I. THE KRAKEN MODEL

Consider two particles, with charges $q_1$ and $q_2$, circulating in the $z$ direction, horizontally and longitudinally displaced by $(x_1, z_1)$ and $(x_2, z_2)$ from the center of the same bunch. The equation of motion of the second particle is

$$\frac{d^2 x_2}{dt^2} + k(s) x_2 = \frac{q_1 q_2}{m \gamma} W_1(s, z_1 - z_2) x_1$$

where $k(s)$ represents quadrupole focusing, $m$ is the particle mass, $\gamma$ is the Lorentz factor, and $\beta \equiv 1$ is assumed. The “transverse wake field” $W_1$ characterizes the way that the first particle interacts with the environment, to modify the transverse motion of the second particle. It is always positive for particles that are very close together, $W_1(s, +\epsilon) > 0$, defocusing a trailing particle that is in phase with the source particle. Causality demands that $W_1 = 0$ if $z_1 < z_2$, and if multi-turn wakes are neglected.

When a wake field generating device at $s = 0$ is short, it is natural to talk of its “transverse wake potential”,

$$V_1(x_1, z_1) = \int_{\text{device}} W_1(s, x_1, z_2) \, ds$$

since then the equation of motion becomes

$$\frac{d^2 x_2}{dt^2} + K(s) x_2 = \frac{q_1 q_2}{m c^2 \gamma} V_1(x_1, z_1 - z_2) x_1 \delta(s)$$

where a prime denotes differentiation with respect to $s$.

II. GENERAL HEAD-TAIL RESULTS

If the chromaticity $\chi = dQ/d\delta$ is zero (where $\delta$ is the off-momentum parameter, $\Delta p/p$), the motion is stable for increasing $N_b$, until the strong head-tail threshold is passed. Unstable motion above this threshold has a rise time of $\tau \sim T_1$, the time scale on which the macroparticles exchange their “drive” and “response” roles. The strong head-tail instability, also known as the “transverse mode coupling”, “transverse turbulent” or “transverse microwave” instability, has only been observed at electron storage rings.

The transverse motion of two macroparticles can be decomposed into “+” and “−” eigenmodes, in which the particles oscillate in or out of phase. When a small chromaticity $\chi$ is introduced, one eigenmode grows and the other is damped, with a slow timescale $\tau \gg T_1$. This is the head-tail instability.

The situation is conveniently parameterised by the dimensionless quantity

$$T(x, \tilde{z}) = \frac{\beta_D N_b z^2 \tilde{z}^2}{4 A m_w c^2 \gamma} \times$$

$$\int_{-T/2}^{T/2} V_1(x_1, z_2) \exp \left[i \left(2 \chi \tilde{z} T_2 \right) \sin \left(\frac{2 \pi t}{T_2} \right) \right] \, dt$$

where $\beta_D$ is the Twiss function at the device [1]. The longitudinal distance between macroparticles,

$$\Delta z_2 = 2 \tilde{z} \sin \left(\frac{2 \pi t}{T_2} \right)$$

is related to $\beta_D$, the maximum off momentum parameter of a macroparticle, through the relationship

$$\beta_2 \equiv \frac{\tilde{z}}{\delta} = \frac{\eta C T_2}{2 \pi}$$

where $\eta$ is the slip factor and $C$ is the circumference of the accelerator.

Two macroparticles with longitudinal amplitude $\tilde{z}$ do not suffer from strong head-tail instability if

$$Re T(x, \tilde{z}) < 2$$

while the head-tail eigenmode growth rates, per turn, are

$$\tau_2^{-1} = \mp \frac{Im T(x, \tilde{z})}{T_2}$$

These results hold for a general transverse wake field. Fortunately, $\tau_2^{-1}$ is overestimated, and in practice both modes are stabilised (above transition) by a slightly positive chromaticity.
DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.
DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.
**A. STEP FUNCTION WAKE POTENTIAL**

Various authors have reported analytic and simulation head-tail results for the linear (and analytically soluble) case of a step function wake potential[1], [2], [3], [4].

\[ V_1(z) = V_1 \quad z > 0 \]  
\[ V_1(z) = 0 \quad z < 0 \]  

In this case \( T(x) \) is independent of \( z \). Fig. 1 shows the horizontal “+” mode growth rate of simulated motion as a function of chromaticity, and compares it with the prediction of Eqn. 10, when \( Re T(0) = 0.03, \delta = 0.001, \) and \( T_x = 300 \) turns.

![Figure 1](image)

Figure. 1. The head-tail growth rate \( \tau^{-1} \) versus chromaticity \( \chi \), for a step function wake potential.

**III. NEW CRITERIA WITH MOMENTUM DEPENDENT COUPLING**

It is usually implicitly assumed that linear coupling is unimportant in the head-tail effect, by treating only one transverse dimension at a time. However, the Tevatron experience is that linear coupling, and its variation with momentum, are important in routine operation close to the tune diagonal [5], [6]. When coupling is important, it is reasonable to conjecture that head-tail stability is only guaranteed if the eigenchromaticities are positive for all momenta within the beam [7], [8]. A weaker conjecture is that only on-momentum particles must have positive eigenchromaticities, or that it is the average eigenchromaticity over one synchrotron period that must be positive.

The extreme values for the eigenchromaticities \( \chi_+ \) and \( \chi_- \) (for the worst possible combination of momentum, skew quadrupole, and normal quadrupole settings) are

\[ \chi_{\text{extreme}} = \frac{1}{2}(\chi_x + \chi_y) \pm \frac{1}{2}\sqrt{k^2 + (\chi_x - \chi_y)^2} \]  

where the “skew chromaticity” vector \( k \) parameterises the variation of the closest approach of eigentunes, \( \Delta Q_{\text{min}} \), as a function of momentum. This is analogous to the way that the “normal chromaticities”, \( \chi_x \) and \( \chi_y \), parameterise the tune versus momentum, far from the tune diagonal.

Insisting that both of the extreme eigenchromaticities are positive leads to the **new and strong criteria**[7], [8] that

\[ \chi_x + \chi_y > 0 \]  
\[ 4\chi_x \chi_y > k^2 \]  

If true, neither eigenchromaticity can ever become negative. As such, these criteria are “sufficient but often not necessary”. Both \( \chi_x \) and \( \chi_y \) must be positive to meet the criteria, even when \( k = 0 \), thereby recovering the standard uncoupled head-tail result (above transition).

**IV. RESISTIVE WALL WAKE**

The transverse resistive wall wake potential is given by

\[ V_1(z) = \frac{8L}{r^4 \varepsilon_0 (4\pi)^{3/2}} \quad 0 < z < z_e \]  
\[ V_1(z) = \frac{2L}{\pi r^3} \sqrt{\frac{c}{4\pi \varepsilon_0 \sigma}} \quad z > z_e \]  

where \( L, r, \) and \( \sigma \) are, respectively, the length, radius, and conductivity of the beam pipe [1]. The critical length \( z_e \), given by

\[ z_e = \left( \frac{\varepsilon_0 c}{\sigma} \right)^{1/3} r^{2/3} \]  

tends to be very short. For example, \( z_e = 0.12 \) mm for the dominant (cold) beampipe in RHIC, with \( L = 2955 \) m, \( r = 0.0346 \) m, and \( \sigma = 2.0 \Omega^{-1} \text{m}^{-1} \).
Table I

RHIC parameters during proton injection.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>units</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bunch population, (N_b)</td>
<td></td>
<td>(1.0 \times 10^{11})</td>
</tr>
<tr>
<td>Lorentz factor, (\gamma)</td>
<td></td>
<td>31.17</td>
</tr>
<tr>
<td>Transition gamma, (\gamma_T)</td>
<td></td>
<td>22.89</td>
</tr>
<tr>
<td>Average device beta, (\beta_D)</td>
<td>m</td>
<td>30.0</td>
</tr>
<tr>
<td>Circumference, (C)</td>
<td>m</td>
<td>3833.8</td>
</tr>
<tr>
<td>Synchrotron period, (T_s)</td>
<td>turns</td>
<td>1414</td>
</tr>
<tr>
<td>RMS momentum error, (\sigma_p/p)</td>
<td></td>
<td>(4.66 \times 10^{-3})</td>
</tr>
<tr>
<td>RMS bunch length, (\sigma_s)</td>
<td>m</td>
<td>0.353</td>
</tr>
</tbody>
</table>

Table I lists the nominal RHIC parameters for protons at injection, when RHIC is especially vulnerable to head-tail effects. When these values are substituted into Eqn. 6, with \(\delta = \sigma_p/p\) and \(\varepsilon = \sigma_s\), they lead to the variation of \(T\) with chromaticity recorded in Figure 3. Since the maximum value of \(ReT = 0.3\), RHIC is expected to be about an order of magnitude short of strong head-tail instability. Chromaticity values of \(\chi \approx 2\) appear to be optimal. Figure 4 compares the growth rates predicted by Eqn. 10 with the rates measured by KRAKEN. The agreement is good in most cases, except that when \(\chi \approx 1.5\), the anti-damping of the “+” mode is about 50% stronger than expected.

V. SUMMARY AND PLANS

The simulation code KRAKEN confirms analytical predictions of head-tail stability criteria, Eqns. 14 and 15, in the presence of momentum dependent linear coupling. It also confirms that resistive wall transverse wake fields are not a serious threat to strong head-tail stability in RHIC, at the vulnerable stage of proton injection.

Equation 10, derived from the perspective of two macroparticles, potentially offers a very convenient semi-numerical evaluation of the effects of arbitrary transverse wake potentials. It remains to be seen how well the two macroparticle results correlate with simulations using, say, 100 macroparticles.

KRAKEN is still under rapid development. Future plans are to include resonant wakefields, multiple bunches, space charge wakefields, betatron detuning, and a connection to the detailed RHIC impedance database.

References
