A Method for Achieving Constant Rotation Rates in a Micro-Orthogonal Linkage System

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Abstract

Silicon micromachine designs include engines that consist of orthogonally oriented linear comb drive actuators mechanically connected to a rotating gear. These gears are as small as 50 μm in diameter and can be driven at rotation rates exceeding 300,000 rpm. Generally, these engines will run with non-uniform rotation rates if the drive signals are not properly designed and maintained over a range of system parameters. We present a method for producing constant rotation rates in a micro-engine

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driven by an orthogonal linkage system. We show that provided the values of certain masses, springs, damping factors, and lever arms are in the right proportions, the system behaves as though it were symmetrical. We will refer to systems built in this way as being quasi-symmetrical. We show that if a system is built quasi-symmetrically, then it is possible to achieve constant rotation rates even if one does not know the form of the friction function, or the value of the friction. We analyze this case in some detail.

Key words: Micromachines, actuators, drive signals, feedback

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1 Introduction

An example of an orthogonally driven microengine [1] [2] is shown in Fig. 1. This is driven by linear electrostatic actuators that are mechanically coupled to an output gear. Electrostatically applied forces are transmitted to the gear through mechanical linkages in a way that ideally results in continuous rotation of the output gear [1]. The output gear can be used to apply torque to a load device.

It is almost always a highly desirable goal to try to run these engines so that they rotate with nearly uniform rotation rate. In order to study the motion of these engines an optical technique has been developed that permits the recording of engine position and direction information in real time [3] [4]. Known as the

Micromachine Optical Probe (MOP), the method is based on the measurement of optical radiation scattered from the gear teeth or other physical structures. It uses both forward and backscattered radiation to detect engine direction; however, only one signal is needed to determine uniformity of the engine motion. For the eight-tooth gear of Fig. 1, a typical backscatter signal from the MOP technique is shown in Fig. 2. This particular gear was running at 500 Hz in the counterclockwise (CCW) direction. The peaks correspond to points where the gear tooth was passing under the probe beam. The trace shows eight peaks, which agrees with the eight teeth of the gear. The uneven spacing of the peaks means that the gear was not sweeping uniformly through its rotation. Peaks four through seven make up one-half of the gear rotation, yet they occurred in about one-fifth of the total rotation time. Fig. 3 shows a backscatter signal from a more advanced, nineteen-tooth version of the engine in Fig. 1. Again, the uneven spacing of the nineteen peaks means that the gear was running non-uniformly.

These results show that these engines are far from achieving a uniform rotation rate. This stems from the fact that classical techniques for achieving uniform rotation rates in ordinary sized engines do not carry over to their micro-engine counterparts. In particular, due to the minute size of the microengines, the inertial effect of a flywheel is of very little use. To date, the only technique available for achieving constant rotation rates has been to use drive voltages obtained from what we will refer to as backwards mathematical modeling of the

engines.

To illustrate this process of obtaining drive voltages by backwards mathematical modeling, we consider a particular mathematical model of a microengine. Since the results of this paper are based on this particular model, we will elaborate on these equations in the next section. For the time being we are merely using these equations to illustrate what we mean by backwards mathematical modeling. We will use the equations [5] [6]

$$\pm V_x^2(t) = \frac{k_x r}{\gamma a} \left(\frac{\gamma^2}{\omega_x^2} \left[\left(\ddot{\theta} + 2\delta_x \dot{\theta} \right) \cos(\theta) - \dot{\theta}^2 \sin(\theta) \right] + \gamma^2 \sin(\theta) + \frac{F_r \sin(\theta) + F_l \cos(\theta)}{k_x r} \right) \quad (1a)$$

$$\pm V_y^2(t) = \frac{k_y r}{a} \left(\frac{1}{\omega_y^2} \left[\left(\ddot{\theta} + 2\delta_y \dot{\theta} \right) \sin(\theta) + \dot{\theta}^2 \cos(\theta) \right] + 1 - \cos(\theta) + \frac{-F_r \cos(\theta) + F_l \sin(\theta)}{k_y r} \right) \quad (1b)$$

$$F_l = \mu(F_r) \operatorname{sgn}(\dot{\theta}) F_r \quad (1c)$$

In these equations $\theta(t)$ gives the angle of the gear, $F_r(t)$ the radial force on the hub, and $V_x(t)$ and $V_y(t)$ are the drive voltages. The rest of the parameters k_x , k_y , etc will be defined in the next section. In the equation relating F_l to F_r , we need to include the factor $\operatorname{sgn}(\dot{\theta})$ in order to take into account the fact that the direction of the force of friction is opposite to the direction of movement of the engine relative to the hub. If one is given the drive voltages, one needs to solve these nonlinear ordinary algebraic differential equations in order to obtain the functions $\theta(t)$ and $F_r(t)$. We refer to this as forwards modeling of

the engine. This is in general a very difficult task, and can seldom be done in any other way than to numerically integrate the equations. However, if one substitutes any desired functions $\theta(t)$ and $F_r(t)$ into the right hand side of these equations it is perfectly straightforward (although tedious) to determine what drive voltages would be necessary to achieve these outputs. This is the process of backwards mathematical modeling that we have already alluded to. In particular, we can carry out this process of backwards mathematical modeling for the case where $\theta(t)$ increases linearly with time with slope ω (the uniform rotation rate solution), and $F_r(t)$ is constant in time. In this case we find that $Q_x(t) = \pm \frac{a}{\gamma k_x r} V_x(t)$ and $Q_y(t) = \pm \frac{a}{r k_y} V_y(t) - 1$ are linear combinations of sines and cosines with frequency ω . The amplitude and phase of the functions $Q_x(t)$ and $Q_y(t)$ are in general quite complicated functions of all of the parameters in our system.

Obtaining the drive voltages through backwards modeling has several drawbacks. It assumes that our model accurately models the performance of the engine, and that we know the values of the parameters that go into this model. If we design the drive voltages using one set of parameters, we are not guaranteed that anything even near a uniform rotation rate will be achieved if the true parameters are slightly different than the assumed parameters. In particular the solutions obtained in this way are not guaranteed of being stable. If we have a large uncertainty in any one of the parameters, this whole process becomes very unreliable. In the equations 1, the most uncertain parameter is the coefficient

of friction $\mu(F_r)$ which is typically assumed to be a constant independent of F_r . Not only are we uncertain about its value, but it is most likely not justifiable to assume that there is a linear relation between F_r and F_l .

In this paper we point out that if our system is built symmetrically then its behavior is very simple. In particular, we can get uniform rotation out of our system by choosing the functions $Q_x(t)$ and $Q_y(t)$ to have the same amplitude, and to be 90 degrees out of phase. This state of affairs will produce a constant rotation rate for all values of the parameters in our system (provided it is built symmetrically), and for any form of the friction function. In order to achieve these constant rotation rates it is necessary that the amplitude of the signals $Q_x(t)$ and $Q_y(t)$ be above a critical value that does in fact depend on all of the parameters in our system.

The parameter γ (to be defined in the next section) is the ratio of two lever arms. It is often desirable to have this parameter be something other than unity. This will destroy the symmetry of our system. We show that provided the ratio of some of the parameters in the system are chosen in the right proportions, the system with $\gamma \neq 1$ satisfies the same equations as that with $\gamma = 1$. This means that the system will behave as though it were built symmetrically. We will refer to a system built in this way as being quasi-symmetrical.

We will analyze the behavior of quasi-symmetrical systems and see that if the amplitude of the forcing is below a certain critical level A_1 , then we cannot achieve a constant rotation rate. In this case the motor will eventually approach

a state where it oscillates about a constant state of rotation (a jerky rotation). Once the forcing is bigger than this critical level, it is possible to have a constant rotation rate, but it is still possible that the motor will approach a non-uniform rotation for certain initial conditions. However, once the amplitude of the forcing exceeds a critical value A_2 , we are guaranteed of approaching a constant state of rotation for any initial conditions.

The behavior of quasi-symmetrical systems with $\mu = 0$ can be completely classified. This situation is identical to the analysis given by Andronov and Vitt [13] of the pendulum with a constant torque. One interesting conclusion from this analysis is that if we force the motor with a drive signal of frequency ω , then there are only two possible behaviors.

- The system eventually approaches a state of uniform rotation with frequency ω .
- The system eventually approaches a periodic, but non-uniform state of rotation. In this case the motor will spin about its axis with a frequency that is less than ω

The fact that our system can respond to a forcing at ω by rotating at a lower frequency is somewhat counter intuitive. We believe that these results hold even when $\mu \neq 0$, however, there are some loopholes in our proof that this must be the case. However, when the system is not built quasi-symmetrically, we believe that it is also possible for the system to undergo non-uniform periodic motions

right at the frequency ω . This is a consequence of experimental observation.

2 The Basic Equations

Fig. 4) is a diagram illustrating the parameters that go into our model equations 1. The gear is connected to two comb drives of mass m_x and m_y . The comb drives are connected to springs with spring constants k_x and k_y , and they experience damping forces proportional to δ_x and δ_y . A very important parameter in our discussion is the ratio of the lengths C and L in Fig. 4). We call this ratio

$$\gamma = \frac{C}{L} \quad (2)$$

The comb drives are driven by the voltages V_x and V_y . The signs of the terms $\pm V_x^2$ and $\pm V_y^2$ depend on how the voltages to the comb drives are applied.

Fundamental microengine device properties, such as spring constants, electrostatic force constants, and damping coefficients, have been measured by using the engine response to various applied drive voltages [2] [7] [8]. Frictional forces between the gear and hub have also been measured using time-dependent position measurements of the output gear [5] [6].

The equations 1 are derived by balancing the forces in this system. The equations assume that $r/L \ll 1$, which is usually an excellent approximation. We prefer to write the system of equations as

$$Q_x(t) = \frac{1}{\omega_x^2} \left[\left(\ddot{\theta} + 2\delta_x \dot{\theta} \right) \cos(\theta) - \dot{\theta}^2 \sin(\theta) \right] + \sin(\theta) + \frac{F_r \sin(\theta) + F_l \cos(\theta)}{\gamma^2 k_x r} \quad (3a)$$

$$Q_y(t) = \frac{1}{\omega_y^2} \left[\left(\ddot{\theta} + 2\delta_y \dot{\theta} \right) \sin(\theta) + \dot{\theta}^2 \cos(\theta) \right] - \cos(\theta) + \frac{-F_r \cos(\theta) + F_l \sin(\theta)}{k_y r} \quad (3b)$$

$$F_l = \mu(F_r) \operatorname{sgn}(\dot{\theta}) F_r \quad (3c)$$

$$Q_x = \frac{\pm V_x^2 a}{\gamma r k_x} \quad (3d)$$

$$Q_y = \frac{\pm V_y^2 a}{r k_y} - 1 \quad (3e)$$

In this system of equations we are solving for $\theta(t)$ and $F_r(t)$. If there is a linear relation between F_r and F_l , ($\mu(F_r)$ is constant), then the equations 3 can be reduced to a single second order nonlinear differential equation for θ . In order to do this we consider the equations 3 as two linear equations in the two unknowns F_l and F_r . We can solve for these in terms of $Q_x(t)$, $Q_y(t)$, and θ and its first two derivatives. If we substitute these expressions for F_l and F_r back into one of the equations in 3, we end up with a single second order differential equation for θ . In theory we could do this even if μ were not constant, but it would require solving nonlinear equations for the determination of F_r and F_l . Even in the linear case, this gives us quite a messy equation. Although we do not carry out this procedure, the fact that it can be done shows us that these equations are equivalent to a second order differential equation for θ . It

shows that assuming certain regularity conditions, we should be able to uniquely determine the motion of the gear once we know the initial values of θ and its first derivative.

3 Solutions with Uniform Rotation

In this section we will define precisely what we mean by quasi-symmetrical systems, and we will consider the production of constant rotation rate solutions in these systems. A quasi-symmetric system is one where the parameters are such that

$$\omega_x = \omega_y,$$

$$\delta_x = \delta_y,$$

$$\gamma^2 k_x = k_y,$$

and the inputs have the form

$$Q_x(t) = A \cos(\omega t)$$

$$Q_y(t) = A \sin(\omega t)$$

For such a system we will look for a solution of the form

$$\theta = \omega t + \alpha$$

$$F_r = \text{constant}$$

When we substitute this form into the first of equations 3, the terms on the right hand side will be of the form $B(\alpha, F_r)\cos(\omega t) + C(\alpha, F_r)\sin(\omega t)$, where

$$B(\alpha, F_r) = \cos(\alpha) \left(\frac{2\delta_x \omega}{\omega_x^2} + \frac{\mu(F_r)F_r}{\gamma^2 k_x r} \right) + \sin(\alpha) \left(-\frac{\omega^2}{\omega_x^2} + 1 + \frac{F_r}{k_x \gamma^2 r} \right)$$

$$C(\alpha, F_r) = -\sin(\alpha) \left(\frac{2\delta_x \omega}{\omega_x^2} + \frac{\mu(F_r)F_r}{\gamma^2 k_x r} \right) + \cos(\alpha) \left(-\frac{\omega^2}{\omega_x^2} + 1 + \frac{F_r}{k_x \gamma^2 r} \right)$$

We would like to choose α and F_r such that

$$A\cos(\omega t) = B(\alpha, F_r)\cos(\omega t) + C(\alpha, F_r)\sin(\omega t)$$

This will be satisfied provided

$$A = B(\alpha, F_r) \tag{4a}$$

$$C(\alpha, F_r) = 0 \tag{4b}$$

If these equations are satisfied, the first of the equations 3 will be satisfied by the solution $\theta(t) = \omega t + \alpha$. In general there is no guarantee that the second of the equations will also be satisfied. However, if the system is built quasi-symmetrically, then the terms on the right hand side of the second equation 3 are identical to the terms on the right hand side of the first equation, but phase shifted by 90 degrees. This means that in this particular situation, if

$Q_y(t)$ is $A\sin(\omega t)$, then θ determined by the solution to the first of eqns. 3 will automatically satisfy the second of eqns. 3 as well. It follows that we will get uniform rotation no matter what the parameters are, provided only that α and F_r satisfy 4.

Assuming we can satisfy the equations 4, this setup should give us uniform rotation even when F_l is related to F_r in a nonlinear fashion. In this case we will get a more complicated system of equations for the determination of α and F_r . However, we are still guaranteed that the second of equations 3 will be satisfied provided that the first are.

We now consider the equations 4 in the particular case where μ is a constant. In this case we can decouple the equations so that we have a single nonlinear equation for α .

$$A(\cos(\alpha) - \mu\sin(\alpha)) = \frac{2\delta_x\omega}{\omega_x^2} - \mu\left(1 - \frac{\omega^2}{\omega_x^2}\right) \quad (5a)$$

$$\frac{F_r}{k_x\gamma^2 r} = A\sin(\alpha) + \frac{\omega^2}{\omega_x^2} - 1 \quad (5b)$$

The first of these equations can be solved independently of the second. Once we have determined α from the first equation, F_r is found explicitly. We will write the first of these equations as

$$\frac{A}{A_0}(\cos(\alpha) - \mu\sin(\alpha)) = 1 \quad (6a)$$

$$A_0 = \frac{2\delta_x \omega}{\omega_x^2} - \mu \left(1 - \frac{\omega^2}{\omega_x^2} \right) \quad (6b)$$

The parameter A_0 can be either positive or negative. If the damping δ is small, the sign of A_0 will depend on whether the frequency ω is above or below the resonance frequency ω_x . If we introduce

$$\mu = \tan(\psi) \quad (7)$$

We can write these equations as

$$\cos(\alpha + \psi) = \frac{A_1}{A} \quad (8a)$$

where

$$A_1 = A_0 \cos(\psi) \quad (8b)$$

We see that if $|A_1/A| > 1$, then there are no solutions to this equation. If $0 < |A_1/A| < 1$, then there will be two solutions (with $|\alpha| < \pi$). This shows that in the case where μ is constant, we are guaranteed of having two solutions to the equations 5 provided

$$A > A_1 \quad (9)$$

One of the solutions will be on a part of the curve where $\cos(\alpha) - \mu \sin(\alpha)$ is increasing with α , the other will be on a part of the curve where it is decreasing with α . This means that one of the solutions will have $A(\sin(\alpha) + \mu \cos(\alpha))$ positive, the other will have this quantity negative. In a later section we will show that the solution with $A(\sin(\alpha) + \mu \cos(\alpha)) > 0$ is stable with respect to small disturbances, and the other solution is unstable.

We have shown that when μ is constant, we will have two solutions to eqns. 4 provided A is greater than some critical value A_1 , and none if A is below that value. If μ is a weak function of F_r we are guaranteed of having a similar situation. When μ is a stronger function of F_r this situation most likely persists, but one has to treat it on a case by case basis.

4 The Time Dependent Equations for Quasi-symmetrical Systems

We will now derive a simple differential equation that describes the behavior of quasi-symmetrical systems. We will also make the assumption that μ is a constant. In this case 3 can be written as

$$A \cos(\omega t) = \frac{1}{\omega_y^2} \left[\left(\ddot{\theta} + 2\delta_y \dot{\theta} \right) \cos(\theta) - \dot{\theta}^2 \sin(\theta) \right] + \sin(\theta) + \frac{F_r \sin(\theta) + F_l \cos(\theta)}{\gamma^2 k_z r} \quad (10a)$$

$$A \sin(\omega t) = \frac{1}{\omega_y^2} \left[\left(\ddot{\theta} + 2\delta_y \dot{\theta} \right) \sin(\theta) + \dot{\theta}^2 \cos(\theta) \right] - \cos(\theta) + \frac{-F_r \cos(\theta) + F_l \sin(\theta)}{k_y r} \quad (10b)$$

$$F_l = \mu \operatorname{sgn}(\dot{\theta}) F_r \quad (10c)$$

We would like to eliminate F_r from these equations. To do this we multiply the first of these equations by $\cos(\theta)$, the second by $\sin(\theta)$, and add the results. This gives us

$$A \cos(\omega t - \theta) = \frac{1}{\omega_x^2} (\ddot{\theta} + 2\delta_x \dot{\theta}) + \frac{\text{sgn}(\dot{\theta}) \mu F_r}{\gamma^2 k_x r}$$

If we multiply the first of eqns. 10 by $\sin(\theta)$, the second by $-\cos(\theta)$, and add the results we get

$$-\sin(\omega t - \theta) = \left(1 - \frac{\dot{\theta}^2}{\omega_0^2}\right) + \frac{F_r}{\gamma^2 k_x r}$$

Eliminating F_r from these last two equations we find

$$A \left(\cos(\omega t - \theta) + \mu \text{sgn}(\dot{\theta}) \sin(\omega t - \theta) \right) = \frac{1}{\omega_x^2} (\ddot{\theta} + 2\delta_x \dot{\theta}) - \mu \text{sgn}(\dot{\theta}) \left(1 - \frac{\dot{\theta}^2}{\omega_x^2}\right) \quad (11)$$

If we make the substitution $\phi = \theta - \omega t$ we get the equation

$$A \left(\cos(\phi) - \mu \text{sgn}(\dot{\phi} + \omega) \sin(\phi) \right) = \frac{1}{\omega_x^2} (\ddot{\phi} + 2\delta_x (\omega + \dot{\phi})) - \mu \text{sgn}(\dot{\phi} + \omega) \left(1 - \frac{(\dot{\phi} + \omega)^2}{\omega_x^2}\right) \quad (12)$$

This is the final form of the equation that we will use to analyze the transient behavior of our solutions. It should be recalled that these equations assume that the system is built so that it is quasi-symmetrical.

5 The Stability of the Constant Rotation Rate Solutions

The equation 12 has the equilibrium solution $\phi = \alpha$ where α satisfies the first of eqns. (5). The stability of this solution can be computed by assuming that $\phi = \alpha + \xi(t)$. Assuming that $\xi \ll 1$ we get the linearized equations

$$\ddot{\xi} + \dot{\xi}(2\delta_x + 2\mu\omega) + \omega_x^2 A(\mu\cos(\alpha) + \sin(\alpha))\xi = 0 \quad (13)$$

The solution will be stable if and only if

$$A(\mu\cos(\alpha) + \sin(\alpha)) > 0.$$

s follows from the fact that an equation of the form

$$\ddot{\xi} + a\dot{\xi} + b\xi = 0$$

where $a > 0$ will have growing solutions if and only if $b < 0$. In order to see this note that the solutions of this equation can be written as linear combinations of solutions of the form $e^{\lambda t}$ where λ is a root of the equation

$$\lambda^2 + a\lambda + b = 0$$

The quadratic equation gives two roots to this equation,

$$\lambda_{\pm} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}.$$

If $b < 0$, then the discriminant $a^2 - 4b$ will always be positive, and we will have $\sqrt{a^2 - 4b} > 0$. It follows that the root λ_+ will be positive, and hence we will have growing solutions when $b < 0$. On the other hand, if $b > 0$, then the discriminant can be either positive or negative. If it is negative, then $\lambda_{\pm}$ will be complex conjugates of each other, and each of these roots will have a real part of $-a/2$. Thus when the discriminant is negative both roots will lead to decaying solutions. If $b > 0$ and the discriminant is positive, then $\sqrt{a^2 - 4b} < a$. In this case both roots will be real and negative. We conclude that we will have a growing solution if and only $b < 0$.

Recall that provided A is large enough, there will be two equilibrium solutions. One and only one of these will have $A(\mu \cos(\alpha) + \sin(\alpha)) > 0$. It follows that one of the solutions will be stable, and the other will be unstable.

We have two possible solutions to this equation. One of them is in the first quadrant, the other is in the third quadrant. The solution in the first quadrant satisfies the stability condition, the one in third quadrant does not. As we decrease A eventually the two solutions will merge. The point at which they merge is exactly the point where $\mu \cos(\alpha) + \sin(\alpha) = 0$.

We should emphasize that we have shown that the stable solution is stable to small disturbances, but not necessarily to large amplitude disturbances. For a given value of A , It is possible that for some initial conditions we will approach the uniform rotation rate solution, and for other initial conditions we will not. We will discuss this further in the next section.

6 The Case with $\mu = 0$

We will now analyze the equations 12 in the special case where μ is identically zero. In this case the equations of motion simplify to the extent that we can completely classify all of the possible behavior of the system. When $\mu = 0$ eqns. 12 simplify to

$$\ddot{\phi} + 2\delta_z(\omega + \dot{\phi}) - \omega_z^2 A \cos(\phi) = 0$$

If we make the substitution

$$s = \omega t$$

$$\beta = A\omega_z^2/\omega^2$$

$$\delta = \delta_z/\omega$$

these equations can be written as

$$\frac{d^2\phi}{ds^2} + 2\delta(1 + \frac{d\phi}{ds}) = \beta \cos(\phi) \quad (14)$$

We prefer to write this as a second order system

$$\dot{\phi} = z \quad (15a)$$

$$\dot{z} = -2\delta(1 + z) + \beta \cos(\phi) \quad (15b)$$

These equations are identical to those for a pendulum with a constant torque of -2δ being applied [13]. The behavior of this system (but in the more general case where the torque is constant but not necessarily equal to the damping factor) has been analyzed in [13].

The equation 15 is a nonlinear autonomous second order differential equation. The Poincare-Benedixon theorem [14] shows that the long time behavior of any bounded solution to such an equation is very simple. The solutions are either approaching an equilibrium solution (where the solution does not change with time), approaching a periodic solution, or approaching an orbit that infinitely slowly passes from one unstable equilibrium to another. This last situation is a very special situation and is non-generic (has a zero probability of occurring in a real system) .

If ϕ is approaching an equilibrium solution, then the motor will be approaching a state of uniform rotation ω . If $\phi(t)$ is approaching a periodic solution, then the motor will be approaching a non-uniform periodic motion that has a frequency of ω . We will see that when $\mu = 0$ this is not possible. When $\mu \neq 0$ we can show that this is not possible unless the motor actually runs in reverse for part of the cycle. We believe that this is highly unlikely physically, and that more refined methods of proof would show that it is impossible.

Solutions where ϕ remains bounded are not the only physically relevant solutions. It is possible to have $(\cos(\phi(t)), \sin(\phi(t)))$ be periodic, but have $\phi(t)$

on the average grow linearly in time. Equations (such as ours) of the form

$$\dot{\phi} = F(\phi, z)$$

$$\dot{z} = G(\phi, z)$$

where both F and G are periodic functions of ϕ can be analyzed by mapping the solution curves using a cylinder. The solution curve $(\phi(t), z(t))$ can be drawn on the cylinder using z as the axial position and ϕ as the angle around the cylinder (Fig(5)). As long as z remains bounded, there is only one additional behavior that we must add to the ones already described by the Poincare-Bendixon theorem. Our solutions can asymptotically approach a solution that is periodic on the cylinder. This means that $(\cos(\phi(t)), \sin(\phi(t)))$ will be a periodic function, but $\phi(t)$ will on the average increase linearly with time.

Physically solutions that are periodic on the cylinder will correspond to the motor running in a periodic but non-uniform motion. The frequency of rotation will in general be different than the frequency of the input signals.

Once again, the equations 15 are analyzed in [13] to describe the motion of a pendulum with constant torque. We will give some details of their analysis in the appendix, but for now we will merely describe their results. It is helpful for the reader to visualize a pendulum that is attached to a rotating shaft that through the forces of friction imparts a constant torque to the pendulum Fig(6). The angle ϕ measure the angle of the pendulum from the horizontal. Equilibrium solutions correspond to the pendulum coming to rest in a position

where the torque due to gravity balances the torque from the shaft. Periodic motions correspond to the pendulum undergoing oscillatory behavior, but not periodically winding about the shaft. Solutions that are periodic on the cylinder correspond to motions where the pendulum periodically winds around the shaft.

Physically it is possible to have a motion where the pendulum oscillates without winding about the shaft, but this requires a nonlinear frictional force that we are not including in our equations [13].

The analysis of the pendulum depends crucially on the steady state solutions to this equation. A steady state solution must satisfy

$$z = 0$$

$$2\delta = \beta \cos(\phi)$$

Clearly, we can only have steady state solutions if

$$\left| \frac{2\delta}{\beta} \right| < 1$$

If this is satisfied, then there will be two equilibrium solutions with $|\phi| < \pi$, one of which is stable to infinitesimal disturbances. Assuming some fixed point exists, there will be an infinite number of fixed points differing from it by a multiple of 2π . The stable equilibrium points will be stable to small disturbances, but not necessarily to arbitrarily large disturbances. When a solution is stable to arbitrarily large disturbances, we say that it is globally stable.

In [13] it is shown that the global stability of the equilibrium points can be determined by the behavior of the curves coming out of the unstable fixed points (see Fig 7). If the curve coming out of an unstable fixed point with $z < 0$ reaches the next unstable fixed point with $z < 0$, then we will say that the phase portraits are of type *A*. If they reach the ϕ axis before reaching the next unstable fixed point, we will say they are of type *B*. The behavior of our system can be classified as follows

- If $|\frac{2\delta}{\beta}| > 1$, then asymptotically the pendulum will approach a state where it periodically wraps around the shaft. In this state $(\cos(\phi(t)), \sin(\phi(t)))$ will be a periodic function of time, but $\phi(t)$ will decrease by 2π every period.
- If $|\frac{2\delta}{\beta}| < 1$, then if β is small enough, the phase portraits will be of type *A*. In this case we will be able to find some initial conditions so that the pendulum asymptotically approaches the stable equilibrium solution, but there will be some initial conditions such that the pendulum asymptotically wraps around the shaft in a periodic manner.
- If $|\frac{2\delta}{\beta}| < 1$, and β is large enough, then the phase portraits will be of type *B*. In this case the pendulum will eventually approach a stable equilibrium for any initial condition (except for the one where it begins precisely at the unstable equilibrium).

We see that if β is small enough, then there will be no equilibrium solutions,

and the pendulum ends up wrapping around the shaft in a periodic manner. Once β is raised to a critical value β_1 , it is still possible for the pendulum to wrap around in a periodic manner, but it is also possible for the pendulum to approach a steady state. However, once β exceeds a second critical value β_2 , the pendulum will always asymptotically approach an equilibrium solution. The critical value β_2 can be found numerically by determining the value of β such that the two saddle points (unstable equilibria) are connected to each other.

For our purposes it is relevant to note that when the pendulum periodically wraps around the shaft, the average value of $\dot{\phi}$ must be less than 0 but greater than -1 .

When we apply these results to the equations for the micro-engine, we see that the solutions where ϕ becomes unbounded correspond to the engine undergoing a periodic non-uniform rotation at an average rate that is less than the driving frequency ω . There are no solutions where the motor turns around at the driving frequency except for those that have a uniform rate. We see that if A is below a critical level A_1 , then the only possible motion of the motor is a non-uniform rotation. When A is increased beyond A_1 , it is possible to have a uniform rotation rate, but some initial conditions will lead to the non-uniform behavior. When A is above a critical value A_2 , any initial state of the motor will eventually settle down to a uniform rotation state.

A code was written to determine the global stability point A_2 as a function of the damping δ . For a given value of δ , the parameter A was adjusted so that the

curves coming out of the unstable equilibrium point connected to the unstable equilibrium point just to its left. The curves near the equilibrium points were computed analytically by taking the first few terms in a series expansion. Away from the equilibrium point the curves were computed using an ODE solver. The parameter A was adjusted using Newton's method so that the curves coming out of these equilibrium points met.

We introduce the parameter ϵ such that

$$A_2 = A_1/\epsilon$$

The parameter ϵ tells how far above the first critical value A_1 we must go to reach the second critical value of A_2 . Fig. 8 shows a plot of $\epsilon(\delta)$. We see that if the damping is large, then the two critical values are close to each other. However, for very small values of the damping, the two critical values can be drastically different.

7 Conclusions

We have shown that the behavior of the micro-engines can be analyzed much easier when the system is built so that it is quasi-symmetrical. By this we mean that the parameters are chosen so that $\omega_x = \omega_y$, $\delta_x = \delta_y$, $\gamma^2 k_x = k_y$, $Q_x(t) = A \cos(\omega t)$, and $Q_y(t) = A \sin(\omega t)$. In this case we are guaranteed of having solutions withh uniform rotation rate provided A is above a critical value A_1 . Quasi-symmetrical systems have the advantage over more general systems

in that the phase of the inputs $Q_x(t)$ and $Q_y(t)$ necessary to achieve a uniform rotation rate are independent of the parameters in the system. This means that we can achieve uniform rotation rate even if we do not know the value of μ , or even if μ is a function of F_r .

They have shown that provided A is below a critical level A_1 , the engine cannot approach a uniform rate no matter what the initial conditions are. Once A exceeds this critical level there will be two possible solutions with uniform rotation rate. One of these will be stable to small disturbances, and one will be unstable. However, it is still possible that for some initial conditions the engine will not asymptotically approach a uniform rate of rotation. We have analyzed the case where $\mu = 0$ and have proven that in this case there is a second critical value A_2 such that if A is above this value, one of the uniform rate solutions will be globally stable. This means that no matter what the initial conditions are, the engine will eventually approach a state of uniform rotation.

The case with $\mu = 0$ shows that if the engine does not approach a uniform rotation rate, it will eventually approach a state where it rotates with a non-uniform rotation rate that is on the average slower than the frequency of the input signals. We have not been able to rigorously prove that these results completely characterize the behavior when $\mu \neq 0$, but numerical experiments indicate that this behavior holds for all quasi-symmetrical systems. In particular numerical simulations have never shown non-uniform rotation with the same frequency as the driving force. We can in fact show that if such a behavior were to exist,

then the motor would have to run in reverse for part of the cycle. This seems very unlikely physically.

8 Appendix

In this appendix we will outline the proof of the results given for the case with $\mu = 0$, and suggest how these results may be extended to the case where $\mu \neq 0$. The differential equation 14 is a second order autonomous differential equation. The Poincare-Bendixson [14] theorem shows that any bounded solution to such an equation is either approaching an equilibrium solution, a periodic solution, or slowly passing from one unstable equilibrium to another. This last possibility is very degenerate, and can only occur when the solution curves emerging from an unstable equilibrium point happen to coincide with those emerging from a different unstable equilibrium (a saddle point connection). In our situation this will happen at precisely the value of A where we get global stability. It follows that other than this exceptional case, any bounded solution to 15 is either approaching an equilibrium solution or a periodic solution. If we have an equation of the form

$$\dot{\phi} = F(z, \phi)$$

$$\dot{z} = G(z, \phi)$$

then Benedixson's criterion [13] shows that we cannot have any periodic solu-

tions if

$$\Gamma(\phi, z) = \frac{\partial F}{\partial \phi} + \frac{\partial G}{\partial z} < 0$$

More generally, there cannot be any periodic solutions in any region where Γ does not change sign. When we have $\mu = 0$ we have

$$\Gamma(\phi, z) = -2\delta$$

It follows that in this case we cannot have any periodic bounded solutions to our differential equation. It follows that any solution $\phi(t)$ to 14 is either approaching a constant value, or it is eventually becoming unbounded.

We will use some very general properties of differential equations of the form

$$\dot{\phi} = z$$

$$\dot{z} = F(\phi, z)$$

where $F(\phi + 2\pi, z) = F(\phi, z)$. In this case it can be shown that

- If $\phi(t)$ becomes unbounded, and $\dot{\phi}$ remains bounded, then $\phi(t)$ must approach a solution that is periodic on the cylinder.
- Any solution that is periodic on the cylinder must have z be of a single sign.
- If z remains bounded, and there is any solution that wraps around the cylinder once while increasing the magnitude of z , then there must be solutions that are periodic on the cylinder.

All of these properties can be derived by considering the topology of solution curves on a cylinder. These properties become clear if one tries to draw solution curves on a cylinder keeping in mind that no solution curve can cross itself, and ϕ must be increasing if $z > 0$, and decreasing if $z < 0$.

In our case, it can quite simply be shown that the function $z(t)$ must remain bounded, and that it must asymptotically lie in the interval $-1 - A/\delta < z < -1 + A/\delta$. Once we know that z remains bounded it follows that $\phi(t)$ must eventually approach a periodic solution on the cylinder. Any solution that is periodic on the cylinder must have $z < 0$. This follows from multiplying 14 by $\dot{\phi}$ and integrating over a single period. If we do this we find

$$\phi(t+T) - \phi(t) = - \int_t^{t+T} \dot{\phi}^2 dt$$

This shows that ϕ must be decreasing every time we go through one period.

An equilibrium solution will be globally stable if and only if there are no periodic solutions on the cylinder. If our phase portraits are of type A then there must be a solution that is periodic on the cylinder. This follows since we have a solution curve that wraps around the cylinder once while increasing the magnitude of z . On the other hand, if the phase portraits are of type B, then Fig. 9) shows that any initial condition lying in the band M_0 will eventually cross the line $z = 0$, and hence could not be a periodic solution. Similarly, any solution in the band M_1 will also eventually have $z < 0$. Continuing in this manner we can get a sequence of bands $M_1, M_2, \dots$ and we see that any

solution that has any value of z must eventually end up with $z > 0$. It follows that any initial condition must eventually end up with $z > 0$, hence we cannot have any solutions that are periodic on the cylinder.

It remains to show that as we increase A we go from a situation where the phase portraits are of type A to one that are of type B . In [13] they consider the equation.

$$\ddot{\phi} + 2\delta\dot{\phi} - \beta\cos(\phi) = -\alpha$$

This is identical to our equation if we set $\alpha = 2\delta$. They show that as we increase δ from an initial value of zero, we will go from having phase portraits of type A to having phase portraits of type B . We would like to show that as we increase β we go from phase portraits of type A to those of type B . Our situation is somewhat different here since we are keeping δ and α fixed while changing β . We have not been able to prove that this is the case, but numerical experiments seem to prove this quite conclusively.

If $\mu \neq 0$, then we can use Benedixons criterion to show that we cannot have any periodic solutions with ϕ bounded provided $\text{sgn}(\dot{\theta}) = \text{sgn}(1 + \dot{\phi})$ does not change sign. This means that we cannot have any periodic solutions with frequency ω unless the engine actually reverses itself during part of the cycle. We can also show that z must remain bounded. This shows that if ϕ becomes unbounded then it must be approaching a periodic solution on the cylinder. Even when $\mu \neq 0$ it is still possible to compute the second critical value A_2 by adjusting A so that two neighboring equilibrium points connect to each other.

Fig. 1. Microengine diagram.

Fig. 2. Backscatter signal for eight-tooth gear running at 500 Hz.

Fig. 3. Backscatter signal for nineteen-tooth gear running at 500 Hz.

Fig. 4. A schematic of the orthogonal linkage system.

Fig. 5. A schematic of a cylindrical phase space. The solution is drawn on a cylinder using the z axis to represent the time derivative $\dot{\phi}$, and the azimuthal angle to locate the position ϕ .

Fig. 6. A schematic of the pendulum at constant torque. The pendulum is attached to a rotating shaft that applies a constant torque to the cylinder through the force of friction.

Fig 7. a) This illustrates a type A phase portrait. The solution leaving the unstable fixed point with $z < 0$ reaches the unstable fixed point to its left with $z < 0$. In this case the solution curves will wrap around the cylinder so that ϕ increases without bound.

b) This illustrated a type B phase portrait. The solution leaving the unstable fixed point with $z < 0$ never reaches the next unstable fixed fixed point. In this case we cannot have any solutions that wrap around the cylinder. Instead, all solutions must have ϕ remain bounded.

Fig. 8. The function $\epsilon(\delta)$ that relates the first critical value A_1 to the second critical value A_2 by $A_2(\delta) = A_1(\delta)/\epsilon(\delta)$.

Fig. 9. If the portraits are of type B then any initial condition will eventually end up with $z > 0$.

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Figure 1

Back Scatter

Figure 2

Back Scatter

Figure 3

Figure 4

Figure 5

Figure 6

Figure 7a. Type A phase portraits.

Figure 7b. Type B phase portraits.

Figure 8:

Figure 9