SOME PROPERTIES OF TRANSFINITE CARDINAL

AND ORDINAL NUMBERS (2)

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1. Introduction: Some Properties of Sets in General

Notion of set.--We shall regard the notion of set (or class or aggregate) as a primitive idea, e. g., the set of books in a certain library, the set of pupils in a certain class, a set of definitions, or even a set of sets. According to Cantor, a set, or aggregate, is a collection of definite distinct objects which is regarded as a single whole.[1]

We think of a set not as a group of things that can be specified by enumerating its members one after the other, but as something that can be determined by a property, which can be used to test the claim of any object to be a member of the set.

Examples of sets are (1) the set of even integers, determined by the property of being twice some other integer; (2) the set of algebraic numbers, determined by the property of satisfying a polynomial equation with rational coefficients.

The letters A, B, C, . . . will be used to denote sets. A typical member of element of A will be denoted by a, of B by b, of C by c, ... The constitution of the set A may be denoted by the equation $A = \{a\}$. $a \in A$ denotes that a is an element of A.

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[1]This definition is given by G. Cantor, "Beiträge zur Begründung der transfiniten Mengenlehre," Math. Annalen, Vol. XLVI (1895), p. 481, as follows: "Unter einer 'Menge' verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objecten m unsrer Anschauung oder unseres Denkens (welche die 'Elemente' von M genannt werden) zu einem Ganzen."

$a \bar{\in} A$ denotes that a is not an element of A. A set A is defined when it can be said of every element x whether $x \in A$ or $x \bar{\in} A$.

The set which has only the single element a is denoted by (a). Thus $x \in (a)$ means $x = a$.

<u>Null-set</u>.—If we were to regard a set as a collection of things, a set with no elements would be a rather shadowy or even paradoxical entity; but the mysterious quality disappears if statements about sets are interpreted as statements about properties. Thus, the <u>null-set</u> (denoted by 0) is the set which has no elements. Let p be called a null-property if it is not possessed by any object. Examples of null-properties are (1) being a real number greater than three and less than two; (2) being a zero of e^z. Any two null-properties are "formally equivalent," i. e., no object has one property without having the other; and therefore all of these properties determine the same set, which we call the null-set.

<u>Comparison of sets</u>.—The set A is said to be a <u>subset</u> of the set B, $A \subseteq B$, or $B \supseteq A$, if every element of A is an element of B. The set A and the set B are <u>identical</u>, $A = B$, if every element of A is an element of B, and conversely. The set A is a <u>proper subset</u>, or part, of the set B, $A \subset B$, if every element of A is an element of B, and there exists an element of B which is not an element of A.

<u>The null-set is a subset of every set</u> ($0 \subseteq A$). To arrive at this statement, it is necessary to consider more closely the definition of $A \subseteq B$. $A \subseteq B$ was defined to mean that every element, say x, of A was an element of B. This means that $x \in B$ unless $x \bar{\in} A$,

i. e., $x \in B$ is true, or $x \in A$ is false. This final form may be taken as the basic meaning of $A \subseteq B$, and from it, it is clear that $0 \subseteq A$, since, for every element x, $x \in 0$ is false, the statement, " $x \in A$ is true, or $x \in 0$ is false," is true, whatever the set A may be.

The following properties of sets can be accepted as self-evident:[2]

1) $A \subseteq A$,

2) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$,

3) $A = B$ if, and only if, $A \subseteq B$ and $B \subseteq A$.

3) could be regarded as the definition of equality of two sets.

2. Algebra of Sets

Addition of sets.--The logical sum of two sets A and B, $A + B$, is the set of all elements in at least one of the sets A and B. The logical sum of the sets $A, B, C, \ldots, A + B + C + \ldots$, is the set of all elements in at least one of the sets $A, B, C, \ldots$. The number of sets $A, B, C, \ldots$ need not be "finite." The sum set is independent of the order in which the sets are taken.

The commutative and associative laws of algebra hold for the addition of sets, i. e.,

1) $A + B = B + A$,

2) $A + (B + C) = (A + B) + C$.

Also, the following laws hold for the addition of sets:

1) $A + A = A$,

2) $A \subseteq A + B$,

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[2] M. H. A. Newman, Topology of Plane Sets, p. 4.

3) If $A \subseteq C$ and $B \subseteq C$, then $A + B \subseteq C$,

4) $A + 0 = A$ (from 3)).

Product of sets.--The logical product, or the common part, or the intersection, of two sets A and B, $A\,B$ or $A \cdot B$, is the set of all points which are in A and B, i. e., " $x \in A\,B$ " means " $x \in A$ " and " $x \in B$." The usual convention concerning parentheses is adopted, i. e., $A\,B + C$ means $(A\,B) + C$.

The commutative and associative laws of multiplication and the distributive law with respect to addition hold for the multiplication of sets, i. e.,

1) $A\,B = B\,A$,

2) $A\,(B\,C) = (A\,B)\,C$,

3) $A\,(B + C) = A\,B + A\,C$.

Also, the following laws hold for the multiplication of sets:

1) $A\,A = A$,

2) $A\,B \subseteq A$,

3) If $A \supseteq C$ and $B \supseteq C$, then $A\,B \supseteq C$,

4) $A\,0 = 0$ (from 3)).

Two sets A and B are said to intersect if they have at least one common element. It follows from the definition of the null-set that a necessary and sufficient condition that A and B intersect is that $A\,B \neq 0$. Two sets A and B are said to be mutually exclusive if $A\,B = 0$.

Subtraction of sets.--The logical difference between two sets A and B, $A - B$, is the set of elements belonging to A but not to B, i. e., " $x \in (A - B)$ " means " $x \in A$ " but " $x \bar{\in} B$." Evidently $A - A = 0$, and $A - 0 = A$.

The agreeable similarity so far observable between the algebra of sets and ordinary algebra breaks down with the introduction of subtraction; for

$$A + (A - A) = A + 0 = A \ ,$$

$$(A + A) - A = A - A = 0 \ .$$

This is due to the fact that $A - B$ is not necessarily a solution of the equation $B + X = A$, which may have an infinity of solutions (e. g., if $A = B$), or none (if $A = 0$). It is possible, however, to maintain a workable algebra by operating with complements with respect to a fixed set S.

If $A \subseteq S$, the set $S - A$ is called the <u>complement</u> of A with respect to S. If S is supposed fixed, $S - A$ may be denoted by $\overline{A}$.

Besides the following properties, which are obvious,

1) $\overline{S} = 0$; $0 = \overline{S}$,

2) $A + \overline{A} = S$; $A\overline{A} = 0$,

3) Complement of $\overline{A} = A$,

4) If $A \subseteq B$, then $\overline{B} \subseteq \overline{A}$,

5) If $A + X = S$, and $AX = 0$, then $X = \overline{A}$,

the complement has the important property of interchanging $+$ and $\cdot$,

6) $\overline{A + B} = \overline{A}\ \overline{B}$.

We now return to the difference, $A - B$, between two sets. If S is any set containing A and B, the property " x belongs to A but not to B," defining $A - B$, is evidently equivalent to " x belongs to A and S, but not to B," i. e., to ($x \in A$) and ($x \in S$ but $x \bar{\in} B$). The first parenthesis is the determining property of A, the second that of $\overline{B}$. Hence the seventh property,

7) If complements are taken with respect to any set containing both A and B, $A - B = A\bar{B}$.

By means of 7) all differences occurring in any formula can be expressed in terms of complements with respect to a fixed set, say S, containing all the sets involved, and the properties 1)-6) applied.

For example, if all complements are formed with respect to an arbitrary set S, containing A, B and C,

1) $A(B - C) = B(A - C) = AB - C = AB\bar{C}$,

2) $B - A = B - AB = B\,\overline{(AB)} = B\bar{A} + B\bar{B} = B\bar{A}$.

__Duality.__--The algebra of sets so far developed has a duality property probably already observed by the reader. If in any theorem of the algebra of sets all differences are expressed in terms of complements with respect to a fixed set S, and then the symbols $+$ and $\cdot$ are everywhere interchanged, the result is also a true theorem of the algebra of sets.[3]

__Sum and product notations.__--If M is a set of sets, the elements of M are usually denoted by a subscript notation, A_χ. The subscript χ may range through any set B, e. g., the integers from 1 to k, all the positive integers, all the real numbers, and so on. When this notation is used for the elements of M, the set M itself is denoted by $\{A_\chi\}$.

The sum-set $\sum_{\chi\in B} A_\chi$ is the set of all elements of the sets A_χ, i. e., "$z \in \sum_{\chi\in B} A_\chi$" means "for some element x of B, $z \in A_\chi$." The product-set $\prod_{\chi\in B} A_\chi$ is the set of elements that belong to all the

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[3]D. Hilbert und W. Ackermann, __Grundzüge der Theoretischen Logik__, p. 13.

A_χ, i. e., " $z \in \prod_{\chi \in B} A_\chi$ " means for every element x of B, $z \in A_\chi$." The notations for sum-set and product-set may be abbreviated to $\sum_\chi A_\chi$ and $\prod_\chi A_\chi$, or even ΣA and ΠA, when the meaning is clear. When the subscripts are positive integers the sum is denoted by $\sum_1^k A_n$ or $\sum_1^\infty A_n$, and the product similarly.

The following are some formal properties of Σ and Π:

1) If $b \in B$, $\prod_{\chi \in B} A_\chi \subseteq A_b \subseteq \sum_{\chi \in B} A_\chi$.

2) If, for every element b of B, $A_b \subseteq C$, then $\Sigma A_\chi \subseteq C$, i. e., if $b \in B$ implies $A_b \subseteq C$, then " $z \in A_b$ and $b \in B$ " implies $z \in C$.

3) If, for every element b of B, $A_b \supseteq C$, then $\Pi A_\chi \supseteq C$.

4) If for every x, $A_\chi \subseteq B_\chi$, then $\Sigma A_\chi \subseteq \Sigma B_\chi$ and $\Pi A_\chi \subseteq \Pi B_\chi$.

5) $\Sigma (A_\chi + B_\chi) = \Sigma A_\chi + \Sigma B_\chi$.

6) $\Pi (A + B_\chi) = A + \Pi B_\chi$.

7) $\Sigma (A B_\chi) = A \Sigma B_\chi$.

8) If S contains all the sets A_χ, then ΣA_χ and $\Pi(S - A_\chi)$ are complementary sets in S, i. e., $\overline{\Sigma A_\chi} = \Pi \overline{A}_\chi$.

3. Denumerable and Non-Denumerable Sets

Correspondence.--The notion of correspondence underlies the process of tallying.[4] The elements of one set may be made to stand in some logical relation with those of another set, so that a definite element of one set is regarded as correspondent to a definite element of another set.

The correspondence may be complete, in the sense that, to every

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[4] E. W. Hobson, The Theory of Functions of a Real Variable, Vol. I, p. 2.

element of either set there corresponds at least one element of the other set; or the correspondence may be incomplete, in which case at least one of the sets has one or more elements to which no elements in the other set correspond.

A correspondence between two sets is defined when specifications or rules are laid down which suffice to decide which elements of one set correspond to each element of the other set; so that, in the case of complete correspondence, no element of either set is without a corresponding one in the other.

A biunique correspondence between two sets A and B, or a (1,1)-mapping, say f, of a set A on a set B, is determined if with every element x of A there is associated an element $f(x)$ of B, the image of x, in such a way that each element y of B is the image of just one element of A (which is called $f^{-1}(y)$). The condition is symmetrical between A and B, and f^{-1} is a (1,1)-mapping of B on A. This is a complete correspondence in the sense above. The set A is equivalent to the set B, $A \sim B$, if a biunique correspondence can be set up between A and B. It must be noticed that the idea of equivalence contains no reference to order. Clearly,

1) $A \sim A$ (reflexive),

2) If $A \sim B$, then $B \sim A$ (symmetrical),

3) If $A \sim B$ and $B \sim C$, then $A \sim C$ (transitive).

A finite set is a set which is either the null-set or, for some positive integer n, is equivalent to the set $I_n = \{1, 2, \ldots, n\}$. Thus, the elements of a non-null finite set can be named $a_1, a_2, \ldots, a_m$. An infinite set is a set which is not a finite set.

Two equivalent sets are evidently both finite or both infinite. The following obvious properties of finite sets are assumed implicitly in the simplest everyday use of numbers:

1) Every subset of a finite set is finite;

2) The sum of a finite set of finite sets is finite.

The set of all positive integers is an infinite set. For, if $I = \{1, 2, \ldots, m, \ldots\}$ is the set of all positive integers, obviously I is not equivalent to the finite set (1). Assume that I is not equivalent to the finite set I_k. Suppose that I_{k+1} is equivalent to I, with $f(k+1) = m$. Then I_k is equivalent to $I - (m)$. Now mate biuniquely the elements of $I - (m)$ and I so that every element ν of I that is less than m is mated to the element ν of $I - (m)$, and every element ν of I greater than or equal to m is mated to the element $\nu + 1$ of $I - (m)$. Then I_k would be equivalent to I, which is a contradiction. Thus I_{k+1} cannot be equivalent to I. Hence I must be infinite.

Denumerable sets.--A set which can be put into biunique correspondence with the set of all natural numbers is said to be denumerable (enumerable, countable, "abzälbar," or "dénombrable"). Hence the elements of a denumerable set can be enumerated as an infinite sequence, $a_1, a_2, \ldots,$ with increasing subscripts. Conversely, the set of all terms of an infinite sequence is denumerable.

Every subset of a denumerable set, if not finite, is obviously denumerable, since any subset of a sequence may be arranged as a sequence with increasing indices. Thus the set of all odd numbers, all prime numbers, all squares are each denumerable.

<u>The sum of a finite set and a denumerable set is a denumerable set.</u> For, the sum of the finite set

$$a_1, a_2, \ldots, a_\nu$$

and the set

$$b_1, b_2, \ldots, b_\nu, \ldots$$

may be written as the infinite sequence

$$a_1, a_2, \ldots, a_\nu, b_1, b_2, \ldots, b_\nu, \ldots.$$

Also, <u>the sum of a denumerable set of finite sets is either finite or denumerable.</u> Let $A = A_1 + A_2 + \ldots$, where

$$A_1 = \{a_{11}, a_{12}, \ldots, a_{1m_1}\}$$

$$A_2 = \{a_{21}, a_{22}, \ldots, a_{2m_2}\}$$

$$\vdots$$

writing the sequence

$$a_{11}, a_{12}, \ldots, a_{1m_1}, a_{21}, a_{22}, \ldots, a_{2m_2}, \ldots,$$

which contains the elements of A, possibly with repetitions, if, for example, A_1 and A_2 contain elements in common. Therefore the set A is finite or denumerable.

<u>Theorem 3.1. The sum of a finite or denumerable set of denumerable sets is denumerable.</u>

For, if $A_1, A_2, \ldots$ be a sequence of denumerable sets, the elements of the set

$$S = A_1 + A_2 + \ldots$$

may be written down as a double sequence

$$\begin{array}{llllll}
A_1 : & a_{11}, & a_{12}, & a_{13}, & a_{14}, & \cdots \\
A_2 : & a_{21}, & a_{22}, & a_{23}, & a_{24}, & \cdots \\
A_3 : & a_{31}, & a_{32}, & a_{33}, & a_{34}, & \cdots \\
A_4 : & a_{41}, & a_{42}, & a_{43}, & a_{44}, & \cdots \\
\vdots & & & & & \\
A_n : & a_{n1}, & a_{n2}, & a_{n3}, & a_{n4}, & \cdots \\
\vdots & & & & & .
\end{array}$$

where $a_{k1}, a_{k2}, \ldots$ are the elements of A_k. Arranging the elements of the double sequence according to the "diagonal" method of enumeration, i. e., such that the <u>nth</u> group consists of all elements a_{kl}, where $k + l = n + 1$, we obtain the sequence

$$a_{11}, a_{12}, a_{21}, a_{31}, a_{22}, a_{13}, a_{14}, a_{23}, a_{32}, a_{41}, \ldots,$$

containing all elements of S. Hence the set S is finite or denumerable. But S contains the infinite subset A and is accordingly itself infinite. It follows that S is denumerable.

<u>The set of all integers (positive, negative and zero) is denumerable.</u>

<u>Corollary</u> 3.1.1. <u>The set of all rational numbers is denumerable.</u>

For, the set $R = A_1 + A_2 + \ldots$, where

$$A_1 = \{0, 1, 1/2, 1/3, 1/4, \ldots\}$$
$$A_2 = \{-1, -1/2, -1/3, -1/4, -1/5, \ldots\}$$
$$A_3 = \{2, 1, 2/3, 2/4, 2/5, \ldots\}$$
$$\vdots$$

By Theorem 3.1, R is denumerable.

It follows that <u>the set of all positive rational numbers and the set of all negative rational numbers are each denumerable.</u>

Corollary 3.1.2. The set of rational points, i. e., points with rational coordinates, in n-space is denumerable.

It will suffice to prove the theorem for two-space. If

$$R = \{r_1, r_2, \ldots\}$$

is the set of all rational numbers, the set of rational points in two-space may be expressed by

$$S = A_1 + A_2 + \ldots,$$

where $$A_\nu = \{(r_\nu, r_1), (r_\nu, r_2), \ldots\}.$$

Therefore S is denumerable.

It follows that the set of all points with integral coordinates in n-space is denumerable.

Corollary 3.1.3. The set of all algebraic numbers is denumerable.

The set S of all polynomials with rational coefficients is denumerable, since a biunique correspondence may be established between the set S_n of polynomials of degree less than or equal to n and the set of rational points in $(n + 1)$-space, by mating the polynomial $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ with $(a_0, a_1, a_2, \ldots, a_n)$, and $S = S_1 + S_2 + \ldots$. A polynomial has at most a finite number of distinct roots. Hence the set of algebraic numbers is finite or denumerable. But A is not finite, since every integer n is algebraic (is in fact a root of $x - n = 0$). Hence A is denumerable.

Non-denumerable sets.—A set which is neither finite nor denumerable is said to be a non-denumerable set. For example, the set S of all infinite sequences of natural numbers is non-denumerable. For, if S were denumerable, say $S = \{S_1, S_2, \ldots\}$, it could be written as the double sequence:

$$
\begin{aligned}
S_1 &: n_{11}, n_{12}, \ldots, \\
S_2 &: n_{21}, n_{22}, \ldots, \\
S_3 &: n_{31}, n_{32}, \ldots, \\
&\vdots
\end{aligned}
$$

But the infinite sequence $n_{11}+1, n_{22}+1, \ldots$ differs from each of the above sequences and so is not an element of S, which is contrary to the hypothesis that S consists of all infinite sequences of natural numbers.

<u>Theorem</u> 3.2. <u>The set of all real numbers is non-denumerable.</u>

Given a biunique correspondence between the set of all natural numbers and any set S of real numbers a, $0 < a < 1$, we shall show that there is at least one real number, say b, $0 < b < 1$, which is not in S. Since the numbers in S are mated biuniquely to the natural numbers, we may write them as a sequence of real numbers $a_1, a_2, \ldots$. Since every real number can be expressed as an infinite decimal, express the numbers in the above sequence in decimal notation, thus

$$
\begin{aligned}
a_1 &= .a_{11}\, a_{12} \ldots, \\
a_2 &= .a_{21}\, a_{22} \ldots, \\
&\vdots
\end{aligned}
$$

Let the number $b = .b_1 b_2 \ldots$ be formed as follows: $b_n = 1$ if $a_{nn} \neq 1$, $b_n = 2$ if $a_{nn} = 1$. Since b is different from every a_n in the given sequence, the set of all real numbers is non-denumerable.

<u>If a finite or denumerable set be removed from a non-denumerable set, the remaining set is non-denumerable.</u> For, let P be a

non-denumerable set, Q a finite or denumerable set contained in P, and R the remainder of P. Since $P = Q + R$, if R were finite or denumerable, then P would be finite or denumerable, contrary to assumption. Therefore R is non-denumerable.

After removing from the set of real numbers the set of rational numbers, there remains the non-denumerable set of irrational numbers. Similarly, after removing from the set of real numbers the set of algebraic numbers, there remains the non-denumerable set of transcendental numbers.

4. Cardinal Number of Sets

Notion of cardinal number.--Sets which are equivalent to one another are said to have the same cardinal number (or potency or power).

A cardinal number is accordingly characteristic of a class of equivalent sets. The question whether two defined sets have, or have not, the same cardinal number is thus equivalent to the question whether it is, or is not, possible to establish a biunique correspondence between the elements of the two sets. It follows from the definition that to every set corresponds a cardinal number.

The law of correspondence which can be established between an element of A and an element of B is in general of a character which admits of a certain arbitrariness. The cardinal number is accordingly regarded as independent of the notion of order in the set.

The power, or potency, or cardinal number, of a set A has been defined by Cantor as the concept which is obtained by abstraction when the nature of the elements of A, and the order in which they are given, are entirely disregarded. The cardinal number of the set A is sometimes denoted by $\overline{\overline{A}}$.

We shall use the Greek letters α, β, γ, . . . to denote cardinals. When α and A occur simultaneously it is to be understood that A has cardinal α.

Since Cantor regards the cardinal number of A as independent of the precise nature of the elements of A, we may, in accordance with this view, substitute for each element the number unity. We have thus a new set which is a collection of elements each of which is the number 1, and is equivalent to A; this new set is regarded by Cantor as a symbolical representation of the cardinal number α.

<u>The cardinal numbers</u> $\aleph_0$ <u>and</u> c.--Cardinal numbers different from the natural numbers are called <u>transfinite cardinal numbers</u>. There exist different transfinite cardinals, e. g., $\aleph_0$ and c. The cardinal number corresponding to the class of all denumerable sets is denoted by $\aleph_0$ and the one corresponding to the class of all sets that can be mated biuniquely with the set of all real numbers by c. Sets with cardinal number c are said to have the cardinal number, or potency, of the continuum.

<u>Comparison of cardinals and sets</u>.-- $\alpha = \beta$ is defined to mean that $A \sim B$. The set A has <u>greater cardinal number</u> than the set B, in notation, $A \succ B$, $B \prec A$ or $\alpha > \beta$, $\beta < \alpha$, is defined to mean that

1) B can be put into biunique correspondence with a subset of A;
2) A cannot be put into biunique correspondence with a subset of B.

Evidently, if A and B are any two sets, not more than one of the relations $A \sim B$, $A \succ B$, $A \prec B$ holds; but we cannot, as yet, state that the cardinal numbers α, β of any two sets whatever must satisfy one of the three relations $\alpha = \beta$, $\alpha > \beta$, $\alpha < \beta$, since there is the possibility that neither A nor B is equivalent to a subset of the other. Two sets which are such that their cardinal numbers α, β stand to one another in one of the relations $\alpha = \beta$, $\alpha < \beta$, $\alpha > \beta$ may be said to be comparable with one another. Otherwise they are incomparable with one another. We leave the question of incomparability to be settled later (Corollary 6.5.1).

Addition of cardinals.--Given two mutually exclusive sets A and B, the sum of α and β, $\alpha + \beta$, is the cardinal number of the set $A + B$. Clearly, the commutative and associative laws hold for addition of cardinals, i. e.,

1) $\alpha + \beta = \beta + \alpha$,

2) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

A similar definition applies to the case of the cardinal number of the sum of any number of sets, no two of which have an element in common.

It follows from Section 3 that

1) $n + \aleph_0 = \aleph_0$,

2) $\aleph_0 + \aleph_0 = \aleph_0$,

3) $\aleph_0 + \aleph_0 + \aleph_0 + \ldots$ ($\aleph_0$ terms) $= \aleph_0$.

Consider the set C of all real numbers, and let A be the set of all rational numbers and B the remainder. Then

$$c = \aleph_0 + \beta = \beta + \aleph_0,$$

$$c + \aleph_0 = (\beta + \aleph_0) + \aleph_0 = \beta + (\aleph_0 + \aleph_0) = \beta + \aleph_0 = c.$$

For n, a natural number,

$$c + m = (c + \aleph_0) + m = c + (\aleph_0 + m) = c + \aleph_0 = c .$$

The set of all real numbers x satisfying the inequality $a < x < b$ can be put into biunique correspondence with the set of all real numbers y. For, consider the mating

$$y = \frac{x - \frac{a+b}{2}}{\frac{b-a}{2} - \left| x - \frac{a+b}{2} \right|} .$$

Thus the set of all real numbers x, $a < x < b$, has cardinal number c. This cardinal number will not change if we add a finite number of elements to the set. Hence, for every a and $b > a$, the set of all real numbers x, $a \leq x \leq b$ or $a < x \leq b$ or $a \leq x < b$, has cardinal number c.

The set E_1 of all real numbers x satisfying the inequality $0 \leq x < 1$, the set E_2 of all real numbers x satisfying the inequality $1 \leq x < 2$, and the set S of all real numbers x satisfying the inequality $0 \leq x < 2$ all have cardinal number c. But E_1 and E_2 are mutually exclusive sets and $E_1 + E_2 = S$; therefore

$$c + c = c .$$

Similarly, it can be shown that

$$c + c + c + \ldots = c .$$

Let A have cardinal $\alpha > \aleph_0$, D have cardinal $\aleph_0$. Then D is equivalent to a proper subset A_1 of A, and A_1 has cardinal $\aleph_0$;

$$A + D = (A - A_1 + A_1) + D = (A - A_1) + (A_1 + D)$$

and $A_1 + D$ has cardinal $\aleph_0$. Also

$$A = (A - A_1) + A_1 .$$

Mate $A - A_1$ with $A - A_1$; $A_1 + D$ with A_1. This mates $A + D$

with A ; hence

$$\alpha + \aleph_0 = \alpha \quad \text{if} \quad \alpha > \aleph_0 .$$

It follows that <u>if a (finite or) denumerable set be taken from a set of cardinal</u> $\alpha > \aleph_0$, <u>a set of cardinal</u> α <u>remains</u>.

<u>Multiplication of cardinals</u>.--Given two sets A and B , the product of α and β , $\alpha\beta$, is the cardinal number of the set consisting of all pairs of elements (a, b) , where $a \in A$ and $b \in B$.

This definition of multiplication may be extended. Let E be a finite or infinite class of sets. The product of the cardinal numbers of the sets of E is the cardinal number of the class of all sets consisting of one and only one element from each set in E .

It is easily seen that the commutative and associative laws hold for the multiplication of cardinal numbers, i. e.,

1) $\alpha\beta = \beta\alpha$,

2) $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.

If m be a finite cardinal and β any cardinal number, we have

$$m\beta = \beta + \beta + \cdots + \beta \text{ (to } m \text{ terms)} .$$

For, let A be the finite set of natural numbers 1, 2, . . . , m ; B a set with cardinal β ; and S the set of all pairs (a, b) , where $a \in A$ and $b \in B$. S has cardinal $\alpha\beta$. Denoting for a given k the set of all pairs (k, b) by S_k , the cardinal of S_k is obviously β , and $S = S_1 + S_2 + \cdots + S_m$, where the sets S_k are mutually exclusive. In particular, $m\aleph_0 = \aleph_0$ and $mc = c$. Similarly, it can be shown that, for every cardinal number β , $\aleph_0\beta = \beta + \beta + \cdots$, and thus $\aleph_0\aleph_0 = \aleph_0 + \aleph_0 + \cdots = \aleph_0$ and $\aleph_0 c = c + c + \cdots = c$.

To obtain the product cc , let S be the set of all pairs

(x, y) where $0 < x \leq 1$, $0 < y \leq 1$. Let (x, y) be an element of S. Develop x and y as non-terminating decimal fractions, e. g.,

$$x = .x_1 x_2 \ldots \quad (0 \leq x_i \leq 9),$$

$$y = .y_1 y_2 \ldots \quad (0 \leq y_i \leq 9).$$

Divide the digits to the right of each decimal point into groups by means of a stroke after each digit not equal to zero; we thus get two infinite sequences of groups. Place the groups of the second sequence between the successive groups of the first sequence and we get a new sequence of groups. Omitting the strokes, we get an infinite sequence of digits, the decimal representation of a real number z, which we put in correspondence with the pair (x, y). We have a biunique correspondence between the elements of the set S and those of $C:(0 < z \leq 1)$. Hence S has cardinal c, and we have

$$c\,c = c.$$

It follows from the above that <u>the set of all pairs (x, y) of real numbers has the same cardinal number as the set of all real numbers.</u> Geometrically, this means that the set of all points in the plane has the same cardinal number as the set of all points in a straight line and, therefore, the same cardinal as the set of all points in a finite segment.

From the definition of the product of cardinal numbers, $\prod_{\nu=1}^{\infty} \alpha_\nu$ is the cardinal number of the set S of all the infinite sequences

$$a_1, a_2, \ldots,$$

where $a_k \in A_k$ for $k = 1, 2, \ldots$, A_k being any set of cardinal α_k. In particular, let each A_k be the set consisting of the numbers 0 and 1. The set S will, therefore, be the set of all infinite sequences

$$e_1, e_2, \ldots,$$

consisting of 0's and 1's. Denote by Q the set of these sequences in which there is an infinite number of 1's, and by R the remainder of S. R consists, therefore, of all those sequences in which, from a certain stage onwards, there are only 0's and so has the same cardinal number as the set of all finite sequences consisting of 0's and 1's, which is a denumerable set. The set Q, however, has the same cardinal number as the set X of all positive real numbers less than or equal to 1, since a biunique correspondence between the elements of Q and those of X may be established if the sequence $q_1, q_2, \ldots,$ belonging to Q, is put into correspondence with the number $q_1/2 + q_2/2^2 + \ldots$, which obviously belongs to X.[5] Hence, the cardinal number of S is c, and so

$$c = 2 \cdot 2 \cdot \ldots.$$

<u>Exponentiation</u>.--Let A and B be two sets, where $\beta \neq 0$. Let (B|A) be the set of all correspondences between A and B in which every element of B corresponds to a unique element of A, repetitions and omissions of elements of A being allowed. An example will make the meaning easily intelligible. Suppose that A consists of a_1 and a_2, B of b_1, b_2, b_3. Let us denote the set of correspondences $b_1 \to a_p, b_2 \to a_q, b_3 \to a_r$ for brevity by (p,q,r). Then the elements constituting (B|A) are:

(1,1,1), (1,1,2), (1,2,1), (2,1,1), (1,2,2), (2,1,2), (2,2,1), (2,2,2).

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[5] W. Sierpiński, <u>General Topology</u>, p. 218.

Since, if $A \sim A'$, $B \sim B'$, then $(B|A) \sim (B'|A')$, it is evident that the cardinal number of $(B|A)$ is independent of the choice of A and B and depends only on α and β. Thus we shall define <u>exponentiation</u> as follows: α^β is the cardinal number of the set $(B|A)$. The genesis of the definition in common sense is, of course, "the total number of ways of distributing β things among α persons, where any number of the β things may be given to one person." α^0 is defined to be 1, for $\alpha \neq 0$ (0^0 is not defined). Thus, from above, $2^3 = 8$. Furthermore, $2^{\aleph_0} = 2 \cdot 2 \cdot \ldots$ and we conclude that $2^{\aleph_0} = c$.

For three arbitrarily chosen sets A, B and C, it is easily seen that

1) $\alpha^\beta \cdot \alpha^\gamma = \alpha^{\beta+\gamma}$,

2) $\alpha^\gamma \cdot \beta^\gamma = (\alpha\beta)^\gamma$,

3) $(\alpha^\beta)^\gamma = \alpha^{\beta\gamma}$,

provided 0^0 does not occur.

<u>If</u> $\alpha < \beta$, <u>then</u> $\alpha^\gamma \leq \beta^\gamma$. For, since $A \sim B_1 \subset B$, $(C|A) \sim (C|B_1) \subset (C|B)$ or $\alpha^\gamma \leq \beta^\gamma$. Thus

$$c = 2^{\aleph_0} \leq n^{\aleph_0} \leq \aleph^{\aleph_0} \leq c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = c;$$

therefore

$$n^{\aleph_0} = \aleph_0^{\aleph_0} = c^{\aleph_0} = c, \ (n \geq 2).$$

<u>Theorem</u> 4.1. <u>If A is equivalent to a proper subset B_1 of B and B is equivalent to a proper subset A_1 of A, then A is equivalent to B.</u>[6]

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[6]Due to E. Schröder, "Über G. Cantorsche Sätze," <u>Jahresbericht der deutsch. Math.-Vereinigung</u>, Vol. V (1896), pp. 81-82, and F. Bernstein (see Émile Borel, <u>Leçons sur la Théorie des Fonctions</u>, p. 103). See also E. Zermelo, "Über die Addition transfiniter Cardinalzahlen," <u>Göttinger Nachrichten</u>, 1901, p. 34.

By the first correspondence, $A \sim B_1 \subset B$; by the second $B \sim A_1 \subset A$. Let $S_1 = B - B_1$ and $R_1 = A - A_1$. Then $A_1 \sim B_2 \subset B_1$ and $B_1 \sim A_2 \subset A_1$; let $S_2 = B_1 - B_2$, $R_2 = A_1 - A_2$ and continue this process. By the first correspondence we get $R_1 \sim S_2$, $R_3 \sim S_4$, . . . ; by the second corresondence we get $S_1 \sim R_2$, $S_3 \sim R_4$, ... Then

$$A = P + R_1 + R_2 + R_3 + R_4 + \cdots, \text{ where } P = \prod_{m=1}^{\infty} A_m.$$

$$B = Q + S_1 + S_2 + S_3 + S_4 + \cdots, \text{ where } Q = \prod_{m=1}^{\infty} B_m.$$

$P \sim Q_1 \subseteq Q$ by the first correspondence. Let q be any element of Q. Then q is mated to some element a of A by this correspondence. If a were not in P, it would be in some R_μ and consequently q would be in some S_ν and hence not in Q. Therefore $a \in P$ and $q \in Q_1$. Hence $P \sim Q$ by the first correspondence. We can therefore set up a biunique correspondence between A and B by mating P, R_1, R_3, . . . to Q, S_2, S_4, . . . by the first correspondence above, R_2, R_4, . . . to S_1, S_3, . . . by the second.

It is, however, still unknown whether there are cardinal numbers β satisfying the inequality $\aleph_0 < \beta < c$. The assumption that there are no such cardinal numbers is known as the "Hypothesis of the Continuum." The assumption that there is no cardinal number between β and 2^β, whatever be the transfinite number β, is known as the "Cantor Aleph-Hypothesis." It can be shown that every cardinal number β satisfies the inequality $2^\beta > \beta$; in other words, the set of all subsets of a given set has cardinal number greater than that of the set.[7] From the inequality $2^\beta > \beta$, we get at once the infinite sequence of inequalities

[7] W. Sierpiński, op. cit., p. 221.

$$\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < 2^{2^{2^{\aleph_0}}} < \ldots ,$$

which shows that there is an infinite number of transfinite cardinal numbers.

__The Axiom of Choice.__--The Axiom of Choice (or Multiplicative Axiom or "Auswahl Prinzip" of Zermelo) was stated by Ernst Zermelo in 1904, and although it is not accepted by all mathematicians, it has not been disproved.[8] The axiom is as follows: __For every class, C, consisting of sets E, non-null and mutually exclusive, there exists at least one set T containing one, and only one, element of each set E of C.__

The meaning of this Axiom may be explained by the following examples:

1) Divide all real numbers into sets assigning two numbers to the same set if and only if their difference is rational. We thus get an aggregate C of mutually exclusive, non-null sets. By the Axiom of Choice, there exists a set T containing one and only one number of each set E. No one, however, has been able so far to construct the set T, for it is impossible in this case to put down a law of selection which would pick out a certain element of each set E. This has lead some mathematicians to doubt the truth of the axiom;

2) Divide all denumerable sets of points on a straight line which are not symmetrical with respect to the point O into classes, assigning to the same class those sets which are symmetrical images of each

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[8]See E. Zermelo, "Beweis, dass jede Menge wohlgeordnet werden kann," __Math. Ann.__, Vol. LIX (1904), pp. 514-516, and "Neuer Beweis für die Möglichkeit einer Wohlordnung," __Math. Ann.__, Vol. LXV (1908), p. 110.

other with respect to the point 0 . There will obviously be two sets in each class. By the Axiom of Choice there exists a set T containing one set only of each pair, but we cannot devise any rule which would enable us to select this set.[9] The existence of the set T is, therefore, deduced only on the basis of the Axiom of Choice.

If, however, all sets on a straight line be divided into classes, assigning to the same class two sets if, and only if, they are mutually exclusive and their sum gives the whole line, then the set T may be actually constructed; for it is sufficient to assign to T that set of each class which contains the point 0 .

There are other forms of the Axiom of Choice, e. g., the following by D. Hilbert: _There exists a correspondence which mates to each property W possessed by at least one object a certain element $\tau(W)$ possessing the property._[10]

This axiom leads to the so-called "general principle of selection." E being any set, denote by W_E the property of belonging to the set E . If E is not the null-set, there exists obviously at least one object which has the property W_E , and $\tau(W_E)$ will be an element of E . Hence, there exists a correspondence which assigns to every non-null set an element of that set. There exists, therefore, for every given set a mating which assigns to every non-null subset of the given set a certain element belonging to that subset.

5. Order-Types

The order-types of simply ordered sets.—A _simple order_ is defined

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[9] W. Sierpiński, _op. cit._, p. 222.   [10] _Ibid._, p. 223.

to be a mathematical system $\mathcal{S}$ consisting of an arbitrary set S and a binary relationship R such that

1) if a and b are any two distinct elements of S, then one and only one of the following relationships holds:

$$a\,R\,b \quad \text{or} \quad b\,R\,a \quad \text{(asymmetry)} ;$$

2) if $a\,R\,b$ and $b\,R\,c$, then $a\,R\,c$ (transitivity).

In what follows $a\,R\,b$ will be expressed by saying that a precedes b, or b follows a; in notation, $a \prec b$, $b \succ a$.

If a set is given, it may be possible to order the set in a variety of essentially distinct ways. If the set is finite, the ordering of it may be accomplished by arbitrarily assigning to each element its rank relative to the others. In case the set is an infinite one, the ordering of it consists in the setting up of some general rule which suffices logically to assign the relative order of any two elements.

If there is an element of S which is not preceded by any other, it is said to be the "first" element; if there is one which is not followed by any other, it is called the "last" element of the set S.

Two simply ordered sets A and B are said to be similar, $A \simeq B$, when a biunique correspondence can be established, in accordance with some law, such that, to any two distinct elements a, a', of A, there correspond two elements b, b', of B, in such a manner that the relative order of a, a', in A, is the same as that of the corresponding elements b, b', in B.

Every simply ordered set is similar to itself. Two simply ordered sets which are similar to a third set are similar to each other. Therefore, the relation of similarity is symmetrical and transitive.

All simply ordered sets which are similar to one another are said to have the <u>same-order type</u>. An order-type is accordingly characteristic of a class of similar sets.

The order-type of a simply ordered set A is defined by Cantor as the concept which is obtained by abstraction when the nature of the elements of A is disregarded, their order being retained. The order-type of A is sometimes denoted by $\overline{A}$. That similar sets have the same order-type is regarded by Cantor as a deduction from this definition.

If, in $\overline{A}$, we further disregard the order of the elements, we obtain $\overline{\overline{A}}$, the cardinal number of A.

The order-type of A is, from Cantor's point of view, regarded as a simply ordered set similar to A, such that each element is the number 1. If any order-type be denoted by α, the corresponding cardinal number is denoted $\overline{\alpha}$. $\alpha = \beta$ indicates $A \simeq B$.

Corresponding to any given transfinite cardinal number, there is a multiplicity of order-types which form a class; each such class of order-types is characterized by the common cardinal number of all the order-types of the class.

If the order of every pair of elements in a simply ordered set A be reversed, the set in the new order is denoted by A^*. If the order-type of A is denoted by α, then the order-type of A^* is denoted by α^*. It may happen that $\alpha^* = \alpha$; this is the case for every finite order-type.

The order-type of the set of all positive integers in their natural order $(1, 2, \ldots, n, \ldots)$ is denoted by ω. This is therefore the order-type of every aggregate $(a_1, a_2, \dots, a_n, \dots)$ which is similar to $(1, 2, \dots n, \dots)$. The set $(\dots, a_n, \dots, a_2, a_1)$ has order type ω^*.

The set of natural numbers apart from its usual order may be also ordered according to the following convention: of two numbers the one with the least number of distinct prime factors will come first; and in case of an equal number of distinct prime factors the one of smaller value will come first. It is easily seen that this agreement orders the set of natural numbers, since conditions 1) and 2) of the definition are satisfied. Hence we get

1, 2, 3, 4, 5, 7, 8, 9, . . . ; 6, 10, 12, . . . ; 30, 42, . . . ; . . .

Addition of order-types.—If A, B denote two simply ordered sets, and if the set $(A;B)$ be formed, in which all the elements of both A and B occur, and which is such that any two elements of A have the same relative order as in A, and that any two elements of B have the same relative order as in B, and further that each element of A has a lower rank than all the elements of B, then the new simply ordered set $(A;B)$ is said to be the sum of the two simply ordered sets A and B. It is clear that if $A \simeq A'$, $B \simeq B'$, then $(A;B) \simeq (A';B')$, and thus that the order-type of $(A;B)$ depends only on the order-types of A and B.

If α, β are the order-types of A, B, respectively, the sum $\alpha + \beta$ is defined to be the order-type of the set $(A;B)$ defined above.

It is easily seen that the addition of order-types does not obey the commutative law. For if α, β are the order-types of A, B, respectively, then $\alpha + \beta$ is the order-type of $(A;B)$, but $\beta + \alpha$

is the order-type of $(B;A)$; and the order-types of $(A;B)$ and $(B;A)$ are in general different from one another.

If n denotes a finite integer, $\omega + n$ is the order-type of the simply ordered set

$$(e_1, e_2, \ldots, f_1, f_2, \ldots, f_m),$$

whereas $n + \omega$ is the order-type of

$$(f_1, f_2, \ldots, f_m, e_1, e_2, \ldots).$$

It thus appears that

$$n + \omega = \omega$$

but

$$\omega + n \neq \omega.$$

It follows from the definition of the types ω and ω^* that $\omega^* + \omega$ is the order-type of the set of all integers ordered according to magnititude, i. e.,

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots,$$

while the sum $\omega + \omega^*$ is the order-type of the class containing the set of the reciprocals of all the integers (zero excluded) ordered according to magnitude, i. e.,

$$-1/1, -1/2, -1/3, \ldots, 1/3, 1/2, 1/1.$$

The order-types $\omega^* + \omega$ and $\omega + \omega^*$ are different, for the first one does not contain a first nor last element whereas the second has both but has a gap, i. e., there is no last element in ω and no first element in ω^*. Hence

$$\omega^* + \omega \neq \omega + \omega^*.$$

If we put $\xi = \omega + \omega^*$, we find that $n + \xi = \xi + n$, where n is any natural number, since each sum is equal to ξ.

Furthermore it is easily seen that the relation

$$(\alpha + \beta)^* = \beta^* + \alpha^*$$

is true for every two order-types α and β.

The definition of the sum of order-types may be extended immediately to any finite number of types, and such a sum is easily seen to satisfy the associative law, i. e., $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.

<u>Multiplication of order-types.</u>--In the simply ordered set B, let us suppose that, in the place of each element, there is substituted a simply ordered set similar to A, whereby a new simply ordered set is formed; this may be denoted by (A.B). It is clear that if $A \simeq A'$, $B \simeq B'$, then $(A.B) \simeq (A'.B')$; thus the order-type of (A.B) depends only on the order types of A and B.

The <u>product</u> $\alpha\beta$ is defined to be the order-type of (A.B), as just defined.

It is easily seen that the product $\alpha\beta$ is in general different from $\beta\alpha$, and thus that the multiplication of order-types does not obey the commutative law. For example, $\omega \cdot 2$ is the order-type of the set formed by substituting in (a_1, a_2) for each of the two elements a set of type ω; $\omega \cdot 2$ is therefore the order-type of a set

$$(b_1, b_2, \ldots, c_1, c_2, \ldots),$$

in which there is no last element, and no element immediately preceding c_1. Its order-type is $\omega + \omega$. On the other hand $2 \cdot \omega$ is the order type obtained by substituting for each element in $(a_1, a_2, \ldots)$, a set consisting of two elements; and $2 \cdot \omega$ is thus the order-type of the denumerable set

$$(a_{11}, a_{12}, a_{21}, a_{22}, \ldots),$$

which is similar to

$$(b_1, b_2, \ldots) .$$

It has thus been shown that

$$2 \cdot \omega = \omega ;$$

similarly, for every natural number n,

$$n \cdot \omega = \omega .$$

The multiplication of order-types is, however, associative, and distributive if the second factor is a sum. Thus

$$(\alpha\beta)\gamma = \alpha(\beta\gamma) ,$$

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma ,$$

but

$$(1 + 1) \cdot \omega \neq 1 \cdot \omega + 1 \cdot \omega .$$

We have obviously for every order-type α and every natural number n

$$\alpha \cdot n = \alpha + \alpha + \alpha + \ldots + \alpha \quad (n \text{ terms}).$$

Similarly,

$$\alpha \cdot \omega = \alpha + \alpha + \alpha + \ldots .$$

<u>The structure of simply ordered sets</u>.--An examination of the structure of a simply ordered set A can, in general, only be attempted by considering the nature of its subsets, in each of which the order of the elements is taken to be the same as that of the same elements in the whole set. The simplest transfinite part of a simply ordered set is that which has one of the types ω, ω^*. These we speak of as <u>ascending sequences</u>, and <u>descending sequences</u>, respectively, contained in A.

Suppose that, in a simply ordered set A, there is an element a

which satisfies the following conditions, with respect to an ascending sequence $\{a_n\}$ contained in A :

1) For every n , $a_n < a$;

2) For every element x of A which precedes a , there exists an integer n such that $a_n > x$;

then the element a is said to be the <u>limiting element</u>, or <u>limit</u> of $\{a_n\}$ in A ; the limiting element of a descending sequence contained in A is similarly defined.

<u>A sequence contained in A can not have more than one limiting element in A</u> . It is clear that, if A , A' are similar simply ordered sets, an ascending, or a descending sequence in A corresponds to a sequence of the same kind in A' . To every limiting element in A there corresponds a limiting element in A' .

A simply ordered set which is such that every element is a limiting element is said to be <u>dense in itself</u>.

An ascending (descending) sequence $\{a_n\}$ is said to be <u>bounded</u> if there exists an element, say e , such that, for every n , a_n is less than (greater than) e .

If, in a simply ordered set, every bounded ascending or descending sequence which is contained therein has a limiting element in the set, then the simply ordered set is said to be a <u>closed set</u>.

A simply ordered set which is both dense in itself and closed, is said to be <u>perfect</u>.

A simply ordered set which is such that, between any two whatever of its elements, there is at least one element of the set, and therefore, an infinite number of them, is said to be <u>everywhere dense</u>.

The properties of a simply ordered set, thus defined, are also properties of any similar set; hence the terms may be applied to the order-types which are symbolised by replacing the elements of the simply ordered set by 1 . The continuum is an example of a simply ordered set which has all the properties of being dense in itself, closed, perfect, and everywhere dense.

A cut of a simply ordered set A is a division of all the elements of the set into two non-null classes C and D such that every element of the class C precedes every element of the class D . Such a division is denoted by [C,D] .

If in a given cut [C,D] the class C has a last element and the class D a first element, then this cut is said to give rise to a jump. Thus in the set of natural numbers each cut supplies a jump. Obviously, in order that a simply ordered set be every where dense it is necessary and sufficient that none of its cuts gives rise to a jump.

If in a cut [C,D] the class C has no last term and the class D no first term, the cut is said to produce a gap. Thus in the set of all rational numbers different from zero the cut into the class of negative rational numbers and the class of positive rational numbers produces a gap. The order-type $\omega + \omega^*$ has a gap.

A simply ordered set which has neither jumps nor gaps is said to be continuous. The linear continuum is an example.

The order-type η .—Let R represent the set of all rational numbers in the order of increasing magnitude. Let η be the order-type of the simply ordered set R .

It will be shown that the order-type η is characterised by the following properties:

1) $\overline{\eta} = \aleph_0$,

2) There is in η no first or last element,

3) η is everywhere dense.

In fact, every simply ordered set A, which has these three characteristics, is similar to the set R.

To prove this, we first observe that, on account of the condition 1), the order of the elements in both A and R can be so altered that each of them is reduced to the order-type ω. Let this be done; and denote by A_0, R_0 the new simply ordered sets

$$A_0 = (a_1, a_2, a_3, \ldots),$$
$$R_0 = (r_1, r_2, r_3, \ldots).$$

We have to show that $A \simeq R$; to do this we have to show how to establish the required correspondence between the elements a of A, and r of R. Let r_1 be made to correspond to a_1. Suppose the elements $r_1, r_2, r_3, \ldots, r_m$, of R, correspond to the elements $a_1, a_{\epsilon_2}, a_{\epsilon_3}, \ldots, a_{\epsilon_m}$, of A, where from the remaining elements of A at each step, a_{ϵ_ν} is the element with the smallest subscript as it appears in A which has the same relation to all a_{ϵ_i}, $i < \nu$, as regards order in A, that r_ν has to all r_i, $i < \nu$. We proceed then to choose in the same manner the element $a_{\epsilon_{m+1}}$, which is to be made to correspond to r_{m+1}, and thus we obtain by induction for every r_m, the corresponding a_{ϵ_m}. It must, however, be shown that this process exhausts all the elements a of A, that is to say, that in the sequence $1, \epsilon_2, \epsilon_3, \ldots, \epsilon_m, \ldots$ every integral number p occurs in some definite place. This can be proved by the method of induction. Let us assume that the elements $a_1, a_2, a_3, \ldots, a_m$ all occur in the correspondence that has been set up between the whole of R and at least a part of A ; then we shall prove than a_{m+1} also occurs. Upon this assumption, let λ be so great that, among the elements $a_1, a_{\epsilon_2}, a_{\epsilon_3}, \ldots, a_{\epsilon_\lambda}$ all the elements $a_1, a_2, a_3, \ldots, a_m$ occur. Then, if a_{m+1} is not also among those elements, choose out of $r_{\lambda+1}, r_{\lambda+2}, r_{\lambda+3}, \ldots$ that element $r_{\lambda+s}$, with the smallest subscript, which has the same relation to $r_1, r_2, r_3, \ldots, r_\lambda$, as regards order in R, that a_{m+1} has relative to $a_1, a_{\epsilon_2}, a_{\epsilon_3}, \ldots, a_{\epsilon_\lambda}$, as regards order in A. Then the element a_{m+1} has the same relation to $a_1, a_{\epsilon_2}, a_{\epsilon_3}, \ldots, a_{\epsilon_{\lambda+s-1}}$, as regards order in A, that $r_{\lambda+s}$ has to $r_1, r_2, r_3, \ldots, r_{\lambda+s-1}$ as regards order in R. But a_{m+1} is the element with the smallest subscript having this property; hence $a_{\epsilon_{\lambda+s}} = a_{m+1}$; i. e., the element a_{m+1} occurs in the correspondence which has been established between A and R. It follows that A and R are similar simply ordered sets.

Examples of the order-type η are

1) the set of all rational numbers which lie between two real numbers a and b,

2) the set of all algebraic numbers in their natural order in the continuum, or of all these numbers which lie between two real numbers a and b.

The rational numbers x, $0 \leqslant x \leqslant 1$, form a set of the order-type $1 + \eta + 1$.

It is easily seen that

$$\eta + \eta = \eta.$$

and, in fact,

$$\eta = \eta + \eta + \eta + \cdots .$$

Also

$$(\eta + 1) + \eta = \eta + (1 + \eta) = \eta .$$

<u>The order-type</u> λ.—We next consider the order-type λ of the set of all real numbers ordered according to their magnitude.

It will be shown that any simply ordered set A is similar to the set X of all real numbers in their natural order, provided

1) A is perfect,

2) in A, a set S, with the cardinal number $\aleph_0$, is contained, which is so related to A, that between any two elements a_0, a_1, of A, there are elements of S,

3) A has no first or last element.

It is obvious that S is everywhere dense and without a first or last element; thus S is of type η.

Since $S \simeq R$, we may mate the elements of S to the elements of R in an order-preserving manner. Any element a, of A, which does not belong to S, is the limiting element of a sequence $\{a_n\}$ of elements of S. To this sequence $\{a_n\}$, there corresponds a sequence $\{r_n\}$ in X, all the elements of which belong to R; and this sequence $\{r_n\}$ has a limiting element x, in X, not belonging to R, because it is a bounded monotone sequence of real numbers; we take therefore a, in A, to correspond to x, in X. If we take a different sequence $\{a'_n\}$ of elements of S, which has the same limiting element a as before, in A, then there corresponds to it a sequence $\{r'_n\}$ in R, which has the same limiting element x as before, in X. It follows that a biunique correspondence between the elements of A and X is established. It will now be shown that this correspondence is order-preserving. This clearly holds of any two elements of A which are also elements of S . Concider next two elements a and s , of A , the first of which does not, and the second of which does, belong to S ; let x , r be the corresponding elements of X . If $r < x$, there exists an ascending sequence in R , of which x is the limiting element, such that all its elements are greater than r ; then, to this sequence there corresponds an ascending sequence in S , all the elements of which are greater than s , and of which a is the limiting element; hence $s < a$. If $r > x$, it can be shown in a similar manner that $s > a$. The proof that, corresponding to any two elements a_1, a_2, of A , which do not belong to S , the elements x_1, x_2, of X , are such that $a_1 \gtrless a_2$, according as $x_1 \gtrless x_2$, is of precisely similar character as that just given. It has thus been shown that A and X are similar sets, and that the order-type λ is characterized by the conditions 1), 2) and 3).

6. Normally Ordered Sets

Notion of normally ordered sets.--The order-type of a simply ordered set is, as we have already seen, such that the structure of the set, as revealed by an examination of the sequences contained in it, may be of the most varied character; the various sequences may be ascending or descending ones, and may, or may not, have a limiting element within the set.

Of all the possible order-types, those are of especial importance which have been defined by Cantor as the order-types of normally ordered sets (or well-ordered sets or "wohlgeordnete Mengen").

A set S is <u>normally ordered</u> if S is simply ordered and every non-null subset of S has a first. This definition, of course, implies that the set S, if non-null, has a first element. An element of S (other than the first) does not necessarily have an immediate predecessor.

Every finite simply ordered set is normally ordered. Sets whose order-types are ω, $\omega+1$, $\omega+\omega$, $\omega\cdot\omega$ are evidently normally ordered; but the sets whose order-types are ω^*, η, λ are not normally ordered. Evidently, every subset of a normally ordered set (in the original order) is a normally ordered set.

The set of all elements preceding a given element e of S is called an <u>initial segment of S</u> and will be denoted by $S(e)$.

<u>Theorem</u> 6.1. (<u>Inductive Principle for Normal Orders</u>) <u>If W be any normal order and P be any property valid for any element e of W provided it is valid for every element of $S(e)$, then P is valid for every element of W</u>.

Let T be the set of elements of W for which P is valid. Then T contains the first element e_1 of W since $S(e_1)$ is the null-set. The subset of elements of W which are not in T contains a first, unless it is the null set. Let f be this first element. Then P holds for every element of $S(f)$, and would thus hold for f. This is a contradiction. Thus $T = W$.

<u>Properties of normally ordered sets</u>.--By means of the Axiom of Choice, Zermelo has proved the following theorem:

__Theorem__ 6.2. __Every set can be normally ordered.__[11]

__Theorem__ 6.3. __A simply ordered set A is normally ordered if, and only if, it contains no subset of order-type__ ω^* .

If A is not normally ordered, at least one non-null subset must have no first element, and, by induction, we can choose from this subset a sequence whose order-type is ω^* . On the other hand, if A is normally ordered, every non-null subset contains a first and hence does not have order-type ω^* .

A set which has both a first and a last element, and is such that each element except the last has one that immediately succeeds it, is not necessarily normally ordered, even if each element except the first has one immediately preceding it. This can be seen by considering the order-type $\omega + \omega^*$.

__Theorem__ 6.4. __A normally ordered set cannot be similar to an initial segment of itself.__

Let W be any normally ordered set. Suppose W is similar to an initial segment W' of itself. Consider $W - W'$. Any element g of this set must be mated to an element of W', which is an element preceding g in W . Let f be the first element of W mated to an element of W', say e , which precedes it in W . Since the mating is biunique, e , as an element of W , cannot be mated to e in W'; since the mating is order-preserving, the mate of e in W' is not $\succ$ e . Therefore, e is mated to some element $d \prec e$, which contradicts the

[11] __Math. Ann.__, Vol. LIX (1904), pp. 514-516, and Vol. LXV (1908), pp. 107-128. Compare P. E. B. Jourdain, "On the Transfinite Cardinal Numbers of Well-ordered Aggregates," __Phil. Mag.__ (6), Vol. VII (1904), pp. 61-75, and "On a Proof that every Aggregate can be well-ordered," __Math. Ann.__, Vol. LX (1905), pp. 465-470.

fact that f was the first element with this property. The theorem follows.

Corollary 6.4.1. Two different initial segments of a normally ordered set cannot be similar.

For one of these initial segments is an initial segment of the other.

Corollary 6.4.2. There is only one way of putting the elements of two similar normally ordered sets into biunique, order-preserving correspondence.

For, if, in two ways of placing the sets in correspondence, two elements f, f', of one set A, correspond to one element e of the other set A', the segments of A determined by f, f' are each similar to the segment of A' determined by e; but it has been shown to be impossible that A can have two different initial segments which are similar to one another.

Theorem 6.5. If A and B are any two normally ordered sets, then 1) either A is similar to an initial segment of B or B is similar to an initial segment of A or A is similar to B; 2) the possibilities in 1) are mutually exclusive.

Consider

$$A : a_1, a_2, a_3, \ldots, a_n, \ldots, a_\beta, \ldots$$

$$B : b_1, b_2, b_3, \ldots, b_n, \ldots, b_\lambda, \ldots$$

Mate the first element of A with the first element of B. Assuming that every element of $S(a_\beta)$ is properly mated with an element of B, mate a_β with the first element of B not previously used, unless, of course, they have all been used.

The mating so defined is order preserving. For let a_λ and a_μ be two elements of A with $a_\lambda \prec a_\mu$. Denote by b' and b'' the mates in B of a_λ and a_μ respectively. Suppose $b' \succ b''$. But this contradicts the fact that a_λ was mated with the first element of B not used as a mate of an element of $S(a_\lambda)$. Evidently $b' \neq b''$. Therefore, $b' \prec b''$.

Either every element of A finds a mate or there is at least one element of A which does not. In the first case, A is evidently mated in order-preserving fashion with an initial segment of B or to B itself. In the second case let e be the first element of A not mated. If any elements of B remained unused, there would be a first to which, by our mating system, e would be mated. Thus B is mated to an initial segment S(e) of A.

Part 2) follows immediately from Theorem 6.4.

Corollary 6.5.1. Any two sets are comparable in cardinal number.

7. Ordinal Numbers of Sets

Notion of ordinal number.--The order-types of normally ordered sets are called ordinal numbers. Thus ordinal number applies only to normally ordered sets. Two normally ordered sets A and B have the same ordinal number if they have the same order type, i. e., are similar.

The null-set and the set $\{x_1, x_2, \ldots, x_n\}$, consisting of n elements, are said to have ordinal number 0, n respectively.

Comparison of ordinal numbers.--- $\alpha > \beta$, or $\beta < \alpha$ is defined to mean that B is similar to an initial segment of A. It follows from Theorem 6.5 that, if α, β are any two ordinal numbers whatever, they satisfy one, and only one, of the relations $\alpha = \beta$, $\alpha > \beta$, $\alpha < \beta$. Further, it is seen that, if $\alpha < \beta$ and $\beta < \gamma$, then $\alpha < \gamma$; hence the set of all ordinal numbers is a simply ordered set.

<u>Theorem</u> 7.1. <u>A normally ordered set of type ϕ is similar to the set of all ordinal numbers less than ϕ (0 included) ordered according to increasing magnitude.</u>

Let W be a normally ordered set of type ϕ. Let, further, a be an element of W, and $\psi(a)$ be the order-type of the initial segment $S(a)$, where $\psi(a) = 0$ if a is the first element of W; we shall have obviously $\psi(a) < \phi$ and $\psi(a_1) < \psi(a_2)$, for $a_1 \prec a_2$. Hence, to every element of W there corresponds an ordinal number $\psi < \phi$, and to a later element corresponds a larger ordinal number. Conversely, every ordinal number $\psi < \phi$ corresponds to some element of W; in fact, if $\psi < \phi$, then any set W, of type ψ is similar to a certain initial segment $S(a)$ of W, and so $\psi = \psi(a)$. Hence the theorem.

The elements of a normally ordered set may, therefore, be denoted by the symbol a_ψ, where the subscripts $\psi = \psi(a)$ are ordinal numbers (including 0 which is the subscript of the first element a_0). Thus, the n elements of a finite set may be denoted by

$$a_0, a_1, a_2, a_3, \dots, a_{n-1},$$

the elements of a set of type ω by

$$a_0, a_1, a_2, a_3, \dots,$$

the elements of a set of type $\omega + n$, n a natural number, by

$$a_0, a_1, a_2, \dots, a_\omega, a_{\omega+1}, \dots, a_{\omega+n-1}, a_{\omega+n}.$$

Generally, the elements of a normally ordered set of ordinal ϕ

may be written down as a transfinite sequence of type ϕ, i. e.,

$$a_0, a_1, a_2, \ldots, a_\omega, \ldots, a_\xi, \ldots \quad (\xi < \phi).$$

Theorem 7.2. If N is any class of normally ordered sets, no two of which are similar, arranged according to ascending magnitude (as defined) of their ordinal numbers, then N is a normal order.

If A and B are any two distinct sets of N, we have one and only one of the relations $\alpha < \beta$, $\beta < \alpha$. If A, B and C are any three sets of N, of course $\alpha < \beta$, $\beta < \gamma$ implies $\alpha < \gamma$. It remains to show that every non-null sub-class of N has a first. Let N' be a non-null sub-class and let S be any normally-ordered set in N'. If e is any element of S, either e is always used in the attempted order-preserving mating of S to an arbitrary set of N', or there is at least one set of N' to which an initial segment of S may be mated without using e. Let f be the first[12] element of the latter type and let T be a set of N' such that $T \simeq S_1$, an initial segment of S not containing f. Then T is the first element of N', since $U \simeq T_1 \subset T$ would imply $U \simeq S_2 \subset S_1$, which contradicts the assumption that f was the first element of S of the second type.

Addition of ordinal numbers.--The sum $\alpha + \beta$ of two ordinal numbers is, in accordance with the general definition of the sum of two order-types, the order-type of the ordered set (A;B).

Since neither A nor B contains a part of type ω^*, the same is evidently true of (A;B). Hence the set (A;B) is normally

[12]If the set of elements of this type is the null-set, evidently S is the first element of N'.

ordered and $\alpha + \beta$ is an ordinal number. Since A is an initial segment of $(A;B)$, we see that $\alpha < \alpha + \beta$. $(\beta \neq 0)$.

The addition of ordinal numbers obeys the associative law, but not in general the commutative law; thus $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ but $\alpha + \beta$ is in general not the same as $\beta + \alpha$.

From the above, for every ordinal number α,

$$\alpha + 1 > \alpha .$$

It is easily seen that there is no ordinal number β satisfying the inequalities $\alpha < \beta < \alpha + 1$. Hence, every ordinal number has an immediate successor. But not every ordinal number has an immediate predecessor, e. g., the number ω. Zero and those ordinal numbers which possess immediate predecessors, i. e., those of the form $\alpha + 1$, are said to be of the <u>first kind</u>, those which do not are of the <u>second kind</u>.

<u>Multiplication of ordinal numbers</u>.--Clearly, the product $\alpha\beta$, of two ordinal numbers is an ordinal number. In general $\alpha\beta \neq \beta\alpha$, i. e., multiplication is not necessarily commutative. It is easily seen that $\alpha\beta > \alpha$, provided $\alpha, \beta > 1$; and that, if $\alpha\beta = \alpha\gamma$, then $\beta = \gamma$. Multiplication of ordinal numbers is associative and distributive, i. e., $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ and $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

<u>Subtraction of ordinal numbers</u>.--If α, β are two ordinal numbers such that $\alpha < \beta$, there evidently exists an ordinal number γ such that $\alpha + \gamma = \beta$; this number γ is denoted by $\beta - \alpha$. For, if B has the ordinal number β, there is an initial segment of B, say B_1, with ordinal α. If $B = (B_1; S)$, $\beta - \alpha$ is the ordinal number of S.

Cantor's classes of ordinal numbers.--Every finite simply ordered set is normally ordered, and its order-type is the ordinal number of the set. All finite ordinal numbers (zero included) are said to be numbers of Cantor's first class; to each such ordinal number there corresponds a single cardinal number, and the properties of the finite ordinal numbers are identical with those of the finite cardinal numbers, the terms ordinal and cardinal simply defining the two uses of the same number. In the case of transfinite sets there is no such identity between ordinal and cardinal numbers; in fact the arithmetic of the one kind of numbers is essentially different from that of the other kind.

Corresponding to a single transfinite cardinal number there is an infinity of transfinite ordinal numbers; all those transfinite ordinal numbers which correspond to sets that have one and the same cardinal number α form a class $Z(\alpha)$.

The ordinal numbers of all those normally-ordered sets which have the cardinal $\aleph_0$ are said to be of Cantor's second class $Z(\aleph_0)$.

Theorem 7.8. The ordinal number ω is the smallest number of the second class.

Let A denote the set $(a_1, a_2, a_3, \ldots, a_n, \ldots)$. Then A has ordinal number ω, and $\overline{\omega} = \aleph_0$. Any number β, which is less than ω, must be the order-type of an initial segment of A, and A has only initial segments $(a_1, a_2, a_3, \ldots a_n)$ with finite ordinal numbers n; thus β must be a finite number; and therefore the only ordinal numbers less than ω are finite ones.

<u>Theorem</u> 7.4. <u>The set</u> E <u>of all ordinal numbers of the first and second classes is non-denumerable</u>.

E, being a set of ordinal numbers, can be normally ordered according to ascending magnitude. Let γ be its ordinal number. If E were denumerable, γ would be an element of E and $S(\gamma)$ would be an initial segment of E with ordinal γ. But E can not be similar to an initial segment of E. Hence the set of all ordinal numbers of the first and second classes is non-denumerable.

The cardinal number of the set of all ordinal numbers of the first and second classes is denoted by $\aleph_1$, i. e., $\aleph_1$ is the cardinal number associated with γ.

<u>Theorem</u> 7.5. $\aleph_1$ <u>is the first (smallest) cardinal number greater than</u> $\aleph_0$.

Let S be any infinite set with cardinal less than $\aleph_1$. Denote the cardinal number of S by $\aleph$. Let S be normally ordered and denote the resulting ordinal number by σ. If σ were equal to γ, we would have S mated biuniquely (and in order preserving fashion) with the set of ordinal numbers of classes one and two, i. e., with a set of cardinal $\aleph_1$. Thus we would have $\aleph = \aleph_1$, contrary to assumption. Hence $\sigma < \gamma$ or $\sigma > \gamma$. Suppose $\sigma > \gamma$. Then we would have the set of ordinal numbers of classes one and two mated biuniquely (and with preservation of order) to a proper subset (in fact, to a proper initial segment) of S. Hence $\aleph > \aleph_1$, contrary to assumption. Therefore $\sigma < \gamma$. Thus S can be mated in biunique (and order preserving) manner with a proper initial segment, say $S(\sigma)$, of the set of ordinal numbers of classes one and two. Hence S has cardinal $\aleph_0$.

All ordinal numbers which are order-types of normally ordered sets of cardinal number $\aleph_1$ constitute <u>Cantor's third class</u>. The smallest of them is easily seen to be γ. It can be shown that the set of all ordinals of the first, second and third classes has cardinal number greater than $\aleph_1$; its cardinal number is denoted by $\aleph_2$.

In general, $\aleph_\lambda$ is the cardinal number of the set of all ordinal numbers having cardinal $\aleph_\beta$, where $\beta < \lambda$. In particular, $\aleph_\omega$ is the cardinal number of the set of all ordinal numbers having one of the cardinal numbers n, $\aleph_0$, $\aleph_1$, $\aleph_2$, $\aleph_3$, ... , $\aleph_n$, ...

BIBLIOGRAPHY

Borel, Émile, Leçons sur la Théorie des Fonctions, Paris, Gauthier-Villars et Fils, 1898.

Cantor, G., Beiträge zur Begründung der transfiniten Mengenlehre," Mathematische Annalen, Vol. XLVI (1895), pp. 381-513.

Fine, H. B., College Algebra, Boston, Ginn and Co., 1904.

Hardy, G. H., Pure Mathematics, Cambridge, University Press, 1938.

Hausdorff, F., Grundzüge der Mengenlehre, Leipzig, Veit und Co., 1914.

Hausdorff, F., Mengenlehre, Berlin, Walter de Gruyter und Co., 1927.

Hilbert, D. und Ackermann, W., Grundzüge der Theoretischen Logik, Berlin, J. Springer, 1938.

Hobson, E. W., The Theory of Functions of a Real Variable, Vol. I, Cambridge, University Press, 1927.

Jourdain, P. E. B., "On a Proof that every Aggregate can be well-ordered," Mathematische Annalen, Vol. LX (1905), pp. 465-470.

Jourdain, P. E. B., "On the Transfinite Cardinal Numbers of Well-ordered Aggregates," Philosophical Magazine, Series 6, Vol. VII (1904), p. 61-75.

Kamke, E., "Allgemeine Mengenlehre," Enzyklopädie der Mathematischen Wissenschaften, Band I1, Heft 2, am 28. Juni 1939, 5,1 - 5,56.

Littlewood, J. E., The Elements of the Theory of Real Functions, Cambridge, W. Heffer and Sons, 1926.

Moore, R. L., Foundations of Point Set Theory, American Mathematical Society Colloquium Publications, Vol. XIII, New York, American Mathematical Society, 1932.

Newman, M. H. A., Elements of the Topology of Plane Sets of Points, Cambridge, University Press, 1939.

Osgood, W. F., Functions of Real Variables, New York, G. E. Stechert and Co., 1938.

Russell, Bertrand, Principles of Mathematics, New York, W. W. Norton and Co., 1938.

Schröder, Ernst, "Über G. Cantorsche Sätze," Jahresbericht der deutschen Mathematiker-Vereinigung, Vol. V (1896), pp. 81-82.

Sierpiński, Waclaw, Hypothèse du Continu, Warszawa, Lwów, Z. Subwencji Funduszu Kultury Narodowej, 1934.

Sierpiński, Waclaw, Introduction to General Topology, Translated by C. Cecilia Krieger, Toronto, University of Toronot Press, 1934.

Titchmarsh, E. C., The Theory of Functions, Oxford, Clarendon Press, 1932.

Zermelo, Ernst, "Beweis, dasz jede Menge wohlgeordnet werden kann," Mathematische Annalen, Vol. LIX (1908), pp. 514-516.

Zermelo, Ernst, "Neuer Beweis für die Möglichkeit einer Wohlordnung," Mathematische Annalen, Vol. LXV (1908), pp. 107-128.

Zermelo, Ernst, "Über die Addition transfiniter Cardinalzahlen," Göttinger Nachrichten, 1901, pp. 34-41.