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RESULTS OF VON NEUMANN ANALYSES FOR REPRODUCING KERNEL SEMI-DISCRETIZATIONS

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Abstract. *The Reproducing Kernel Particle Method (RKPM) has many attractive properties that make it ideal for treating a broad class of physical problems. RKPM may be implemented in a "mesh-full" or a "mesh-free" manner and provides the ability to tune the method, via the selection of a dilation parameter and window function, in order to achieve the requisite numerical performance. RKPM also provides a framework for performing hierarchical computations making it an ideal candidate for simulating multi-scale problems. Although RKPM has many appealing attributes, the method is quite new and its numerical performance is still being quantified with respect to more traditional discretization methods. In order to assess the numerical performance of RKPM, detailed studies of RKPM on a series of model partial differential equations has been undertaken. The results of von Neumann analyses for RKPM semi-discretizations of one and two-dimensional, first and second-order wave equations are presented in the form of phase and group errors. Excellent dispersion characteristics are found for the consistent mass matrix with the proper choice of dilation parameter. In contrast, the influence of row-sum lumping the mass matrix is shown to introduce severe lagging phase errors. A "higher-order" mass matrix improves the dispersion characteristics relative to the lumped mass matrix but delivers severe lagging phase errors relative to the fully integrated, consistent mass matrix.*

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1 INTRODUCTION

The accurate simulation of wave propagation or advection dominated processes using discrete numerical schemes hinges upon having a clear understanding of the constraining numerical errors, the grid resolution to minimize such errors, and sufficient computational resources to effect solutions with the requisite grid scale. Examples of this may be seen when attempting to simulate wave propagation in an acoustic medium, or compute turbulent flow fields via direct numerical simulation (DNS) or large eddy simulation (LES). In physical problems with a dominant hyperbolic character, controlling the dispersive errors, i.e., phase and group speed errors, to within 5% can require approximately 8 to 10 grid points per wavelength with traditional grid-based methods. Thus, the computation of wave propagation problems is limited by the wavelength, or frequency, that the grid can accurately represent. A failure to respect the so-called Nyquist limit of the grid introduces deleterious aliasing effects that can corrupt the fidelity of the simulation. Similarly, the calculation of turbulent flows via DNS and LES is limited by the range of wavelengths that the grid can accurately resolve. For problems with complex geometry, graded, unstructured meshes are frequently limited by the conservative estimates made for the wavelengths that the mesh can resolve.

An alternative to traditional grid-based approaches is the class of methods based on moving least-squares, reproducing kernels, and partitions of unity. An overview of the development of this class of numerical methods is presented by Belytschko, et al.¹ In specific, the class of methods based on reproducing kernels promises to deliver enhanced numerical performance on a broad range of physical problems. Here, numerical performance is defined to include the following: truncation error, consistency and stability, rate of convergence, dispersive character, and spatial adaptivity.

Liu and his co-workers have been developing reproducing kernel particle methods for a number of years and have demonstrated applications ranging from structural acoustics to large deformation mechanics problems.²⁻⁵ In addition, Liu, et al.^{6,7} have combined reproducing kernel ideas with multi-resolution analysis using wavelets, permitting the decomposition of discrete solutions into multiple scale responses. The application of RKPM to structural dynamics has been demonstrated by Liu, et al.⁸ in addition to showing that the reproducing kernel interpolation functions satisfy consistency conditions. Uras, et al.⁹ have applied RKPM to acoustics problems, and demonstrated the effect of grid refinement through the use of the dilation parameter. In a series of papers by Liu and Li¹⁰⁻¹² moving least squares reproducing kernel methods are developed beginning with the basic formulation and continuing through a Fourier analysis and the incorporation of wavelet packets. The possibility for RKPM to deliver synchronized rates of convergence for the discrete functions and their derivatives has also been explored by Li and Liu.¹³ The application of RKPM for nearly incompressible, hyper-elastic solids was considered by Chen, et al.,¹⁴ while the treatment of large deformation problems has been explored by Liu and his colleagues.^{15,16} The enrichment of finite element computations with RKPM has also

been addressed permitting local regions of the computational domain to be treated with RKPM while the global problem is treated with a standard finite element formulation.^{17,18}

The application of RKPM has been demonstrated on a broad class of problems, however, its numerical performance has not been quantified with respect to more traditional methods. In order to apply numerical methods such as RKPM to wave propagation or advection dominated problems and faithfully represent the physics, a clear understanding of the phase and group errors associated with the numerical method is necessary. Thus, the focus of this work is on characterizing the dispersive nature of RKPM semi-discretizations of both the first and second-order wave equations.

In general, the application of discrete methods to hyperbolic partial differential equations can result in solutions that are dispersive even though the physical model for wave propagation is non-dispersive. Dispersion errors are typically characterized by the differences between the apparent, i.e., numerical, phase and group speed of waves and their exact counterparts. Phase and group speed errors represent some of the most constraining numerical errors for simulating wave propagation and advection dominated flows. In the context of linear acoustics, the phase speed is the speed at which individual waves propagate. In the absence of dispersion, i.e., for a perfect acoustic fluid, this is simply the sound speed. In a dispersive medium, the phase speed is a function of the frequency or wavelength of the propagating wave. In this context, phase error may be viewed as a measure of the influence of numerical dispersion on the apparent sound speed relative to the true sound speed or advective velocity.

In contrast to the phase speed, the group speed describes the propagation of wave packets that are comprised of short wavelength signals modulating at slowly varying, longer wavelength envelopes. Thus, the group speed refers to the speed of a wave group, and is sometimes referred to as the speed of modulation. The energy associated with a wave packet travels with the packet and often the group speed is referred to as the "energy" velocity. For a non-dispersive acoustic medium the phase and group speeds are identical. However, in discrete solutions, the group speed may be used to study and explain the propagation of the short wavelength, spurious oscillations that are typically $2\Delta x$ in wavelength, where Δx is the characteristic mesh spacing. Vichnevetsky¹⁹ has demonstrated that the spurious $2\Delta x$ oscillations, which may be induced by rapid changes in mesh resolution or physical boundaries, propagate at a group speed associated with a $2\Delta x$ wavelength.

The investigation into the dispersive errors associated with discrete solutions is not new and has been used by numerous researchers to characterize the performance of numerical methods in the past. A brief review of this work is presented, before proceeding with a discussion of the RKPM formulation.

The effects of consistent, lumped and higher-order mass matrices on the phase speed for linear and quadratic finite elements were investigated by Belytschko and Mullen²⁰ for linear elastic wave propagation in one dimension. Here, the compensatory interaction between the time integrator and mass matrix, i.e., trapezoidal rule and consistent mass

versus central difference and lumped mass, was verified, but shown to depend upon the Courant number. In a subsequent effort, Mullen and Belytschko²¹ characterized the phase speed errors for finite element semi-discretizations of the second-order wave equation.

Fourier analysis has also been applied to finite element discretizations in order to understand the dispersive nature of elastic wave propagation in bars and locking phenomena in beams.²² This analysis technique was applied by Park and Flaggs²³ in an effort to understand and ameliorate locking phenomena in C^0 plate elements. Alvin and Park²⁴ have also used Fourier analysis to tailor the frequency response of beams and bars discretized with the finite element method.

Vichnevetsky²⁵⁻²⁷ has investigated the dispersive nature of both finite difference and finite element methods for the first-order wave equation. Vichnevetsky²⁸ has also considered the dispersive errors introduced by nonuniform grid spacing and "hard" boundary conditions, and the possibility of using artificial viscosity to damp short wavelength spurious waves. Similar analysis techniques have been applied to wave propagation in periodic domains.²⁹ Trefethen³⁰ has performed a detailed study of the group speed associated with finite difference discretizations of the first and second-order wave equations in one and two space dimensions. Karni³¹ has characterized the group speed errors associated with symmetric upwind schemes for pure advection, i.e., a first order wave equation.

More recently, Shakib and Hughes³² have applied Fourier analysis to the space-time Galerkin least-squares method for advection-diffusion problems. Harari and Hughes³³ present the phase error associated with the Galerkin least-squares discretization for the second-order wave equation in a finite domain. Deville and Mund³⁴ have used Fourier analysis to investigate the spectral behavior of the iteration matrix for finite element preconditioning. Thompson and Pinsky³⁵ extended the concepts of Fourier analysis in order to treat p-version finite element discretizations. This work provides practical guidelines for the number of elements per wavelength in terms of the spectral order. Similarly, Grosh and Pinsky³⁶ have applied Fourier dispersion analysis to fluid loaded plates for structural acoustics simulations.

The following discussion begins with an overview of the dispersion analysis and a summary of the formulae for computing the phase and group speed for RKPM semi-discretizations of the first and second-order wave equations. In section 3, the dispersion characteristics for RKPM are presented. The phase and group speed for both consistent, lumped, and higher-order mass matrices are presented. Finally, the results of the dispersion analysis are summarized and conclusions are drawn.

2 FORMULATION

This section begins with a brief overview of the RKPM method followed by a derivation of the formulae for computing the normalized phase and group speeds associated with RKPM semi-discretizations of the model hyperbolic partial differential equations. A detailed presentation of RKPM is beyond the scope of this work, and the reader is directed to the literature for details concerning the method.²⁻¹⁸

2.1 Reproducing Kernel Particle Formulation

For the sake of clarity, the following overview is limited to one spatial dimension although the formulation may be easily extended to higher dimensions.^{3,4,14} The RKPM formulation begins with the notion of a kernel approximation of a function, U on a domain Ω ,

$$U^h(x) = \int_{\Omega} U(\xi) \varphi(x - \xi) d\xi, \quad (1)$$

where φ is the kernel function and U^h is an approximation to U .^{2,3} In order to address discrete problems, numerical quadrature (e.g., trapezoidal integration) is used to evaluate Eq. (1) as

$$U^h(x) = \sum_{i=1}^{Nnp} U(x_i) \varphi(x - x_i) \Delta x_i, \quad (2)$$

where Nnp is the total number of nodes in Ω .³

One of the most commonly used kernel functions is the cubic spline. In one-dimension, the cubic spline kernel function is

$$\varphi(z) = \begin{cases} \frac{2}{3r\Delta x} \left[1 - \frac{3}{2}z^2 + \frac{3}{4}z^3 \right] & |z| < 1 \\ \frac{1}{6r\Delta x} [2 - z]_3 & 1 \leq |z| < 2, \\ 0 & |z| \geq 2 \end{cases} \quad (3)$$

where $z = (x - x_i)/(r\Delta x)$ and Δx is the nodal spacing.^{1,3,4} The refinement parameter, r , in Eq. (3) permits dilation of the kernel function. The effect of dilation is the modification of the domain of influence of the kernel function, e.g., $r = 0.5$ and $r = 1.0$ results in support over 3 and 5 nodes respectively for the cubic-spline. An optimal dilation parameter of $r = 1.14$, as established by Liu and Chen,⁷ is used in this work. This parameter was designed by Liu and Chen to minimize aliasing error in terms of an energy error ratio.

It is possible to generate kernel functions for multiple dimensions employing tensor products of Eq. (3),

$$\varphi\left(\frac{x - x_i}{r\Delta x}, \frac{y - y_i}{r\Delta y}\right) = \varphi\left(\frac{x - x_i}{r\Delta x}\right) \varphi\left(\frac{y - y_i}{r\Delta y}\right), \quad (4)$$

resulting in a two-dimensional kernel function with rectangular support.^{1,4} The tensor-product kernel function in Eq. (4) is used for the two-dimensional dispersion analysis whose results are presented in the section 3.

In general, Eq. (2) will not exactly reproduce an arbitrary polynomial. The accurate reproduction of polynomials to order S is ensured by introducing a modified window function,

$$\bar{\varphi}(x - x_i) = \sum_{k=0}^S \beta_k(x) \varphi(x - x_i) (x - x_i)^k, \quad (5)$$

where $\beta_k(x)$ is a set of correction functions that vary within Ω .^{3,4} The modified window function replaces φ in Eq. (2) yielding

$$U(x) = \sum_{i=1}^{Nnp} \bar{\varphi}(x - x_i) U(x_i) \Delta x_i. \quad (6)$$

The correction functions are determined by substituting Eq. (5) into Eq. (6) and requiring that the resulting kernel approximation reproduce polynomials to the desired order. For example, if polynomials $U(x) = 1$ and $U(x) = x$ are to be reproduced then

$$\sum_{i=1}^{Nnp} [\beta_0(x) + \beta_1(x)(x - x_i)] \varphi(x - x_i) \Delta x_i = 1 \quad (7)$$

and,

$$\sum_{i=1}^{Nnp} [\beta_0(x) + \beta_1(x)(x - x_i)] \varphi(x - x_i) x_i \Delta x_i = x \quad (8)$$

are required. Equations (7) and (8) constitute two equations from which $\beta_0(x)$ and $\beta_1(x)$ may be calculated for linear consistency.

In the reproducing kernel particle method, shape functions, $N_i(x)$, arise by considering Eq. (6) to be an expansion in terms of $N_i(x)$ where

$$N_i(x) = \bar{\varphi}(x - x_i) \Delta x_i. \quad (9)$$

These shape functions may be used with a Bubnov-Galerkin procedure to obtain the weak form of the first and second-order wave equations yielding the mass, stiffness and advection matrices. In fact, if the hat function,

$$\varphi(z) = \begin{cases} \frac{1-z}{r\Delta x} & |z| < 1 \\ 0 & |z| \geq 1, \end{cases} \quad (10)$$

is used (with $r = 1$) instead of the cubic-spline of Eq. (3), the resulting shape functions are simply the linear finite element shape functions, yielding the usual form of the mass and stiffness operators.

In the formulation of the semi-discrete equations, the spatial derivatives of $N_i(x)$ are required. Calculation of these derivatives requires the computation of the derivatives of $\beta_k(x)$ as well. Although the calculation of these derivatives is rather straight forward, the algebra required is significant. For this reason, the reader may wish to consult the work of Liu, et al.^{3,4,14}

2.2 von Neumann Analysis

With the RKPM formulation outlined, the weak forms of two model partial differential equations are presented with a brief description of the Fourier analysis. To begin, the two-dimensional first and second-order wave equations are, in Cartesian coordinates,

$$\frac{\partial U}{\partial t} + c_x \frac{\partial U}{\partial x} + c_y \frac{\partial U}{\partial y} = 0 \quad (11)$$

and,

$$\frac{\partial^2 U}{\partial t^2} - c^2 \left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right] = 0, \quad (12)$$

where t is time, U is the unknown variable, c is the wave velocity, and $c_x = c \cos(\theta)$ and $c_y = c \sin(\theta)$ are the advection velocity components in the x and y -directions respectively.

The semi-discrete forms of Eq. (11) and (12) are required for the following analysis. The details for obtaining the weak form of these equations are well known,³⁷ and are not repeated here. The semi-discrete forms of the first and second-order wave equations are,

$$\mathbf{M}\dot{\mathbf{U}} + \mathbf{A}\mathbf{U} = 0 \quad (13)$$

and,

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{K}\mathbf{U} = 0, \quad (14)$$

where $\mathbf{A}$ is the advection operator, and $\mathbf{K}$ is the stiffness matrix. The generalized mass matrix is defined as

$$\mathbf{M} = \alpha \mathbf{M}^c + (1 - \alpha) \mathbf{M}^l, \quad (15)$$

where $\mathbf{M}^c$ and $\mathbf{M}^l$ are the consistent and row-sum-lumped mass matrices respectively, and α is the lumping parameter.

Proceeding with the Fourier analysis, a plane wave solution is placed on an infinite span (alternatively, on a finite domain with periodic boundary conditions) in order to compare the exact and semi-discrete solutions. The plane wave solution to Eq. (11) and (12) may be expressed as

$$U(x, y, t) = U_0 \exp[ik(x \cos(\theta) + y \sin(\theta)) - i\omega t], \quad (16)$$

where U_0 is the amplitude and θ is the propagation direction of the plane wave measured from the x -axis.^{19,20,38} For a non-dispersive medium, the wave frequency and number, ω and k respectively, are related to c as

$$c = \frac{\omega}{k}. \quad (17)$$

Now, considering a mesh with nodes equally spaced at intervals of Δx and Δy , any node $(i + m, j + n)$ at coordinates (x_{i+m}, y_{j+n}) may be located relative any other node (i, j) as $x_{i+m} = x_i + m\Delta x$, $y_{j+n} = y_j + n\Delta y$.

Solutions to the semi-discrete equations are sought in the form,

$$U_{i+m,j+n} = U_{i,j} \exp[ik(m\Delta x \cos \theta + n\Delta y \sin \theta)], \quad (18)$$

where $U_{i,j} = U_0 \exp[-i\omega t]$. Thus, from Eq. (18) $\dot{U}$ and $\ddot{U}$ may be expressed as

$$\dot{U}_{i+m,j+n} = -i\omega U_{i,j} \exp[ik(m\Delta x \cos \theta + n\Delta y \sin \theta)] \quad (19)$$

and,

$$\ddot{U}_{i+m,j+n} = -\omega^2 U_{i,j} \exp[ik(m\Delta x \cos(\theta) + n\Delta y \sin(\theta))]. \quad (20)$$

Given an arbitrarily wide kernel function, the semi-discrete forms of the first and second-order wave equations may be rewritten for node (i, j) as

$$\sum_{l=-n}^n \sum_{m=-n}^n \left\{ M_{(i,j),(i+l,j+m)} \dot{U}_{i+l,j+m} + A_{(i,j),(p,q)} U_{i+l,j+m} \right\} = 0 \quad (21)$$

and,

$$\sum_{l=-n}^n \sum_{m=-n}^n [M_{(i,j),(i+l,j+m)} \ddot{U}_{i+l,j+m} + K_{(i,j),(i+l,j+m)} U_{i+l,j+m}] = 0, \quad (22)$$

where $M_{(i,j),(p,q)}$, $K_{(i,j),(p,q)}$ and $A_{(i,j),(p,q)}$ are the mass, stiffness and advection matrix entries on the row associated with the (i, j) node and the column associated with node (p, q) .

Substituting Eq. (18) through (20) into Eq. (21) and (22) and canceling terms we have,

$$\begin{aligned} & -i\omega \sum_{l=-n}^n \sum_{m=-n}^n [M_{(i,j),(i+l,j+m)} \exp(ik(l\Delta x \cos(\theta) + m\Delta y \sin(\theta)))] + \\ & \sum_{l=-n}^n \sum_{m=-n}^n A_{(i,j),(i+l,j+m)} \exp(ik(l\Delta x \cos(\theta) + m\Delta y \sin(\theta))) = 0 \end{aligned} \quad (23)$$

for the first-order wave equation, and

$$\begin{aligned} & -\omega^2 \sum_{l=-n}^n \sum_{m=-n}^n [M_{(i,j),(i+l,j+m)} \exp(ik(l\Delta x \cos(\theta) + m\Delta y \sin(\theta)))] + \\ & \sum_{l=-n}^n \sum_{m=-n}^n [K_{(i,j),(i+l,j+m)} \exp(ik(l\Delta x \cos(\theta) + m\Delta y \sin(\theta)))] = 0 \end{aligned} \quad (24)$$

for the second-order wave equation.

The computation of the normalized phase and group speed, proceeds by solving for the circular frequency, ω , and making use of Eq. (17). The normalized phase speed

associated with either semi-discrete equation is $\psi = \bar{c}/c$ where $\bar{c}$ is the numerical phase speed. Rearranging Eq. (23) and (24) yields

$$\psi = \frac{1}{ick} \frac{\sum_{l=-n}^n \sum_{m=-n}^n [A_{(i,j),(i+l,j+m)} \exp(ik(l\Delta x \cos(\theta) + m\Delta y \sin(\theta)))]}{\sum_{l=-n}^n \sum_{m=-n}^n [M_{(i,j),(i+l,j+m)} \exp(ik(l\Delta x \cos(\theta) + m\Delta y \sin(\theta)))]} \quad (25)$$

for the first-order wave equation, and

$$\psi = \frac{1}{ck} \sqrt{\frac{\sum_{l=-n}^n \sum_{m=-n}^n [K_{(i,j),(i+l,j+m)} \exp(ik(l\Delta x \cos(\theta) + m\Delta y \sin(\theta)))]}{\sum_{l=-n}^n \sum_{m=-n}^n [M_{(i,j),(i+l,j+m)} \exp(ik(l\Delta x \cos(\theta) + m\Delta y \sin(\theta)))]}} \quad (26)$$

for the second-order wave equation.

The one-dimensional phase speed may be obtained from the two-dimensional results by setting $\theta = 0$. Assuming that the kernel function is symmetric so that $\mathbf{M}^c$ and $\mathbf{K}$ are also symmetric and $\mathbf{A}$ is anti-symmetric, the phase speed for the first and second-order, one-dimensional semi-discretizations are

$$\psi = \frac{f_a}{ck f_m}, \quad (27)$$

and

$$\psi = \frac{1}{ck} \sqrt{\frac{f_k}{f_m}} \quad (28)$$

where,

$$f_a = A_{i,i} + 2 \sum_{l=1}^n \sin(kl\Delta x) A_{i,i+l} \quad (29)$$

$$f_m = M_{i,i} + 2 \sum_{l=1}^n \cos(kl\Delta x) M_{i,i+l}, \quad (30)$$

and

$$f_k = K_{i,i} + 2 \sum_{l=1}^n \cos(kl\Delta x) K_{i,i+l}. \quad (31)$$

The normalized group speed, in one-dimension, is defined as $\zeta = v_g/c$, where $v_g = \partial\omega/\partial k$. Consideration of the normalized group velocity for the two-dimensional, semi-discretizations introduces significant complexities that make such analysis beyond the scope of this work.

Using Eq. (27) and (28), the normalized group speed in one-dimension is

$$\zeta = \frac{g_a f_m - f_a g_m}{c f_m^2}, \quad (32)$$

and

$$\zeta = \frac{\sqrt{f_m}}{2c\sqrt{f_k}} \frac{g_k f_m - f_k g_m}{f_m^2} \quad (33)$$

for the first and second-order wave equations respectively. Here,

$$g_a = \partial f_a / \partial k = 2\Delta x \sum_{l=1}^n l \cos(kl\Delta x) A_{i,i+l} \quad (34)$$

$$g_m = \partial f_m / \partial k = -2\Delta x \sum_{l=1}^n l \sin(kl\Delta x) M_{i,i+l}, \quad (35)$$

and

$$g_k = \partial f_k / \partial k = -2\Delta x \sum_{l=1}^n l \sin(kl\Delta x) K_{i,i+l}. \quad (36)$$

Unless otherwise noted, the normalized phase and group speeds defined above are referred to simply as phase and group speed in the remaining text.

Remark

There have been no restrictions placed on the form or type of shape functions used to obtain the mass, stiffness or advection operators. Thus, Eq. (25) through (33) are equally valid for Galerkin formulations that use either the RKPM shape functions or the usual finite element shape functions.

3 RESULTS

This section summarizes the results of the von Neumann analyses in terms of phase and group speed for RKPM semi-discretizations of the one-dimensional model hyperbolic equations followed by phase speed associated with the two-dimensional equations. Both the one and two-dimensional RKPM formulations use the cubic spline kernel functions in Eq. (3). Further, the two dimensional formulation uses the tensor product of Eq. (4) to produce a two dimensional kernel function. Both spatial formulations use the procedure outlined in Section 2.1 to generate modified window functions to ensure that linear ($U(x) = 1 + x$; one-dimensional) and bi-linear ($U(x, y) = 1 + x + y + xy$; two-dimensional) functions are reproduced exactly.

For the purpose of comparison, results are presented for linear and bi-linear FEM semi-discretizations. The FEM phase and group speed are calculated using the formulae presented in Section 2.2 with linear finite element shape functions. Here, the linear and bi-linear finite element shape functions were chosen for comparison as they provide the same order of consistency as the RKPM discretizations used.

3.1 One-Dimensional Hyperbolic Equations

In this section, the phase and group speed for both the semi-discrete, one-dimensional first-order and second-order wave equation are presented. In the discussion that follows, the phase and group speed results are presented as functions of non-dimensional wave number, $k\Delta x/\pi = 2\Delta x/\lambda$.

3.1.1 First-Order Wave Equation

Phase and group speed for the linear finite element semi-discretizations of the first-order wave equation are presented in Figure 1. Results are plotted for fully integrated, consistent (CF), lumped (LF) and higher-order (HF) mass matrix formulations. The higher-order mass matrix implies $\alpha = 0.5$, cf. Eq. (15). As shown, all the FEM formulations introduce strictly lagging phase speed for all wavelengths considered with the CF formulation delivering smaller phase errors up to the $2\Delta x$ limit. All three mass matrices result in a phase speed of zero at $2\Delta x/\lambda = 1$, i.e., wavelengths of $2\Delta x$ are stationary on the grid.

Similar to the phase speed, the finite element discretizations yield strictly lagging group speed for all three mass matrices. However, the lumped mass matrix yields a zero group speed for $4\Delta x$ wavelengths while both the CF and HF mass matrices have zero group speed at longer wavelengths. The CF formulation performs better than the LF and HF formulations, i.e., yields smaller group errors for $\lambda \leq 3\Delta x$. All three formulations yield negative group speeds for short wavelengths, $2\Delta x$ wavelength packets will propagate in the opposite direction of the longer wavelength signals. Surprisingly, the LF formulation yields a smaller, albeit still negative group speed in the limit of $2\Delta x$ wavelengths.

Figure 2 shows the phase and group speed for the one-dimensional RKPM semi-discretizations of the first-order wave equation. Again, fully integrated consistent (CF), lumped (LF) and higher-order (HF) mass matrix formulations are presented. In addition, results are shown for the consistent mass matrix formulation with point-wise integration of the advection and mass matrices (CT). In point-wise (or trapezoidal) integration, nodal points are used as quadrature points, eliminating the need for a background integration mesh.

As with the FEM results of Figure 1, the RKPM method introduces lagging phase errors over the discrete spectrum of wavelengths. The consistent mass (CF) formulation performs the best and delivers significantly better phase speed relative to the FEM results of Figure 1. In order to quantify the increased performance of the RKPM CF method, consider a phase error, $\epsilon = |1 - \psi|$, of 5% or less to be appropriate for engineering purposes. For the FEM CF method, this criterion corresponds to $4\Delta x$ or 5 nodes per wavelength. In contrast, the RKPM CF and CT methods require $2 - 3\Delta x$, or approximately 3 – 4 nodes per wavelength. While both the RKPM CT and CF methods perform quite well, the lumped and higher-order formulations introduce severe lagging phase errors relative to their FEM counterparts.

In terms of the group speed, both the CT and CF RKPM formulations are far superior

Figure 1: Phase (a) and group (b) speed for the one-dimensional, linear finite element semi-discretization of the first-order wave equation with fully integrated, consistent (CF), higher-order (HF), and lumped (LF) mass matrix formulations.

Figure 2: Phase (a) and group (b) speed for the one-dimensional Reproducing Kernel Particle semi-discretization first-order wave with a full-integration consistent (CF), higher-order (HF), lumped (LF), and trapezoidal integration consistent (CT) mass matrix formulations.

to the LF and HF formulations. Similar to the phase speed, the CT formulation yields lagging group errors sooner than the CF formulation. However, the trapezoidal mass matrix, CT, avoids the large negative group speed associated with the fully-integrated, CF, matrix. In comparison to the FEM CF results, the RKPM CT formulation does not yield negative group speeds for wavelengths greater than $3\Delta x$, while the CF formulation does not produce negative group speed for wavelengths greater than about $2.5\Delta x$. However, the group errors associated with $2\Delta x$ wavelengths for the CF formulation are over 3 times larger than for the CF FEM results. From these results it is apparent that the RKPM CT and CF formulations exhibit very good dispersive behavior with linear consistency comparable to the FEM formulation.

3.1.2 Second-Order Wave Equation

Phase and group speeds for the linear finite element semi-discretizations of the second-order wave equation are presented in Figure 3 for the fully integrated, consistent, lumped

Figure 3: Phase (a) and group (b) speed for the finite element semi-discretization of the one-dimensional, second-order wave equation using a full-integration consistent (CF), higher-order (HF) and lumped (LF) mass matrix formulation.

and higher-order mass matrix formulations. The consistent mass formulation introduces leading phase errors while the lumped and higher-order methods exhibit strictly lagging phase errors. Additionally, both the LF and HF methods demonstrate lagging group speed for all wavelengths considered while the CF group speed is leading for $2\Delta x/\lambda < 0.85$.

Figure 4 shows the phase and group speeds for the one-dimensional, second-order wave RKPM semi-discretization using the CF, CT, LF and HF formulations. Relative to the FEM results of Figure 3, the consistent mass matrix (CF) provides better phase and group speed. Surprisingly, the trapezoidal mass formulation (CT) yields zero phase speed for $2\Delta x$ wavelengths, i.e., these wavelengths are stationary on the grid. Additionally, the CT formulation results in large, lagging group errors for wavelengths shorter than $3\Delta x$. In contrast, the FEM semi-discretizations do not yield any negative group speeds.

Employing the 5% phase error criterion introduced earlier, the FEM HF method requires approximately 4 nodes per wavelength while only 3 nodes are required for the RKPM CF method. As with the RKPM discretization of the first-order wave equation, the lumped and higher-order formulations introduce severe lagging phase and group errors relative to both the FEM counterparts and the CT and CF mass matrices.

3.2 Two-Dimensional Hyperbolic Equations

This section presents the phase speed results for the semi-discrete, two-dimensional, hyperbolic equations. Results are plotted as functions of the propagation angle, θ , and non-dimensional wave number. For this analysis, the nodal spacing is uniform with a unit aspect ratio, $\Delta y/\Delta x = 1$. As with the one-dimensional analyses, a refinement parameter of $r = 1.14$ based upon a minimum energy error is used in the RKPM formulation. In order to highlight the directional dependence of the phase error, the phase speed is presented with both polar and Cartesian plots.

Figure 4: Phase (a) and group (b) speed for the Reproducing Kernel Particle semi-discretization of the one-dimensional, second-order wave equation using the full-integration, consistent (CF), higher-order (HF), lumped (LF), and trapezoidal integration consistent (CT) mass matrix formulations.

3.2.1 First-Order Wave Equation

Phase speed plots for the semi-discrete first-order wave equation using the fully integrated bi-linear finite element and a consistent mass matrix are shown in Figure 5. The polar plot of Figure 5a shows phase speed as a function of direction, θ , for several values of non-dimensional wavelengths, $2\Delta x/\lambda$. The non-circular phase speed contours emphasize the anisotropic nature of wave propagation on the discrete mesh. Figure 5b presents the results of Figure 5a at five propagation angles, θ . It is apparent from Figure 5 that a minimum error in phase speed occurs when the wave propagation direction is $\pi/4$ from the x-axis. It is also apparent that the anisotropy becomes more pronounced for $2\Delta x/\lambda > 0.4$ (cf. Figure 5a).

Figure 5: Polar (a) and Cartesian (b) plots of the phase speed for the FEM semi-discretization of the two-dimensional, first-order wave equation using a full-integration, consistent mass matrix formulation.

Phase speed results for the fully integrated “bi-linear” reproducing kernel particle method using a consistent mass matrix are shown in Figure 6. As with the FEM formulation, the RKPM semi-discretization leads to strictly lagging phase speed with minimum phase speed errors occurring for $\theta = \pi/4$. However, unlike FEM, RKPM shows

Figure 6: Polar (a) and Cartesian (b) plots of the phase speed for the RKPM semi-discretization of the two-dimensional, first-order wave equation with full-integration, and a consistent mass matrix.

negligible phase error in this direction. Further, relative to the finite element method, the anisotropic behavior has been significantly reduced, with wave propagation being effectively independent of wavelength and propagation direction for $2\Delta x/\lambda \leq 0.8$, i.e., for wavelengths greater than about $2 - 3\Delta x$.

Figure 7 shows polar and Cartesian plots of the phase speed for the “bi-linear” RKPM formulation using trapezoidal integration and a consistent mass matrix. Again, the phase speed is lagging and anisotropic, with minimum errors occurring in the $\theta = \pi/4$ direction. Although the phase speed appears anisotropic for short wavelength signals, this formulation delivers nearly isotropic wave propagation for $2\Delta x/\lambda \leq 0.6$, i.e., wavelengths greater than $3 - 4\Delta x$.

3.2.2 Second-Order Wave Equation

Figure 8 shows phase speed results for the second-order wave semi-discretization using a fully integrated bi-linear finite element method with a consistent mass matrix. The results indicate that the finite element formulation introduces strictly leading phase errors. The finite element semi-discretization results in anisotropic wave propagation, with a minimum phase error occurring in the $\theta = \pi/4$ propagation direction. However, the anisotropy is not as pronounced as it is for the first-order equation.

The fully integrated “bi-linear” reproducing kernel particle method (consistent mass matrix) yields almost negligible phase errors as shown in Figure 9. Further, as phase errors are quite small for all θ , wave propagation is nearly perfectly isotropic. Some slight leading phase speed errors are evident for wavelengths approaching $2\Delta x$. However, these errors are less than about 2.5% with a minimum in phase error occurring in the $\theta = \pi/4$ propagation direction.

Finally, Figure 10 shows the phase speed results for “bi-linear” RKPM semi-discretization using trapezoidal integration with a consistent mass matrix. Unlike the fully integrated results, anisotropic dispersion errors are quite evident for $2\Delta x/\lambda > 0.6$. However, for

Figure 7: Polar (a) and Cartesian (b) plots of the phase speed for the RKPM semi-discretization of the two-dimensional, second-order wave equation with a consistent mass matrix and trapezoidal integration.

Figure 8: Polar (a) and Cartesian (b) plots of the phase speed for the FEM semi-discretization of the two-dimensional, second-order wave equation using a full-integration, consistent mass matrix formulation.

Figure 9: Polar (a) and Cartesian (b) plots of the phase speed for the RKPM semi-discretization of the two-dimensional, second-order wave equation using full-integration and a consistent mass matrix.

Figure 10: Polar (a) and Cartesian (b) plots of the phase speed for the RKPM semi-discretization of the two-dimensional, second-order wave equation using a consistent mass matrix and trapezoidal integration.

$2\Delta x/\lambda \leq 0.6$ phase errors are negligible and are significantly better than for the FEM case (cf. Figure 8). Similar to the fully-integrated RKPM semi-discretization, the phase errors are minimized in the $\pi/4$ propagation direction, but with nearly perfect phase speed for wavelengths longer than $3 - 4\Delta x$.

4 SUMMARY AND CONCLUSIONS

This paper presents a dispersion analysis for RKPM semi-discretizations of two model hyperbolic partial differential equations. The semi-discretizations considered incorporate linear or bi-linear consistency (one or two-dimensional forms respectively) and use variations on matrix integration (full Gauss quadrature or trapezoidal integration) and type (consistent, lumped or higher order). All formulations incorporate a cubic-spline kernel function with the optimal refinement parameter of $r = 1.14$.

The results of the analyses presented here indicate that, for the formulations considered, the consistent mass RKPM formulations display better dispersion properties than the finite element method with similar consistency constraints. In a one-dimensional sense, phase errors of less than 5% are ensured with meshes of 3 to 4 nodes per wavelength while the FEM formulations required 4 to 5 nodes for similar semi-discretizations. Incredibly, RKPM semi-discretizations of the second-order wave equation require only 3 nodes per wavelength (the Nyquist limit) for phase errors of less than 2.5 percent. In addition, wave propagation with the consistent mass RKPM formulation in two-dimensions is nearly isotropic in terms of angular dependence of the phase speed and in terms of the amplitude of the phase errors.

While the consistent mass matrix RKPM formulations perform quite well, the lumped and higher order mass formulations introduce severely lagging phase and group speeds. Thus, the performance of these formulations is quite poor relative to their FEM counterparts.

Finally, the consistent mass RKPM results indicate that minimal losses in phase and group speed error result when trapezoidal integration of the matrices is employed in

place of full (Gauss) quadrature. With the sacrifice of negative group speeds and a slight increase in phase speed errors, the use of point-wise integration may significantly reduce computational cost by reducing the number of quadrature points needed. Further, the method should be simpler to implement as the background integration mesh can be eliminated. However, further direct testing with trapezoidal integration is required.

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