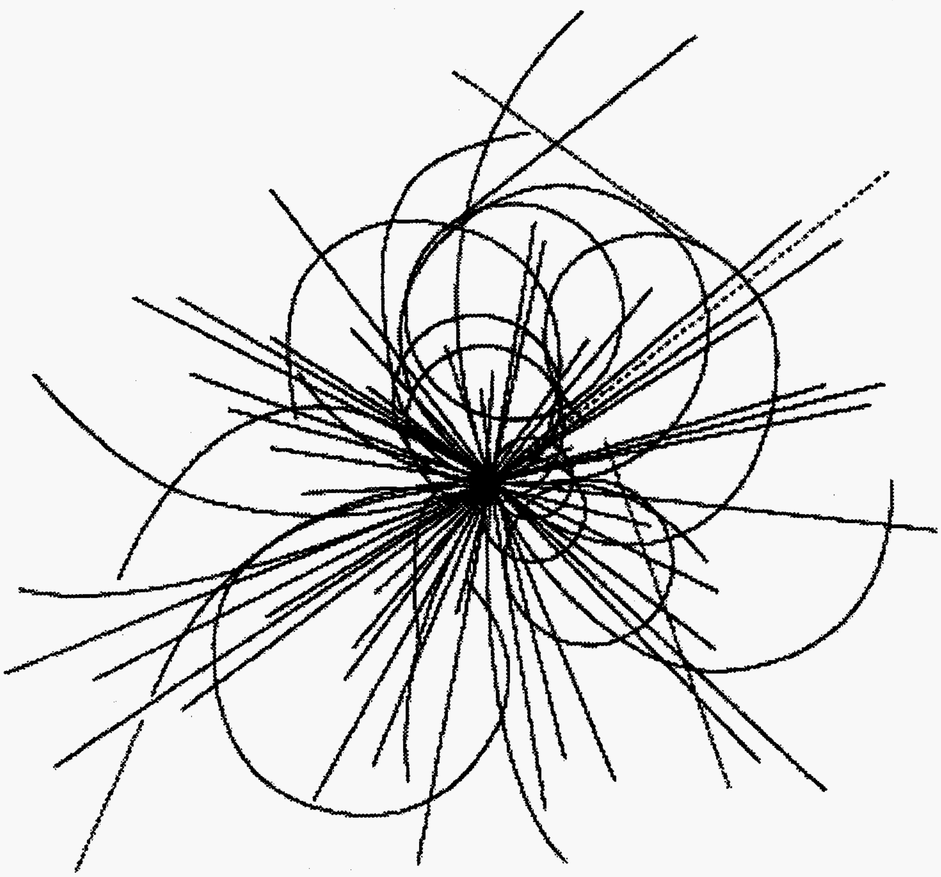

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Emittance Growth Due to Noise and its Suppression with the Feedback System in Large Hadron Colliders*

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EMITTANCE GROWTH DUE TO NOISE AND ITS SUPPRESSION WITH THE FEEDBACK SYSTEM IN LARGE HADRON COLLIDERS

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The problem of emittance growth due to random fluctuation of the magnetic field in hadron colliders is considered. Based on a simple one-dimensional linear model, a formula for an emittance growth rate as a function of the noise spectrum is derived. Different sources of the noise are analyzed and their role is estimated for the Superconducting Super Collider (SSC). A theory of feedback suppression of the emittance growth is developed which predicts the residual growth of the emittance in the accelerator with a feedback system.

1 INTRODUCTION

Future large hadron colliders such as LHC and especially the Superconducting Super Collider (SSC) will have a circumference of tens of kilometers. That means that the revolution frequency in these machines comes into the range of several kilohertz. Typically, the level of the external noise (such as ground motion, current ripple, *etc.*) increases when one goes to smaller frequencies so that noise effects that probably were not an issue of concern for smaller storage rings might seriously degrade the performance of large machines. For the proton colliders with the beam energy of tens of TeV, the synchrotron radiation is still weak enough to counteract the noise effects.

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Depending on the frequency of the noise, we can distinguish two mechanisms of the beam perturbation. At low frequencies (much less than the revolution frequency), the noise produces a distortion of the closed orbit of the beam. For the SSC, these effects have been previously considered in References 1–5. However, if the spectrum of the noise extends up to the resonant betatron frequencies $f_0 \, |\nu - n|$, where f_0 is the revolution frequency, ν is the tune and n is an integer, it resonantly drives the betatron oscillations of the beam. Due to decoherence, these oscillations rapidly translate into the growing emittance of the beam with the growth rate that is proportional to the noise spectrum at the resonant frequency.[6–9]

An effective way to suppress the emittance growth caused by the noise is based on the use of a feedback system that monitors the amplitude of the betatron oscillations and tries to damp them applying appropriate kicks to the beam. If the feedback system damps beam oscillations faster than they decohere, the emittance growth will be strongly suppressed compared to the case without feedback.

In this paper, we first address the problem of emittance growth due to random fluctuation of the dipole magnetic field B. Basing on a simple one-dimensional linear model, in Section 2, we derive a formula for an emittance growth rate as a function of the noise spectrum. In Section 3, we analyze different sources of the noise and estimate their role for the SSC. For completeness, in this section, we also included results of the consideration of the emittance growth due to the fluctuation of the gradient of the magnetic field B'. In Section 4, we develop a theory of feedback suppression of the emittance growth and calculate the residual growth of the emittance in the accelerator with a feedback.

In a subsequent paper, we will present results of computer simulations which confirm the predictions of the analytical theory of this paper and also take into account additional effects not covered by the simple theory of the present paper.

2 EMITTANCE GROWTH DUE TO EXTERNAL NOISE

2.1 *General Considerations*

Considering particle motion in an accelerator, we will be neglecting coupling between vertical and horizontal degrees of freedom and will use the following variables:

$$x = \frac{X}{\sqrt{\beta}}, p = \beta\frac{d}{ds}\frac{X}{\sqrt{\beta}}, \tag{1}$$

where X stands for the particle deviation with respect to the closed orbit, β is the beta function and s is the path length along the orbit.

Assume that the magnetic field in one of the magnets is perturbed by an amount $\delta B(t)$ and varies with time. This perturbation may be due to random (time dependent) displacements of a quadrupole or fluctuations of the current in a dipole magnet.

Each turn when a particle passes through the magnet, the particle experiences a kick that changes its momentum from p to p',

$$p' = p + \Delta p_n, \tag{2}$$

where the change of the momentum Δp_n is related to the perturbation of the magnetic field,

$$\Delta p_n = \sqrt{\beta_0}\frac{el\delta B(nT)}{Pc}. \tag{3}$$

In Eq. (3) P is the particle longitudinal momentum, l is the length of the magnet, B_0 is the beta function at the position of the magnet, T is the revolution period, and n is the turn number. Taking into account that free betatron oscillations are described by the following transformation of the variables x and p,

$$x' = x\cos\theta + p\sin\theta, \quad p' = -x\sin\theta + p\cos\theta, \tag{4}$$

where θ stands for the betatron phase, we can write down the result of N successive passes through the magnet,

$$x_N = \sum_{n=0}^{N-1} \Delta p_n \sin\mu(N-n) + x_0\cos(\mu N + \theta_0), \tag{5}$$

where $\mu = 2\pi\nu$, ν is the tune and x_0 and θ_0 are the initial amplitude and phase of the oscillations.

Throughout this paper, we will be assuming that $\delta B(t)$ is a stationary random function characterized by its correlation function $K_{\delta B}(\tau)$,

$$K_{\delta B}(\tau) = \langle \delta B(t)\delta B(t-\tau)\rangle, \tag{6}$$

where the angular brackets denote the averaging. Using Eq. (3) we will also define the correlation function of the kicks,

$$K_{\Delta p}(\tau) = \langle \Delta p(t)\Delta p(t-\tau)\rangle = \beta_0\left(\frac{el}{Pc}\right)^2 K_{\delta B}(\tau), \tag{7}$$

so that

$$\langle \Delta p_n \Delta p_m\rangle = K_{\Delta p}(T(n-m)). \tag{8}$$

Related to the correlation function $K(\tau)$ is the corresponding spectral density (or, briefly, spectrum) $S(\omega)$. It is a positive even function of the frequency ω, $S(\omega) \geq 0$, $S(\omega) = S(-\omega)$. According to the Wiener-Khintschin theorem, the spectrum can be found as a Fourier transform of the correlation function:

$$K(\tau) = \int_{-\infty}^{\infty} d\omega S(\omega)e^{-i\omega\tau} = 2\int_0^{\infty} d\omega S(\omega)\cos\omega\tau, \quad S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau K(\tau)e^{i\omega\tau}. \tag{9}$$

Note the concept (frequently used in many publications) of white noise that corresponds to the noise spectrum and does not depend on the frequency, $S(\omega) = \text{const}$.

2.2 *Growth of the Amplitude of the Betatron Oscillations*

Using Eq. (5) we can now calculate the averaged square of x_N, $\langle x_N^2 \rangle$. In doing so, we will utilize the condition

$\langle \Delta p_n \rangle = 0$,*

$$\langle x_N^2 \rangle = x_0^2 \cos^2(\mu N + \theta_0) + \sum_{n,m=0}^{N-1} K_{\Delta p}(T(n-m)) \sin\mu(N-n) \sin\mu(N-m). \tag{10}$$

To facilitate the summation in Eq. (10) we express $K_{\Delta p}$ in terms of the spectral density $S_{\Delta p}$ using Eq. (9),

$$\langle x_N^2 \rangle = x_0^2 \cos^2(\mu N + \theta_0) + \frac{1}{2} \int_{-\infty}^{\infty} S_{\Delta p}(\omega) \sum_{n,m=0}^{N-1} e^{i\omega T(n-m)} \sin\mu(N-n) \sin\mu(N-m). \tag{11}$$

The summation in Eq. (11) can now be performed explicitly. We are interested here in the limit of large N, formally $N \to \infty$. As is shown in the Appendix, in this limit Eq. (11) reduces to the following one

$$\langle x_N^2 \rangle = \frac{1}{2} \Sigma\ (\nu) N\Omega + x_0^2 \cos^2(\mu N + \theta_0), \tag{12}$$

where Ω is the revolution frequency, $\Omega = 2\pi/T$, and

$$\Sigma(\nu) = \sum_{n=-\infty}^{\infty} S_{\Delta p}((\nu-n)\Omega) = \beta_0 \left(\frac{el}{Pc}\right)^2 \sum_{n=-\infty}^{\infty} S_{\delta B}((\nu-n)\Omega). \tag{13}$$

A special discussion is needed if one applies Eq. (13) for the white noise. Formally, putting $S(\omega) = \text{const}$ into Eq. (13) gives the infinity on the right-hand side. This happens because the usual definition of the white noise assumes that it has the correlation function $\propto \delta\ (\tau)$ and, hence, the infinite value of $\langle \delta B^2 \rangle$. For our problem, it is more convenient to change this definition so as to understand white noise as the one with the correlation time

*This condition can always be met by the proper choice of the equilibrium closed orbit.

much less than the revolution period. In other words, the white noise in a magnet produces uncorrelated kicks on the beam, $\langle \Delta p_n \Delta p_m \rangle = 0$ if $n \neq m$, and

$$K_{\Delta p}(T(n-m)) = \langle \Delta p_n^2 \rangle \delta_{n,m}, \tag{14}$$

where $\delta_{n,m}$ is the Kroneker symbol. Putting this equation into Eq. (10) and performing the summation, one easily arrives at Eq. (12) with the following $\Sigma(\nu)$,

$$\Sigma(\nu) = \frac{1}{\Omega}\langle \Delta p_n{}^2 \rangle = \beta_0 \left(\frac{el}{Pc}\right)^2 \frac{1}{\Omega}\langle \delta B^2 \rangle. \tag{15}$$

From Eq. (13) it follows that due to the presence of the noise the average square of the single particle displacement linearly grows with time. This time dependence is typical for diffusion processes, in our case the diffusing quantity being the amplitude of the betatron oscillations. The only spectral components that contribute to the growth of the amplitude have the frequency equal to that of the betatron sidebands. For the SSC,[10] the revolution frequency $f_0 = \Omega/2\pi = 3.4$ kHz and the nominal fractional part of the tune, $\{\nu\}$, is equal to 0.28. This gives the lowest resonant frequency of the noise equal to $\{\nu\}f_0 = 960$ Hz.

2.3 *Tune Spread and the Emittance Growth*

In the above derivation the motion of only one particle of the beam has been considered. Since the first term on the right-hand side of Eq. (5) does not depend on the particle initial amplitude and phase, random dipole kicks will drive coherent oscillations of the beam as a whole. However, the tune spread in the beam causes phase mixing of different particles and results in decohering of betatron oscillations on a time scale equal to the inverse spread of the betatron frequencies.

There are several sources of the tune spread in hadron accelerators. First, because of the finite energy spread of the beam, it stems from the chromaticity of the machine. Second, the tune spread is generated by nonlinear elements in the lattice such as sextupoles and octupoles as well as higher-order multipoles in the magnets. Finally, in colliders, the dominant contribution to the tune spread usually comes from the beam-beam interactions. In our further considerations we will assume that the tune spread is determined by the collisions of the beams. In this case, for round beams and the Gaussian particle distribution function, the rms tune spread can be related to the so called interaction parameter ξ,

$$\xi = \frac{e^2 N_{particle}}{4\pi P c \varepsilon}, \tag{16}$$

which, in the limit $\xi \ll 1$, coincides with the betatron frequency shift. In accordance with Reference 8, the rms betatron frequency spread is equal to

$$\Delta\nu_{rms} \equiv \sqrt{\overline{\Delta\nu^2}} \approx 0.2\xi \ , \tag{17}$$

where the bar denotes averaging over the particle distribution function. In Eq. (16), $N_{particle}$ is the number of particles in the bunch and ε is the beam emittance. For the SSC, $\xi = 1.8 \cdot 10^{-3}$ for two interaction regions and two times as much if four interaction regions will be operating. This gives decoherence time $\tau_{decoh} = 1/f_0 \Delta\nu_{rms}$ $\approx 0.8 - 0.4$ s.

Decoherence rapidly translates dipole beam oscillations driven by external noise to the growing emittance of the beam. In accordance with the standard definition of the beam emittance ε, it is equal to the square of the offset

$\overline{x^2}$ averaged over the particle distribution function of the beam.* With so defined ε one can rewrite Eq. (12) in terms of emittance,

$$\langle\varepsilon(t)\rangle = \frac{t}{4\pi}\Sigma(\nu)\Omega^2 + \varepsilon(0). \tag{18}$$

An important characteristic of the noise effect is the derivative $d\langle\varepsilon\rangle/dt$ which gives the rate of the emittance growth,

$$\left(\frac{d\langle\varepsilon\rangle}{dt}\right)_0 = \frac{1}{4\pi}\Sigma(\nu)\Omega^2. \tag{19}$$

We put the index 0 on the left-hand side of Eq. (19) to emphasize that this growth of emittance occurs without a feedback system in the machine.

Two remarks should be made in connection with Eq. (19). First, as follows from Eq. (13), only the noise at a discrete set of frequencies contributes to the emittance growth. However, due to the betatron frequency spread in the beam, the resonant frequency of different particles deviates from the frequency $(n - \nu)\Omega$, resulting in a finite resonance width $\Delta\omega_{res} \approx \Delta\nu\Omega$. Hence, more rigorously, rather than having exactly the frequency $(n - \nu)\Omega$ the resonant noise occupies the frequency range $\Delta\omega_{res}$ in the vicinity of this value. One can neglect the resonance width and use Eqs. (19) and (13) if the noise spectrum does not change appreciably within the width of the resonance $\Delta\omega_{res}$.

Second, it turns out that the above derivation is only valid if the tune is not too close to nonlinear resonances of the machine. Analytical study of these resonances in the presence of the external noise is a cumbersome problem.

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*This definition of emittance is equivalent to $\varepsilon = \frac{1}{2}\left(\overline{x^2} + \overline{p^2}\right)$, because in our case $\overline{x^2} = \overline{p^2}$.

We will address this issue in a subsequent paper devoted to results of computer simulation of the emittance growth.

3 SOURCES OF EXTERNAL NOISE AND ESTIMATES OF THEIR EFFECT

The main sources of the noise in accelerators which produce transverse kicks on the beam are quadrupole vibrations and fluctuations of the magnetic field in the bending magnets and kickers. In this section, we will estimate a tolerable level of the noise in a collider without a feedback system.

3.1 *Vibrations of Quadrupoles*

Quadrupole vibrations are caused by both ambient seismic ground motion and man-made noise that is inevitably generated on the machine site by flows of coolant, operating vacuum pumps, *etc.* In many cases, it is impossible to trace the exact origin of these vibrations. Measurements in different sites[11–14] show that the amplitude and spectrum of the ground motion may differ by several orders of magnitude depending on local conditions of the site and the time of day. Some examples of such spectra are shown in Figure 1. It clearly demonstrates that the spectral density of the ground motion strongly decreases with the frequency. This feature explains why the noise effects are typically negligible for small rings and might be important for machine with a low revolution frequency such as SSC and LHC.

To estimate the emittance growth arising from random vibrations of the quadrupoles, note that in this case, δB is proportional to the average over the length of the magnet displacement of the quadrupole d,

$$\delta B(t) = B' d(t), \tag{20}$$

where B' stands for the gradient of the magnetic field in the quadrupole. Using the definition of the focal length of the quadrupole f_q, $f_q = cP/elB'$, one can easily express $\Sigma(\nu)$ in terms of the spectrum of the displacement $S_d(\omega)$

$$\Sigma(\nu) = \frac{\beta_0}{f_q^2} \sum_{n=-\infty}^{\infty} S_d((\nu - n)\Omega). \tag{21}$$

To obtain a rough estimate of a tolerable level of magnet vibrations we assume that the vibration spectrum is the same for all of the quadrupoles and there is no correlation between displacements of the different quadrupoles in the ring (this is a reasonable assumption taking into account that the relevant frequencies are in the range of a kilohertz and above). Let us also assume FODO lattice, so that the quadrupoles are located at the positions of maximum and minimum values of the beta function. Summing Eq. (21) over all quadrupoles of the ring one finds

$$\Sigma(\nu) = N_{cells} \frac{\beta_{\max} + \beta_{\min}}{f_q^2} \sum_{n=-\infty}^{\infty} S_d((\nu - n)\Omega), \tag{22}$$

where N_{cell} is the total number of cells and $\beta_{\max}, \beta_{\min}$, are the maximum and minimum vales of the beta function in the cell. Note also that for the FODO lattice $(\beta_{\max} + \beta_{\min})/f_q^2 = 8\tan(\Delta\mu/2)/L$ where $\Delta\mu$ is the phase advance through a full FODO period and L is the length of the half cell. For the SSC, $\Delta\mu = 90°$, $L = 90$ m, $N_{cells} = 392$, (the number of cells in the arcs) and the nominal value of the emittance $\varepsilon = 4.7 \cdot 10^{-9}$ cm. Requiring that the emittance doubling time be larger than 20 hours (that is approximately twice the synchrotron radiation cooling time) one finds the following limitation on the noise spectrum:

$$\sum_{n=-\infty}^{\infty} S_d((\nu - n)\Omega) \leq 6 \cdot 10^{-13} \mu\text{m}^2/\text{Hz}. \tag{23}$$

For comparison, note that for the white noise this level of vibrations corresponds to the rms quadrupole displacement $d_{rms} = 1.1 \cdot 10^{-4} \mu$m.

Since the noise spectra rapidly falls down with the frequency, the dominant term on the left-hand side of Eq. (23) is the one with the lowest frequency. As is said above, for the nominal tune in the SSC, this frequency is equal to 960 Hz and the inequality (23), in fact, gives an upper limit for the noise at this frequency.

In the kilohertz range of frequencies, a quadrupole having the length of several meters does not oscillate as a rigid body but rather experiences bending and torching deformations around its axes. If the wavelength of such deformations is much smaller than the length of the magnet, the averaged along the orbit perturbation of the magnetic field is suppressed in comparison with the case when the magnet displaces as a whole. In other words, the displacement d in Eq. (20) is becoming effectively smaller than the amplitude of the ground motion. However, analysis shows[15] that such a suppression is counteracted by multiple mechanical resonances of the magnet body which can significantly amplify the vibrations at the resonant frequencies of the magnet.

Note also a specific mechanism of the magnetic field perturbation if the vacuum tube has an inner liner with sufficiently high conductivity. Since high frequency perturbations of the magnetic field are frozen in the liner, its vibrations will perturb the magnetic field on the orbit even if the quadrupole itself is at rest.

3.2 *Dipole Field Fluctuations*

The effect of the bending magnetic field fluctuations can be easily estimated with the use of Eq. (13) for the function Σ and assuming that the fluctuations are independent and have the same spectrum in all of the bending magnets. For the sake of simplicity, we will also assume that these fluctuations are represented by the white noise and will make use of Eq. (15). For the SSC parameters, with the total number of bending magnets $N_{magnet} = 4200$ and the requirement that the emittance doubling time be less then 20 hours, this gives the following tolerable level of fluctuations in a magnet

$$\frac{\delta B_{rms}}{B} \leq 6.9 \cdot 10^{-10}. \tag{24}$$

Note that magnetic field fluctuation will be somewhat suppressed by the skin effect in the walls of the vacuum chamber (or/and liner) which can prevent the ac magnetic field component from penetrating the chamber. For the SSC, attenuation due to this effect is expected to be about 10–20, which substantially loosens the tolerances for the current fluctuation in the dipoles. On the other hand, freeze of the magnetic field into the beam pipe walls might also have a detrimental effect as a result of vibration modes that change the shape of the beam pipe cross section (*e.g.*, making the cross section elliptical rather than circular). Such modes perturb the dipole field inside the beam pipe modulating it with the frequency of the vibrations.

To avoid confusion, we have to emphasize here that magnetic field fluctuations can also be produced by the power supply ripple. However, in contrast to a wide-band noise, the main components of the ripple are usually concentrated at several well defined frequencies and one can, in principle, avoid their detrimental influence by detuning the working point of the machine away from these frequencies. On the other hand, any random noise in the power ripple will add to the perturbation of the magnetic field caused by other sources.

3.3 *Quadrupole Field Fluctuations and Vibrations of Sextupoles*

We have considered above fluctuations of the dipole magnetic field in the beam orbit. The time dependent perturbations of higher order magnetic multipoles can also blow up the beam emittance. For the quadrupole field perturbation that can be generated either by vibration of the sextupoles or current fluctuations in the quadrupoles, the problem has been studied in References 16 and 17. Here we present the results of these papers and give the estimates of the effect for the SSC.

For a Gaussian density distribution of the beam, the emittance growth rate caused by random fluctuations of the gradient of the magnetic field in a magnet of length l is given by the following formula:[17]

$$\frac{d\langle\varepsilon\rangle}{dt} = \frac{\varepsilon}{2\pi}\Omega^2\left(\frac{l\beta_0 e}{Pc}\right)^2 \sum_{n=-\infty}^{\infty} S_{B'}((2\nu - n)\Omega), \tag{25}$$

where β_0 is the beta function at the position of the magnet and $S_{B'}$ is the spectral density of the fluctuation of the gradient of the magnetic field B'.

Note, that in contrast with Eq. (13), resonant frequencies that contribute to the emittance growth are the sidebands of the double betatron frequency. This feature has a simple physical explanation: fluctuations of the quadrupole field bring about modulation of the tune and make the particle motion unstable via the parametric resonance. As is known, this resonance occurs at the double frequency of the oscillator.

One can use Eq. (25) to make an estimate of a tolerable level of fluctuations of the gradient of the magnetic field in quadrupoles. For the SSC, assuming white noise and taking into account that the number of quadrupoles in the ring is $N_{quad} \approx 800$ each of which has the focal length $f_q \approx 60$ m. Half of the quadrupoles are located at the positions with the local maximum of β, $\beta_0 = 305$ m giving the dominant contribution to the emittance growth. Again, requiring the emittance doubling time to be larger than 20 hours, one finds from Eq. (25),

$$\frac{\delta B'_{rms}}{B'} \leq 6{\cdot}10^{-7}. \tag{26}$$

The gradient of the magnetic field on the orbit can also be perturbed by vibrations of the sextupole magnets. As a feed down of an offset of a sextupole by the distance d one finds the following perturbation of the gradient of the magnetic field

$$\delta B' = B''d. \tag{27}$$

Using Eq. (27), one can rewrite Eq. (24) in terms of the spectral density of the vibrations

$$\frac{d\varepsilon}{dt} = \frac{\varepsilon}{8\pi}\Omega^2\left(\frac{B''l\beta_0 e}{Pc}\right)^2 \sum_{n=-\infty}^{\infty} S_d((2\nu - n)\Omega). \tag{28}$$

For the SSC, according to collider specifications, the product $B''l$ for about 400 sextupoles located near the focusing quadrupoles is equal to $2.4 \cdot 10^3$ T/m. Being located at the positions with the local maximum of the beta function, $\beta_0 = 305$ m, these sextupoles make a dominant contribution to Eq. (28). Taking the nominal collider parameter $Pc = 20$ TeV and requiring the emittance doubling time be more that 20 hours one finds,

$$\sum_{n=-\infty}^{\infty} S_d((2\nu - n)\Omega) \leq 1.5 \cdot 10^{-5} \mu\mathrm{m}^2/\mathrm{Hz}. \tag{29}$$

As was mentioned above, for a rapidly decreasing spectrum of the noise, the dominant term on the left-hand side of Eq. (29) is the one with the lowest frequency. For the nominal tune in the SSC, this frequency is equal to 1.52 kHz and the inequality (29) gives, in fact, an upper limit for the noise level at this frequency. Comparing it with Eq. (23) we notice that this limitation is much less stringent than that originating from the quadrupole vibrations.

4 EMITTANCE GROWTH SUPPRESSION WITH A FEEDBACK SYSTEM

4.1 *Basics of the Feedback Theory*

A transverse feedback system allows one to suppress the emittance growth due to excitation of the betatron oscillations by external noise. The system monitors the offset of the beam centroid and tries to correct it by kicks that are proportional to this offset applied a quarter of the betatron wavelength downstream. In this section, we develop a simple model of such a feedback which finally will allow us to predict the level of the residual emittance growth in the accelerator.

We consider here a model of an idealized feedback system with a sufficiently broad frequency band so that it can resolve the motion of each bunch in the machine. Let X_1 denotes the beam displacement at the position of the pick-up electrode (point 1). The kicker located a quarter of the betatron wavelength downstream (point 2) deflects the beam by an angle α,

$$\alpha = g \frac{X_1}{\sqrt{\beta_1 \beta_2}}, \tag{30}$$

where β_1 and β_2 are the values of the beta function at the positions of the pick-up and the kicker electrodes and g is the dimensionless amplification factor (gain) of the feedback system. Using the definition of momentum p according to Eq. (1), one can find the change in the momentum Δp_2, produced by the kick,

$$\Delta p_2 = \sqrt{\beta_2}\alpha = g\frac{X_1}{\sqrt{\beta_1}} = g x_1. \tag{31}$$

Noting that because the points 1 and 2 have 90° phase difference, $x_1 = -p_2$, and we can express Δp_2 in terms of the beam momentum at the point 2,

$$\Delta p_2 = -g p_2. \tag{32}$$

Since the kicker does not perturb the beam coordinate x_2, it is easy to write down, in matrix notation, the transformation of the variables x and p resulting from a passing through the feedback system,

$$\begin{pmatrix} x_2' \\ p_2' \end{pmatrix} = F\begin{pmatrix} x_2 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1-g \end{pmatrix}\begin{pmatrix} x_2 \\ p_2 \end{pmatrix}, \tag{33}$$

where x_2 and p_2 refer to the initial and x'_2 and p'_2—to the final state of the beam and F is the transformation matrix. To obtain a complete transformation M that includes the beam motion along the ring outside the feedback system one has to combine the transformation (33) with that of Eq. (4), multiplying the matrix F by the standard rotation matrix R,

$$M = RF = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1-g \end{pmatrix} = \begin{pmatrix} \cos\mu & (1-g)\sin\mu \\ -\sin\mu & (1-g)\cos\mu \end{pmatrix} . \tag{34}$$

The influence of the feedback on the beam motion is characterized by the eigenvalues $\lambda_{1,2}$ of the matrix M. A simple calculation yields,

$$\lambda_{1,2} = \left(1 - \frac{1}{2}g\right)\cos\mu \ \pm\ i\sqrt{(1-g)\sin^2\mu - \frac{g^2}{4}\cos^2\mu}\,. \tag{35}$$

Analysis shows[8] that $|\lambda_{1,2}| < 1$ only if $g < 2$. That means that the feedback system damps oscillations for $g < 2$ and would amplify them if $g > 2$. If the gain g satisfies the inequality,

$$\frac{g}{\sqrt{1-g}} < 2\tan\mu,$$

both eigenvalues have equal modula smaller than one, $|\lambda_1| = |\lambda_2| = \sqrt{1-g} < 1$.

The damping decrement γ in general case can be found as

$$\gamma = f_0 \min\{(1 - |\lambda_1|), (1 - |\lambda_2|)\},$$

where f_0 is the revolution frequency. In the limit $g \ll 1$, this gives

$$\gamma \approx \frac{1}{2} f_0 g. \tag{36}$$

Remembering that the beam decoheres on a time scale $\tau_d = (f_0 \Delta\nu_{rms.})^{-1}$, in order for the feedback system to suppress the beam oscillations before they decohere, we have to require the decrement γ be much larger than the inverse decoherence time τ_d^{-1}, that is,

$$g \gg \Delta\nu_{rms}. \tag{37}$$

4.2 *Suppression of Emittance Growth by a Feedback System*

A feedback system is able to damp a single noise kick but for the steady state noise producing many kicks on each turn of the beam, a residual level of the oscillations still survive. These residual oscillations combined with the decoherence cause the emittance growth, although with growth rate lower than the original emittance growth given by Eq. (19).

A key problem for the theory of feedback is the prediction of the residual emittance growth of the beam in the accelerator. We will calculate it using a simple model and considering, first, the motion of the beam particles after a single noise kick that displaces the beam distribution function, as a whole, by $\Delta x, \Delta p$ in the phase plane. Without the feedback, decoherence eventually translates this initial displacement into the incremental increase of the emittance, $\Delta\varepsilon_0 = \left(\Delta x^2 + \Delta p^2\right)/2$. Below, we will calculate the increase of the emittance with the feedback.

Denote the coordinate of the centroid of the beam by $\bar{x}$ and the averaged momentum of the beam by $\bar{p}$. In the absence of the feedback they satisfy the following equations,

$$\frac{d\bar{x}}{d\theta} = \bar{p}, \quad \frac{d\bar{p}}{d\theta} = \bar{x}. \tag{38}$$

Eqs. (38) describe the betatron oscillations of the centroid neglecting the effect of decoherence.

Assuming $g \ll 1$, it follows from Eq. (36) that the damping occurs on a time scale of many turns around the ring. In this case, instead of considering the transformation with the matrix M, we adopt here an approach based on a differential equation for $\bar{x}$ and $\bar{p}$. The feedback adds kicks according to Eq. (32) to the motion described by Eqs. (38). In the limit of small g, these kicks must be added to the second of Eqs. (38) as follows

$$\frac{d\bar{p}}{d\theta} = -\bar{x} - g\bar{p}\tilde{\delta}(\theta), \tag{39}$$

where the periodic delta function,

$$\tilde{\delta}(\theta) = \sum_{n=-\infty}^{\infty} \delta(\theta - \mu n), \tag{40}$$

accounts for periodicity of the kicks. Neglecting higher order harmonics produced by the kicks we will average the $\tilde{\delta}$-function over θ and keep only the time independent component,

$$\tilde{\delta}(\theta) \rightarrow \frac{1}{\mu}. \tag{41}$$

With this substitution, combining first of the equations (38) with Eq. (39) yields,

$$\frac{d^2\bar{x}}{d^2\theta} + \frac{g}{\mu}\frac{d\bar{x}}{d\theta} + \bar{x} = 0. \tag{42}$$

This is the equation of a damped oscillator. Since we are assuming that $g \ll 1$ and, hence, $g/\mu \ll 1$ we can find an approximate solution of Eq. (42) in the following form

$$\bar{x} = e^{-g\theta/2\mu}(\Delta x\cos\theta + \Delta p\sin\theta), \quad \bar{p} = e^{-g\theta/2\mu}(-\Delta x\sin\theta + \Delta p\cos\theta), \tag{43}$$

where Δx and Δp are the initial values of $\bar{x}$ and $\bar{p}$, respectively, produced by a noise kick. These equations demonstrate that an initial perturbation exponentially dies out with the decrement equal to that obtained from the matrix analysis, Eq. (36).

Now when we have found the dependence $\bar{p}$ versus θ and know the interaction with the feedback system, we can solve for the motion of each particle of the beam. To this aim, we will solve the equation of motion for a particle having the tune which slightly differs (by the amount $\Delta\nu$) from the tune ν of the centroid of the beam. In our notation, these equations have a form:

$$\frac{dx}{d\theta} = p, \quad \frac{dp}{d\theta} = -\left(1 + \frac{\Delta\nu}{\nu}\right)^2 x - g\bar{p}\tilde{\delta}(\theta). \tag{44}$$

In the limit $g \to 0$, they govern the motion of a linear oscillator with the tune $\nu + \Delta\nu$, and the g-term accounts for the interaction with the feedback. Analogous to Eq. (42), using Eqs. (43) and (44) one finds

$$\frac{d^2x}{d^2\theta} + \left(1 + \frac{\Delta\nu}{\nu}\right)^2 x = -\frac{g}{\mu}e^{-g\theta/2\nu}(-\Delta x \sin\theta + \Delta p \cos\theta). \tag{45}$$

Integrating this equation with the initial values

$$x(t = 0) = x_0 + \Delta x, \quad p(t = 0) = p_0 + \Delta p,$$

where x_0 and p_0 give the position of the particle in the phase plane before the displacement, one finds the following result in the limit of large θ,

$$x = \left(x_0 + \frac{4\pi\Delta\nu}{g}\Delta p\right)\cos\left[\left(1 + \frac{\Delta\nu}{\nu}\right)\theta\right] + \left(p_0 - \frac{4\pi\Delta\nu}{g}\Delta x\right)\sin\left[\left(1 + \frac{\Delta\nu}{\nu}\right)\theta\right]. \tag{46}$$

From these equations we see that if $\Delta\nu \neq 0$, the feedback does not restore the initial values x_0 and p_0 which the particle had before the noise kick, but rather slightly changes them. This residual perturbation will eventually evolve, on much larger time scale associated with the decoherence process, to an incremental increase of the emittance $\Delta\varepsilon$. To find $\Delta\varepsilon$, one has to average $(x^2 + p^2)/2$ over the particle distribution function. In doing so one finds that the linear terms in Δx and Δp cancel and the result is

$$\Delta\varepsilon = \frac{16\pi^2\overline{\Delta\nu^2}}{g^2}\Delta\varepsilon_0, \tag{47}$$

where $\Delta\varepsilon_0 = (\Delta x^2 + \Delta p^2)/2$ is the increase of the emittance that would occur if there were no feedback in the system.

After we have considered one kick and shown that the increase of the emittance with the feedback is suppressed according to Eq. (47) we can generalize this result for many independent uncorrelated kicks. If inequality (37) holds, that is the damping time is much smaller than the decoherence time, one can consider that the feedback reacts independently to each noise kick on the beam. In this case increases in the emittance due to single noise kicks are simply summed up giving the following emittance growth

$$\frac{d\langle\varepsilon\rangle}{dt} = \frac{16\pi^2\overline{\Delta\nu^2}}{g^2}\left(\frac{d\langle\varepsilon\rangle}{dt}\right)_0, \tag{48}$$

where $\left(d\langle\varepsilon\rangle/dt\right)_0$ is given by Eq. (19).

Now let us add the effect of the errors associated with the measurements of the position of the beam. For a wide-band feedback system, one can expect these errors to be a white noise with the mean square equal to X^2_{noise}. This noise generates additional kicks on the beam according to Eq. (31),

$$\Delta p^2_{noise} = \frac{g^2}{\beta_1}X^2_{noise}, \tag{49}$$

which, in turn, without feedback would cause the emittance growth

$$\frac{d\varepsilon_{noise}}{dt} = \frac{f_0 g^2}{2\beta_1}X^2_{noise}. \tag{50}$$

With the feedback, this term must be added to $\left(d\langle\varepsilon\rangle/dt\right)_0$ in Eq. (48) giving,

$$\frac{d\langle\varepsilon\rangle}{dt} = \frac{16\pi^2\overline{\Delta\nu^2}}{g^2}\left[\left(\frac{d\langle\varepsilon\rangle}{dt}\right)_0 + \frac{f_0 g^2}{2\beta_1}X^2_{noise}\right]. \tag{51}$$

As an illustration of the using of Eq. (51) we show in Figure 2 how the emittance doubling time $T = \varepsilon_0\left(d\langle\varepsilon\rangle/dt\right)^{-1}$ in an accelerator with the feedback depends on the level of external noise (measured in terms of the emittance doubling time without feedback, $T_0 = \varepsilon_0\left(d\langle\varepsilon\rangle/dt\right)_0^{-1}$) and the accuracy of the Beam Position Monitor (BPM). Figure 2 clearly demonstrates that in order to suppress the emittance growth one has to have a good resolution of the BPM (small values of X_{noise}), otherwise turning on of the feedback system might even decrease the emittance doubling time in the machine.

5 CONCLUSION

In this paper we presented a derivation and analysis of emittance growth due to external random noise in a hadron accelerator. The theory predicts that the rate of the emittance growth is proportional to the values of the noise spectrum at the betatron sideband frequencies $f_0|\nu - n|$. It is important to emphasize here that the mechanism which is responsible for growth of the emittance of the beam is the tune spread, though formally $\Delta\nu$ does not enter Eq. (19).

We also considered the work of the feedback system and derived the residual emittance growth rate. If the gain of the feedback is much larger than the tune spread in the beam, $g \gg \Delta\nu_{rms}$, one finds a decrease in the growth rate proportional to g^2 according to Eq. (51). In the opposite limit, $g \ll \Delta\nu_{rms}$, the feedback would not suppress the emittance growth because decoherence proceeds faster than damping. Internal noise of the feedback has been also be included in the theory in terms of the accuracy of the BPM.

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APPENDIX

To calculate the sum

$$\Re = \sum_{n,m=0}^{N-1} e^{i\omega T(n-m)} \sin\mu(N-n)\sin\mu(N-m) \tag{A.1}$$

from Eq. (11) in the limit $N \to \infty$, first, we express sine in terms of the exponential and perform the summation:

$$\Re = -\frac{1}{4}e^{2i\mu N}\frac{\left(e^{i(\omega T-\mu)N}-1\right)\left(e^{-i(\omega T+\mu)N}-1\right)}{\left(e^{i(\omega T-\mu)}-1(e^{-i(\omega T+u)}-1\right.} + \frac{1}{4}\frac{\left(e^{i(\omega T+\mu)N}-1\right)\left(e^{-i(\omega T+\mu)N}-1\right)}{\left(e^{i(\omega T+\mu)}-1(e^{-i(\omega T+u)}-1\right.} + c.c. =$$

$$= \frac{1}{4}e^{i\mu(N+1)}\frac{\sin \tfrac{1}{2}N(\omega T-\mu)}{\sin \tfrac{1}{2}(\omega T-\mu)}\frac{\sin \tfrac{1}{2}N(\omega T+\mu)}{\sin \tfrac{1}{2}(\omega T+\mu)} + \frac{1}{4}\left(\frac{\sin \tfrac{1}{2}N(\omega T+\mu)}{\sin \tfrac{1}{2}(\omega T+\mu)}\right)^2 + c.c., \tag{A.2}$$

where $c.c.$ denotes the complex conjugate. In the limit $N \to \infty$, the following identities are valid

$$\frac{\sin \tfrac{1}{2}N\xi}{\sin \tfrac{1}{2}\xi} \to 4\pi\tilde{\delta}(\xi), \quad \frac{\sin^2 \tfrac{1}{2}N\xi}{\sin^2 \tfrac{1}{2}\xi} \to 2\pi N\tilde{\delta}(\xi), \tag{A.3}$$

where

$$\tilde{\delta}(\xi) = \sum_{n=-\infty}^{\infty} \delta(\xi - 2n\pi). \tag{A.4}$$

Now, note that $\tilde{\delta}(\omega T-\mu)\tilde{\delta}(\omega T+\mu)$ is identically equal to zero, unless ν is equal to half an integer—the possibility excluded for an accelerator. The only term that is left in Eq. (A.2) is

$$\Re = \pi N\tilde{\delta}(\omega T-\mu). \tag{A.5}$$

Putting (A.5) in Eq. (11) gives Eqs. (12) and (13).

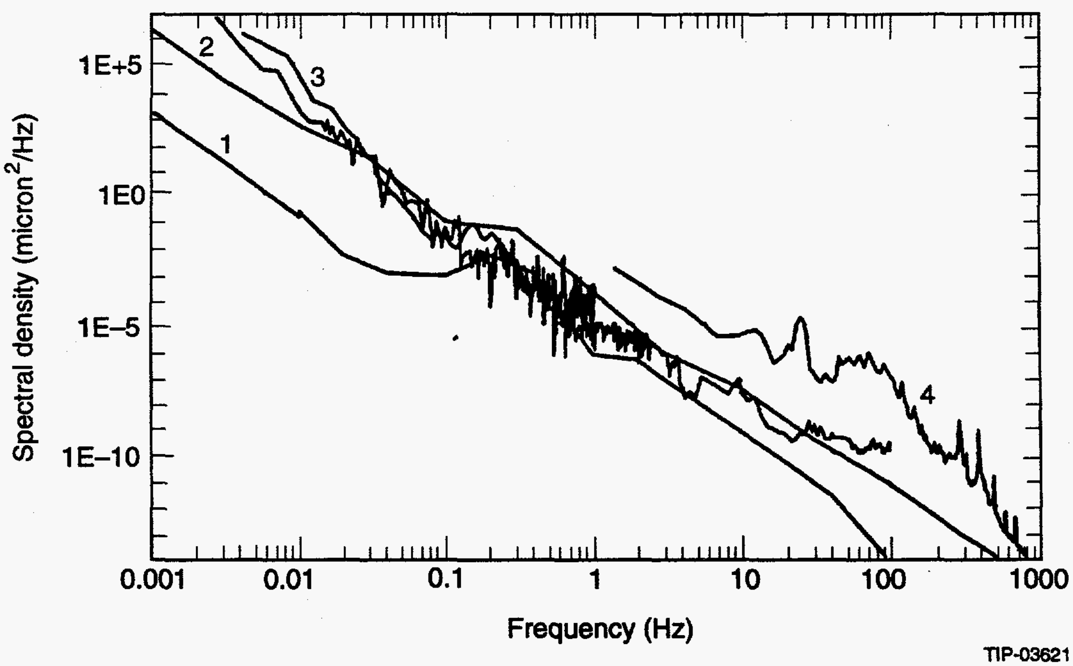

FIGURE 1. Spectra of ground motion at different sites: 1- Ref. 11, 2 - Ref. 12, 3 - Ref. 1, 4 - Ref. 13.

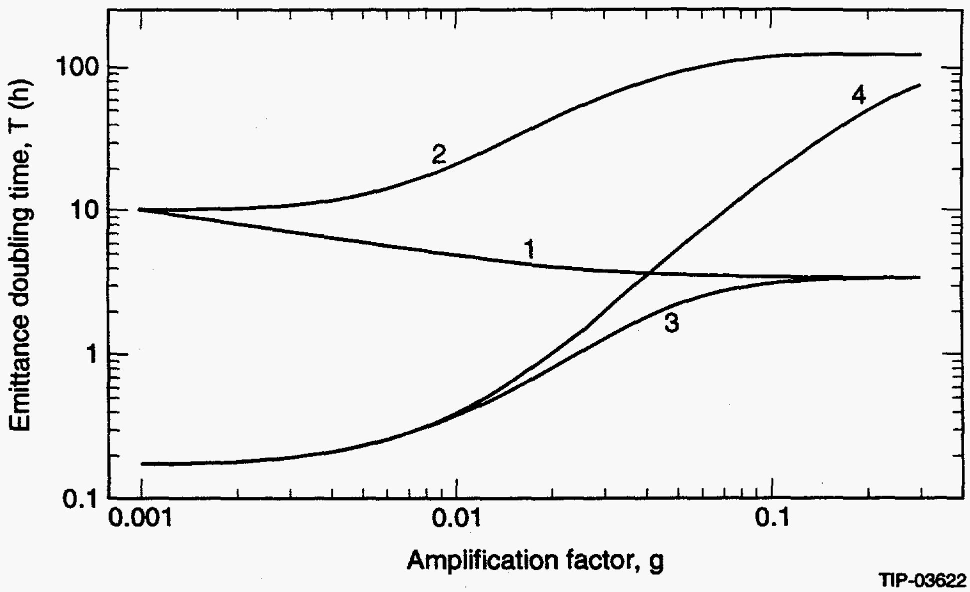

FIGURE 2. Emittance doubling time T as a function of the feedback gain g, for different values of T_0 and X_{noise}: *1*- $T_0 = 10$ h, $X_{noise} = 3.5$ µm, *2*- $T_0 = 10$ h, $X_{noise} = 0.5$ µm, *3*-$T_0 = 10$ min, $X_{noise} = 3.5$ µm, *4* - T_0=10 min, X_{noise} = 0.5 µm. Other parameters are: $\beta_1 = 350$ m, $\Delta\nu_{rms} = 1.8\cdot10^{-4}$.