Polynomial Approximation of Shape Function Gradients From Element Geometries¹

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Abstract. A method is presented for the polynomial approximation of shape function gradients based solely on the geometry of finite element boundaries. The method is founded on a least squares approach which leads to an integration scheme satisfying a necessary condition for convergence. In its simplest form the method reduces to the well-known uniform strain approach for finite elements. The method is applicable to a broad class of problems such as connecting dissimilar meshes, mesh adaptivity and transitioning, and the construction of finite elements with variable topologies. Finite elements based on the polynomial approximations are shown to pass patch tests of various orders. In contrast to standard elements, higher-order patch tests are passed without the need for nodes internal to the element boundaries. Less sensitivity to volumetric locking under plane strain conditions is demonstrated through comparisons with a standard element formulation.

Key Words. Shape functions, uniform strain, volumetric locking, dissimilar meshes, patch tests.

¹Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under Contract DE-AL04-94AL8500.

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1. Introduction

Finite elements based on the uniform strain approach [1] have the important features of simplicity and versatility. It could be argued that such elements are the simplest possible that pass first-order patch tests. Averaged shape function gradients for the uniform strain approach are given explicitly by derivatives of element areas (volumes) with respect to nodal coordinates [1]. Thus, for an isoparametric formulation, only the geometry of the element boundary must be known. This fact enables the approach to serve many functions. Indeed, even when restricted to the scope of the first author, the approach has formed the basis for various triangular and tetrahedral elements [2], transition elements [3], node-based elements [4], and techniques for connecting dissimilar meshes [5].

The question naturally arises as to whether uniform strain concepts can be extended easily to higher-order approximations. If so, then the desirable features of the approach would also apply to higher-order elements. Such an extension is presented here. The approach is based on minimizing the squared difference between polynomial fitting functions and the shape function gradients as inferred from the geometry of the element boundary. An important result of the approach is an integration scheme which satisfies a necessary condition for passing patch tests. The ability of the approximations to reproduce polynomials of a certain degree, another patch test requirement, is achieved through an auxiliary set of constraints.

In the development that follows, it is shown that the element formulation proposed herein passes not only first-order patch tests, but higher-order patch tests as well, regardless of the element geometry. In contrast, conventional isoparametric elements generally cannot pass second-order patch tests unless they involve internal shape functions (e.g. as associated with nodes in the element interior). The standard eight-node serendipity quadrilateral, for example, is known to pass second-order patch tests only if it assumes the shape of a parallelogram.

Chief among the desirable features of the proposed element formulation is its freedom from restrictions on both the element geometry, and the number and arrangement of the nodes. This feature enormously simplifies the task of automatic generation of unstructured meshes. With a similar motivation in mind, specialized transition elements can be devised without abandoning the conventional isoparametric format; see section 3.11 of [6]. Whereas such elements are useful for facilitating transitions in mesh pattern or density, they are still subject to the usual restrictions on element geometry. Further, the piecewise definition of the shape functions typically used in transition elements can lead to a somewhat awkward numerical integration rule.

The element formulation proposed herein bears many similarities to the recently-developed Variable-Element-Topology Finite Element Method (VET-FEM) [7]. In particular, the VET-FEM exhibits the same flexibility with regard to element geometry and nodal arrangement as does the present method. The primary difference between the two methods lies in the fact that the present method provides for polynomial approximations of the shape function *gradients*, whereas in the VET-FEM, the shape functions themselves are given by polynomial approximants over the element. Although these two methods differ in the details of their respective shape-function formulations, both lead to small systems of linear equations

for each element that must be solved for the shape-function polynomial coefficients. These linear equations, in turn, emerge from suitably-defined minimization problems that represent different, but each effective and valid, conceptions of optimal shape-function design. In this connection, it bears mention that the VETFEM satisfies first-order patch tests only in general, whereas the proposed formulation is capable of passing higher-order patch tests as well.

Both the present method and the VETFEM [7] are properly regarded as nonconforming finite element methods, in that they both involve basis functions that are not globally C^0 continuous. In the VETFEM, whereas the shape functions exhibit the Kronecker-delta property and are otherwise optimized for inter-element compatibility, slight discontinuities at inter-element boundaries exist in general. In the method proposed herein, the nature of the nonconformity is somewhat different: the shape functions within each element are not explicitly constructed; instead, their gradients are approximated by polynomials. These approximants do not, in general, have zero curl, so that they are not integrable in general. Therefore, there exist no global basis functions, continuous or not; only approximations to their gradients. However, convergence of the method is still guaranteed through passage of the patch test. Indeed, it will be seen in the next section that the lack of integrability of the shape function gradients actually facilitates the ability to pass higher-order patch tests in the present context. Also, it bears mention that approximate integrals of the shape-function gradients may be derived for use in, e.g., mass-matrix formulations, as described in the next section.

The development of the proposed method is given in the next section. Notation is standard throughout, with lower-case Latin indices representing coordinate or vector component, and upper-case Latin indices giving the node number for a typical element. Example problems are presented in the third section which demonstrate the advantages of the proposed method over conventional element formulations.

2. Method

Within the domain Ω of a finite element, a field variable u is interpolated according to the equation

$$u = \sum_{I=1}^N u_I \Phi_I \quad (1)$$

where N is the number of nodes in the element, u_I is the value of u at node I , and Φ_I is the element shape function for node I . Differentiating Eq. (1) with respect to spatial coordinate x_i yields

$$u_{,i} = \sum_{I=1}^N u_I \Phi_{I,i} \quad (2)$$

where

$$(\quad)_{,i} = \frac{\partial(\quad)}{\partial x_i} \quad (3)$$

Let $\bar{\Phi}_{I,i}$ denote a degree p polynomial approximation of $\Phi_{I,i}$ expressed as

$$\bar{\Phi}_{I,i} = \sum_{j_1=0}^{r_1} \cdots \sum_{j_d=0}^{r_d} a_{j_1 \dots j_d}^{Ii} x_1^{j_1} \cdots x_d^{j_d} \quad (4)$$

where d is the spatial dimension, $a_{j_1 \dots j_d}^{Ii}$ are constants, and

$$r_{k+1} = r_k - j_k, \quad r_1 = p \quad (5)$$

Equations (4) and (5) represent a complete d -dimensional polynomial of degree p . The $\bar{\Phi}_{I,i}$ notation is symbolic and not operational. That is, $\bar{\Phi}_{I,i}$ is not the partial derivative of $\bar{\Phi}_I$ with respect to x_i . It is convenient to group the terms $a_{j_1 \dots j_d}^{Ii}$ and $x_1^{j_1} \cdots x_d^{j_d}$ into the vectors a_{Ii} and x so that

$$\bar{\Phi}_{I,i} = a_{Ii}^T x \quad (6)$$

Consider the function g defined by

$$g = \int_{\Omega} (\bar{\Phi}_{I,i} - \Phi_{I,i})^2 d\Omega \quad (7)$$

Substitution of Eq. (6) into Eq. (7) and minimization of g with respect to a_{Ii} yields

$$\left(\int_{\Omega} x x^T d\Omega \right) a_{Ii} = \int_{\Omega} x \Phi_{I,i} d\Omega \quad (8)$$

The integral on the right hand side of Eq. (8) is rewritten using the divergence theorem as

$$\int_{\Omega} x \Phi_{I,i} d\Omega = \int_S n_i x \Phi_I dS - \int_{\Omega} x_{,i} \Phi_I d\Omega, \quad (9)$$

where S is the boundary of the element, $\mathbf{n} = \sum_{j=1}^d n_j \mathbf{e}_j$ is the unit outward normal to S , and $\mathbf{e}_j$ is a unit vector in coordinate direction j . It follows from the previous two equations that

$$C a_{Ii} = s_{Ii} - v_{Ii} \quad (10)$$

where

$$C = \int_{\Omega} x x^T d\Omega \quad (11)$$

$$s_{Ii} = \int_S n_i x \Phi_I dS \quad (12)$$

$$v_{Ii} = \int_{\Omega} x_{,i} \Phi_I d\Omega \quad (13)$$

The coefficient matrix C is symmetric and positive definite, provided that the element area (volume) is positive. Dimensions of the matrix C are provided in Table 1 for various values of p . The integral in Eq. (12) can be calculated given the nodal coordinates and definitions of shape functions on element boundaries. Likewise, this information can also be used to

calculate the integral in Eq. (11) after application of the divergence theorem. The examples presented in this study for $d = 2$ use standard, one-dimensional shape functions on the element edges.

The vector x_i in Eq. (13) can be expressed as

$$x_i = E_i \bar{x} \quad (14)$$

where each row in E_i contains at most one nonzero integer, and the vector $\bar{x}$ contains all d -dimensional monomials of degree $p - 1$. Substitution of Eqs. (13) and (14) into Eq. (10) and solving for a_{Ii} yields

$$a_{Ii} = C^{-1}(s_{Ii} - E_i b_I) \quad (15)$$

where

$$b_I = \int_{\Omega} \bar{x} \Phi_I d\Omega \quad (16)$$

Recall that the overall objective of the method, as expressed by minimization of g (Eq. (7)), is to approximate the gradients of shape functions about which only partial information is available. Equation (15), together with (11), (12), and (16), illuminate the nature of the information about each shape function that is required to fix these gradient approximants. Specifically, the matrix C given in (11) embodies the crucial aspects of the element's geometry, whereas (12) indicates the manner in which the boundary variation of each shape function influences the gradient approximants. Finally, the b_I as defined in (16) contain all information about the shape-function behavior in the *interior* of the element that is required to formulate gradient approximants of a given polynomial degree. However, unlike C and s_{Ii} , no information about the behavior of the shape functions on the element interior can be deduced merely from the element's geometry. In order to proceed, then, the b_I must be set in some reasonable, if somewhat arbitrary, manner.

Several different methods for choosing b_I were considered. These included 1.) minimizing the curl of the gradient approximations, 2.) minimizing the difference between approximated and known shape functions over the element boundary, 3.) satisfying the Kronecker delta property of shape functions at element nodes in a least squares sense, and 4.) using the minimum norm solution of the completeness requirement given by Eq. (21). Each method had its advantages and shortcomings, but we opted for the one described below.

Consider the function h_I defined as

$$h_I = \sum_{i=1}^d \int_{\Omega} (\bar{\Phi}_{I,i})^2 d\Omega \quad (17)$$

Substitution of Eqs. (6), (11) and (15) into Eq. (17) and minimization of h_I with respect to b_I leads to

$$\left(\sum_{i=1}^d E_i^T C^{-1} E_i \right) b_I = \sum_{i=1}^d E_i C^{-1} s_{Ii} \quad (18)$$

Let $\hat{b}_I$ denote the solution of Eq. (18). Although setting b_I equal to $\hat{b}_I$ minimizes the norm of the gradient approximations as measured by h_I , it does not, in general, lead to satisfaction

of a completeness condition. Specifically, in order to represent an arbitrary polynomial of degree q exactly on Ω , the element shape functions must satisfy

$$\sum_{I=1}^N x_{1I}^{m_1} \cdots x_{dI}^{m_d} \Phi_I = x_1^{m_1} \cdots x_d^{m_d} \quad (19)$$

for all possible values of m_i such that $0 \leq \sum_{i=1}^d m_i \leq q$, where x_{iI} is coordinate i of node I . The degree- q completeness condition (19), together with Eq. (16), lead to the necessary condition

$$\sum_{I=1}^N x_{1I}^{m_1} \cdots x_{dI}^{m_d} b_I = \int_{\Omega} x_1^{m_1} \cdots x_d^{m_d} \tilde{x} d\Omega \quad (20)$$

which is rewritten as

$$XB^T = F \quad (21)$$

where the entries in X contain products of nodal coordinates, the entries of F are contained in C , and

$$B = \begin{bmatrix} b_1 & \cdots & b_N \end{bmatrix} \quad (22)$$

Let

$$B = \hat{B} + RX \quad (23)$$

where

$$\hat{B} = \begin{bmatrix} \hat{b}_1 & \cdots & \hat{b}_N \end{bmatrix} \quad (24)$$

Substituting Eq. (23) into Eq. (21) and solving for R yields

$$R = (F^T - \hat{B}X^T)(XX^T)^{-1} \quad (25)$$

Selecting B according to Eq. (23) attempts to minimize h_I while satisfying the completeness requirement. The B thus obtained is not necessarily identical to the exact minimizer of the sum of h_I subject to Eq. (21), but it can be calculated in a simple and straightforward manner. It is noted that the number of nodes on an element boundary must be large enough to satisfy the completeness requirement. For example, at least six nodes are required for $d = 2$ and $q = 2$. In addition, mid-edge nodes for quadratic and higher-order elements must be geometrically centered in order to pass higher-order patch tests.

Observe from Eqs. (15) and (25) that the calculation of a_{Ii} requires factoring the matrices C and XX^T . The size of the matrix C depends on the value of p , whereas the size of XX^T depends on q . Both these matrices also depend on the spatial dimension d (see Table 1). The value of p corresponds to the order of the integration scheme, whereas q specifies the polynomial degree of the finite element.

Formation of element stiffness matrices often requires calculating integrals of the form $\int_{\Omega} \Phi_{I,i} \Phi_{J,j} d\Omega$. It follows from Eqs. (6), (10) and (11) that

$$\int_{\Omega} \bar{\Phi}_{I,i} \bar{\Phi}_{J,j} d\Omega = (s_{Ii} - v_{Ii})^T a_{Jj} \quad (26)$$

Equation (26) is used in the calculation of element stiffness matrices for all the example problems. This equation is applicable because all the examples are concerned with the linear response of elements consisting of homogeneous materials. A numerical integration scheme applicable to a broader class of problems is described subsequently. Minimum values of p required to avoid spurious zero energy modes for plane elasticity are presented in Table 2 for elements with different numbers of edges.

The accuracy of numerical integration rules required for convergence is discussed in Ref. 8. For second-order differential equations such as the ones considered here, the integral of $\Phi_{I,i}$ must be evaluated exactly in order to pass first-order patch tests. For second-order patch tests, integrands of the form $x_j \Phi_{I,i}$ must be integrated exactly as well. Observe from Eqs. (8) and (11) that shape function gradient integrals are given exactly by the product $Ca_{I,i}$. Thus, the requirements for passage of patch tests are met. It is demonstrated numerically in the following section that the method passes both first and higher-order patch tests.

A standard numerical integration rule used by many elements is Gaussian quadrature. This rule is commonly used for elements involving a mapping from a parent square (cube) and only requires the evaluation of the integrand at specific points. Because such a mapping is not used in the present formulation, Gaussian quadrature cannot be applied directly. Fortunately, all the essentials of a numerical integration scheme are contained in the matrix C . It follows from Eq. (11) that

$$\int_{\Omega} (a^T x)(b^T x) d\Omega = a^T C b. \quad (27)$$

where a and b are vectors of constants. Notice that Eq. (26) is simply a special case of Eq. (27) with $a = a_{I,i}$ and $b = a_{J,j}$. To obtain $\int_{\Omega} x_j^r \Phi_{I,i} d\Omega$, one simply sets $a = a_{I,i}$ and all the entries of b equal to zero except for a value of unity in the location of the vector x corresponding to x_j^r . For nonlinear problems or elements with nonhomogeneous materials, it may be necessary to evaluate integrals of the form $\int_{\Omega} f(x_1, \dots, x_d) d\Omega$. One feasible option is to construct a and b in Eq. (27) from the Taylor series expansion of f . It is noted that Taylor series expansions are inherent to Gaussian quadrature as well.

The need for mass matrices in dynamic analysis motivates the approximation of the shape functions and not just their gradients. To this end, integration of Eq. (6) with respect to x_i leads to

$$\bar{\Phi}_I^i = r_I + a_{I,i}^T \hat{x}_i, \quad (28)$$

where r_I is a constant and

$$\hat{x}_{i,i} = x \quad (29)$$

The superscript i in Eq. (28) indicates that the result was obtained from integration with respect to x_i . Define

$$\hat{k}_m = \begin{cases} 0 & \text{if } k_m = 0 \\ 1 & \text{if } k_m > 0 \end{cases} \quad (30)$$

Because the curl of the gradient approximations is not necessarily zero, the coefficients of monomials $x_1^{k_1} \dots x_d^{k_d}$ with $\sum_{m=1}^d k_m > 1$ may not be equal in $\bar{\Phi}_I^i$ and $\bar{\Phi}_I^j$ for $i \neq j$.

Consequently, an averaging of these terms is done. The result is

$$\bar{\Phi}_I = r_I + \sum_{i=1}^d (Y_i a_{Ii})^T \hat{x}_i \quad (31)$$

where the matrix Y_i is diagonal. The diagonal entry in row and column n of Y_i is given by $1/\sum_{m=1}^d \hat{k}_{mn}$ where $x_1^{k_{1n}} \dots x_d^{k_{dn}}$ is the monomial appearing in row n of $\hat{x}_i$. The constant r_I can be chosen to minimize the squared difference between $\bar{\Phi}_I$ and Φ_I integrated over the element boundary. The shape function approximation $\bar{\Phi}_I$ can then be used with standard procedures to obtain the element mass matrix. The mass matrix is scaled by a suitable constant to ensure that the sum of all entries divided by d equals the element mass. Existing methods for lumping mass at the nodes can be used for explicit transient dynamic calculations that require a diagonal mass matrix.

3. Examples

First-order patch tests:

Consider the VETFEM mesh of a unit square shown in Figure 1 bounded by $x_1 = 0$, $x_1 = 1$, $x_2 = 0$ and $x_2 = 1$. The displacement boundary conditions for the problem are specified as

$$u_1(0, x_2) = 0, \quad u_2(0, 0) = 0 \quad (32)$$

The traction boundary conditions on all edges are zero except for

$$\sigma_{11}(1, x_2) = 1 \quad (33)$$

Plane stress conditions are assumed. The exact solution to the problem has all stresses equal to zero except for $\sigma_{11}(x_1, x_2) = 1$ and the displacements are given by

$$u_1(x_1, x_2) = x_1/E, \quad u_2(x_1, x_2) = -\nu x_2/E \quad (34)$$

where E is Young's modulus and ν is Poisson's ratio. The calculated solution is in perfect agreement, to machine precision, with the exact solution. This statement also holds if the initial positions of all nodes internal to the mesh boundary are moved randomly.

Second-order patch tests:

The second example is identical to the first with the exception that the traction on the edge $x_1 = 1$ is given by

$$\sigma_{11} = 1/2 - x_2 \quad (35)$$

The exact solution to the problem has all stresses equal to zero except for $\sigma_{11}(x_1, x_2) = 1/2 - x_2$ and the displacements are given by

$$u_1(x_1, x_2) = x_1(1/2 - x_2)/E, \quad u_2(x_1, x_2) = [x_1^2 + \nu((x_2 - 1/2)^2 - 1/4)]/(2E) \quad (36)$$

Again, the calculated solution is in perfect agreement with the exact solution, however, this statement only holds if all mid-edge nodes are initially centered. The same caveat also holds for conventional quadratic elements such as the six-node triangle.

Identical boundary conditions were also applied to the mesh shown in Figure 2. Again, the calculated solution passed the patch test. In contrast, this patch test is not passed with standard eight-node quadrilateral elements because all elements are not parallelograms [9]. This is noteworthy, granted that the same element connectivity and number of nodes is used in both cases.

Displacement boundary conditions for the next patch test are specified as

$$u_1(1,0) = 0, \quad u_2(1,0) = 0; \quad u_1(1,1) = 0 \quad (37)$$

The traction boundary conditions on all edges are zero except for

$$\sigma_{11}(1, x_2) = -1 \quad (38)$$

Plane stress conditions are assumed and the mesh in Figure 1 is subjected to a constant body force of unit value in the x_1 direction. Equivalent nodal forces in the x_1 direction for the body force are obtained using the terms $\int_{\Omega} \Phi_I d\Omega$ contained in the matrix B . The exact solution to the problem has all stresses equal to zero except for $\sigma_{11}(x_1, x_2) = -x$ and the displacements are given by

$$u_1(x_1, x_2) = [(1 - x_1^2) - \nu x_2(x_2 - 1)]/(2E) \quad (39)$$

$$u_2(x_1, x_2) = \nu(1 - x_1 + 2x_1x_2)/(2E) \quad (40)$$

The calculated solution obtained is in perfect agreement with the exact solution.

Third-order patch tests:

Displacement boundary conditions specified on the outer edges of the mesh in Figure 3 are given by

$$u_1(x_1, x_2) = (x_1^3 - 3x_1x_2^2)10^{-6}, \quad u_2(x_1, x_2) = (x_2^3 - 3x_2x_1^2)10^{-6} \quad (41)$$

There are no traction boundary conditions since displacement boundary conditions are specified on the entire mesh boundary. The exact solution for either plane stress or plane strain conditions is also given by Eq. (41). The calculated solution for either of these conditions is in perfect agreement with the exact solution. That is, the displacements of nodes internal to the mesh and element strains vary in accordance with the exact solution.

The same boundary conditions were applied to the mesh in Figure 3, but the analysis was done using standard twelve-node quadrilateral elements. In contrast to results for the present method, the patch test was not passed even though the same element connectivity and number of nodes were used.

Bending of Cantilevered Beam:

Consider the VETFEM and structured meshes of a beam in Figure 4 bounded by $x_1 = 0$, $x_1 = 10$, $x_2 = 0$ and $x_2 = 1$. The displacement boundary conditions for the problem are specified as

$$u_1(0, x_2) = 0, \quad u_2(0, x_2) = 0 \quad (42)$$

The traction boundary conditions on all edges are zero except for

$$\sigma_{12}(10, x_2) = 1 \quad (43)$$

Plane strain conditions are assumed. Comparisons of transverse tip displacements of the upper right corner of the beam are shown in Table 3. Results are normalized with respect to a standard finite element analysis of the structured eight-node quadrilateral mesh. Excellent agreement between the results is evident.

Slotted Panel in Compression:

Consider the VETFEM and structured meshes of a slotted panel in Figure 5 bounded by $x_1 = 0$, $x_1 = 90$, $x_2 = 0$ and $x_2 = 90$. The displacement boundary conditions for the problem are specified as

$$u_1(0, x_2) = 0, \quad u_2(0, 0) = 0 \quad (44)$$

The traction boundary conditions on all edges are zero except for

$$\sigma_{11}(90, x_2) = -1 \quad (45)$$

Again, plane strain conditions are assumed. Comparisons of horizontal displacements of the upper right corner of the panel are shown in Table 4. Results are normalized with respect to a standard finite element analysis of the structured eight-node quadrilateral mesh. Very good agreement is obtained for values of ν away from $1/2$, but notice the larger differences for $\nu = 0.49999$. These results indicate that the present method leads to a more flexible solution than a standard finite element analysis. The number of elements in the structured mesh of Figure 5 was quadrupled by dividing each existing element into four. Normalization of results with respect to a standard analysis of the new structured mesh leads to 1.0205 in Table 4 instead of 1.1134 for $\nu = 0.49999$.

These results prompted the investigation of eigenvalues for a skewed eight-node quadrilateral element with vertices $(0,0)$, $(1,0)$, $(1,3/2)$, and $(0,1)$. Mid-edge nodes are centered between the vertex nodes. Ratios of the sixth largest eigenvalue of the element stiffness matrix for the standard and present approaches are presented in Table 5 for plane strain conditions. Notice that the ratio increases as ν approaches $1/2$. These results help to explain the results in Table 4.

4. Conclusions

The method presented for the polynomial approximation of shape function gradients from element geometries retains the simplicity and versatility of the uniform strain approach while extending its applicability to higher-order elements. Example problems demonstrated

the flexibility of the method for both standard and nonstandard finite elements. Finite elements based on the method were shown to pass both second and third-order patch tests. In contrast, the corresponding serendipity elements with the same connectivities and number of nodes were unable to pass the tests because of their non-parallelogram shapes. It was shown that the method is less sensitive to volumetric locking in plane strain than a standard formulation for eight-node quadrilaterals. The method is also applicable to problems such as mesh transitions and connecting dissimilar meshes. For example, transition elements between quadratic hexahedral and tetrahedral elements can be obtained from straightforward application of the method.

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Table 1: Dimensions of square matrices C and XX^T for different values of p , q and d .

C			XX^T		
p	$d = 2$	$d = 3$	q	$d = 2$	$d = 3$
0	1	1	1	0	0
1	3	4	2	1	1
2	6	10	3	3	4
3	10	20	4	6	10
4	15	35	5	10	20

Table 2: Minimum values of p required to avoid spurious zero energy modes of individual elements for plane elasticity. The number of element edges is denoted by $nedge$. Note for $q = 3$ that a cubic deformation field cannot be represented exactly by a triangular element ($nedge = 3$) with only boundary nodes.

$nedge$	$q = 1$	$q = 2$	$q = 3$
3	0	1	2*
4	1	2	3
5	1	2	3
6	2	3	4
7	2	3	4
8	3	4	5

Table 3: Comparison of transverse tip displacements for cantilevered beam example. Results are normalized with respect to a standard finite element analysis of the structured eight-node quadrilateral mesh in Figure 4.

ν	VETFEM mesh	Structured Mesh
0.3	1.0007	1.0001
0.49	1.0037	1.0008
0.49999	0.9995	0.9957

Table 4: Comparison of horizontal displacements for slotted panel example. Results are normalized with respect to a standard finite element analysis of the structured eight-node quadrilateral mesh in Figure 5. Note for $\nu = 0.49999$ that a standard finite element analysis with eight-node quadrilateral elements shows signs of volumetric locking.

ν	VETFEM mesh	Structured Mesh
0.3	0.9905	1.0002
0.49	0.9923	1.0014
0.49999	1.1244*	1.1134*

Table 5: Ratios of sixth largest eigenvalues of a skewed eight-node quadrilateral element for standard and present methods. Plane strain conditions are assumed. The standard method uses 3 by 3 Gauss quadrature (2 by 2 quadrature leads to a spurious zero energy mode). The present method uses $p = 2$.

ν	ratio of sixth largest eigenvalue
0.47	1.09
0.48	1.41
0.49	2.36
0.499	19.62
0.4999	181.89
0.49999	293.21

Figure 1: Example VETFEM mesh (see [7]) where circles denote mid-edge nodes.

Figure 2: Example mesh that passes second-order patch tests for the present method but not for standard eight-node quadrilateral elements.

Figure 3: Mesh used for third-order patch tests.

Figure 4: Structured and VETFEM meshes of a beam.

Figure 5: Structured and VETFEM meshes of a slotted panel.