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FINAL REPORT

The Use of Symbolic Computation in Radiative, Energy, and Neutron Transport Calculations

PI/PD: J.I. Frankel, Ph.D.
Associate Professor

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Monitor: Dr. F. A. Howes
Program Manager
Scientific Computing Staff
ER-7, GTN
DOE
Washington, DC 20545

Abstract:

This final report summarizes the results obtained during the entire research effort covering 1992-1994.

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Summary:

This investigation used sysmbolic manipulation in developing analytical methods and general computational strategies for solving both linear and nonlinear, regular and singular integral and integro-differential equations which appear in radiative and mixed-mode energy transport.

Contained in this report are seven (7) papers which present the technical results as individual modules. Below are listed, for easy reference, the Abstracts of the individual papers:

1. "*A Galerkin Solution to a Regularized Cauchy Singular Integro-Differential Equation*" by J.I. Frankel (**Quarterly of Applied Mathematics,** in press).

Abstract

This paper presents a Galerkin approach for solving a regularized version the Cauchy singular, linear integro-differential equation

$$\frac{d\theta}{dx}(x) - f(x) = \lambda \int_{y=0}^{1} \frac{\theta(y)}{x-y} dy, \quad x \epsilon (0,1)$$

subject to $\theta(0) = \theta(1) = 0$. This equation has appeared in both combined infrared gaseous radiation and molecular conduction, and elastic contact studies. A regularized formulation is produced which suggests the use of an expansion technique where the orthogonal basis functions are chosen as the Chebyshev polynomialsof the first kind. Accurate results, requiring a minimal computational cost, are formally documented and compared to a purely numerical solution.

2. "*Several Symbolic Augmented Chebyshev Expansions for Solving the Equation of Radiative Transfer*" by J.I. Frankel, (**Journal of Computational Physics,** in press)

Abstract

Three expansion methods are described using Chebyshev polynomials of the first kind for solving the integral form of the equation of radiative transfer in an isotropically scattering, absorbing, and emitting plane-parallel medium. With the aid of symbolic computation, the unknown expansion coefficients associated with this novel choice of basis functions are shown to permit analytic resolution. A unified and systematic solution treatment is offered using the methods of collocation, Ritz-Galerkin, and Weighted-Galerkin for determining the unknown expansion coefficients. Numerical results are presented contrasting the three expansion methods and comparing them with existing benchmark results. Theoretical considerations are also presented illustrating rigorous error bounds, residual characteristics, accuracy, and convergence rates. This paper presents the foundation for future works involving fully coupled, nonlinear heat transfer studies.

3. "*A Note on the Integral Formulation of Kumar and Sloan*" by J.I. Frankel, (**Journal of Computational and Applied Mathematics**, in press).

Abstract

In this note, we present a methodology for developing a posteriori error estimates for the recently proposed method of Kumar and Sloan. Kumar and Sloan proposed a formulation which converts a Hammerstein equation into a conducive form for approximation by a collocation method. Symbolic computation is used in performing the numerous analytic manipulations leading to the establishment of the error estimates. Finally, some remarks on the generalization of the method of Kumar and Sloan to higher-dimensional systems are offered.

4. "*Chebyshev Series Solution for Radiative Transport in a Medium with a Linearly Anisotropic Scattering Phase Function*" by T. LaClair and J.I. Frankel (**International Journal for Numerical Methods in Heat and Fluid Flow** , in press).

Abstract

One-dimensional radiative heat transfer is considered in a plane-parallel geometry for an absorbing, emitting, and linearly anisotropic scattering medium subjected to azimuthally symmetric incident radiation at the boundaries. The integral form of the transport equation is used throughout the analysis. This formulation leads to a system of weakly-singular Fredholm integral equations of the second kind. The resulting unknown functions are then formally expanded in

Chebyshev series. These series representations are truncated at a specified number of terms, leaving residual functions as a result of the approximation. The collocation and the Ritz-Galerkin methods are formulated, and are expressed in terms of general orthogonality conditions applied to the residual functions. Error bounds are obtained for the approximating functions by developing equations relating the residuals to the errors and applying functional norms to the resulting set of equations. The collocation and Ritz-Galerkin methods are each applied in turn to determine the expansion coefficients of the approximating functions. The effectiveness of each method is interpreted by analyzing the errors which result from the approximations.

5. "*A New Orthogonal Collocation Solution to a Nonlinear, Weakly-Singular Partial Integro-Differential Equation*", by J.I. Frankel, (***BETECH 94,*** Orlando, FL March 1994).

Abstract

A new formulation is offered for a nonlinear, weakly-singular, partial integro-differential equation of mathematical physics. This new formulation highlights the use of an intermediate dependent variable and promotes the use of an expansion method by which highly accurate numerical results can be achieved in a computationally rapid fashion. The intention of this debut investigation is to demonstrate merit, accuracy and future potential. The expansion method makes use of orthogonal collocation in the spatial variable where Chebyshev polynomials of the first kind are used as the basis functions while the temporal variable is resolved by an initial value method. Some error analysis is presented and comparisons are made with existing solutions from the literature. All computational and graphical aspects of the study are performed with the aid of *Mathematica*^{TM}.

6. "*A Cumulative Variable Formulation for Transient Conductive and Radiative Transport in Participating Media*", by J.I. Frankel, (**AIAA Journal of Thermophysics and Heat Transfer,** in review).

A new mathematical formulation is proposed for transient conductive and radiative transport in a participating gray, isotropically scattering plane-parallel medium. The methodology can be easily extended to include numerous additional effects. A systematic and unified treatment is presented using cumulative variables which allows for high-order integration using standard initial-value methods in the temporal variable while allowing for an effective orthogonal collocation method to be implemented in the spatial variable. A spectral approach is incorporated in the present context where Chebyshev polynomials of the first kind are used as the basis functions. This paper illustrates the methodology and presents some comparisons with previously reported works.

7. "*Generalization of the Method of Peters to Cauchy Singular Integro-Differential Equations*" by J.I. Frankel (unpublished technical document).

Abstract

An analytic methodology is presented for solving linear Cauchy singular integro-differential equations. A representative equation is studied detailing the approach. Peters' notion, conceived when studying Cauchy singular integral equations of the airfoil type, is generalized to include Cauchy singular integro-differential equations. The final outcome from the analytic preconditioning suggests the use of Chebyshev polynomials as the basis functions for developing the approximate solution. The proposed analytic procedure is augmented with symbolic computation for performing algebraic manipulations. Results indicate that the approach has merit and deserves additional consideration.

As indicated by the number of accepted manuscripts, the work performed under the auspices of DOE proved to be successful and of interest to the research community. Papers # 2,3,5,6 are of particular merit owing to novel error and convergence analyses, and to the novel generalization of the method of Kumar and Sloan to nonlinear, weakly singular integro-differential equations.

Complete papers containing mathematical details are attached.

A GALERKIN SOLUTION TO A REGULARIZED CAUCHY SINGULAR INTEGRO-DIFFERENTIAL EQUATION

by

Jay I. Frankel
Associate Professor
Mechanical and Aerospace Engineering Department
Florida Institute of Technology
Melbourne, FL (USA) 32901

ABSTRACT

This paper presents a Galerkin approach for solving a regularized version of the Cauchy singular, linear integro-differential equation

$$\frac{d\theta}{dx}(x) - f(x) = \lambda \,⨍_{y=0}^{1} \frac{\theta(y)}{x-y}\, dy, \qquad x \in (0,1),$$

subject to $\theta(0) = \theta(1) = 0$. This equation has appeared in both combined infrared gaseous radiation and molecular conduction, and elastic contact studies. A regularized formulation is produced which suggests the use of an expansion technique where the orthogonal basis functions are chosen as the Chebychev polynomials of the first kind. Accurate results, requiring a minimal computational cost, are formally documented and compared to a purely numerical solution.

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1. INTRODUCTION

The mathematical formulation of physical phenomena often involves Cauchy-type (or more severe) singular integral equations. Integral and integro-differential equations containing strongly singular kernels appear in studies involving elastic contact [1], stress analysis [1], fracture mechanics [2-4], airfoil theory [5-10], and combined infrared radiation and molecular conduction [11,12]. Owing to their common appearance in practice, there exists a growing need to develop accurate approximate analytical and numerical solutions to a large variety of singular integral and integro-differential equations. Simulation techniques which take advantage of recent hardware and software developments are particularly appealing amid the rapidly changing computational environment.

Over the past thirty years, substantial progress has been made in developing innovative approximate analytical and purely numerical solutions to a large class of singular integral equations. The books edited by Golberg [13,14] are particularly enlightening and should be consulted since they contain fairly extensive literature surveys on both approximate analytical and purely numerical techniques. The interested reader should also consult the fine expositions by Linz [15], Baker [16], Delves and Mohamed [17], and Atkinson [18] for numerical methods, and consult the books by Tricomi [19], Hochstadt [20], Green [21], and Porter and Stirling [22] for information concerning analytical solution methods.

In formulating physical problems, it is worth noting that in many instances the initial formulation may not lend itself easily to approximation [23,24]. Thus, it may be necessary to recast the original formulation into a form conducive to approximation. This preconditioning often permits additional interpretation and insight into choosing a well-suited numerical method. If analytical preconditioning is not performed, a less satisfactory but more expedient brute force technique is usually opted for, generally yielding marginal results. The present work illustrates that analytic preconditioning leads to a natural approximation process which requires little computational cost.

This paper uses a classical expansion method for solving a linear, Cauchy singular integro-differential equation. Cess and Tiwari [11] derived

$$\mu\frac{d\theta}{dx}(x) - f(x) = \lambda ⨍_{y=0}^{1} \frac{\theta(y)}{x - y}dy, \quad x \in (0,1), \tag{1a}$$

where

$$f(x) = -\left(x - \frac{1}{2}\right), \tag{1b}$$

with the auxiliary conditions

$$\theta(0) = \theta(1) = 0. \tag{1c}$$

This mathematical formulation was originally developed in the context of combined gaseous infrared radiation and molecular conduction in a plane-parallel geometry. Due to physical symmetry [11], we know $\theta(x) = \theta(1 - x)$. Here $\theta(x)$ represents the unknown temperature, x represents the spatial variable, and λ is a constant composed of several physical properties. Equation

(1) represent the large path length limit to a more general equation presented in [11] (when $\mu = 1$). Here $⨍$ denotes integration in the Cauchy principal value sense. We suppose that $\theta(x)$, $x \in [0,1]$ is continuous and that the derivative $\theta'(x)$ exists and is continuous on the interval $x \in (0,1)$, then $⨍_{y=0}^{1} \theta(y)(x-y)^{-1}dy$ exists in the principal value sense. Cess and Tiwari [11] developed a purely numerical solution to Eq. (1) for $\theta(x)$. Unfortunately, no details were furnished describing their numerical implementation.

A similar equation to that expressed by Eq. (1a) has also appeared in the study of elastic contact [1]. Sankar et al. [1] developed a power series solution method which included a term that moderated the solution tendencies at the endpoints. If $\mu = 0$ in Eq. (1a), then the derivative term vanishes and classical airfoil equation is recovered. Peters [25,26] developed a simple yet insightful solution method to the airfoil equation based on reducing the airfoil equation to the solution of an Abel integral equation. More recently Golberg [13,14], and Porter and Stirling [22] have expounded on the merits of Peters' approach. In fracture mechanics [27,28], Cauchy-type kernels (and stronger, i.e., $(x - y)^{-\alpha}$, $\alpha > 1$) often arise. Kaya and Erdogan [27,28] have reported several interesting findings based on the evaluation of finite-part integrals (i.e., in the sense of Hadamard integration). Additionally, noteable papers (too many to cite) by Ioakmikis

[29,30] have considered numerous approximate analytic and purely numeric methods for resolving Fredholm integral equations with Cauchy type singularities.

The present work describes a Galerkin approach for solving Eq. (1) for the case when $\mu = 1$. The approach begins by transforming the initial integro-differential equation into a alternative form in which the unknown function $\theta(x)$ is receptive to approximation by an expansion technique. The basis functions chosen for the expansion are based on Chebychev polynomials. This regularized form permits the use of several integral and algebraic relations, and the inclusion of the orthogonality property associated with Chebychev polynomials for determining the unknown expansion coefficients of the basis functions. Results indicate that the approach yields accurate results with a minimal computational cost.

This paper is divided into three main sections. Section 2 presents the alternative formulation based on Peters' [25,26] notion of the simplifying operator. Use of this concept allows us to transform the original integro-differential equation into a regularized form which is conducive to approximation by an orthogonal series representation. Section 3 describes the Chebychev series expansion and describes the Fourier method used in determining the unknown expansion coefficients. Section 4 presents some representative numerical results.

2. FORMULATION

In this section, we present a regularized form of Eq. (1). The approach taken here uses the concept proposed by Peters [25,26] in studying the airfoil equation. Potter and Stirling [22] have coined the phrase "simplifying operator" to denote the effect of the transformation.

Letting $\mu = 1$ and following the well-documented approach offered by Peters [25,26], we arrive at the regularized form

$$\theta(x) = \frac{C_o}{\pi\sqrt{x}\sqrt{1-x}} + \frac{1}{\lambda\pi^2} ⨍_{z=0}^{1} \frac{dz}{z-x} \frac{\sqrt{z}\sqrt{1-z}}{\sqrt{x}\sqrt{1-x}}\left[\frac{d\theta}{dz}(z) - f(z)\right],$$

$$x \in (0,1), \qquad (2a)$$

where C_o is given by

$$C_o = \int_{y=0}^{1} \theta(y)dy. \qquad (2b)$$

An alternative expression for the constant C_o can be developed by first multiplying Eq. (2a) by $\sqrt{x}\sqrt{1-x}$ and then following up by evaluating the resulting form at $x = 0$. Doing so, we arrive at

$$C_o = -\frac{1}{\lambda\pi}\int_{z=0}^{1} \sqrt{\frac{1-z}{z}}\left[\frac{d\theta}{dz}(z) - f(z)\right]dz. \qquad (3)$$

Next, we substitute Eq. (3) into Eq. (2a) for C_o to obtain

$$\theta(x) = \frac{1}{\lambda\pi^2} ⨍_{z=0}^{1} \frac{d\theta}{dz}(z) - f(z)]\sqrt{\frac{x(1-z)}{z(1-x)}}\frac{dz}{z-x}, \qquad x \in (0,1). \qquad (4)$$

We shall refer to Eq. (2a) as our Chebychev form while we refer to Eq. (4) as our Jacobi form. The rationale for this terminology will become clear in the next section.

It is interesting to note that if we substitute Eq. (2b) into the left-hand-side of Eq. (3) for C_o and substitute Eq. (1a) into the right-hand-side of Eq. (3), we find

$$\int_{z=0}^{1} \theta(z)dz = -\frac{1}{\pi}\int_{z=0}^{1} \sqrt{\frac{1-z}{z}}[⨍_{y=0}^{1} \frac{\theta(y)}{z-y}dy]dz, \qquad (5a)$$

or upon interchanging the orders of integration, we arrive at

$$\int_{z=0}^{1} \theta(z)dz = -\frac{1}{\pi}\int_{y=0}^{1} \theta(y)[⨍_{z=0}^{1}\sqrt{\frac{1-z}{z}} \frac{dz}{z-y}]dy, \qquad (5b)$$

thereby identifying the value of the inner integral as

$$⨍_{z=0}^{1}\sqrt{\frac{1-z}{z}} \frac{dz}{z-y} = -\pi. \qquad (5c)$$

Additionally, if we multiply Eq. (2a) by $\sqrt{x}\sqrt{1-x}$ and evaluate the resulting expression at x = 1, and make use of $\theta(1) = 0$, we obtain yet another expression for C_o, namely

$$C_o = \frac{1}{\lambda\pi}\int_{z=0}^{1} \sqrt{\frac{z}{1-z}}[\frac{d\theta}{dz}(z) - f(z)]dz. \qquad (6a)$$

Similar logic can be used to show that

$$⨍_{z=0}^{1}\sqrt{\frac{z}{1-z}} \frac{dz}{z-y} = \pi, \qquad (6b)$$

which is identical to the contour integration result reported by Lu [31].

In closing this section, we state some alternative formulations. The first two forms are based on simple calculus manipulations which are intended to reduce the severity of the singularity. Clearly, an alternative formulation for Eq. (1a) can be expressed as

$$\frac{d\theta}{dx}(x) - f(x) = \lambda \int_{y=0}^{1} \frac{d\theta}{dy}(y) \log|x-y|dy, \quad x \in [0,1], \qquad (7a)$$

where $\theta(x)$ can be recovered by direct integration, namely

$$\theta(x) = \int_{y=0}^{x} \frac{d\theta}{dy}(y)\, dy, \quad x \in [0,1], \tag{7b}$$

since $\theta(0) = 0$. If one desires a formulation in terms of $\theta(x)$ directly, we can express Eq. (1a) in the alternative form

$$\theta(x) = g(x) + \lambda \int_{y=0}^{1} \theta(y) \log\left|\frac{x - y}{y}\right| dy, \quad x \in [0,1], \tag{8a}$$

where

$$g(x) = \int_{y=0}^{x} f(y)dy. \tag{8b}$$

Both Eqs. (7a) and (8a) permit the use of product integration [13,15-18,24], and singularity subtraction [13,17,18,24].

As an alternative to "decreasing" the order, as illustrated by Eq. (8a), we present the contrasting notion of "increasing" the order. To illustrate this concept, we begin by taking the derivative of Eq. (7a) with respect to x to get

$$\frac{d^2\theta}{dx^2}(x) = \frac{df}{dx}(x) + \lambda \int_{y=0}^{1} \frac{d\theta}{dy}(y) \frac{dy}{x - y}, \quad x \in (0,1). \tag{9}$$

In operator (or symbolic) form, we can express Eq. (1a) when $\mu = 1$, and Eq. (9) as

$$\theta' = f + \lambda K\theta, \tag{10a}$$

$$\theta'' = f' + \lambda K\theta', \tag{10b}$$

respectively. Substituting Eq. (10a) into Eq. (10b) for θ' yields

$$\theta'' = f' + \lambda Kf + \lambda^2 K^2\theta, \tag{10c}$$

or explicitly

$$\frac{d\theta^2}{dx^2}(x) = \frac{df}{dx}(x) + \lambda ⨍_{y=0}^{1} \frac{f(y)}{x - y}dy + \lambda^2 ⨍_{z=0}^{1} \frac{dz}{x - z} ⨍_{y=0}^{1} \theta(y)\frac{dy}{z - y},$$

$$x \in (0,1), \tag{10d}$$

subject to $\theta(0) = \theta(1) = 0$. Using the Hardy-Poincaré-Bertrand formula [19] reduces Eq. (10d) to

$$\frac{d\theta^2}{dx^2}(x) + \lambda^2\pi^2\theta(x) = h(x) + \lambda^2\int_{y=0}^{1} M(x,y)\theta(y)dy, \quad x \in (0,1), \tag{11a}$$

where

$$h(x) = \frac{df}{dx}(x) + \lambda \fint_{y=0}^{1} \frac{f(y)}{x - y}dy \ , \tag{11b}$$

and with

$$M(x,y) = \frac{1}{x - y} \log[\frac{x(1 - y)}{y(1 - x)}] = M(y,x). \tag{11c}$$

Notice that $M(x,x) = [x(1 - x)]^{-1}$. It is interesting to note that one can approximate the kernel M(x,y) in a separable form [32]. Conversion of Eq. (11a) into an equivalent integral equation can be accomplished with the aid of the Green's function method. Using this formulation, one can develop a solution based on solving for the moments of the unknown function. This approach has been explored and some results have been documented. It should be noted that the approach presented in the next section appears to be the most acceptable solution method studied by the present author.

3. METHOD OF SOLUTION

This section is dedicated to developing an approximate solution of Eq. (2) subject to the auxiliary condition expressed by Eq. (1b). Before proceeding further, it is convenient to map the physical domain from $x \in [0,1]$ to $\eta \in [-1,1]$, thereby arriving at

$$\theta(\eta)\sqrt{1-\eta^2} = \frac{1}{\pi}\int_{\xi=-1}^{1} \theta(\xi)d\xi + \frac{2}{\lambda\pi^2} \fint_{\xi=-1}^{1} \frac{\sqrt{1-\xi^2}}{\xi-\eta}\left[\frac{d\theta}{d\xi}(\xi) - \frac{f(\frac{\xi+1}{2})}{2}\right]d\xi, \quad \eta \in (-1,1), \tag{12a}$$

subject to

$$\theta(-1) = \theta(1) = 0. \tag{12b}$$

Here $f(\frac{\xi+1}{2})$ reduces to $-\xi/2$. Therefore, Eq. (12a) can be expressed as

$$\theta(\eta)\sqrt{1-\eta^2} - \frac{1}{\pi}\int_{\xi=-1}^{1} \theta(\xi)d\xi + \frac{2}{\lambda\pi^2} \fint_{\xi=-1}^{1} \frac{\sqrt{1-\xi^2}}{\eta-\xi} \frac{d\theta}{d\xi}(\xi)d\xi$$

$$= -\frac{1}{2\lambda\pi^2}\fint_{\xi=-1}^{1} \frac{\xi\sqrt{1-\xi^2}}{\eta-\xi}d\xi, \quad \eta \in (-1,1). \tag{13}$$

The manipulations leading to Eq. (13) illustrate the merit of analytic preconditioning since this new formulation naturally suggests the use of a Chebychev series expansion for the unknown function $\theta(\eta)$. Let $T_m(\eta) = \cos[m(\cos^{-1}\eta)]$, $\eta \in [-1,1]$, $m = 0,1,\ldots$ denote the Chebychev polynomials of the first kind while $U_m(\eta) = \sin[(m+1)(\cos^{-1}\eta)]/\sin(\cos^{-1}\eta)$, $m = 0,1,\ldots$ denote the Chebychev polynomials of the second kind. It is well known that $\{T_m(\eta)\}$, $m = 0,1,\ldots$ form an orthogonal sequence of functions with respect to the weight function $(1-\eta^2)^{-\frac{1}{2}}$ while $\{U_m(\eta)\}$, $m = 0,1,\ldots$ form an orthogonal sequence of functions with respect to the weight function $(1 - \eta^2)^{-\frac{1}{2}}$.

It appears reasonable to attempt a series expansion to $\theta(\eta)$ in Eq. (13) in terms of Chebychev polynomials of the first kind. This choice is not arbitrary since one can identify a portion of the integrand as the weight function associated with $T_m(\eta)$. It now becomes apparent why we have coined the phrases Chebychev and Jacobi forms when speaking of Eq. (2a) and Eq. (4), respectively.

We assume that the function $\theta(\eta)$ has the expansion of the form (other choices also exist which are acceptable)

$$\theta(\eta) = \sum_{m=0,2,...}^{\infty} a_m T_m(\eta) - 1, \quad \eta \in (-1,1), \tag{14a}$$

where $T_m(\eta)$ represents the m^{th} Chebychev polynomial of the first kind. The unknown expansion coefficients a_m, $m = 0,2,...$ are to be determined by some means, say either by a collocation or Galerkin method. Notice that only even powers of the Chebychev polynomials are preserved. This is done in light of the physical fact (i.e., symmetry) that $\theta(\eta) = \theta(-\eta)$. Additionally, we impose the auxiliary condition shown in Eq. (12b), namely $\theta(\pm 1) = 0$. Imposing this constraint yields

$$\sum_{m=0,2,...}^{\infty} a_m = 1, \tag{14b}$$

since $T_m(\pm 1) = 1$, $m = 0,2,...$ [33]. Thus, we can express Eq. (14a) as

$$\theta(\eta) = \sum_{m=0,2,...}^{\infty} a_m[T_m(\eta) - 1], \quad \eta \in [-1,1]. \tag{14c}$$

The rationale for this auxiliary profile condition is apparent due to the integro-differential form of the equation. The derivative of Eq. (14c) with respect to η becomes

$$\frac{d\theta}{d\eta}(\eta) = \sum_{m=2,4,\ldots}^{\infty} m a_m U_{m-1}(\eta), \quad \eta \in [-1,1], \tag{15}$$

since $\frac{dT_m}{d\eta}(\eta) = mU_{m-1}$ [33]. Here $U_m(\eta)$ denotes the m^{th} Chebychev polynomial of the second kind. Substituting Eq. (14c) and Eq. (15) into Eq. (13) yields

$$\sqrt{1-\eta^2} \sum_{m=0,2,\ldots}^{\infty} a_m[T_m(\eta) - T_0(\eta)] - \frac{1}{\pi}\sum_{m=0,2,\ldots}^{\infty} a_m \int_{\xi=-1}^{1}[T_m(\xi) - T_0(\xi)]d\xi +$$

$$\frac{2}{\lambda\pi^2}\sum_{m=2,4,\ldots}^{\infty} m a_m \fint_{\xi=-1}^{1} \frac{\sqrt{1-\xi^2}}{\eta - \xi} U_{m-1}(\xi)d\xi = -\frac{1}{4\lambda\pi^2}\fint_{\xi=-1}^{1}\frac{\sqrt{1-\xi^2}}{\eta-\xi}U_1(\xi)d\xi, \tag{16}$$

where we have made the substitutions $1 = T_0(\eta) = T_0(\xi)$, and $U_1(\xi) = 2\xi$.

For convenience, we state some well-known algebraic and integral relations associated with the Chebychev polynomials.

<u>ORTHOGONALITY PROPERTY [33]:</u>

$$\int_{\xi=-1}^{1} (1-\xi^2)^{-\frac{1}{2}} T_m(\xi)T_n(\xi)d\xi = \begin{cases} 0, & m \neq n, \\ \pi, & m = n = 0, \\ \frac{\pi}{2}, & m = n > 0, \end{cases} \tag{17}$$

<u>CLOSED FORM INTEGRAL RELATIONS [27,28]:</u>

$$\fint_{\xi=-1}^{1} \frac{U_n(\xi)\sqrt{1-\xi^2}}{\xi - \eta}d\xi = -\pi T_{n+1}(\eta), \quad n = 0,1,\ldots \tag{18a}$$

$$\int_{\xi=-1}^{1} T_n(\xi)d\xi = \begin{cases} \frac{2}{1-n^2} & , n = 0,2,\ldots \\ 0 & , n = 1,3,\ldots \end{cases} \tag{18b}$$

ALGEBRAIC RELATION [34]:

$$T_m(\xi)T_n(\xi) = \frac{1}{2}[T_{m+n}(\xi) + T_{|m-n|}(\xi)], \quad m=0,1,\ldots \quad n=0,1,\ldots \tag{19}$$

THREE-TERM RECURRENCE [33,35]:

$$T_{n+1}(\xi) - 2\xi T_n(\xi) + T_{n-1}(\xi) = 0, \quad n = 1,2,\ldots \tag{20a}$$

$$U_{n+1}(\xi) - 2\xi U_n(\xi) + U_{n-1}(\xi) = 0, \quad n = 1,2,\ldots \tag{20b}$$

The above expressions contain all the necessary ingredients for determining the unknown expansion coefficients a_m, $m = 0,2,\ldots$

Using Eq. (18), Eq. (16) reduces to

$$\sqrt{1-\eta^2}\sum_{m=0,2,\ldots}^{\infty} a_m[T_m(\eta) - T_0(\eta)] - \frac{1}{\pi}\sum_{m=0,2,\ldots}^{\infty} a_m\left[\frac{2}{1-m^2} - 2\right] +$$

$$\frac{2}{\lambda\pi}\sum_{m=2,4,\ldots}^{\infty} m a_m T_m(\eta) = -\frac{1}{4\lambda\pi}T_2(\eta), \quad \eta \in [-1,1]. \tag{21}$$

The salient property required for determining the unknown expansion coefficients is derived from the orthogonality relation shown in Eq. (17). Thus, our approach is to follow a classical Fourier technique for determining the unknown expansion coefficients. Dividing Eq. (21) by $\sqrt{1-\eta^2}$ and then operating on the result with

$$\int_{\eta=-1}^{1} T_n(\eta)d\eta,$$

we arrive at

$$\sum_{m=0,2,\ldots}^{\infty} a_m\int_{\eta=-1}^{1}[T_m(\eta) - T_0(\eta)]T_n(\eta)d\eta -$$

$$\frac{1}{\pi}\sum_{m=0,2,\ldots}^{\infty} a_m\left[\frac{2}{1-m^2} - 2\right]\int_{\eta=-1}^{1}\frac{T_n(\eta)}{\sqrt{1-\eta^2}}d\eta + \frac{2}{\lambda\pi}\sum_{m=2,4,\ldots}^{\infty} m a_m\int_{\eta=-1}^{1}\frac{T_m(\eta)T_n(\eta)}{\sqrt{1-\eta^2}}d\eta$$

$$= -\frac{1}{4\lambda\pi}\int_{\eta=-1}^{1}\frac{T_2(\eta)T_n(\eta)}{\sqrt{1-\eta^2}}d\eta, \quad n = 0,2,\ldots \quad \eta \in [-1,1], \quad \lambda > 0. \tag{22}$$

Making use of the previously reported closed-form integral relations, Eq. (22) then gives rise to the remarkably compact form

$$\sum_{m=0,2,...}^{\infty} a_m c_{mn} = -\frac{1}{8\lambda}\delta_{2,n} \qquad n = 0,2,... \tag{23a}$$

where c_{mn} is defined as

$$c_{mn} = \frac{1}{1-(m+n)^2} + \frac{1}{1-(m-n)^2} - \frac{2}{1-n^2} + \frac{n}{\lambda}\delta_{m,n}\,, \tag{23b}$$

$$m = 0,2,... \quad n = 0,2,...,$$

and where $\delta_{m,n}$ represents the Kronecker delta. Thus, Eq. (23a) leads to an infinite system of linear algebraic equations for the unknown expansion coefficients a_m, $m = 0,2,...$ It is clear from viewing the above system of equations that $c_{0n} = 0$, $n = 0,2,....$ Thus in order to assure a unique solution for the unknown Chebychev coefficients, we replace the first equation in the system shown in Eq. (23a), i.e., $n = 0$, by the auxiliary profile condition expressed in Eq. (14b).

Practically speaking, we must truncate the series represention shown in Eq. (14c) after some finite number of terms, say N (even), such that

$$\theta(\eta) \cong \theta^N(\eta) = \sum_{m=0,2,...}^{N} a_m[T_m(\eta) - 1], \quad \eta \in [-1,1]. \tag{24}$$

Thus, we are left with determining only [(N/2) + 1] coefficients. We refer to $\theta^N(\eta)$ as the approximate solution to $\theta(\eta)$ in terms of a finite Chebychev series. Results from several numerical experiments are presented in the next section.

4. RESULTS AND DISCUSSION

In this section, we present some representative numerical results illustrating the effectiveness of the regularized formulation/Galerkin solution discussed in the preceeding sections. Being a formal presentation, convergence [7,8,30] and accuracy are demonstrated by empirical means. A further study delving into theoretical issues is presently under consideration.

The physical parameter λ, known as the radiation-conduction number for the large path length limit, represents the single parameter of the dimensionless system shown in Eq. (1). As λ increases, the effect of molecular conduction diminishes relative to infrared radiation. Also, the contribution of the integral term tends to increase as λ increases. Through the approximation process, we must also consider the effect of the number of terms $(N/2 + 1)$ retained in the expansion on the accuracy of the series representation for the unknown function $\theta(\eta)$. The function $\theta(\eta)$ can be reconstructed through Eq. (24) once the expansion coefficients a_m, $m = 0,2,\ldots,N$ have been resolved. This approximation process naturally lends itself to a massively parallel computation since one can obtain numerical results for $\theta(\eta)$ at some predefined set of spatial locations in a parallel fashion.

Table 1 illustrates the effect of coupling among the expansion coefficients a_m, $m = 0,2,\ldots,N$ as a function of the radiation-conduction number λ. Columns 3-5 contain only the first eleven (dominant) terms. Clearly, as λ increases the strength of the coupling among the corresponding coefficients becomes more pronounced. As λ increases, we observe that the leading term becomes even more dominant relative to the next term. Additionally, as λ increases, we see that the coefficients a_m, m = 0,2,...,N decrease in magnitude at a more moderate rate. For $\lambda = 0.1$, we find $a_{100} = O(10^{-11})$ for $\lambda = 1,10$ we find that $a_{100} = O(10^{-10})$.

Tables 2 and 3 present a comparison between the approximate solution based on Eq. (2a) that was formally programmed using Eqs. (23) and (24) to that of the purely numerical solution based on Eq. (7). The purely numerical solution was obtained with the aid of singularity subtraction [13,24] and simple trapezoidal integration [36].

Table 2 displays results from the approximate solution as a function of λ at six spatial location $\eta \in [0,1]$ for different values of N. Here (N/2 + 1) denotes the actual number of terms retained in the series representation shown in Eq. (24). Only half of the physical domain is displayed due to symmetry, $\theta(\eta) = \theta(-\eta)$. As indicated by this table, as N increases, convergence to at least 5 places of accuracy appears to occur for the three cases illustrated ($\lambda = 0.1,1,10$). Knowing the behaviour of the coefficients, as depicted in Table 1, and since $|T_m(\eta)| \leq 1$, m = 0,2,...N, we get a qualitative feel that convergence is close at hand. Numerical results for the approximate solution were obtained using both an IBM-XT with a math co-processor, and an Ardent Titan II Graphics Supercomputer. Complete simulations (CPU time and I/O) required less than 8 seconds per test case on the IBM while complete simulations required less than 0.21 seconds on the Titan II (single processor, i.e. serial mode). The developed computer code was not optimized in any manner. The results presented in Tables 2 and 3 were obtained using the Ardent Titan II Graphics Supercomputer. The generated results shown in Table 3 took substantially longer run times when compared to the results presented in Table 2.

Results from the numerical solution based on singularity subtraction and trapezoidal integration are shown in Table 3. The number of panels used in the trapezoidal rule are indicated as M - 1. If the integrand is smooth, it is well known that the global truncation error associated with trapezoidal integration is $O(\Delta\eta^2)$ [36]. However, in the presence of a weakly singular kernel, this error is not anticipated [18]. The global truncation error of the implemented trapezoidal rule can be empirically approximated [18] if one assumes that the results presented in Table 2 (N = 100) are numerically "exact" to the indicated places. In the present study, we found that the global truncation error is $O(\Delta\eta^r)$ where $1.7 < r < 1.8$ which is in line with our expectations, i.e., $r < 2$. As the number of panels in the integration rule is increased, results from the purely numerical solution appear to be approaching the finite Chebychev series results shown in Table 2.

5. CONCLUSIONS

This paper demonstrates that analytic preconditioning of the Cess and Tiwari equation, using the method of Peters, allows for the development of an accurate approximate solution based on a finite Chebychev series representation. The tantalizing results offered here illustrate that Peters' approach can be easily extended to Cauchy singular integro-differential equations to produce a formulation highly receptive to approximation by an expansion technique. Extensions to and theoretical issues concerning the presented approach are under consideration. A symbolic (computation) augmentation is also being pursued which will be useful in error analysis.

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Expansion coefficients for finite Chebychev series, a_j, $j = 0,2,\ldots,N$				
λ	j	N = 20	N = 40	N = 80
0.1	0	0.1058220E+01	0.1058220E+01	0.1058220E+01
	2	-0.5786490E-01	-0.5786490E-01	-0.5786490E-01
	4	-0.3217863E-03	-0.3217860E-03	-0.3217860E-03
	6	-0.2560451E-04	-0.2560443E-04	-0.2560443E-04
	8	-0.5216013E-05	-0.5215983E-05	-0.5215981E-05
	10	-0.1600168E-05	-0.1600155E-05	-0.1600154E-05
	12	-0.6211032E-06	-0.6210982E-06	-0.6210976E-06
	14	-0.2814815E-06	-0.2814814E-06	-0.2814810E-06
	16	-0.1424587E-06	-0.1424632E-06	-0.1424629E-06
	18	-0.7832737E-07	-0.7834016E-07	-0.7834000E-07
	20	-0.4590472E-07	-0.4595899E-07	-0.4595889E-07
1	0	0.1036350E+01	0.1036351E+01	0.1036351E+01
	2	-0.3454941E-01	-0.3454930E-01	-0.3454929E-01
	4	-0.1549988E-02	-0.1549966E-02	-0.1549965E-02
	6	-0.1889990E-03	-0.1889926E-03	-0.1549922E-03
	8	-0.4055971E-04	-0.4055730E-04	-0.4055712E-04
	10	-0.1242804E-04	-0.1242701E-04	-0.1242692E-04
	12	-0.4786996E-05	-0.4786594E-05	-0.4786544E-05
	14	-0.2153864E-05	-0.2153846E-05	-0.2153816E-05
	16	-0.1083504E-05	-0.1083847E-05	-0.1083828E-05
	18	-0.5923305E-06	-0.5933078E-06	-0.5932955E-06
	20	-0.3429042E-06	-0.3468224E-06	-0.3468144E-06
10	0	0.1008119E+01	0.1008120E+01	0.1008120E+01
	2	-0.6608189E-02	-0.6607234E-02	-0.6607175E-02
	4	-0.1026673E-02	-0.1026240E-02	-0.1026213E-02
	6	-0.2974672E-03	-0.2972585E-03	-0.2972450E-03
	8	-0.1059435E-03	-0.1058395E-03	-0.1058324E-03
	10	-0.4285136E-04	-0.4280009E-04	-0.4279608E-04
	12	-0.1914318E-04	-0.1912122E-04	-0.1911885E-04
	14	-0.9299324E-05	-0.9296901E-05	-0.9295431E-05
	16	-0.4847851E-05	-0.4863665E-05	-0.4862725E-05
	18	-0.2666322E-05	-0.2710667E-05	-0.2710053E-05
	20	-0.1465065E-05	-0.1595284E-05	-0.1594883E-05

Table 1. First eleven Chebychev expansion coefficients for λ = 0.1,1,10. Here [(N/2) + 1] represents the number of terms retained in the series expansion.

Approximate Analytic Solution, Eq. (11b) for $\theta(\eta)$				
λ	η	N = 40	N = 80	N = 100
0.1	1	0	0	0
	0.8	0.04230519	0.04230520	0.04230520
	0.6	0.07467375	0.07467375	0.07467375
	0.4	0.09757693	0.09757693	0.09757693
	0.2	0.1112420	0.1112420	0.1112420
	0	0.1157850	0.1157850	0.1157850
1	1	0	0	0
	0.8	0.02809824	0.02809828	0.02809828
	0.6	0.04718276	0.04718280	0.04718280
	0.4	0.05984623	0.05984626	0.05984626
	0.2	0.06712799	0.06712802	0.06712802
	0	0.06950770	0.06950773	0.06950773
10	1	0	0	0
	0.8	0.007278492	0.007278589	0.007278593
	0.6	0.01060173	0.01060179	0.01060179
	0.4	0.01253844	0.01253848	0.01253848
	0.2	0.01358764	0.01358768	0.01358768
	0	0.01392214	0.01392220	0.01392220

Table 2. Approximate analytic solution for $\theta(\eta) = \theta(-\eta)$ using the finite Chebychev series representation shown in Eq. (33). Here [(N/2) + 1] denotes the actual number of terms retained in the series expansion. (Note that when $\lambda = 10$, $\theta^*(0.8) = 0.007278595$ for N = 150, 200.)

Numerical Solution, Eq. (16) for $\theta(\eta)$				
λ	η	M = 51	M = 101	M = 201
0.1	1	0	0	0
	0.8	0.04231206	0.04230719	0.04230576
	0.6	0.07468139	0.07467594	0.07467436
	0.4	0.09758491	0.09757920	0.09757756
	0.2	0.1112502	0.1112444	0.1112427
	0	0.1157932	0.1157874	0.1157857
1	1	0	0	0
	0.8	0.02813853	0.02811001	0.02810163
	0.6	0.04722369	0.04719451	0.04718609
	0.4	0.05988672	0.05985778	0.05984948
	0.2	0.06716808	0.0671394	0.06713119
	0	0.06954765	0.06951906	0.06951088
10	1	0	0	0
	0.8	0.007343878	0.00729894	0.007284601
	0.6	0.01065554	0.01061828	0.01060661
	0.4	0.01258667	0.01255319	0.01254276
	0.2	0.01363321	0.01360155	0.01359171
	0	0.01396694	0.01393582	0.01392616

Table 3. Results for $\theta(\eta)$ as obtained by singularity subtraction and trapezoidal integration. Here M - 1 denotes the number of panels used.

Appendix C:

Paper entitled: "Generalization of the Method of Peters to Cauchy Singular Integro-Differential Equations", by J.I. Frankel, Ph.D., **Proceedings of the Royal Society of London**, Series A (in review).

SEVERAL SYMBOLIC AUGMENTED CHEBYSHEV EXPANSIONS FOR SOLVING THE EQUATION OF RADIATIVE TRANSFER

by

Jay I. Frankel, Ph.D.

Associate Professor

Mechanical and Aerospace Engineering Department

University of Tennessee, Knoxville

Knoxville, Tennessee 37996-2210

Running title: Chebyshev Solutions for the Transport Equation

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ABSTRACT

Three expansion methods are described using Chebyshev polynomials of the first kind for solving the integral form of the equation of radiative transfer in an isotropically scattering, absorbing, and emitting plane-parallel medium. With the aid of symbolic computation, the unknown expansion coefficients associated with this choice of basis functions are shown to permit analytic resolution. A unified and systematic solution treatment is offered using the projection methods of collocation, Ritz-Galerkin, and Weighted-Galerkin. Numerical results are presented contrasting the three expansion methods and comparing them with existing benchmark results. New theoretical results are presented illustrating rigorous error bounds, residual characteristics, accuracy, and convergence rates.

1. INTRODUCTION

In radiative [1,2] and neutron [3] transport theories, utilization of the equivalent integral form of the Boltzmann transport equation has often been called upon when considering highly anisotropic scattering in either a plane-parallel [4,5] or spherical medium [6]. Peierl's equation produces (i) a reduction in dimensionality leading to a pure (smoothing) integral form, and (ii) a direct relation with several key physical quantities of interest [4,5]. Unfortunately, the equivalent integral form also leads to a system of weakly-singular Fredholm integral equations of the second kind. The appearance of the first exponential integral function, as a kernel function, introduces a logarithmic singularity as its argument vanishes.

Recently, Frankel [4] alluded to the natural implementation of symbolic computation to the integral form of the transport equation. Unfortunately, no symbolic implementation was actually performed. Indeed, only a crude numeric solution, based on singularity subtraction [7-9] and trapezoidal integration [10], was used. It is interesting to note that accurate results still resulted.

Owing to a lack of satisfactory closure on several fronts [4], this paper addresses three particular issues not previously resolved. In order to illustrate several fundamental points, this study focuses on radiative transport in an isotropically scattering plane-parallel medium. Extension to the general anisotropic scattering case will be evident. Thus, the purpose of the present exposition is threefold; (i) to develop three expansion methods which make use of Chebyshev polynomials of the first kind as the set of orthogonal basis functions, (ii) to implement and demonstrate the utility of symbolic computation, such as offered by the packages Mathematica™ and Maple, for augmenting the solution methodology, and (iii) to present some informative residual/error and convergence analyses which are intended to indicate performance and accuracy of the methods.

This paper is divided into three major sections. In Section 2, we formulate the problem of interest. In Section 3, we introduce a series representation for the zeroth Legendre moment of the intensity and develop the three solution methods for finding the unknown expansion coefficents. In Section 4, we present results and discuss the merit of the proposed approach.

2. FORMULATION

In this section, we present the necessary formulation of the equations governing the zeroth Legendre moment of the radiative intensity in a plane-parallel, isotropically scattering, absorbing, and emitting medium. The transformation from the integro-differential form of the equation of radiative transfer to the pure integral form can be found in the fine expositions by Ozisik [1] and Duderstadt and Martin [3].

2.1 Peierl's Equation

We begin by considering the integral form of the transport equation [1,3,4], namely

$$G(\eta) = f^{\alpha}(\eta) + \frac{\lambda}{2}\int_{\xi=-1}^{1} K(\alpha|\eta - \xi|)G(\xi)d\xi, \quad \eta \in [-1,1], \tag{2.1a}$$

where $G(\eta)$ is the zeroth Legendre moment of the radiative intensity, defined as [4]

$$G(\eta) \equiv 2\pi\int_{\mu=-1}^{1} I(\eta,\mu)d\mu, \quad \eta \in [-1,1]. \tag{2.1b}$$

Here $I(\eta,\mu)$ is the radiation intensity, $K(\alpha|\eta - \xi|)$ is the kernel which is explicitly given as

$$K(\alpha|\eta - \xi|) = E_1(\alpha|\eta - \xi|), \tag{2.1c}$$

where the exponential integral function is given as [11, p. 228]

$$E_n(z) = \int_{t=0}^{1} t^{n-2}e^{-(\frac{z}{t})}dt, \quad n > 0, \tag{2.1d}$$

and where η is the optical variable and μ is the cosine of the angle between the positive η-direction and the direction of the beam. The optical thickness and single scattering albedo are denoted by L and ω, respectively. Here the two real parameters seen in Eq. (2.1a) are expressible in terms of the single scattering albedo and optical thickness as $\alpha = L/2$ and $\lambda = \omega\alpha$. The forcing function, $f^{\alpha}(\eta)$ contains the imposed boundary constraints in terms of surface intensities and/or internal sources. For purpose of demonstration, we will consider a classical problem (similar to the slab albedo problem in neutron transport) where numerous citation exist. Thus, we will assume that no internal sources are present and that transparent surfaces are present at both $\eta = -1$ and $\eta = 1$. Further, we assume that the front surface at $\eta = -1$ is irradiated by an externally symmetric source while the back surface at $\eta = 1$ is free of any external source. Thus, the forcing function reduces to the nonsingular form

$$f^{\alpha}(\eta) = 2\pi E_2(\alpha(1 + \eta)), \quad \eta \in [-1,1]. \tag{2.2}$$

Equation (2.1a) represents a linear, weakly-singular Fredholm integral equation of the second kind. The weakly-singular kernel displayed in Eq. (2.1c) is quadratically integrable [12] in the square $\eta \in [-1,1]$ and $\xi \in [-1,1]$, and is also symmetric. Purely numeric solutions [13,14] and approximate analytic solutions [15-22] have been presented in the literature. Analytic approaches have typically been based on expansion methods. Legendre polynomials [18,19] and simple monomials [20-22] have been popular choices for the basis function.

2.2 Physical Quantities

Following the notation of Thynell and Ozisik [5], and Frankel [4], we can define two important surface properties.

Using the definitions presented in [4], one can express the refectivity, R as

$$R = \frac{\lambda}{2\pi}\int_{\eta=-1}^{1} G(\eta)E_2(\alpha(1 + \eta))d\eta, \tag{2.3}$$

and transmissivity, T as

$$T = 2E_3(2\alpha) + \frac{\lambda}{2\pi}\int_{\eta=-1}^{1} G(\eta)E_2(\alpha(1 - \eta))d\eta. \tag{2.4}$$

These properties (and/or exiting surface heat fluxes) are often used as the sole basis for demonstrating accuracy.

3. ANALYSIS

Traditional expansion methods for solving the integral form of the transport equation have used Legendre polynomials [18,19] and monomials [20-22] as the basis functions. In the present work, we illustrate that Chebyshev polynomials of the first kind, augmented with symbolic computation, can be added to the solution arena. It should be noted that Kamiuto [23] developed a Chebyshev collocation solution for the spherical harmonics equation. In that work, no error or convergence analysis was provided.

3.1 Chebyshev Expansions

Assume that our real-valued function $G(\eta)$ can be expressed in the form

$$G(\eta) = \sum_{m=0}^{\infty} a_m^* T_m(\eta), \qquad \eta \in [-1,1] \tag{3.1}$$

where $\{T_m(\eta)\}_{m=0}^{\infty}$ are the Chebyshev polynomials of the first kind [24] and follow several well-known relations [11]. The Chebyshev polynomials are defined as

$$T_m(\eta) = \cos[m(\cos^{-1}\eta)], \quad m = 0,1,\ldots$$

where $|T_m(\eta)| \leq 1$ for $m \geq 0$. Here the coefficients $\{a_m^*\}_{m=0}^{\infty}$ are to be determined by some practical means. It is well known [24,25] that $\{T_m(\eta)\}_{m=0}^{\infty}$ form an orthogonal sequence of functions with respect to the weight function $\sqrt{1 - \eta^2}$. Thus, when implementing an expansion method, the main goal lies in determining the unknown expansion coefficients $\{a_m^*\}_{m=0}^{\infty}$. Chebyshev polynomials have several exploitable features [11,24,25] and have been the topic of much research and interest with regard to spectral methods [26] boundary value problems [27], nonsingular integral equations [28], and the solution of Cauchy singular integral and integro-differential equations [27-32].

In general, we seek an approximate solution to $G(\eta)$ by truncating the infinite series displayed in Eq. (3.1) at a certain order N, namely

$$G_N(\eta) = \sum_{m=0}^{N} a_m^N T_m(\eta), \quad \eta \in [-1,1], \tag{3.2}$$

where a_m^N is an approximation to a_m^* for each fixed m. Thus, we may express Eq. (2.1a) as

$$R_N(\eta) = G_N(\eta) - f^\alpha(\eta) - \frac{\lambda}{2}\int_{\xi=-1}^{1} E_1(\alpha|\eta - \xi|)G_N(\xi)d\xi,$$
$$\eta \in [-1,1], \ \lambda > 0, \ \alpha > 0, \tag{3.3}$$

where we have introduced the local residual function, $R_N(\eta)$. Unless the true solution is a linear combination of $\{T_m(\eta)\}_{m=0}^{N}$, we cannot choose $\{a_m^N\}_{m=0}^{N}$ to make $R_N(\eta)$ vanish for all $\eta \in [-1,1]$. However, suitable expansion coefficients can be obtained by making the residual $R_N(\eta)$ small in some sense.

Let us define the inner product of two real functions $g_1(t)$ and $g_2(t)$ as

$$\langle g_1, g_2 \rangle_{w_k} \equiv \int_{t=-1}^{1} w_k(t)g_1(t)g_2(t)dt, \tag{3.4a}$$

and the corresponding norm as

$$\|g_1\|_{w_k} = \sqrt{\int_{t=-1}^{1} w_k(t)g_1^2(t)dt} \tag{3.4b}$$

where $w_k(t)$ is a non-negative, real and integrable weight function.

A particular expansion method is defined by any restrictions imposed on the residual function displayed in Eq. (3.3). Our aim is to determine the unknown expansion coefficients $\{a_m^N\}_{m=0}^N$ in such a manner that some measure of $R_N(\eta)$ is small. A systematic way of expressing this is to require that the orthogonality condition

$$\langle R_N(\eta), \Psi_k(\eta)\rangle_{w_k} = 0, \quad k=0,1,\ldots,N, \tag{3.5}$$

be enforced for k = 0,1,...,N. In other words, we will require that the residual $R_N(\eta)$ be orthogonal to the first (N+1) $\Psi_k(\eta)$ functions with respect to the weight function $w_k(\eta)$.

Let the local error in the approximation be defined as

$$\epsilon_N(\eta) = G(\eta) - G_N(\eta), \tag{3.6}$$

and its size may be measured by means of some functional norm. Unfortunately, the error is as inaccessible as the exact solution. However, the residual $R_N(\eta)$ is a computable measure of how well $G_N(\eta)$ is to $G(\eta)$. One can develop a corresponding weakly-singular Fredholm integral equation of the second kind for the error $\epsilon_N(\eta)$ in terms of the residual $R_N(\eta)$, namely

$$R_N(\eta) = -\epsilon_N(\eta) + \frac{\lambda}{2}\int_{\xi=-1}^{1} K(\alpha|\eta - \xi|)\epsilon_N(\xi)d\xi, \quad \eta \in [-1,1]. \tag{3.7}$$

Defining the p^{th} functional norm of $R_N(\eta)$ as $\|R_N\|_p$, and the p^{th} operator norm of κ as $\|\kappa\|_p$ then one may derive in symbolic form the rigorous error bound [33]

$$\frac{\|R_N\|_p}{1 + \frac{\lambda}{2}\|\kappa\|_p} \le \|\epsilon_N\|_p \le \frac{\|R_N\|_p}{1 - \frac{\lambda}{2}\|\kappa\|_p}, \tag{3.8}$$

(when $1 - \frac{\lambda}{2}\|\kappa\|_p > 0$). Note that κ denotes an integral operator, such as defined by

$$\kappa g = \int_{\xi=-1}^{1} K(\eta,\xi)g(\xi)d\xi,$$

where $K(\eta,\xi)$ is the kernel and g represents some unknown function.

This estimate will be discussed further in the next section. It should be noted that we can obtain an analytic expression for the residual, $R_N(\eta)$.

Substituting Eq. (3.2) into Eq. (3.3) yields

$$R_N(\eta) = \sum_{m=0}^{N} a_m^N [T_m(\eta) - \frac{\lambda}{2} I_m^\alpha(\eta)] - f^\alpha(\eta), \quad \eta \in [-1,1], \ \lambda > 0, \tag{3.9a}$$

where

$$I_m^\alpha(\eta) = \int_{\xi=-1}^{1} E_1(\alpha|\eta - \xi|) T_m(\xi) d\xi, \quad m=0,1,\ldots,N, \quad \alpha > 0 \tag{3.9b}$$

or explicitly

$$I_m^\alpha(\eta) = 2 \sum_{j=0}^{[\frac{m}{2}]} \frac{E_{2+2j}(0) T_m^{(2j)}(\eta)}{\alpha^{2j+1}} + \sum_{j=0}^{m} \frac{(-1)^{j+1} E_{j+2}(\alpha(1+\eta)) T_m^{(j)}(-1)}{\alpha^{j+1}}$$

$$- \sum_{j=0}^{m} \frac{E_{j+2}(\alpha(1-\eta)) T_m^{(j)}(1)}{\alpha^{j+1}}, \quad m = 0,1,..,N, \quad \alpha > 0, \tag{3.9c}$$

where $[\frac{m}{2}]$ denotes the integer resultant. Here, we denote the k^{th} spatial derivative on the m^{th} Chebyshev polynomial as $T_m^{(k)}(\eta)$.

Three methods for determining the expansion coefficients $\{a_m^N\}_{m=0}^N$ are developed in accordance to the concept displayed in Eq. (3.5). The three proposed methods use the weight and test functions indicated in Table I. Formally proceeding and requiring $\langle R_N, \Psi_k \rangle = 0$ for $k = 0,1,\ldots N$, we arrive at the general expression

$$0 = \sum_{m=0}^{N} a_m^N \int_{\eta=-1}^{1} w_k(\eta) \Psi_k(\eta) [T_m(\eta) - \frac{\lambda}{2} I_m^\alpha(\eta)] d\eta -$$

$$\int_{\eta=-1}^{1} w_k(\eta) \Psi_k(\eta) f^\alpha(\eta) d\eta, \quad k = 0,1,\ldots,N, \ \lambda > 0, \ \alpha > 0. \tag{3.10}$$

This equation will be made use of throughout the present analysis.

3.2 Collocation

Referring to Table I, the corresponding collocation method requires that $w_k(\eta) = \delta(\eta - \eta_k)$ and $\Psi_k(\eta) = 1$. This implies that the residual $R_N(\eta)$ be zero at (N + 1) discrete collocation points defined by η_k. Thus, we arrive at

$$\sum_{m=0}^{N} a_m^N[T_m(\eta_k) - \frac{\lambda}{2}I_m^\alpha(\eta_k)] = f^\alpha(\eta_k), \quad k = 0,1,\ldots,N, \ \lambda > 0 \quad (3.11)$$

where $I_N^\alpha(\eta)$ is defined in Eq. (3.9c). This provides (N + 1) equations for determining (N + 1) unknown expansion coefficients. This approach is clearly simple to implement and computationally inexpensive in terms of operation count.

3.3 Ritz-Galerkin

Referring to Table I, and using $w_k(\eta) = 1$, $\Psi_k(\eta) = T_k(\eta)$, and requiring that $\langle R_N, \Psi_k \rangle = 0$ for $k = 0,1,\ldots,N$, we arrive at

$$\sum_{m=0}^{N} a_m^N(A_{mk} - \frac{\lambda}{2}C_{mk}^\alpha) = f_k^\alpha, \quad k = 0,1,\ldots,N, \quad (3.12a)$$

where

$$A_{mk} = \int_{\eta=-1}^{1} T_m(\eta)T_k(\eta)d\eta, \quad m=0,1,\ldots,N, \quad k=0,1,\ldots,N, \quad (3.12b)$$

$$C_{mk}^\alpha = \int_{\eta=-1}^{1} I_m^\alpha(\eta)T_k(\eta)d\eta, \quad m=0,1,\ldots,N, \quad k=0,1,\ldots,N, \quad (3.12c)$$

$$f_k^\alpha = \int_{\eta=-1}^{1} f(\eta)T_k(\eta)d\eta, \quad k=0,1,\ldots,N. \quad (3.12d)$$

Explicit expressions for these constants have been derived and can be found in Appendix A. Thus, we are left with a system of (N + 1) linear equations for the (N + 1) unknown expansion coefficients.

3.4 Weighted-Galerkin

Some discussion is warranted with regard to this approach. At first glance, it appears that several insurmountable obstacles are present owing to the appearance of the singular weight function displayed in Table I. We can quickly overcome these apparent impediments by expressing the exponential integral function $E_1(\alpha|\eta - \xi|)$ in terms of its standard series representation [11]

$$E_1(\alpha|\eta - \xi|) = -\log|\eta - \xi| - \sum_{j=0}^{\infty} b_j^\alpha|\eta - \xi|^j, \tag{3.13a}$$

where

$$b_0^\alpha = \gamma + \log(\alpha), \tag{3.13b}$$

$$b_j^\alpha = \frac{(-1)^j\alpha^j}{j(j!)}, \quad j = 1,2,\ldots, \tag{3.13c}$$

where γ is Euler's constant. We now express Eq. (3.9a) in the explicit form

$$R_N(\eta) = \sum_{m=0}^{N} a_m^N[T_m(\eta) + \frac{\lambda}{2}\int_{\xi=-1}^{1} T_m(\xi)\log|\eta - \xi|d\xi$$
$$+ \frac{\lambda}{2}\sum_{j=0}^{\infty} b_j^\alpha\int_{\xi=-1}^{1} |\eta - \xi|^j T_m(\xi)d\xi] - f^\alpha(\eta), \quad \eta \in [-1,1]. \tag{3.14}$$

Forming the inner product, in accordance to Eq. (3.5), we formally arrive at

$$\sum_{m=0}^{N} a_m^N[N_{mn} + \frac{\lambda}{2}(A_{mn}\beta_n + B_{mn}^\alpha)] = 2\pi\nu_n^\alpha N_{nn}, \quad n = 0,1,\ldots,N, \ \lambda > 0, \tag{3.15}$$

where A_{mn} is defined in Eq. (3.12b) and

$$B_{mn}^\alpha = \sum_{j=0}^{\infty} b_j^\alpha \int_{\eta=-1}^{1}\frac{T_n(\eta)}{\sqrt{1 - \eta^2}} \int_{\xi=-1}^{1}|\eta - \xi|^j T_m(\xi)d\xi d\eta, \quad \alpha > 0. \tag{3.16}$$

In deriving Eq. (3.15), we made use of the following well-known orthogonality condition associated with Chebyshev polynomials of the first kind, namely [24]

$$N_{mn} = \int_{\eta=-1}^{1} \frac{T_m(\eta)T_n(\eta)}{\sqrt{1-\eta^2}}\, d\eta = \begin{cases} 0, & m \neq n, \\ \pi, & m = n = 0, \\ \frac{\pi}{2}, & m = n > 0, \end{cases} \tag{3.17}$$

where N_{mn} is the normalization integral. Additionally, we made use of

$$\int_{\eta=-1}^{1} \frac{T_k(\eta)\log|\eta - \xi|}{\sqrt{1-\eta^2}}\, d\eta = \beta_k T_k(\xi), \tag{3.18a}$$

where

$$\beta_0 = -\pi \log 2, \qquad k = 0, \tag{3.18b}$$

$$\beta_k = -\frac{\pi}{k}, \qquad k > 0. \tag{3.18c}$$

Equation (3.18a) can be derived from [11,34]

$$⨍_{\eta=-1}^{1} \frac{T_k(\eta)}{\sqrt{1-\eta^2}\,(\eta - \xi)}\, d\eta = \begin{cases} \pi U_{k-1}(\xi), & k > 0, \\ \\ 0, & k = 0, \end{cases} \tag{3.19}$$

where $U_{k-1}(\xi)$ is the $(k-1)^{th}$ Chebyshev polynomial of the second kind [24]. Here $⨍$ represents integration in the Cauchy principal value sense [35,36].

In arriving at Eq. (3.15), we expanded the forcing function $f^\alpha(\eta)$ in terms of a Chebyshev series expansion, namely

$$f^\alpha(\eta) = 2\pi E_2(\alpha(1+\eta)) = 2\pi \sum_{j=0}^{\infty} \nu_j^\alpha T_j(\eta). \tag{3.20}$$

This series representation converges fairly rapidly. In Appendix B, we develop the procedure for determining the expansion coefficients, ν_j^α, $j = 0,1,\ldots$

Symbolic computation was called upon to integrate, in an exact fashion, the double integral displayed in Eq. (3.16) for B_{mn}^α, $m = 0,1,\ldots,N$, $n = 0,1,\ldots,N$. Manually, this intermediate level computation is rather tedious. Again, we are now in a position of determining the expansion coefficients from a well-behaved system of linear equations.

4. RESULTS

In this section, we highlight some findings concerning the nature of the Chebyshev polynomials of the first kind and make some comparisons with previously reported results. The rationale for choosing Chebyshev polynomials of the first kind as the basis functions are twofold, namely (i) to develop some new theoretical results, and (ii) to extend the results of Kamiuto's [23] to the integral form of the transport equation. It should be noted that Kamiuto presented only operational results and did not discuss convergence or error estimates in his development of the solution for the integro-differential form of the transport equation.

The two parameter ($\omega, L \rightarrow \lambda, \alpha$) problem posed here has been previously considered by Lii and Ozisik [37]. The symbolic computation software package Mathematica™, implemented on a NeXT Turbostation with 16 MBytes of memory, was used for developing the solutions and graphics (with exception to Fig. 2.) presented here.

Using Eqs. (2.3) and (2.4) for R,T, respectively, and the finite Chebyshev series representation for $G(\eta)$ displayed in Eq. (3.2), we find

$$R = -\frac{\lambda}{2\pi} \sum_{m=0}^{N} a_m^N \sum_{n=0}^{m} \frac{1}{\alpha^{n+1}} [T_m^{(n)}(1) E_{n+3}(2\alpha) - T_m^{(n)}(-1) E_{n+3}(0)], \quad (4.1)$$

and

$$T = 2E_3(2\alpha) + \frac{\lambda}{2\pi} \sum_{m=0}^{N} a_m^N \sum_{n=0}^{m} \frac{(-1)^n}{\alpha^{n+1}} [T_m^{(n)}(1)E_{n+3}(0) - T_m^{(n)}(-1)E_{n+3}(2\alpha)]. \quad (4.2)$$

It is interesting to note that most studies merely compare R and T results to indicate accuracy. In light of this, we illustrate the effectiveness of the simple collocation method previously described using the Chebyshev polynomials of the first kind. Table II indicates that the collocation approach produces acceptable numerical results for R and T when compared with the exact solution [1,37] over a range of optical thicknesses and single-scattering albedos. The collocation points were established from a closed, Gauss-Chebyshev (Lobatto-Chebyshev) [9] rule, i.e., $\eta_k = \cos(\frac{\pi k}{N})$, $k = 0,1,\ldots,N$. This closed rule ensures $R_N(1) = R_N(-1) = 0$.

Lii and Ozisik [37] also included exact and approximate results when the optical thickness, L was 15 and 30 for the cases where the single-scatter albedo approached unity. The orthogonal collocation method outlined here proved to produce excellent numerical results when compared to the exact results reported in [37]. For example, when ω=0.995, L=15 and L=30, the exact results for the reflectivity and transmissivity were reported [37] as R=0.8438, T=0.0450 and R=0.8497, T=0.0071, respectively. Correspondingly, Lii and Ozisik [37] reported that their approximate solution produced R=0.8429, T=0.0453 and R=0.8489, T=0.0070, respectively. The orthogonal collocation method embodied here for the case L=15, ω=0.995 produced R=0.843896, T=0.044943 (when N=13), and R=0.843853, T=0.044956 (when N=15). When L=30,

ω=0.995, the present method produced R=0.850151, T=0.007036 (when N=13), R=0.849919, T=0.0070495 (when N=15), and R=0.849816, T=0.0070556 (when N=17). Clearly, the exact solution is supportive of these representative results. At various other albedos, it has been shown that accurate numerical results are obtained. Additionally, no numerical instabilities emerged from the computations required in arriving at any results reported in this paper.

When ω = 1, Lii and Ozisik [37] did not report any results for the reflectivity and transmissivity and thus no basis of comparison can be made with that study. However, the present methodology has provided accurate results when ω=1 as verified by rigorous error estimates and by physical trends (i.e., $\omega \to 1$) without any special modification to the analytic procedure or computer code.

Being an exploratory investigation, some additional characteristics associated with the approximation should be elucidated. Some additional theoretic considerations can be developed from knowledge of the residual function, $R_N(\eta)$. As remarked early, an analytic expression for $R_N(\eta)$ can be developed, namely

$$R_N(\eta) = - f^{\alpha}(\eta) + \sum_{m=0}^{N} a_m^N [\ T_m(\eta) - \frac{\lambda}{2}\{\ 2\sum_{j=0}^{[\frac{m}{2}]} \frac{E_{2+2j}(0)T_m^{(2j)}(\eta)}{\alpha^{2j+1}} \tag{4.3}$$

$$+ \sum_{j=0}^{m} \frac{(-1)^{j+1}E_{j+2}(\alpha(1+\eta))T_m^{(j)}(-1)}{\alpha^{j+1}} - \sum_{j=0}^{m} \frac{E_{j+2}(\alpha(1-\eta))T_m^{(j)}(1)}{\alpha^{j+1}}\}]$$

$$\eta \in [-1,1],\ \alpha > 0,\ \lambda > 0,$$

which is valid for the three methods previously discussed.

Some clarification concerning the Weighted-Galerkin method must be made here. In order to perform all the integrations analytically, the forcing function $f^{\alpha}(\eta)$, which contained the exponential integral function, $E_2(\alpha(1 + \eta))$, was expanded into an equivalent Chebyshev series (see Appendix B). Thus, the forcing function was approximated to some extent. Appendix B contains some details on the errors associated with this approximation. The Ritz-Galerkin method did not suffer a similar fate since the integration involving the forcing function, weight function, and coordinate function could be performed without approximation.

Figure 1 (a-d) present $G_N(\eta)$ and $R_N(\eta)$ for the three discussed methods (b-collocation, c-Ritz-Galerkin, d-Weighted-Galerkin) where $N = 7$, $\alpha = 1$, and $\lambda = 0.8$. In Fig. 1b, the collocation points were chosen in accordance to $\eta_k = \cos(\frac{k\pi}{N})$, $k = 0,1,...,N$. The oscillatory characteristic associated with the residual are somewhat similar to each other with the exception of the obvious enforcement of $R_N(1) = R_N(-1) = 0$ by the collocation method. The effect of the Chebyshev weight function is also evident when comparing the Ritz- and Weighted-Galerkin methods near $\eta = -1$ and $\eta = 1$. These approaches produced graphically identical $G_N(\eta)$ results as displayed in Fig. 1a. Observe that the approximate solution $G_N(\eta)$ is bounded while the kernel shown in Eq. (2.1c) is not.

It is clear from viewing Fig. 1b that the residuals for the orthogonal collocation method at the endpoints are zero as forced by construction using the collocation points defined by a closed rule. Some numerical methods have indicated that large errors take place at the endpoints in $G(\eta)$. However, the orthogonal collocation method offered here (and also clearly demonstrated by Frankel [44]) illustrates that excellent numerical results can be achieved at these locations. Clearly, the residual plots offer some insight into this phenomena.

From the residual plots, the collocation method appears to produce comparable results to that of the Galerkin methods. Realizing the extensive amount of arithmetic associated with Galerkin methods due to multiple integrals, it appears that the collocation method represents an economical and accurate N^{th} order approximation to $G(\eta)$. Computer times for the collocation method using Mathematica™ on a NeXT workstation typically took less than 30 seconds for determining the expansion coefficients depending on the number of terms retained in the expansion. Most of the CPU time used in a single run was attributed to calculating the norms of the various functions and kernels required in arriving at rigorous error estimates.

As alluded to earlier, the unknown expansion coefficients $\{a_m^N\}_{m=0}^N$ are found by solving a system of coupled linear algebraic equations by matrix means. Owing to the obvious coupling among the coefficients, the effect of N on the accuracy of a_m^N must also be considered. Table III illustrates the effect of the number of terms retained in the collocation expansion on the convergence of the expansion coefficients when $\alpha = 1$ and $\lambda = 0.8$. Clearly, the dominant terms are converging as N grows.

With regard to Tables IV and V, both the Ritz-Galerkin and Weighted-Galerkin methods produce similar convergence trends on the expansion coefficients as compared with the collocation method. Tables III through V also present the resulting numerical values for the reflectivity R and transmissivity T using a finite Chebyshev series representation for the unknown function, $G(\eta)$. From viewing Tables III through V, it is clear that as N grows, accurate numerical results for these two surface properties are being generated. From Table II, the exact values corresponding to Tables III-V for R and T are 0.03280 and 0.01973, respectively. It appears that 4 places of accuracy can be quickly obtained for both R and T even though the expansion coefficients have not converged to a comparable number of places for small N. Some care should be exercised when using $G_N(\eta)$ when N is small.

At this point, it is instructive to define the infinity norm of a function as

$$\|\Theta\|_\infty \equiv \sup_{\eta\in[-1,1]} |\Theta(\eta)|. \tag{4.5a}$$

Correspondingly, the infinity norm of the integral operator K is [7, p. 14]

$$\|K\|_\infty \equiv \sup_{\eta\in[-1,1]} \int_{\xi=-1}^{1} |K(\alpha|\eta - \xi|)|d\xi, \tag{4.5b}$$

and where the infinity norm of the residual function becomes

$$\|R_N\|_\infty \equiv \sup_{\eta\in[-1,1]} |R_N(\eta)|. \tag{4.5c}$$

Observe that, for this physical problem, the Galerkin methods produce their extremal at $\eta = -1$ unlike the collocation method where the extremal of $R_N(\eta)$ is located in the interior of the physical domain. Thus, when calculating the L_∞-norm of the residual for the collocation method, some additional numerical/symbolic analysis is performed in order to locate the corresponding extremal.

A rigorous L_∞-error bound $\|\epsilon_N\|_\infty$ for $G_N(\eta)$ is available from Eq. (3.8). For the three methods, the error bounds are also presented in Tables III through V. The upper-error bound has the superscript U while the lower-error bound has the superscript L. It is clear from viewing Tables III through V that there is more error in $G(\eta)$ for fixed N than is realized in either R or T.

For comparison purposes, a simple numeric solution for solving the weakly-singular Fredholm integral equation for the local error $\epsilon_N(\eta)$, shown in Eq. (3.7) has been carried out using the residual generated by the collocation method when $\alpha = 1$ and $\lambda = 0.8$. Table VI presents the discrete L_∞-error as obtained by direct numerical simulation for this illustrative case. These numerical results were obtained by solving the weakly-singular Fredholm integral equation of the second kind shown in Eq. (3.7) using singularity subtraction and trapezoidal integration [4]. Here, the numerically obtained L_∞ norm of $\epsilon_N(\eta)$ denoted by $\|\epsilon_N\|_\infty^n$ (superscripted with a "n" for numerical) is contrasted to the upper- and lower-bounds as developed by Eq.

(3.8). It is interesting to note that the error tends toward the lower bound for this case rather rapidly. Using the results shown in column 4 for the numerically obtained L_∞- error from the integral equation, we can extract useful information concerning the convergence rate of the method. The convergence rate associated with a solution method is a crucial factor in determining the success of the method. The information presented in Table VI permits empirical interpretation toward estimating the convergence rate. Examination of column 3 in Table VI circumstantially indicates that the convergence rate can be approximated by the expression

$$\|\epsilon_N\| \approx \|\epsilon_N\|_\infty^n = O(\frac{1}{N^2}), \tag{4.6}$$

where N is the order of the expansion. In Appendix C, a pessimistic estimate is developed based on a projection method [7,34,40,41]. This approach produces

$$\|\epsilon_N\|_\infty = O(\frac{\ln(N)}{N^r}), \tag{4.7}$$

where $G \in C^r$ [-1,1]. Note that our observational skill relied on fairly low values of N while the estimate developed in Appendix C required N sufficiently large (greater than our observational data). In general, a viable expansion technique should have rapid convergence.

Finally, Table VII demonstrates the influence of the parameters L, ω on the upper and lower L_∞-error bounds as the number of terms in the Chebyshev series is increased. As the optical thickness increases and the single-scattering albedo decreases, the bounds appear to be tightening (though apparently large in magnitude) rapidly as N increases.

5. CONCLUSIONS

Several expansion methods have been implemented for solving the equation of radiative transfer in a isotropically scattering, absorbing, and potentially emitting media using Chebyshev polynomials of the first kind as the basis functions. The practical analytical/computational procedure offered here was assisted by the implementation of symbolic computation in determining the expansion coefficients. Symbolic manipulation was effective in assisting in developing some of the error analysis presentation within. The collocation technique using this choice of basis augmented by symbolic computation offers an easy and obvious generalization to situations involving anisotropic scatter [4,5], and spatially varying albedos [22,38] in both plane-parallel and spherical geometries [38]. The rate of convergence for the series representation offered by the orthogonal collocation method is deemed moderate. The collocation method appears especially noteworthy with regard to flexability as noted by Frankel [45] when investigating transient, radiative-conductive transport in a participating medium.

LaClair and Frankel [46] extended the present work to include a linear anisotropic scattering phase function. Novel error estimates were also obtained extending the procedure offered here to two coupled, weakly-singular, Fredholm integral equations of the second kind. Again, symbolic computation was called upon for performing numerous and often tedious analytic manipulations. Future generalization to radiative transport involving highly anisotropic scattering, such as in coal-fired combustion [39], now becomes assessible by approximation using a finite series representation.

In closing, the accuracy of a numerical method for the radiative equation of transfer based on R and T comparisions should be carefully examined. In many practical applications, the actual spatial distribution of the intensity or the zeroth (or first) moment of the intensity appear crucial owing to coupling to another dependent variable such as temperature. A clear error estimate (or bounds) on the unknown function(s) certainly serves to permit a proper evaluation of accuracy.

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Appendix A

Explicit expressions for the constants displayed in Eqs. (3.12b)-(3.12d) are now provided:

$$A_{mk} = \begin{cases} 0 & m,k \text{ mixed odd/even} \\ \dfrac{1}{1-(m+k)^2} + \dfrac{1}{1-|m-k|^2} & \text{both } m,k \text{ even or odd} \end{cases} \tag{A.1}$$

and

$$C^{\alpha}_{mk} = 2\sum_{j=0}^{M} \frac{E_{2+2j}(0)}{\alpha^{2j+1}} \int_{\eta=-1}^{1} T_k(\eta)T_m^{(2j)}(\eta)d\eta +$$

$$\sum_{j=0}^{m} \frac{1}{\alpha^{j+1}}\{ (-1)^j\, T_m^{(j)}(-1)\sum_{n=0}^{k} \frac{1}{\alpha^{n+1}}[T_k^{(n)}(1)E_{n+j+3}(2\alpha) - T_k^{(n)}(-1)E_{n+j+3}(0)]$$

$$- T_m^{(j)}(1)\sum_{n=0}^{k} \frac{(-1)^n}{\alpha^{n+1}}[T_k^{(n)}(1)E_{n+j+3}(0) - T_k^{(n)}(-1)E_{n+j+3}(2\alpha)]\}, \tag{A.2}$$

where M = int[m/2] and where the integral term is expressible as

$$\int_{\eta=-1}^{1} T_k(\eta)T_m^{(2j)}(\eta)d\eta = \sum_{n=0}^{m-2j} c_{jknm} \int_{\eta=-1}^{1}T_k(\eta)T_n(\eta)d\eta = \sum_{n=0}^{m-2j} c_{jknm}A_{nk} \tag{A.3}$$

for evaluation purposes. Symbolic software packages such as Mathematica™ or Maple permit the rapid evaluation of the constants c_{jknm} as needed when expressing $T_m^{(2j)}(\eta)$ in terms of a finite Chebyshev sum. Maple has a function already prepared for doing this operation. This tedious procedure can be done by hand by initiating the following process [24]:

$$T_m^{(1)}(\eta) = mU_{m-1}(\eta) = \begin{cases} m[1 + 2 \displaystyle\sum_{j=2,4,\ldots}^{m-1\geq 2} T_j(\eta)], & m \text{ odd} \\ m[\, 2 \displaystyle\sum_{j=1,3,\ldots}^{m-1\geq 1} T_j(\eta)], & m \text{ even} \end{cases}$$

$$T_m^{(2)}(\eta) = \begin{cases} 2m[\sum_{j=2,4,..}^{m-1\geq 2} (2j) \sum_{k=1,3,..}^{j-1\geq 1} T_k(\eta)], & m \text{ odd} \\ 2m[\sum_{j=1,3,..}^{m-1\geq 1} (2j)[\frac{1}{2} + \sum_{k=2,4,..}^{j-1\geq 2} T_k(\eta)]], & m \text{ even} \end{cases}$$

$$T_m^{(3)}(\eta) = \begin{cases} 2m[\sum_{j=2,4,..}^{m-1\geq 2} (2j) \sum_{k=1,3,..}^{j-1\geq 1} (2k)[\frac{1}{2} + \sum_{n=2,4,..}^{k-1\geq 2} T_n(\eta)]], & m \text{ odd} \\ 2m[\sum_{j=1,3,..}^{m-1\geq 1} (2j) \sum_{k=2,4,..}^{j-1\geq 2} (2k) \sum_{n=1,3,..}^{k-1\geq 1} T_n(\eta)], & m \text{ even} \end{cases}$$

$$\ldots\ldots \qquad m = 0,1,\ldots,$$

where $U_n(\eta)$ is the n^{th} Chebyshev polynomial of the second kind [24]. Clearly, a pattern is developing but symbolic manipulation appears to be more prudent and better suited to such computation than the author.

Finally,

$$f_k^\alpha = -2\pi \sum_{n=0}^{k} \frac{1}{\alpha^{n+1}}[E_{3+n}(2\alpha)T_k^{(n)}(1) - E_{3+n}(0)T_k^{(n)}(-1)], \quad k = 0,1,\ldots \quad (A.4)$$

Appendix B

In this appendix, we briefly describe the approximation of the second exponential integral function in terms of a Chebyshev series representation. Consider [11]

$$E_2(\alpha(1+\eta)) = \alpha(1+\eta)\ln(1+\eta) - \sum_{k=0}^{\infty} \beta_k^\alpha (1+\eta)^k, \quad \eta \in [-1,1], \quad \text{(B.1)}$$

where

$$\beta_0^\alpha = -1$$

$$\beta_1^\alpha = \alpha(1 - \gamma - \ln(\alpha))$$

$$\beta_k^\alpha = \frac{(-\alpha)^k}{(k-1)k!}, \quad k > 1.$$

Assuming that $E_2(\alpha(1+\eta))$ can be expressed as

$$E_2(\alpha(1+\eta)) = \sum_{m=0}^{\infty} \nu_m^\alpha T_m(\eta), \quad \eta \in [-1,1], \qquad \text{(B.2a)}$$

where

$$\nu_m^\alpha = \frac{1}{N_{mm}} \int_{\eta=-1}^{1} \frac{E_2(\alpha(1+\eta))T_m(\eta)}{\sqrt{1-\eta^2}}\, d\eta, \quad m = 0,1,\ldots, \quad \text{(B.2b)}$$

where the normalization integral, N_{mn} is defined in Eq. (3.17). Making use of [24]

$$T_m(\eta)T_k(\eta) = \frac{1}{2}[T_{m+k}(\eta) + T_{|m-k|}(\eta)], \qquad \text{(B.3)}$$

and noting that [34, p. 218]

$$h_k(\eta) = \int_{\xi=-1}^{1} \frac{T_k(\xi)\ln|\eta-\xi|}{\sqrt{1-\xi^2}}\, d\xi = \begin{cases} -\pi\ln(2), & k = 0, \\ -\dfrac{\pi T_k(\eta)}{k}, & k \geq 1, \end{cases} \qquad \text{(B.4)}$$

we arrive at

$$\nu_m^\alpha N_{mm} = -\sum_{k=0}^{\infty} \beta_k^\alpha \int_{\eta=-1}^{1} \frac{T_m(\eta)(1+\eta)^k}{\sqrt{1-\eta^2}}\, d\eta \qquad \text{(B.5)}$$

$$+ \frac{\alpha}{2}[2h_m(-1) + h_{m+1}(-1) + h_{|m-1|}(-1)], \quad m = 0,1,\ldots$$

The evaluation of the remaining integral in (B.5) for each fixed m is performed analytically with the aid of Mathematica™.

Table VIII presents the relative error, as defined by

$$|\delta_N| = \left| \frac{E_2(\alpha(1+\eta)) - \sum_{n=0}^{N} \nu_n^\alpha T_n(\eta)}{E_2(\alpha(1+\eta))} \right| \times 100.$$

Clearly, even at N = 11 (i.e., 12 terms in the series representation) a fair amount of error persists. Thus, it appears that a direct numerical approximation for

$$\int_{\eta=-1}^{1} \frac{f^\alpha(\eta) T_k(\eta)}{\sqrt{1 - \eta^2}} \, d\eta, \qquad k = 0,1,\ldots,N$$

is preferable.

Appendix C

A brief discussion concerning the rate of convergence associated with the approximate solution based on the collocation method using the closed-rule collocation points previously described is now presented. The results obtained in this appendix are intended to clarify the observed convergence rate conjectured by Eq. (4.6). Our approach relies on the projection method framework described by Atkinson [7], Baker [40], and others [34,41]. We refer the reader to these fine sources for the particulars.

Let $G \in C^r[-1,1]$, $N > r$, and let the points $\eta_k \in [-1,1]$ be given by $\eta_k = \cos[\frac{k\pi}{N}]$, $k = 0,1,\ldots,N$. Define the interpolatory projection operator P_N such that

$$P_N h = \sum_{m=0}^{N} l_m(\eta) h(\eta_m), \tag{C.1a}$$

where $h(\eta)$ is a real function such that $h \in C^r\,[-1,1]$ and where $l_m(\eta)$ is the Lagrange interpolation polynomial [42]

$$l_m(\eta) = \prod_{k=0(\neq m)}^{N} \frac{(\eta - \eta_k)}{(\eta_m - \eta_k)}, \quad m = 0,1,\ldots,N. \tag{C.1b}$$

For implementation purposes, we express Eq. (C.1b) as

$$l_m(\eta) = \frac{\Omega(\eta)}{\Omega'(\eta_m)(\eta - \eta_m)}, \quad \eta \in [-1,1], \tag{C.1c}$$

where

$$\Omega(\eta) = \prod_{k=0}^{N} (\eta - \eta_k), \tag{C.1d}$$

$$\Omega'(\eta_m) = \prod_{k=0(\neq m)}^{N} (\eta_m - \eta_k). \tag{C.1e}$$

Let us also define s_N such that it is a polynomial of degree $\leq N$. The linear operator P_N has the following property [7]

$$P_N s_N = s_N, \tag{C.2a}$$

from which we can derive the idempotent property [7]

$$P_N^2 = P_N, \tag{C.2b}$$

also note that $\|P_N\|_\infty \geq 1$ for $N \geq 1$. Let the previously defined collocation points also represent the interpolatory points. Also, note that η_k, $k = 0,1,\ldots,N$ are real, distinct, and symmetric around $\eta = 0$. For large N, the points are more dense around the endpoints than toward the center of the interval $\eta \in [-1,1]$. With this in mind, it is clear from its construction that

$$P_N R_N(\eta) = \sum_{k=0}^{N} R_N(\eta_k) l_k(\eta) = 0, \tag{C.3a}$$

and

$$P_N R_N(\eta_m) = \sum_{k=0}^{N} R_N(\eta_k) l_k(\eta_m) = \sum_{k=0}^{N} R_N(\eta_k)\delta_{mk} = 0, \tag{C.3b}$$

where δ_{mk} is the Kronecker delta function; and $R_N(\eta)$ is the local residual function defined in Eq. (3.3) and explicitly expressed in Eq. (4.3). Recall Eq. (3.3) and Eq. (2.1a), respectively, in their corresponding operator form as

$$R_N = G_N - f^\alpha - \frac{\lambda}{2} \kappa G_N, \tag{C.4a}$$

$$0 = G - f^\alpha - \frac{\lambda}{2} \kappa G, \tag{C.4b}$$

where we assume that $\lambda/2$ is a regular value of κ. Operating on Eqs. (C.4a-b) with P_N and noting that G_N is a polynomial of degree N, we find

$$0 = G_N - P_N f^\alpha - \frac{\lambda}{2} P_N \kappa G_N, \tag{C.5a}$$

$$0 = P_N G - P_N f^\alpha - \frac{\lambda}{2} P_N \kappa G. \tag{C.5b}$$

Adding and subtracting G from Eq. (C.5b) then taking the difference between Eqs. (C.5a) and (C.5b) yields

$$(I - \frac{\lambda}{2} P_N \kappa)(G - G_N) = G - P_N G,$$

or

$$G - G_N = (I - \frac{\lambda}{2} P_N \kappa)^{-1} (G - P_N G), \tag{C.6}$$

where

$$\lim_{N\to\infty} \|P_N K - K\|_\infty = 0$$

than for sufficiently large N, $(I - \frac{\lambda}{2} P_N K)^{-1}$ exists [7]. Taking the infinity norm of Eq. (C.6) produces

$$\|G - G_N\|_\infty = \|(I - \frac{\lambda}{2} P_N K)^{-1} (G - P_N G)\|_\infty,$$

or

$$\|G - G_N\|_\infty \le \frac{1}{1 - \frac{\lambda}{2} \|P_N K\|_\infty} \|G - P_N G\|_\infty . \tag{C.7}$$

Introducing the best uniform approximation [40] γ_N (a polynomial of degree $\le$ N), we find

$$\|G - G_N\|_\infty \le \frac{1}{1 - \frac{\lambda}{2} \|P_N K\|_\infty} \|(G - \gamma_N + \gamma_N - P_N G)\|_\infty . \tag{C.8}$$

but since $\gamma_N = P_N \gamma_N$, we find

$$\|G - G_N\|_\infty = O[(1 + \|P_N\|_\infty) \|G - \gamma_N\|_\infty]. \tag{C.9}$$

From Baker [40, p. 93], one can show

$$\|G - \gamma_N\|_\infty = O(\frac{1}{N^r}).$$

These results came about from the use of a Jackson theorem [40,42,43]. Before proceeding, a digression is warranted.

Rivlin [42] and others have investigated a similar case to the present study except that they considered an open-rule set of Chebyshev collocation points. The values of $\|P_N\|_\infty^*$ for N > 0 with collocations points defined by $\eta_j^* = \cos[\frac{(2j - 1)\pi}{2N}]$, j = 1,...,N has been analytically developed in several sources [42,43]. Rivlin [40, pp.93-97] showed

$$\Lambda_N^* = \sum_{m=1}^{N} |l_m(\eta)| < \frac{2}{\pi}\ln(N) + 4, \tag{C.10a}$$

where Λ_N^* is the Lebesgue constant which has at most logarithmic growth. Conforming to our notation, let

$$\|P_N\|_\infty^* = \frac{2}{\pi}\ln(N) + 4. \tag{C.10b}$$

Rivlin [42, p.93] remarks that the proof of this result is "rather long". In fact, Baker [40, p.94] adds that this result is "pessimistic". Baker [40, p. 94] further remarks that for $N < 1000$ $\Lambda_N^* < 5.4$. (Note that a typograpical error exists in his result for Λ_N^* on p. 94.) Baker [40, p.104] does indicate that the closed-rule collocation points should produce a similar relation as indicated in Eq. (C.10a) though the constants involved would be different.

Owing to the numerous applications of the triangle inequality, an empirical approach for bounding $\|P_N\|_\infty$ appears quite reasonable. With these remarks in mind, a contemporary approach representing a compromise between theory and practice is offered with the aid of symbolic manipulation. One can empirically demonstrate, to a high degree of confidence and accuracy, that

$$\|P_N\|_\infty = A\ln(N) + B, \tag{C.11}$$

for sufficiently large N. This is graphically demonstrated in Figure 9. The approximate numerical values for A and B agree well with Reference 24 (page 13). Figure 9 presents a semilog graph which indicates a straightline for sufficiently large N. Numerical results for this figure were generated using the definition of $\|P_N\|_\infty$ based on Eqs. (C.1c-e) and implemented using Mathematica™.

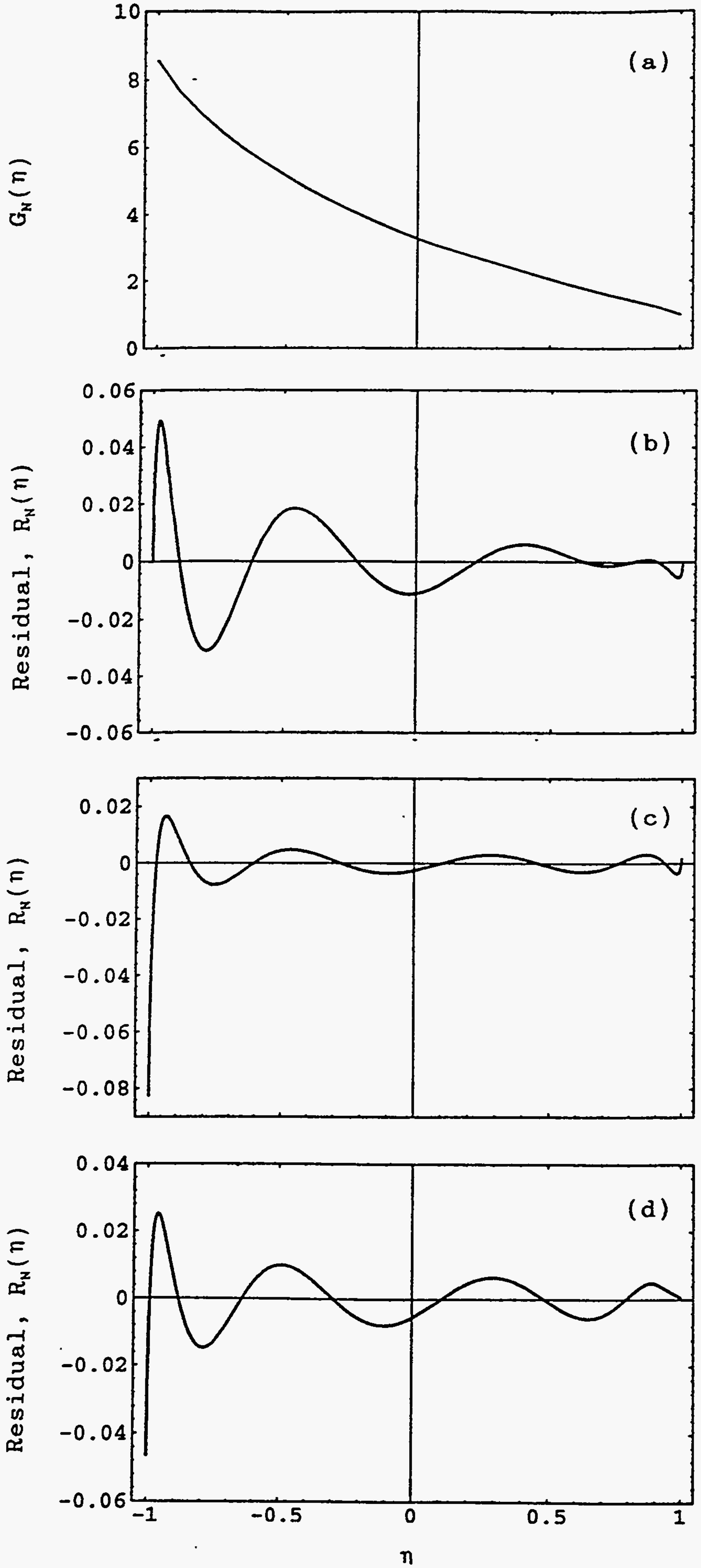
Thus, it appears that from our theoretical development that

$$\|\epsilon_N\|_\infty = \|G - G_N\|_\infty = O(\ln(N)/N^r). \tag{C.12}$$

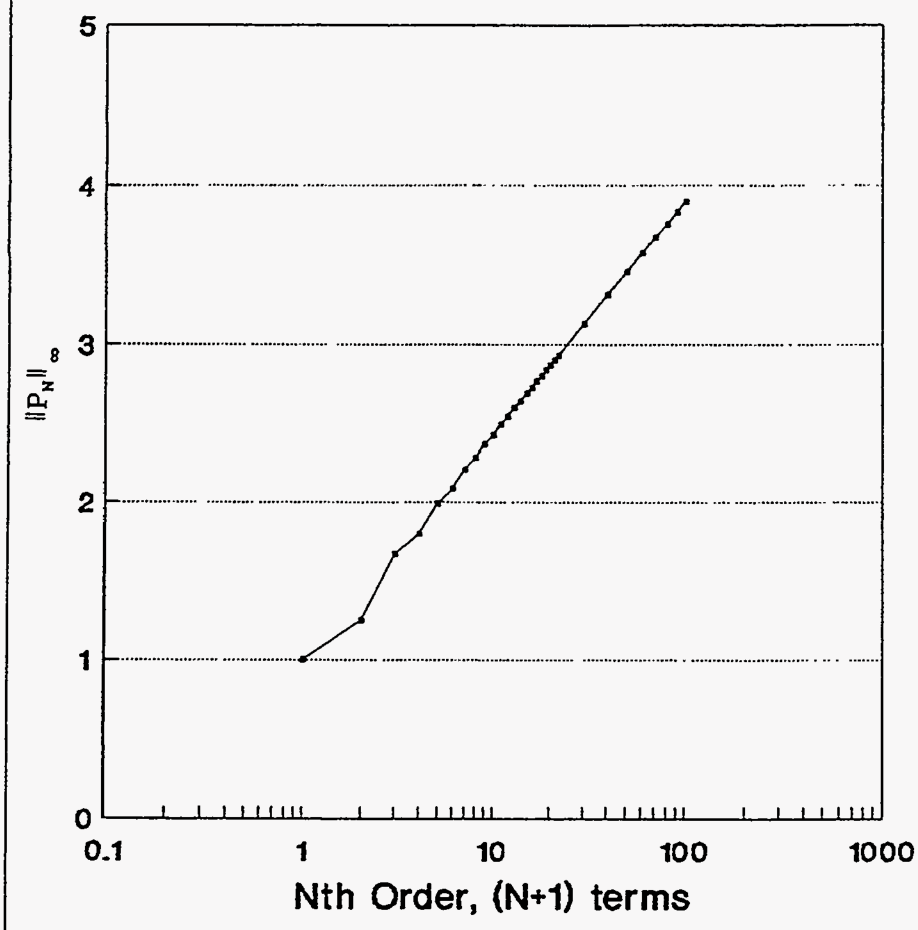
These results indicate that rapid convergence will occur if $G(\eta)$ is sufficiently smooth. This result appears to have direct bearing on regularity of the unknown function $G(\bar{\eta})$. The reported discrepency between Eq. (C.12) and Eq. (4.6) could come about due to i) the conjecture for extracting Eq. (4.6) is based on $N < 9$, (not sufficiently large), and ii) Eq. (C.12) is a pessimistic result due to the bounding processes involved.

In the general anisotropic case or in situations involving mixed-mode, nonlinear heat transfer, this type of analysis appears quite formidable though in some cases it does appear possible.

LIST OF FIGURES



10
8
6
4
2
0
$G_N(\eta)$
(a)
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
Residual, $R_N(\eta)$
(b)
0.02
0
-0.02
-0.04
-0.06
-0.08
Residual, $R_N(\eta)$
(c)
0.04
0.02
0
-0.02
-0.04
-0.06
Residual, $R_N(\eta)$
(d)
-1
-0.5
0
0.5
1
η



5
4
3
2
1
0
$\|P_N\|_\infty$
0.1
1
10
100
1000
Nth Order, (N+1) terms

LIST OF TABLES

$w_k(\eta)$	$\Psi_k(\eta)$	Name of Method
$\delta(\eta - \eta_k)$	1	Collocation
1	$T_k(\eta)$	Ritz-Galerkin
$(1 - \eta^2)^{-1/2}$	$T_k(\eta)$	Weighted-Galerkin

ω	Surface Property	L = 0.5 (N=7)		L = 1 (N=7)		L = 2 (N=9)		L = 5 (N=11)	
		Present	Exact	Present	Exact	Present	Exact	Present	Exact
0.995	R T	0.2932 0.7018	0.2932 0.7018	0.4412 0.5488	0.4412 0.5488	0.5988 0.3815	0.5888 0.3815	0.7636 0.1892	0.7636 0.1892
0.9	R T	0.2475 0.6599	0.2475 0.6599	0.3527 0.4747	0.3527 0.4747	0.4372 0.2656	0.4376 0.2656	0.4764 0.0534	0.4763 0.0534
0.8	R T	0.2056 0.6220	0.2056 0.6220	0.2802 0.4162	0.2806 0.4162	0.3280 0.1973	0.3280 0.1973	0.3417 0.0229	0.3417 0.0229
0.7	R T	0.1690 0.5891	0.1690 0.5891	0.2221 0.3712	0.2221 0.3712	0.2506 0.1551	0.2506 0.1551	0.2566 0.0124	0.2565 0.0124
0.6	R T	0.1365 0.5603	0.1365 0.5603	0.1743 0.3356	0.1743 0.3355	0.1919 0.1269	0.1919 0.1269	0.1948 0.0077	0.1947 0.0077
0.5	R T	0.1077 0.5350	0.1077 0.5350	0.1342 0.3067	0.1342 0.3067	0.1451 0.1071	0.1451 0.1071	0.1466 0.0053	0.1465 0.0053
0.3	R T	0.0584 0.4924	0.0584 0.4925	0.0701 0.2631	0.0701 0.2631	0.0741 0.0814	0.0741 0.0814	0.0745 0.0030	0.0745 0.0030
0.1	R T	0.0178 0.4580	0.0178 0.4578	0.0208 0.2319	0.0207 0.2317	0.0216 0.0659	0.0216 0.0658	0.0217 0.0021	0.0217 0.0020

Table I

a_k^N	$N = 3$	$N = 5$	$N = 7$	$N = 9$	$N = 11$	$N = 13$
a_0^N	4.00079	3.97947	3.97582	3.97469	3.97424	3.97402
a_1^N	-3.58873	-3.50197	-3.49091	-3.48776	-3.48652	-3.48593
a_2^N	0.823741	0.732223	0.722387	0.719784	0.718793	0.718333
a_3^N	-0.242179	-0.257857	-0.239203	-0.235060	-0.233589	-0.232929
a_4^N		0.0976552	0.0774224	0.073513	0.0722306	0.0716811
a_5^N		-0.0490953	-0.0575897	-0.0505265	-0.0484691	-0.0476382
a_6^N			0.0318984	0.0245541	0.0226228	0.0218904
a_7^N			-0.0185276	-0.0233762	-0.0200245	-0.0188587
a_8^N				0.0145546	0.0112132	0.0101349
a_9^N				-0.00890106	-0.0118907	-0.0100779
a_{10}^N					0.0078602	0.00611486
a_{11}^N					-0.00493661	-0.00690141
a_{12}^N						0.00472646
a_{13}^N						-0.00301481
R	0.332869	0.328332	0.328015	0.327968	0.327957	0.327954
T	0.195848	0.197152	0.197253	0.197266	0.197268	0.197269
$\|e_N\|_\infty^U$	0.668451	0.283302	0.153469	0.0954606	0.0648849	0.0468881
$\|e_N\|_\infty^L$	0.126754	0.0537207	0.0291013	0.0181016	0.0123037	0.0088911

	$N = 3$	$N = 5$	$N = 7$	$N = 9$
a_0^N	3.95562	3.96844	3.97149	3.97259
a_1^N	-3.46102	-3.47630	-3.48095	-3.48267
a_2^N	0.677921	0.706724	0.713145	0.715377
a_3^N	-0.204947	-0.222733	-0.227756	-0.229558
a_4^N		0.0584488	0.0661107	0.0685639
a_5^N		-0.0359049	-0.0420111	-0.0440163
a_6^N			0.0154309	0.0184535
a_7^N			-0.0122815	-0.0147768
a_8^N				0.00607399
a_9^N				-0.00501322
R	0.3278841	0.3279440	0.3279493	0.3279502
T	0.1972424	0.1972684	0.1972696	0.1972697
$\|\epsilon_N\|_\infty^U$	0.960897	0.44594	0.258666	0.171938
$\|\epsilon_N\|_\infty^L$	0.182209	0.0845608	0.0490492	0.0326035

	$N = 3$	$N = 5$	$N = 7$	$N = 9$
a_0^N	3.97928	3.97458	3.97395	3.97379
a_1^N	-3.48853	-3.48579	-3.48529	-3.48514
a_2^N	0.728443	0.718416	0.717832	0.717713
a_3^N	-0.236790	-0.232418	-0.232099	-0.232027
a_4^N		0.0725636	0.0710985	0.0709772
a_5^N		-0.0476729	-0.0466972	-0.0466069
a_6^N			0.0214933	0.0210767
a_7^N			-0.0179622	-0.0176301
a_8^N				0.00924411
a_9^N				-0.00861693
R	0.329405	0.328196	0.328024	0.327979
T	0.197923	0.197342	0.197286	0.197275
$\|\epsilon\|_\infty^U$	0.591802	0.259727	0.145653	0.092896
$\|\epsilon\|_\infty^L$	0.112222	0.0492504	0.0276193	0.0176153

N	$\|e_N\|_\infty^L$	$\|e_N\|_\infty^n$	$\|e_N\|_\infty^U$
1	0.5373	2.17	2.834
2	0.2513	0.71	1.325
3	0.1268	0.28	0.6685
4	0.08079	0.16	0.4261
5	0.05372	0.0987	0.2833
6	0.0391	0.069	0.2063
7	0.02913	0.051	0.1535
8	0.02280	0.039	0.1202
9	0.01810		0.09546
10	0.01486		0.07835
11	0.01230		0.06489
12	0.01043		0.05498
13	0.008891		0.04689

N	L	ω	$\|\epsilon_N\|_\infty^L$	$\|\epsilon_N\|_\infty^U$
5	0.1	0.8	0.00530256	0.00699654
	1	0.8	0.0317676	0.105959
	10	0.8	0.186451	1.67066
5	0.1	0.2	0.00981479	0.0105148
	1	0.2	0.088425	0.115948
	10	0.2	0.786421	1.17914
7	0.1	0.8	0.00279293	0.00368517
	1	0.8	0.0169058	0.056388
	10	0.8	0.111123	0.995692
7	0.1	0.2	0.00505042	0.00541063
	1	0.2	0.0456048	0.0597997
	10	0.2	0.414734	0.621843
9	0.1	0.8	0.0025158	0.00331951
	1	0.8	0.0104246	0.0347706
	10	0.8	0.0734615	0.658237
9	0.1	0.2	0.00410602	0.00439887
	1	0.2	0.0277234	0.0363525
	10	0.2	0.254447	0.381513

η	$\lvert\delta_7\rvert$	$\lvert\delta_9\rvert$	$\lvert\delta_{11}\rvert$
-1	1.78971	1.11111	0.757576
-0.8	1.02045	0.302367	0.230138
-0.6	0.395868	0.48912	0.301628
-0.4	1.23546	0.721127	0.229382
-0.2	1.0343	0.229182	0.05697277
0	1.79653	0.873624	0.486842
0.2	1.4602	1.12263	0.746734
0.4	2.77448	1.91536	0.613317
0.6	2.73703	1.74141	0.30272
0.8	2.16854	2.46278	0.563874
1.0	4.24428	3.03923	1.74353

CHEBYSHEV SERIES SOLUTION FOR RADIATIVE TRANSPORT IN A MEDIUM WITH A LINEARLY ANISOTROPIC SCATTERING PHASE FUNCTION

BY

T. LaCLAIR
PROJECT ENGINEER PHILLIP LABORATORY USAF
3550 ABERDEEN AVE. SE, BUILDING 30117
KIRTLAND AFB, NM 87117-5776
(505) 846-0477

AND

J. I. FRANKEL*
DEPARTMENT OF MECHANICAL AND AEROSPACE ENGINEERING
UNIVERSITY OF TENNESSEE, KNOXVILLE
KNOXVILLE, TN (USA) 37996-2210
(615) 974-5129

* To whom correspondence should be addressed.

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I. INTRODUCTION

The transfer of heat which is due to thermal radiation is referred to as radiation heat transfer and is a significant mode of heat transfer in many modern engineering applications. Some specific areas in which radiation heat transfer is important include the design and analysis of energy conversion systems such as furnaces, combustors, solar energy conversion devices, and engines, where high temperatures must exist in order to improve thermodynamic efficiency of the processes, and where the other modes of heat transfer may also be significant. Also, for the processing of materials such as glass, crystals, and metals, in which elevated temperatures are used to remove impurities from the material and temperatures must be controlled to enhance crystal formation for improved properties of the material, radiation is an important consideration. The use of materials such as optical components and fibrous and porous insulations, where the distribution of heat determines the operational performance of the material, requires knowledge of the radiative effects which may influence the temperature within these materials. In both nuclear reactor safety, where the temperatures must constantly be monitored and controlled, and diagnostics such as spectroscopy, remote sensing of atmospheric pollutants, and satellite reconnaissance, where radiation fields are measured and analyzed, radiation heat transfer is a very important factor.[1] Numerous other applications may also be found in the literature.

Radiation heat transfer developed primarily due to activities involving astronomy and astrophysics. Early analytical work was performed by Lord Rayleigh in 1871, Schuster in 1905, and Schwartzschild in 1906.[2] Since that time the importance of thermal radiation has increased in engineering owing to increased high temperature applications. Many analyses of thermal radiation in an absorbing, emitting, and scattering medium have appeared in the literature.[3-8] Complete treatment of this problem, however, was next to impossible prior to the development of modern digital computers, and often several simplifying assumptions were made in order to solve the equations. Essentially identical equations also arise in neutron transport, thus additional investigations for solution of the equations have been made in this field as well.[2]

Numerous analytical and numerical approaches have been offered for solving the linearized Boltzmann transport equation. Case's normal mode expansion [9-12] can supply reliable analytical solutions for idealized problems. The facile method, i.e., F_N method[13], has been applied successfully to produce highly accurate results but has only been implemented on relatively simple geometries. Again, this approach is very useful for obtaining benchmark results but doesn't appear to be tractable to difficult geometries. The conventional P_N method [10,11,14] is an expansion based method which has been used for solving numerous pertinent problems in radiative and neutron transport. Again, irregular geometries may cause some difficulties unless some modifications are made.

Galerkin methods [5,6,15-17] have been developed in the context of an integral formulation where Fredholm integral equations of the second kind are produced in terms of the Legendre moments of the intensity. Power series expansions in the optical variable have been used often in this context [6,15,16]. Legendre polynomials of the first kind were used as basis functions by Cengel and Ozisik[17] in developing a Galerkin solution of radiative transport in a slab geometry. Recently, Frankel[5] illustrated that Chebyshev polynomials of the first kind can be used as the basis functions. Theoretical considerations concerning error bounds and convergence rates were reported in that study when considering an isotropically scattering phase function. The use of Fourier transforms[18] and eigenfunction expansions[19] have been implemented to produce accurate results in a slab geometry. Thynell and Ozisik[19] considered several highly anisotropic scattering phase functions and developed accurate solutions based on eigenfunction expansions. It has been observed[14] that an expansion in the optical variable produces fast convergence. However, little theoretical work has appeared quantifying the rate of convergence and the accurate establishment of error bounds especially with regard to anisotropic scattering.

In practical applications, the most direct solution procedure is based on the discrete-ordinates method [10,11]. This method is well-suited to many physical situations. The transport equation is discretized by using a numerical quadrature for approximating the integrals while a finite difference method is typically used for approximating the spatial variable.

Many of the currently proposed solution methods attempt to determine the radiative intensity distribution throughout the medium of interest and subsequently determine the radiative flux and divergence of the radiative flux to determine heat transfer rates and temperature distributions. However, by first manipulating the radiative transfer equation into an equivalent integral form, we can reduce the number of independent variables and obtain the quantities of interest much more readily than from the radiative intensity. Therefore, this method allows us to both simplify the analysis of the radiation problem itself and gives us the principal quantities of engineering interest without extensive further calculation.

The purpose of the present exposition is threefold; (i) to develop a simple yet elegant expansion method using Chebyshev polynomials of the first kind as a set of orthogonal basis functions, (ii) to present a new and informative residual/error analysis which is useful in assessing performance/accuracy, and (iii) to implement and demonstrate the utility of symbolic computation in arriving at the numerical results.

II. INTEGRAL FORMULATION

In this section, we present the integral form of the radiative transfer equation. This integral formulaton reduces the number of independent variables in the unknown functions and leads to much simpler calculation of many quantities of engineering interest, such as the radiative flux and the divergence of the radiative flux. Furthermore, there appears to be greater stability in the calculations of these quantities as obtained from the integral formulation than from the differential form.

In a plane-parallel, linear-anisotropically scattering, absorbing, and emitting medium of optical thickness τ_D subject to transparent boundary conditions, the appropriate [4,8] integral form becomes:

$$G_0(\tau) = F_0(\tau) + \frac{\omega}{2}\Big[a_0 \int_{\tau_0=0}^{\tau_D} E_1(|\tau-\tau_0|)G_0(\tau_0)d\tau_0$$

$$+ a_1 \int_{\tau_0=0}^{\tau} E_2(\tau-\tau_0)G_1(\tau_0)d\tau_0 - a_1 \int_{\tau_0=\tau}^{\tau_D} E_2(\tau_0-\tau)G_1(\tau_0)d\tau_0\Big], \quad \tau \in [0,\tau_D], \tag{1a}$$

and

$$G_1(\tau) = F_1(\tau) + \frac{\omega}{2}\Big[a_0 \int_{\tau_0=0}^{\tau} E_2(\tau-\tau_0)G_0(\tau_0)d\tau_0$$

$$-a_0 \int_{\tau_0=\tau}^{\tau_D} E_2(\tau_0-\tau)G_0(\tau_0)d\tau_0 + a_1 \int_{\tau_0=0}^{\tau_D} E_3(|\tau-\tau_0|)G_1(\tau_0)d\tau_0\Big], \quad \tau \in [0,\tau_D]. \tag{1b}$$

where

$$F_n(\tau) = \int_{\mu=0}^{1} P_n(\mu)[f_1(\mu)\, e^{-\tau/\mu} + (-1)^n f_2(\mu)\, e^{-(\tau_D-\tau)/\mu}]d\mu$$

$$+ \int_{\tau_0=0}^{\tau_D} s(\tau_0) K_{0,n}(\tau-\tau_0) d\tau_0, \quad n = 0,1,... \tag{1c}$$

and

$$K_{m,n}(\tau-\tau_o) = R_{m,n}(\tau-\tau_o) Q_{m,n}(|\tau-\tau_o|), \tag{1d}$$

with

$$R_{m,n}(\tau-\tau_o) = \begin{cases} 1, & \tau > \tau_o, \\ (-1)^{m+n}, & \tau < \tau_o \end{cases} \tag{1e}$$

where the partial kernel functions $Q_{m,n}(|\tau-\tau_0|)$ reduce to

$$\begin{aligned} Q_{0,0}(|\tau-\tau_o|) &= E_1(|\tau-\tau_o|), \\ Q_{0,1}(|\tau-\tau_o|) &= Q_{1,0}(|\tau-\tau_o|) = E_2(|\tau-\tau_o|), \\ Q_{1,1}(|\tau-\tau_o|) &= E_3(|\tau-\tau_o|) \end{aligned} \tag{1f}$$

Here, the unknown functions, $G_m(\tau)$, m=0,1 are the m^{th} Legendre moments of the intensity and are defined [8] as

$$G_m(\tau) \equiv \int_{\mu'=-1}^{1} P_m(\mu') I(\tau, \mu') d\mu', \tag{2}$$

where $P_m(\mu)$ is the m^{th} Legendre poynomial of the first kind and $I(\tau, \mu)$ is the local radiative intensity. The boundary conditions $f_1(\mu)$ and $f_2(\mu)$ correspond to externally incident, azimuthally-symmetric radiation[9], namely

$$I(0,\mu) = f_1(\mu), \ \mu > 0 \ , \tag{3a}$$

$$I(\tau_{D}, -\mu) = f_2(\mu), \ \mu > 0 \ , \tag{3b}$$

whereas the function $s(\tau)$ is given by

$$s(\tau) = (1-\omega)\frac{n^2\sigma T^4(\tau)}{\pi}, \quad \tau \in [0,\tau_D] , \tag{4}$$

where ω is the single-scattering albedo, n is the index of refraction, σ is the Stefan-Boltzmann constant, and T is the local temperature in the medium at optical depth τ. The exponential integral functions appearing in Eq.(1) are defined as [10]

$$E_n(z) = \int_{\mu=0}^{1} e^{-z/\mu} \mu^{n-2} d\mu, \quad z \geq 0, \; n=1,2,... \tag{5}$$

The constants a_0, a_1 are associated with the linearly anisotropic scattering phase function, namely[9]

$$P(\mu,\mu') = \sum_{m=0}^{1} a_m P_m(\mu) P_m(\mu') = a_0 + a_1 \mu \mu', \tag{6}$$

where $P(\mu,\mu')$ is the scattering phase function under our imposed constraints.

The phase function is normalized by requiring that $a_0=1$. When $a_1=-1$ the scattering is said to be "highly backward", while when $a_1=1$ the scattering is said to be "highly forward". The case when $a_1=0$ represents isotropic scattering. When $|a_1|\geq 1$ in linearly anisotropic scattering, the phase function is actually an approximation of a higher-order phase function which describes highly-forward or highly-backward scattering. This approximation yields accurate results in many important applications.[3] The attributes of the integral formulation are well-documented[4,5,6,8]. Equations (1a, 1b) represent a set of linear Fredholm integral equations of the second kind. The kernel, $E_1(|\tau_0-\tau|)$ shown in Eq. (1a) contains a logarithmic singularity as $\tau_0 \rightarrow \tau$.

At this point, it is convenient to transform the physical domain from $\tau \in [0,\tau_D]$ to $x \in [-1,1]$ in order to introduce our expansion of the unknown functions in terms of a Chebyshev series. Transforming the domain via a linear transformation, we let

$$x \equiv \frac{2\tau - \tau_D}{\tau_D} \quad , \tag{7a}$$

$$x_o \equiv \frac{2\tau_o - \tau_D}{\tau_D} \quad , \tag{7b}$$

and define

$$\alpha \equiv \frac{\tau_D}{2},$$

noting that $x = \dfrac{\tau}{\alpha} - 1$. We therefore define

$$G_n^*(x) \equiv G_n(\alpha x + \frac{\tau_D}{2}) = G_n(\alpha(x+1)), \tag{8a}$$

and

$$F_n^*(x) \equiv F_n(\alpha(x+1)) \text{ , } n = 0,1. \tag{8b}$$

The transformed equations governing the zeroth and first Legendre moment of the intensity become

$$\begin{aligned} G_0^*(x) = F_0^*(x) + \frac{\alpha\,\omega}{2} [a_0 & \int_{x_o=-1}^{1} E_1(\alpha|x - x_o|) G_0^*(x_o) dx_o \\ & + a_1 \int_{x_o=-1}^{x} E_2(\alpha(x - x_o)) G_1^*(x_o) dx_o \\ & - a_1 \int_{x_o=x}^{1} E_2(\alpha(x_o - x)) G_1^*(x_o) dx_o], \; x \in [-1,1], \end{aligned} \tag{9a}$$

and

$$G_1^*(x) = F_1^*(x) + \frac{\alpha\omega}{2}\Big[a_0 \int_{x_0=-1}^{x} E_2(\alpha(x-x_0))G_0^*(x_0)dx_0$$

$$- a_0 \int_{x_0=x}^{1} E_2(\alpha(x_0-x))G_0^*(x_0)dx_0 \tag{9b}$$

$$+ a_1 \int_{x_0=-1}^{1} E_3(\alpha|x-x_0|)G_1^*(x_0)dx_0\Big],\ x \in [-1,1]\ .$$

We are now in a position to formally expand $G_0^*(x)$ and $G_1^*(x)$ in terms of Chebyshev series representations. Thus, we write

$$G_0^*(x) = \sum_{i=0}^{\infty} b_i T_i(x),\ x \in [-1,1], \tag{10a}$$

and

$$G_1^*(x) = \sum_{i=0}^{\infty} c_i T_i(x),\ x \in [-1,1], \tag{10b}$$

where $T_n(x)$ is the n^{th} Chebyshev polynomial of the first kind, given by[20-22]

$$T_n(x) \equiv \cos(n \cos^{-1} x),\ n=0,1,2,... \tag{11}$$

To simplify notation we express Eqs. (9a) and (9b) in operator form,

$$G_0^* = F_0^* + a_0\kappa_0 G_0^* + a_1\kappa_1 G_1^* \tag{12a}$$

and

$$G_1^* = F_1^* + a_0\kappa_1 G_0^* + a_1\kappa_2 G_1^* \tag{12b}$$

where we interpret our symbolic notation as

$$\kappa_j \psi \equiv \int_{y=-1}^{1} k_j(x,y)\psi(y)dy, \ \ x \in [-1,1], \tag{13}$$

where $k_j(x,y)$ are the appropriate kernels and $\psi(y)$ is any arbitrary function.

We seek an approximate solution to $G_m^*(x)$, $m=0,1$, by truncating the infinite series representations for $G_m^*(x)$, $m=0,1$, at a certain order, say N, leaving

$$G_0^N(x) = \sum_{i=0}^{N} b_i^N \, T_i(x), \ \ x \in [-1,1], \tag{14a}$$

and

$$G_1^N(x) = \sum_{i=0}^{N} c_i^N \, T_i(x), \ \ x \in [-1,1], \tag{14b}$$

where b_i^N and c_i^N are approximations to b_i and c_i, respectively. Thus, we may express Eqs. (12a) and (12b) as

$$G_0^N = F_0^* + a_0\kappa_0 G_0^N + a_1\kappa_1 G_1^N - R_0^N, \tag{15a}$$

and

$$G_1^N = F_1^* + a_0\kappa_1 G_0^N + a_1\kappa_2 G_1^N - R_1^N, \tag{15b}$$

respectively, where we have introduced the residual functions R_0^N and R_1^N to account for the error resulting from the approximation.

Substituting Eqs. (14a) and (14b) into Eqs.(15a) and (15b) and, formally interchanging orders at summation and integration produces

$$R_0^N(x) + \sum_{i=0}^{N} b_i^N T_i(x) =$$

$$F_0^*(x) + \frac{\alpha\omega}{2}\Big[a_0 \sum_{i=0}^{N} b_i^N \int_{x_o=-1}^{1} E_1(\alpha|x-x_o|)\, T_i(x_o)dx_o$$

$$+ a_1 \sum_{i=0}^{N} c_i^N \Big\{ \int_{x_o=-1}^{x} E_2(\alpha(x-x_o))\, T_i(x_o)dx_o \tag{16a}$$

$$- \int_{x_o=x}^{1} E_2(\alpha(x_o-x))\, T_i(x_o)dx_o\Big\}\Big], \quad x \in [-1,1],$$

and

$$R_1^N(x) + \sum_{i=0}^{N} c_i^N T_i(x) =$$

$$F_1^*(x) + \frac{\alpha\omega}{2}\Big[a_0 \sum_{i=0}^{N} b_i^N \Big\{ \int_{x_o=-1}^{x} E_2(\alpha(x-x_o))\, T_i(x_o)dx_o$$

$$- \int_{x_o=x}^{1} E_2(\alpha(x_o-x))\, T_i(x_o)dx_o\Big\} \tag{16b}$$

$$+ a_1 \sum_{i=0}^{N} c_i^N \int_{x_o=-1}^{1} E_3(\alpha|x-x_o|)\, T_i(x_o)dx_o\Big], \quad x \in [-1,1].$$

We now notice that, for specified boundary conditions such that $F_0^*(x)$ and $F_1^*(x)$ are known functions given by Eq. (1c), we have reduced the problem to a point where all integrals involved can be determined analytically. These expressions are developed in the next section of analysis. Therefore, we are in an excellent position to perform a numerical analysis to determine the unknown expansion coefficients b_i^N and c_i^N, i=0,1,...,N, by placing some type of restriction on the functions $R_0^N(x)$ and $R_1^N(x)$.

III. COLLOCATION AND RITZ-GALERKIN METHODS

In this section, we present a systematic approach for determining the unknown expansion coefficients shown in Eqs.(16a) and (16b). Projection methods encompass techniques such as collocation, Galerkin methods, and least-squares procedures and have been the topic of much research[23-27]. The proposed numerical methods used here are presented in a weighted-residual framework. An exact solution is given if the residual functions are identically zero for $x \in [-1,1]$. This is not possible for our finite Chebyshev series approximation unless the actual solution is a linear combination of the Chebyshev polynomials $\{T_n(x)\}$, $n = 0,1,\ldots,N$. Therefore, we attempt to minimize the residuals $R_0^N(x)$ and $R_1^N(x)$ in some manner. A particular expansion method is defined by any restrictions imposed on the residual functions shown in Eqs. (16a,b). We wish to determine the unknown expansion coefficients $\{b_n^N\}$ and $\{c_n^N\}$, $n = 0,1,\ldots,N$, in such a manner that some measure of the residual functions is small. A systematic way of expressing this is to require that the orthogonality condition[5]

$$\left\langle R_j^N(x), \varphi_k(x) \right\rangle_{w_i} = 0, \quad j = 0,1 \text{ and } k = 0,1,\ldots,N, \tag{17}$$

be enforced for $i=0,1,\ldots,N$. For the point-collocation method $w_k(x) = \delta(x-x_k)$ and $\varphi_k(x)=1$, where $\delta(x)$ is the Dirac delta function, while for the Ritz-Galerkin method $w_k(x)=1$ and $\varphi_k(x)=T_k(x)$. Note that the inner product of two real functions $g_1(t)$ and $g_2(t)$ is given by

$$\left\langle g_1, g_2 \right\rangle_{w_k} \equiv \int_{t=-1}^{1} w_k(t) g_1(t) g_2(t) dt, \tag{18}$$

where $w_k(t)$ is a non-negative, real, integrable weight function.

Collocation Method

Imposing the orthogonality concept displayed in Eq(17), where $w_k(x)=\delta(x-x_k)$ and $\varphi_k(x)=1$, on Eqs.(16a) and (16b) produces

$$\sum_{i=0}^{N} b_i^N T_i(x_k) = F_0^*(x_k) + \frac{\alpha\omega}{2}\left[a_0 \sum_{i=0}^{N} b_i^N I_{1,i}(x_k) + a_1 \sum_{i=0}^{N} c_i^N I_{2,i}(x_k)\right],\ k = 0,1,\ldots,N, \tag{19a}$$

and

$$\sum_{i=0}^{N} c_i^N T_i(x_k) = F_1^*(x_k) + \frac{\alpha\,\omega}{2}\left[a_0 \sum_{i=0}^{N} b_i^N I_{2,i}(x_k) + a_1 \sum_{i=0}^{N} c_i^N I_{3,i}(x_k)\right],\ k = 0,1,\ldots,N, \tag{19b}$$

where the integral functions $I_{n,i}(x)$, $n=1,2,3$, $i=0,1,2,\ldots,N$, are defined by

$$I_{1,i}(x) \equiv \int_{x_o=-1}^{1} E_1(\alpha|x-x_o|)\, T_i(x_o)dx_o \tag{20a}$$

$$I_{2,i}(x) \equiv \int_{x_o=-1}^{x} E_2(\alpha(x-x_o))\, T_i(x_o)dx_o - \int_{x_o=x}^{1} E_2(\alpha(x_o-x))\, T_i(x_o)dx_o, \tag{20b}$$

$$I_{3,i}(x) \equiv \int_{x_o=-1}^{1} E_3(\alpha|x-x_o|)\, T_i(x_o)dx_o. \tag{20c}$$

Here, x_j represents the collocation points in the finite set $\{x_k\}_{k=1}^{N}$ .Through a lengthy but straightforward set of manipulations, one can show

$$I_{n,i}(x) = 2(-1)^{r_n} \sum_{j=0}^{[\frac{i}{2}-r_n]} \left(\frac{1}{\alpha}\right)^{2j+r_n+1} \frac{1}{2j+r_n+n} T_i^{(2j+r_n)}(x) - \sum_{j=0}^{i} \left(\frac{1}{\alpha}\right)^{j+1} T_i^{(j)}(1)\left[(-1)^i E_{j+n+1}(\alpha(1+x)) + (-1)^{r_n} E_{j+n+1}(\alpha(1-x))\right], \tag{21a}$$

where we have defined

$$r_n \equiv \begin{cases} 1, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases} \tag{21b}$$

Ritz-Galerkin Method

We now proceed with the determination of the expansion coefficients of the Legendre moments of intensity using a Ritz-Galerkin method. As indicated previously, for the Ritz-Galerkin method we use $w_k(x)=1$ and $\varphi_k(x)=T_k(x)$ in the restriction provided by Eq. (17). This provides a uniform weighting of the residual function over the entire interval as opposed to the discrete weighting associated with the collocation method.

From our analysis in the previous section, we can rewrite Eqs. (16a,b) as

$$R_0^N(x) + \sum_{i=0}^{N} b_i^N T_i(x) = F_0^*(x) + \frac{\alpha\omega}{2}\left[a_0 \sum_{i=0}^{N} b_i^N I_{1,i}(x) + a_1 \sum_{i=0}^{N} c_i^N I_{2,i}(x)\right], \tag{22a}$$

and

$$R_1^N(x) + \sum_{i=0}^{N} c_i^N T_i(x) = F_1^*(x) + \frac{\alpha\omega}{2}\left[a_0 \sum_{i=0}^{N} b_i^N I_{2,i}(x) + a_1 \sum_{i=0}^{N} c_i^N I_{3,i}(x)\right], \tag{22b}$$

where $I_{n,i}(x)$, $n=1,2,3$, $i=0,1,\ldots,N$, are given by Eqs. (20a,b). After introducing our chosen functions for $w_k(x)$ and $\varphi_k(x)$ into the orthogonality condition shown in Eq. (17), we find

$$\begin{aligned} \sum_{i=0}^{N} b_i^N A_{i,j} = & \int_{x=-1}^{1} F_0^*(x) T_j(x)\,dx \\ & + \frac{\alpha\omega}{2}\left[a_0 \sum_{i=0}^{N} b_i^N B_{1,i,j} + a_1 \sum_{i=0}^{N} c_i^N B_{2,i,j}\right], \end{aligned} \tag{23a}$$

and

$$\sum_{i=0}^{N} c_i^N A_{i,j} = \int_{x=-1}^{1} F_1^*(x) T_j(x) dx + \frac{\alpha\,\omega}{2}[a_0 \sum_{i=0}^{N} b_i^N B_{2,i,j} + a_1 \sum_{i=0}^{N} c_i^N B_{3,i,j}], \quad j = 0,1,\ldots,N. \tag{23b}$$

where

$$A_{i,j} \equiv \int_{x=-1}^{1} T_i(x) T_j(x) dx, \quad i,j = 0,1,2,\ldots N, \tag{24}$$

which, with the aid of Eq. (11), can be expressed as[5,26]

$$A_{i,j} = \begin{cases} \dfrac{1}{1-(i+j)^2} + \dfrac{1}{1-(i-j)^2}, & i+j \text{ even}, \\ 0, & i+j \text{ odd}. \end{cases} \tag{25}$$

and where

$$B_{n,i,j} \equiv \int_{x=-1}^{1} I_{n,i}(x) T_j(x) dx, \quad i,j = 0,1,\ldots,N, \quad n = 1,2,3. \tag{26}$$

After a straightforward set of manipulations, we arrive at

$$B_{n,i,j} = 2(-1)^{r_n} \sum_{k=0}^{[\frac{i}{2}-r_n]} \left(\frac{1}{\alpha}\right)^{2k+r_n+1} \frac{1}{2k+r_n+n} \sum_{m=0}^{i-2k-r_n} d_{m,i,2k+r_n} A_{m,j} - [(-1)^{i+j} + (-1)^{r_n}] \sum_{k=0}^{i} \left(\frac{1}{\alpha}\right)^{k+1} T_i^{(k)}(1)\, J_{k+n+1,j}(1)\,, \tag{27a}$$

where

$$J_{n,j}(1) = \int_{x=-1}^{1} E_n(\alpha(1-x))T_j(x)dx \ , \tag{27b}$$

and where we expressed $T_i^{(j)}(x)$ as a finite Chebyshev series, namely

$$T_i^{(j)}(x) = \sum_{m=0}^{i-j} d_{m,i,j} T_m(x). \tag{28}$$

We note that $T_i^{(j)}(x)$ is simply a polynomial of order (i-j). With the use of symbolic computation, the calculation of these coefficients is trivial and represents a one-time computation.

In the next section, error analysis is performed to quantify the accuracy of the proposed solutions.

IV ERROR ANALYSIS

After determining the expansion coefficients by the chosen method, the residuals are given by Eqs. (16.a,b). The size of the residuals provides us with an indication of the accuracy of the approximation, but we wish to calculate the errors in the functions $G_0^N(x)$ and $G_1^N(x)$ themselves.

Using our operator notation, Eqs. (15a,b), are expressible as,

$$R_0^N = F_0^* + a_0\kappa_0 G_0^N + a_1\kappa_1 G_1^N - G_0^N \ , \tag{29a}$$

and

$$R_1^N = F_1^* + a_0\kappa_1 G_0^N + a_1\kappa_2 G_1^N - G_1^N \ , \tag{29b}$$

respectively. The exact solution produces no residual, that is,

$$0 = F_0^* + a_0\kappa_0 G_0^* + a_1\kappa_1 G_1^* - G_0^* \ , \tag{30a}$$

$$0 = F_1^* + a_0\kappa_1 G_0^* + a_1\kappa_2 G_1^* - G_1^* \ . \tag{30b}$$

Let us define the error as [5,27]

$$\varepsilon_0^N \equiv G_0^* - G_0^N \ , \tag{31a}$$

$$\varepsilon_1^N \equiv G_1^* - G_1^N . \tag{31b}$$

By subtracting the corresponding equations in Eqs. (29 a,b) from those in Eqs. (30 a,b), and using the definitions of the errors, we obtain a relationship between the errors and the residuals, namely

$$\varepsilon_0^N = R_0^N + a_0\kappa_0\varepsilon_0^N + a_1\kappa_1\varepsilon_1^N \tag{32a}$$

$$\varepsilon_1^N = R_1^N + a_0\kappa_1\varepsilon_0^N + a_1\kappa_2\varepsilon_1^N . \tag{32b}$$

Unfortunately ε_0^N and ε_1^N are as inaccessible as the exact solutions G_0 and G_1, but the sizes of the errors may be measured by means of some functional norm.

We now introduce the concept of the functional norm, in particular the L_∞ norm. We may define this norm as

$$\|\theta\|_\infty \equiv \sup_{x\in[-1,1]} |\theta(x)| \tag{33a}$$

for an arbitrary function $\theta(x)$. The infinity norm corresponds to the maximum absolute value that a function takes on in its domain of existence. The L_∞ norm of the errors expressed in Eqs. (32a,b), therefore, is given as

$$\left\|\varepsilon_j^N\right\|_\infty \equiv \sup_{x\in[-1,1]} |G_j^*(x) - G_j^N(x)|, \quad j = 0,1. \tag{33b}$$

This definition of the error is graphically depicted in Fig. 1, where the maximum absolute error corresponds to the definition displayed in Eq. (33b). The corresponding infinity norm of the integral operator is expressible as[5,26]

$$\|\kappa\|_\infty \equiv \sup_{x\in[-1,1]} \int_{x_0=-1}^{1} |k(x,x_0)|dx_0\,, \tag{33c}$$

where $k(x,x_0)$ is the kernel corresponding to the integral operator κ. Since we will consider only the infinity norm, we shall drop the subscripted "∞" in order to simplify the notation.

We shall use the following elementary results of functional analysis:[28]

$$\begin{aligned} &\|A+B\| \le \|A\|+\|B\|, && \|AB\| \le \|A\|x\|B\|, \\ &\big|\,\|A\|-\|B\|\,\big| \le \|A-B\|, && \|aA\| = |a|\,\|A\|, \end{aligned} \tag{34a-d}$$

where A and B are any arbitrary functions and/or operators and where "a" is a scalar constant. Also, note from our definitions of the norm that $\|A\| \ge 0$. Then we begin our error bound calculations by considering Eqs. (32a,b). Subtracting one from the other, we arrive at

$$\varepsilon_0^N - \varepsilon_1^N = R_0^N - R_1^N + a_0(\kappa_0 - \kappa_1)\varepsilon_0^N + a_1(\kappa_1 - \kappa_2)\varepsilon_1^N\,. \tag{35}$$

We now apply the definition of the infinity norm and, with the aid of Eqs. (34 a-d), we obtain

$$\begin{aligned} \Big|\,\|\varepsilon_0^N\|-\|\varepsilon_1^N\|\,\Big| &\le \|\varepsilon_0^N-\varepsilon_1^N\| \\ &\le \|R_0^N-R_1^N\| + |a_0|\,\|\kappa_0-\kappa_1\|\,\|\varepsilon_0^N\| + |a_1|\,\|\kappa_1-\kappa_2\|\,\|\varepsilon_1^N\|. \end{aligned} \tag{36}$$

The terms $\|\kappa_0-\kappa_1\|$ and $\|\kappa_1-\kappa_2\|$ represent the norms of the differences of the corresponding kernels. Note, however, that these are not equivalent to $\|\kappa_0\|-\|\kappa_1\|$ and $\|\kappa_1\|-\|\kappa_2\|$, respectively. In Eq. (36), we first consider the case that $\|\varepsilon_0^N\| \ge \|\varepsilon_1^N\|$. Then we may write

$$0 \le \|\varepsilon_0^N\|-\|\varepsilon_1^N\| \le \|R_0^N-R_1^N\| + |a_0|\,\|\kappa_0-\kappa_1\|\,\|\varepsilon_0^N\| + |a_1|\,\|\kappa_1-\kappa_2\|\,\|\varepsilon_1^N\| \tag{37a}$$

or

$$\left\|\varepsilon_0^N\right\|\left[1-\left|a_0\right|\left\|\kappa_0-\kappa_1\right\|\right]+\left\|\varepsilon_1^N\right\|\left[-1-\left|a_1\right|\left\|\kappa_1-\kappa_2\right\|\right]\leq\left\|R_0^N-R_1^N\right\| \tag{37b}$$

This then yields the following inequality:

$$\left\|\varepsilon_1^N\right\| \geq \frac{r_1-K_{01}\left\|\varepsilon_0^N\right\|}{-K_{11}}, \tag{38a}$$

where we define

$$K_{01}\equiv\left[1-\left|a_0\right|\left\|\kappa_0-\kappa_1\right\|\right], \qquad K_{11}\equiv 1+\left|a_1\right|\left\|\kappa_1-\kappa_2\right\|,$$
$$r_1\equiv\left\|R_0^N-R_1^N\right\|. \tag{38b-d}$$

Next, we consider the case that $\left\|\varepsilon_0^N\right\| < \left\|\varepsilon_1^N\right\|$. Then Eq. (36) gives us

$$0<\left\|\varepsilon_1^N\right\|-\left\|\varepsilon_0^N\right\| \leq \left\|R_0^N-R_1^N\right\|+\left|a_0\right|\left\|\kappa_0-\kappa_1\right\|\left\|\varepsilon_0^N\right\|+\left|a_1\right|\left\|\kappa_1-\kappa_2\right\|\left\|\varepsilon_1^N\right\|, \tag{39a}$$

or

$$\left\|\varepsilon_0^N\right\|\left[-1-\left|a_0\right|\left\|\kappa_0-\kappa_1\right\|\right]+\left\|\varepsilon_1^N\right\|\left[1-\left|a_1\right|\left\|\kappa_1-\kappa_2\right\|\right]\leq\left\|R_0^N-R_1^N\right\|. \tag{39b}$$

We thus attain

$$\left\|\varepsilon_1^N\right\| \leq \frac{r_2-K_{02}\left\|\varepsilon_0^N\right\|}{K_{12}}, \tag{40a}$$

provided K_{12} is positive,where

$$K_{02}\equiv\left[-1-\left|a_0\right|\left\|\kappa_0-\kappa_1\right\|\right], \qquad K_{12}\equiv\left[1-\left|a_1\right|\left\|\kappa_1-\kappa_2\right\|\right],$$
$$r_2\equiv\left\|R_0^N-R_1^N\right\|. \tag{40b-d}$$

Next, we rearrange Eqs. (30 a.b) in the form

$$R_0^N = \varepsilon_0^N - a_0\kappa_0\varepsilon_0^N - a_1\kappa_1\varepsilon_1^N, \tag{41a}$$

$$R_1^N = \varepsilon_1^N - a_0\kappa_1\varepsilon_0^N - a_1\kappa_2\varepsilon_1^N. \tag{41b}$$

Applying norms and using the relations given by Eqs. (34 a,d), we obtain

$$\left\|R_0^N\right\| \le \left\|\varepsilon_0^N\right\| + |a_0|\,\|\kappa_0\|\,\left\|\varepsilon_0^N\right\| + |a_1|\,\|\kappa_1\|\,\left\|\varepsilon_1^N\right\|, \tag{42a}$$

$$\left\|R_1^N\right\| \le \left\|\varepsilon_1^N\right\| + |a_0|\,\|\kappa_1\|\,\left\|\varepsilon_0^N\right\| + |a_1|\,\|\kappa_2\|\,\left\|\varepsilon_1^N\right\|. \tag{42b}$$

Summing the two inequalities and simplifying, we arrive at

$$\left\|R_0^N\right\| + \left\|R_1^N\right\| \le \left\|\varepsilon_0^N\right\| \left[1+|a_0|\left(\|\kappa_0\|+\|\kappa_1\|\right)\right] + \left\|\varepsilon_1^N\right\| \left[1+|a_1|\left(\|\kappa_1\|+\|\kappa_2\|\right)\right] \tag{43}$$

We may therefore write

$$\left\|\varepsilon_1^N\right\| \ge \frac{r_3 - K_{03}\left\|\varepsilon_0^N\right\|}{K_{13}}, \tag{44a}$$

where

$$K_{03} \equiv \left[1+|a_0|\left(\|\kappa_0\|+\|\kappa_1\|\right)\right], \qquad K_{13} \equiv \left[1+|a_1|\left(\|\kappa_1\|+\|\kappa_2\|\right)\right],$$
$$r_3 \equiv \left\|R_0^N\right\| + \left\|R_1^N\right\|. \tag{44b-d}$$

Once again we begin with Eqs. (5.1.14a,b). Applying the norm to these equations we obtain

$$\left\|\varepsilon_0^N\right\| \le \left\|R_0^N\right\| + |a_0|\,\|\kappa_0\|\,\left\|\varepsilon_0^N\right\| + |a_1|\,\|\kappa_1\|\,\left\|\varepsilon_1^N\right\|, \tag{45a}$$

and

$$\left\|\varepsilon_1^N\right\| \le \left\|R_1^N\right\| + |a_0|\,\|\kappa_1\|\,\left\|\varepsilon_0^N\right\| + |a_1|\,\|\kappa_2\|\,\left\|\varepsilon_1^N\right\|. \tag{45b}$$

Adding the two inequalities yields

$$\begin{aligned}\left\|\varepsilon_0^N\right\| + \left\|\varepsilon_1^N\right\| \le \left\|R_0^N\right\| + \left\|R_1^N\right\| + |a_0|\left(\|\kappa_0\|+\|\kappa_1\|\right)\left\|\varepsilon_0^N\right\| \\ + |a_1|\left(\|\kappa_1\|+\|\kappa_2\|\right)\left\|\varepsilon_1^N\right\|,\end{aligned} \tag{46a}$$

which we may rewrite as

$$\left\|\varepsilon_0^N\right\|\left[1-|a_0|\left(\|\kappa_0\|+\|\kappa_1\|\right)\right] + \left\|\varepsilon_1^N\right\|\left[1-|a_1|\left(\|\kappa_1\|+\|\kappa_2\|\right)\right] \le \left\|R_0^N\right\| + \left\|R_1^N\right\|. \tag{46b}$$

Thus we obtain

$$\left\|\varepsilon_1^N\right\| \le \frac{r_4 - K_{04}\left\|\varepsilon_0^N\right\|}{K_{14}}, \tag{47a}$$

where

$$\begin{aligned} K_{04} &\equiv \left[1-|a_0|\left(\|\kappa_0\|+\|\kappa_1\|\right)\right], \qquad K_{14} \equiv \left[1-|a_1|\left(\|\kappa_1\|+\|\kappa_2\|\right)\right], \\ r_4 &\equiv \left\|R_0^N\right\| + \left\|R_1^N\right\|. \end{aligned} \tag{47b-d}$$

The relations given by Eqs. (38a-d), (40a-d), (44c-d), and (47a-d) therefore provide us with a region in the first quadrant of the plane with abscissa $\left\|\varepsilon_0^N\right\|$ and ordinate $\left\|\varepsilon_1^N\right\|$ in which the errors must lie. This gives us a rigorous error bound for the cases in which the denominators of Eqs. (40a) and (47a) are positive. An exemplary plot of this region will be provided in the results section.

V. RESULTS

Numerical results obtained using the Collocation and Ritz-Galerkin methods applied to the finite Chebyshev series approximations are compared to each other and to previously reported results obtained using other methods. Graphical representations of the Legendre moments of intensity and of the residual functions and errors are presented for the Chebyshev series solutions. Finally, convergence trends of the two methods are presented empirically for a specific example. All calculations and graphics presented here were performed with the symbolic computation software package Mathematica™, version 2.0 for Windows™, and executed on a PC with 8 MBytes of memory.

We consider the case of uniform radiation of unit intensity incident on the boudary at $\tau=0$ and no radiation incident at the boundary $\tau=\tau_D$. This gives us the boundary conditions $I(0,\mu)=1$ and $I(\tau_D,-\mu)=0$, $\mu>0$, corresponding to Eqs. (3a,b). We further assume that no internal sources are present, i.e., $s(\tau)=0$. Therefore, the forcing functions of the integral form of the RTE reduce to $F_0(\tau)=E_2(\tau)$ and $F_1(\tau)=E_3(\tau)$, from Eq. (1c). We consider this case for comparison purposes since it is a classical problem and numerous results corresponding to this problem exist in the literature. For the collocation points, we use a closed, Gauss-Chebyshev rule,[5,,27] i.e. $x_k=\cos(\frac{\pi k}{N})$, k=0,1,...,N which ensures that the residuals vanish at $x=\pm 1$.

The principle quantity of interest in engineering is the radiation heat flux, which can be expressed as $Q(x)=2G_1(x)$. Table 1 compares the radiation heat flux at the boundaries obtained from the finite Chebyshev series approximation for both the collocation and Ritz-Galerkin methods to results obtained by other methods. A Galerkin method similar to that presented here was utilized by Krim[6], and is used as the standard for comparison in this study. Three different phase functions are considered and solutions for different values of the single-scattering albedo ω are given for an optical thickness of $\tau_D=1$. The Chebyshev collocation method is indicated by TC while the Chebyshev Ritz-Galerkin is denoted by TG. The Galerkin approach presented by Krim[6] is denoted by G. Results from the F_N[6] and P_N[14] methods are also included, as are those of the double-spherical harmonics (DP) method.[7] The F_N and P_N methods, in the past, have served as benchmarks for comparison and as such, are included for cases where results were available. The numerical subscripts represent the order of the approximation used in each case. The F_N method has generally been considered to produce the most accurate results of any other method.

It is interesting to note that the collocation method results tend to agree more closely to the weighted Galerkin results G_6 than the higher order Chebyshev Ritz-Galerkin results. This fact makes the TC method quite favorable over the other two due to the computational simplicity of this approach. The TC_9 results seem also to confirm the accuracy of these figures. In light of the similarity between these three approximations, however, the accuracy of the F_9 and P_9 appear to be in error.*

*This was apparently shown by Siewert. This was related to authors by a reviewer.

We see in this table that as the single-scattering albedo ω is increased, the radiative flux decreases at the left boundary while it increases at the right boundary, due to the increased role of the (forward) scattering. The radiant energy is thus scattered forward rather than being absorbed within the medium. Similarly, as a_1 increases, the scattering becomes more highly forward and the back scattering is reduced. This results in less of the incident radiation being scattered back out and more energy passing into and through the medium. Therefore, the radiative flux is increased at both boundaries as compared to the flux for a smaller value of a_1 with fixed ω .

In Table 2, we illustrate the effect of the optical thickness on the radiation heat flux and compare the results to different methods. The results presented correspond to a value of a_1=2.602844 with ω=0.8. From the table, we see that as the optical thickness is increased, the radiative flux decreases at the boundaries, due to absorption and scattering within the medium.

Table 3 compares the zeroth Legendre moment of intensity obtained from the Chebyshev collocation and Ritz-Galerkin methods to results reported by Krim.[6] Again, it is seen that the Chebyshev collocation method gives results which are in excellent agreement with the higher order method G_6 over a large range of values for a_1, τ_D, and ω.

We present in Figure 2 plots of the zeroth Legendre moment of intensity along with the resulting residuals and errors for the case where τ_D=1.0, ω=0.8, N=6, and a_1=0.643833 obtained by the collocation method, and in Figure 3 we present a similar plot for the Ritz-Galerkin method. The error was calculated numerically using a product integration[27] trapezoid rule with 161 equally spaced points. Figures 4 and 5 include plots for the first Legendre moment of intensity and the resulting residuals and errors for the case where τ_D=1.0, ω=0.8, N=6, and a_1=0.643833 for the collocation and Ritz-Galerkin methods, respectively.

The region obtained from the error bounds represented by Eqs. (38a-d), (40a-d), (44a-d), and (47a-d) is plotted for the collocation method in Figure 6 and for the Ritz-Galerkin method in Figure 7, both for the case corresponding to N=6, τ_D=1.0, w=0.8, and a_1=0.643833. For the collocation method, the location of the numerically calculated error norms can be seen to lie roughly in the center of the corresponding region, as indicated in Figure 6, while for the Ritz-Galerkin method, the location of these norms is in the low center portion of Figure 7. From

Figures 3(b) through 6(b) it is clear that the L_∞ norm of the residuals is greater for the Ritz-Galerkin method than for the collocation method. As a result, the L_∞ norm of the errors is expected to be correspondingly greater for the Ritz-Galerkin method. This is indeed the case for $\left\|\varepsilon_0^N\right\|$, based on the numerical calculation of the errors. However, $\left\|\varepsilon_1^N\right\|$ is actually smaller in the case of the Ritz-Galerkin solution. This is clearly due to the smaller residual over the interior portion of the domain in the Ritz-Galerkin method. An analysis of the errors using some other norm, for example the L_2 norm, may more accurately illustrate this effect.

Figures 8 and 9 show the region of the error bound for the collocation and Ritz-Galerkin methods, respectively, for three different values of N in order to show the rates of convergence for the two methods. It is interesting that the polygons which represent the error bounds appear geometrically similar for different values of N, although based on the actual bound inequalities, it was found that the polygons are not truly similar. The convergence rates for the two methods are essentially identical, based on the figures, although the collocation method does have slightly tighter bounds for this case.

From the error plots in Figures (2) through (5), we see that the average error resulting from the Ritz-Galerkin method may be smaller than that arising from the collocation method. The maximum residual occurs at the boundary for the Ritz-Galerkin method, and as a result the maximum error may be expected to occur there as well. In much of the literature, accuracy of results has been measured by comparing the Legendre moments of intensity (or radiative fluxes) which occur at the boundaries. Since the largest errors may occur at the boundaries, this may be a rather poor measurement. The collocation method, on the other hand, produces zero residual at the boundaries if the boundary points are used as collocation points, and as a result the error there is quite small. Therefore, without performing a detailed error analysis, the accuracy of any given method may be improperly interpreted if boundary results are the only means of comparison.

The suitability of one method over another may depend on the purpose of the radiation analysis. For example, if the solution to a radiation problem is required to obtain the boundary condition for a conduction/convection problem, then accurate boundary radiation heat fluxes are required and the collocation method may produce the best results. On the other hand, if one desires the temperature distribution within a medium in which radiation is significant, then accurate boundary results alone will not suffice.

In Table 4, we compare the expansion coefficients which are obtained using the collocation method for the case where τ_D=1.0, ω=0.8, and a_1=0.643833, for different values of N, in order to show convergence trends for this method. In Table 5 the same comparison is made for the Ritz-Galerkin method. It is interesting to note that the leading coefficients-- b_0^N and c_0^N --are within less than one percent of their corresponding values for N=4 and N=10 in the Ritz-Galerkin Method, while the convergence is much slower for the collocation method.

From these results, it is apparent that the collocation method provides excellent results at the boundaries of the domain. However, within the interior the Ritz-Galerkin method may provide somewhat more reliable results. Both methods provide results which are within one percent of the values obtained by other methods and therefore should be acceptable for most engineering applications. Since the collocation method requires much less computational effort, this may be preferable over the other methods.

VI CONCLUSIONS

The problem of one-dimensional radiation in a medium which absorbs, emits, and scatters radiation subjected to uniform incident radiation at one of the boundaries and with no internal emission source was solved for the case of a linearly anisotropic scattering phase function using the integral form of the transport equation. After transforming the domain via a linear transformation, the unknown functions were approximated by a finite Chebyshev series solution. Two separate orthogonality conditions were applied to the residual functions resulting from the approximation to obtain the two separate methods, collocation and Ritz-Galerkin, for determining the unknown expansion coefficients.

Following determination of the expansion coefficients, the residuals were calculated and error bounds were obtained from these residuals. This provided us with a quantitative means of analyzing the accuracy of the approximation. The results obtained were also compared to those obtained using other methods, and the benefits of the different methods were discussed as well as applications for which one method might be more suitable than another. It was found that the collocation method produced quite accurate results by using closed rule Chebyshev-Lobatto collocation points. Due to the relative ease in numerical computation, this method appears to be quite applicable to a variety of applications.

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LIST OF TABLES

Table 1 Total radiation heat flux at the boundaries for different linearly anisotropic phase functions at different values of with $\tau_D = 1$.

a_1	Method	ω					
		0.2		0.5		0.8	
		Q(-1)	Q (1)	Q(-1)	Q(1)	Q(-1)	Q(1)
0.643833	TC_6	0.96562	0.25344	0.89058	032758	0.76103	0.45553
	TG_6	0.96493	0.25356	0.89017	0.32752	0.76083	0.45538
	G_6	0.96564	0.25344	0.88981	0.32757	0.76075	0.45557
	TC_9	0.96564	0.25344	0.89060	0.32757	0.76105	0.45553
	F_9	0.96513	0.25397	0.88976	0.32843	0.76057	0.45588
	P_9	0.96780	0.25435	0.89151	0.32884	0.76192	0.45604
	DP_1	0.9648	0.2568	0.8889	0.3290	0.7587	0.4543
2.319461	TC_6	0.99221	0.27368	0.96735	0.39386	0.90592	0.59525
	TG_6	0.99152	0.27379	0.96695	0.39379	0.90569	0.59508
	G_6	0.99223	0.27368	0.96738	0.39386	0.90594	0.59525
	TC_9					0.90594	0.59525
	F_9					0.90406	0.60251
	P_9					0.90701	0.59596
2.602844	TC_6	0.99697	0.27734	0.98251	0.40719	0.93806	0.62650
	TG_6	0.99628	0.27746	0.98211	0.40712	0.93783	0.62632
	G_6	0.99698	0.27734	0.98254	0.40718	0.93808	0.62650
	TC_9					0.93808	0.62650
	F_9					0.94178	0.63874
	P_9					0.93920	0.62726
	DP_1					0.9336	0.6231

Table 2. Effect of Optical Thickness on the radiative flux Q at the boundaries for a_1=2.602844, ω=0.8.

Method	τ_D					
	0.1		2.0		10.0	
	Q(-1)	Q(1)	Q(-1)	Q(1)	Q(-1)	Q(1)
TC_6	0.98085	0.94234	0.93048	0.41647	0.92177	0.01741
TG_6	0.98087	0.94235	0.92975	0.41593	0.91599	0.01328
G_6	0.98085	0.94234	0.92938	0.41672	0.91005	0.02230
F_1	0.99293	0.95838	0.93336	0.43903	0.92011	0.02216
F_9	0.98692	0.94856	0.92523	0.43029	0.91332	0.02454
P_9	0.98885	0.95020	0.92660	0.43072	0.91464	0.02456
DP_1	0.9777	0.9388	0.9285	0.4171	0.9228	0.0170

Table 3 Zeroth Legendre moment of Intensity at right boundary, $G_0^N(1)$, at different optical depths for a_1=0.643833 and a_1=2.602844 and different values of ω.

a_1	ω	method	τ_D 0.1	1.0	2.0
0.643833	0.2	TC_6	0.74841	0.18021	0.05103
		TG_6	0.75484	0.18318	0.05639
		G_6	0.74841	0.18019	0.05098
	0.5	TC_6	0.79045	0.25117	0.08885
		TG_6	0.78898	0.25444	0.09360
		G_6	0.79046	0.25116	0.08878
	0.8	TC_6	0.83696	0.37749	0.18748
		TG_6	0.83796	0.38179	0.19273
		G_6	0.83696	0.37752	0.18751
2.602844	0.2	TC_6	0.75602	0.19889	0.06146
		TG_6	0.76016	0.20188	0.06678
		G_6	0.75602	0.19888	0.06141
	0.5	TC_6	0.81013	0.31619	0.13775
		TG_6	0.80822	0.31978	0.14294
		G_6	0.81013	0.31618	0.13769
	0.8	TC_6	0.86948	0.52311	0.34235
		TG_6	0.87052	0.52869	0.35022
		G_6	0.86948	0.52316	0.34246

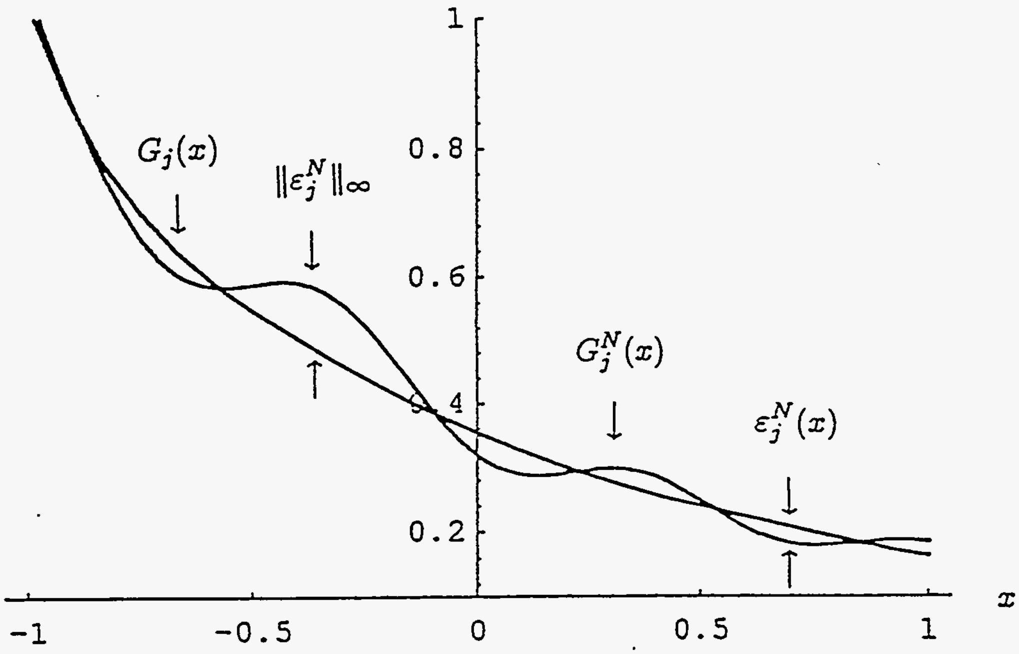
Table 4 Comparison of expansion coefficients obtained from the collocation method for the case where τ_D=1.0, ω=0.8, and a_1=0.643833 for several values of N.

	N=4		N=6		N=8		N=10	
i	b_i^N	c_i^N	b_i^N	c_i^N	b_i^N	c_i^N	b_i^N	c_i^N
0	0.23466	0.22866	0.75549	0.29125	0.78028	0.29372	0.78024	0.29372
1	0.33174	-0.04199	-0.38730	-0.07330	-0.42913	-0.07563	-0.42894	-0.07562
2	-0.38668	-0.00305	0.01328	0.00849	0.04813	0.01020	0.04802	0.01020
3	0.17056	0.00378	0.00649	0.00057	-0.02147	-0.00072	-0.02124	-0.00072
4	-0.04492	-0.00070	-0.01509	-0.00065	0.00524	0.00022	0.00509	0.00034
5			0.00762	0.00043	-0.00534	-0.00004	-0.00499	-0.00004
6			-0.00492	-0.00013	0.00189	0.00004	0.00165	0.00003
7					-0.00276	9.5×10^{-6}	-0.00213	-9.2×10^{-6}
8					0.00067	-0.00004	0.00086	0.00001
9							-0.00138	5.7×10^{-6}
10							0.00034	-0.00002

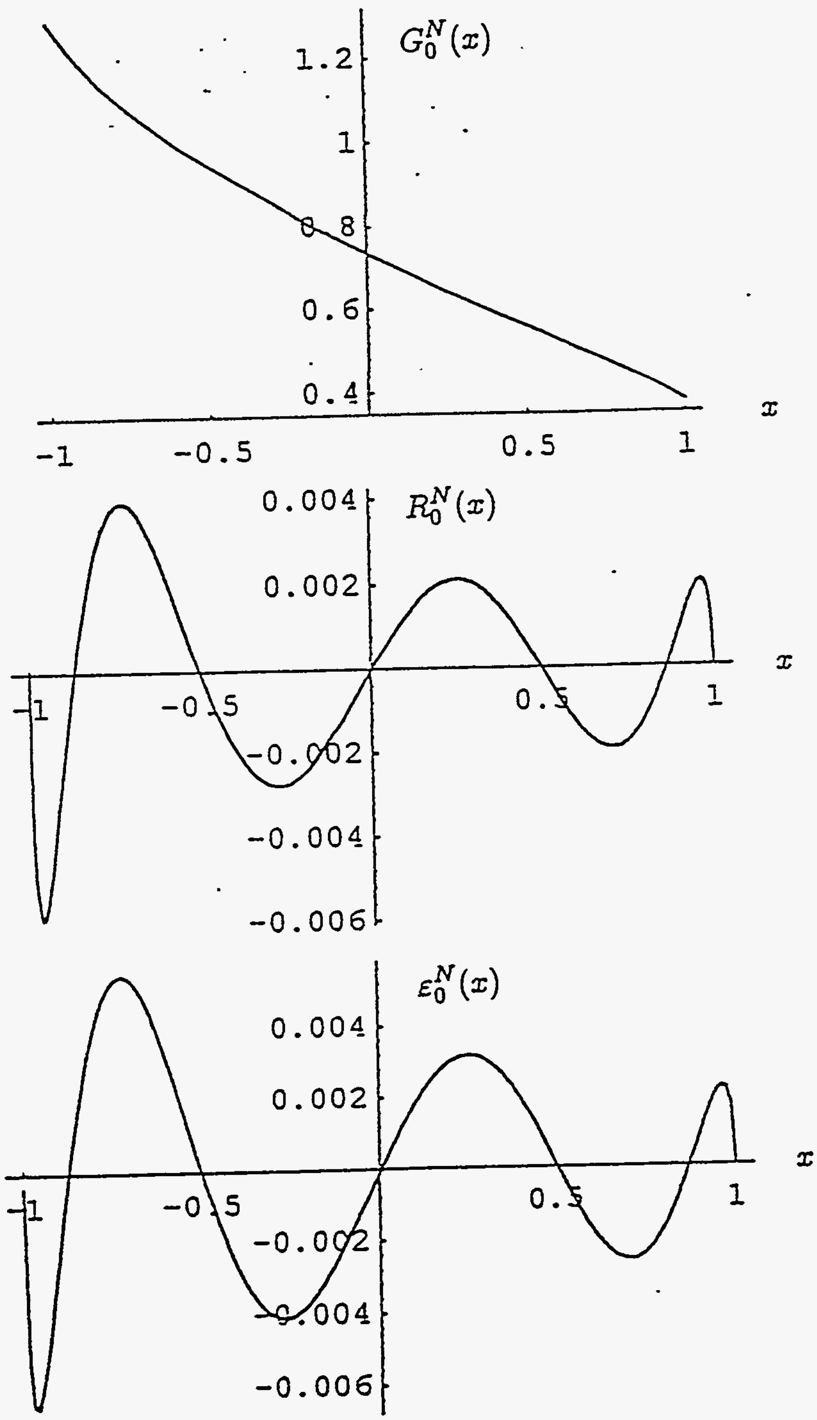
Table 5 Comparison of expansion coefficients obtained from the Ritz-Galerkin method for the case where τ_D=1.0, ω=0.8, and a_1=0.643833 for several values of N.

	N=4		N=6		N=8		N=6	
i	b_i^N	c_i^N	b_i^N	c_i^N	b_i^N	c_i^N	b_i^N	c_i^N
0	0.77980	0.29368	0.78003	0.29371	0.78003	0.29391	0.78002	0.29333
1	-0.42621	-0.07560	-0.42785	-0.07562	-0.42896	-0.07562	-0.42834	-0.07005
2	0.04712	0.01010	0.04760	0.01018	0.04760	0.01059	0.04758	0.00940
3	-0.01816	-0.00070	-0.02007	-0.00071	-0.02127	-0.00072	-0.02060	0.00512
4	0.00406	0.00006	0.00463	0.00018	0.00463	0.00063	0.00460	-0.00067
5			-0.00363	-0.00003	-0.00509	-0.00003	-0.00427	0.00647
6			0.00109	-0.00001	0.00109	0.00055	0.00105	-0.00107
7					-0.00281	-6.1×10^{-6}	-0.00122	0.00812
8					4.4×10^{-6}	0.00102	-0.00014	-0.00203
9							-0.00003	0.01536
10							0.00025	-2.2×10^{-8}

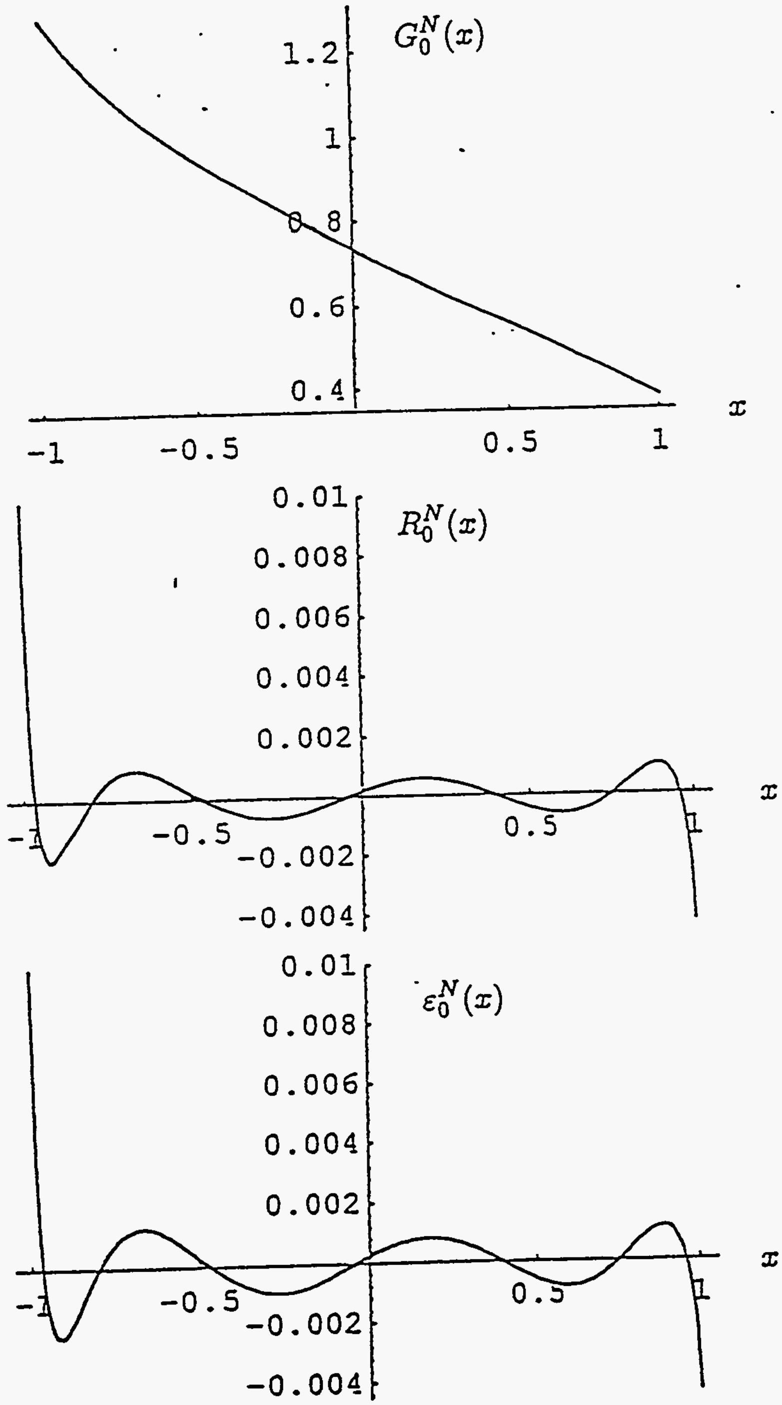
LIST OF FIGURES

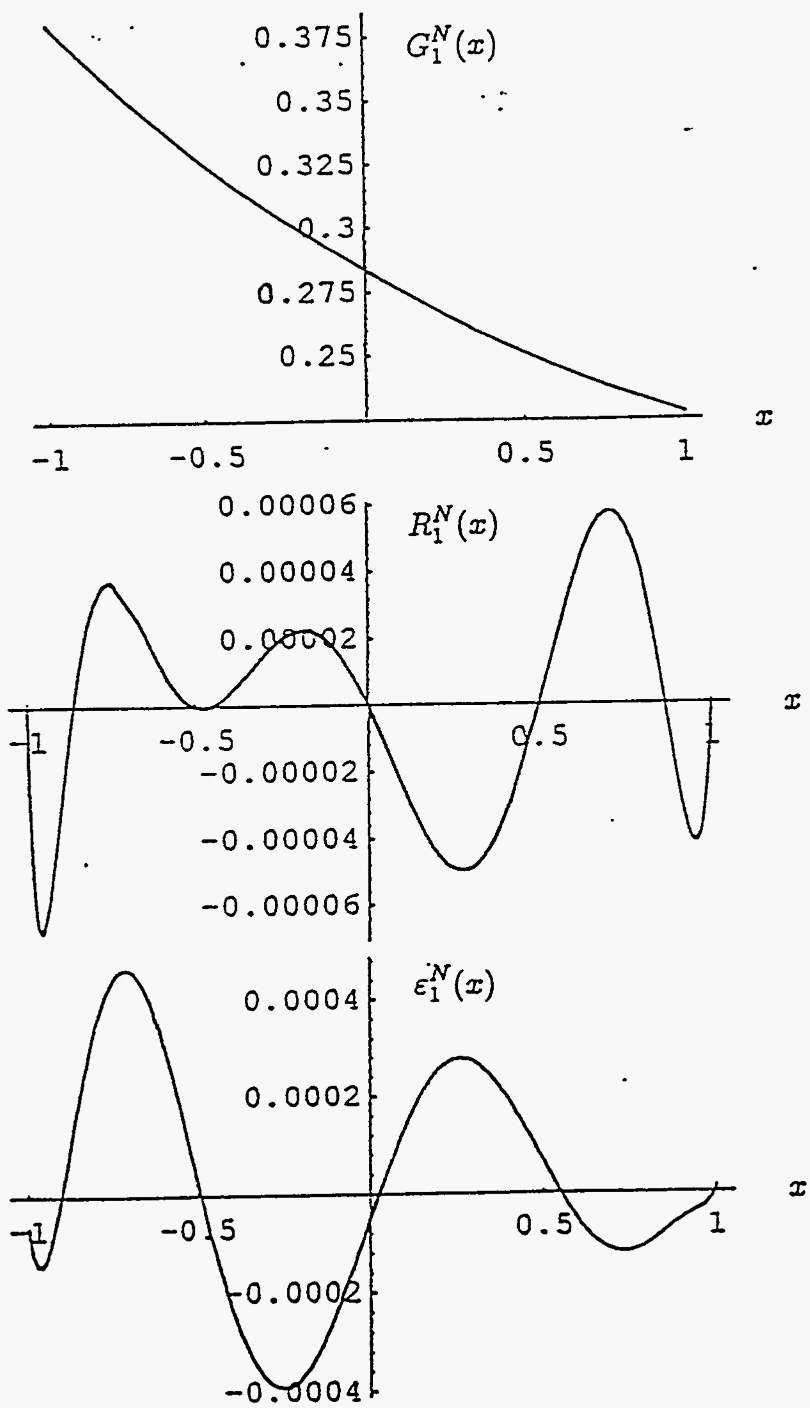


$G_j(x)$
$\|\varepsilon_j^N\|_\infty$
$G_j^N(x)$
$\varepsilon_j^N(x)$
1
0.8
0.6
0.4
0.2
-1
-0.5
0
0.5
1
x

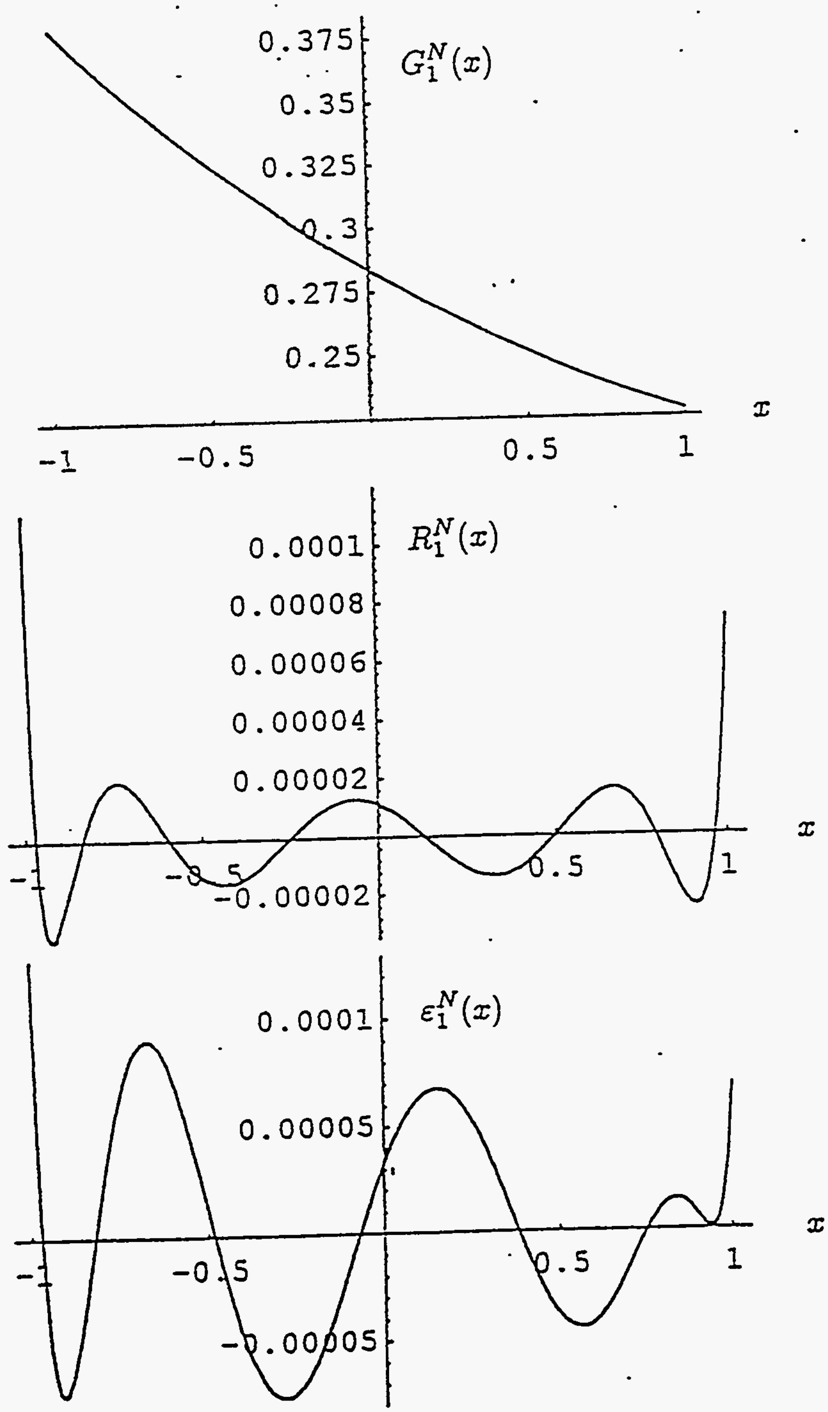


$G_0^N(x)$
1.2
1
0.8
0.6
0.4
x
-1
-0.5
0.5
1
$R_0^N(x)$
0.004
0.002
-0.002
-0.004
-0.006
x
-1
-0.5
0.5
1
$\varepsilon_0^N(x)$
0.004
0.002
-0.002
-0.004
-0.006
x
-1
-0.5
0.5
1

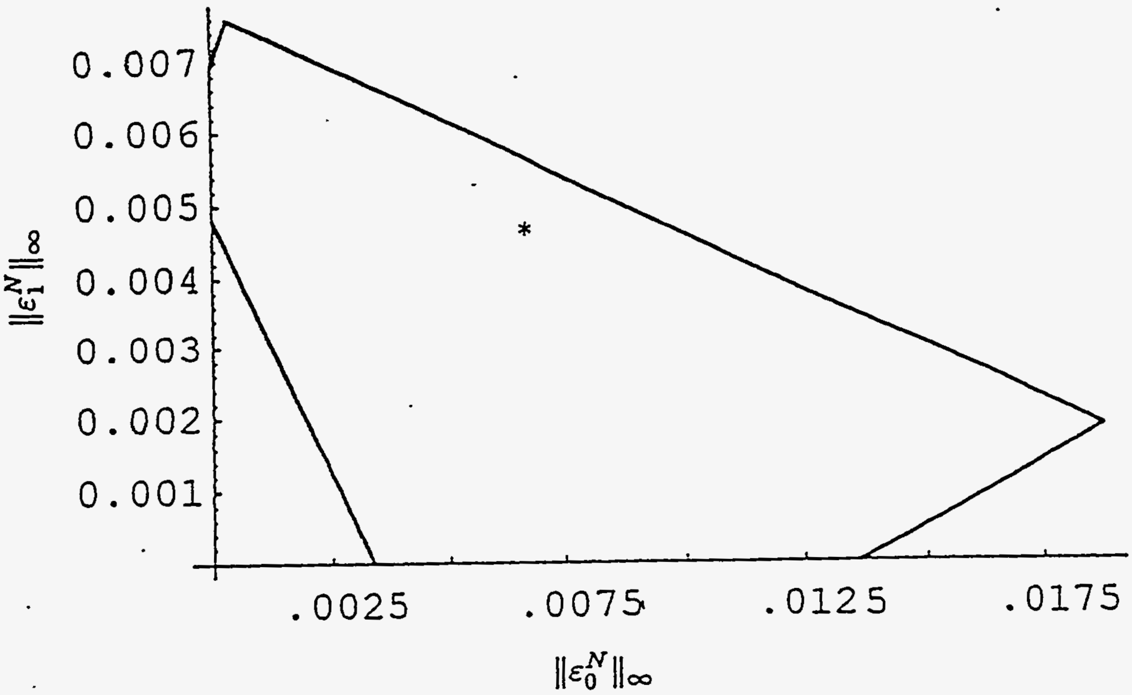

$G_0^N(x)$
1.2
1
0.8
0.6
0.4
x
-1
-0.5
0.5
1
$R_0^N(x)$
0.01
0.008
0.006
0.004
0.002
-0.002
-0.004
x
-0.5
0.5
$\varepsilon_0^N(x)$
0.01
0.008
0.006
0.004
0.002
-0.002
-0.004
x
-0.5
0.5
1



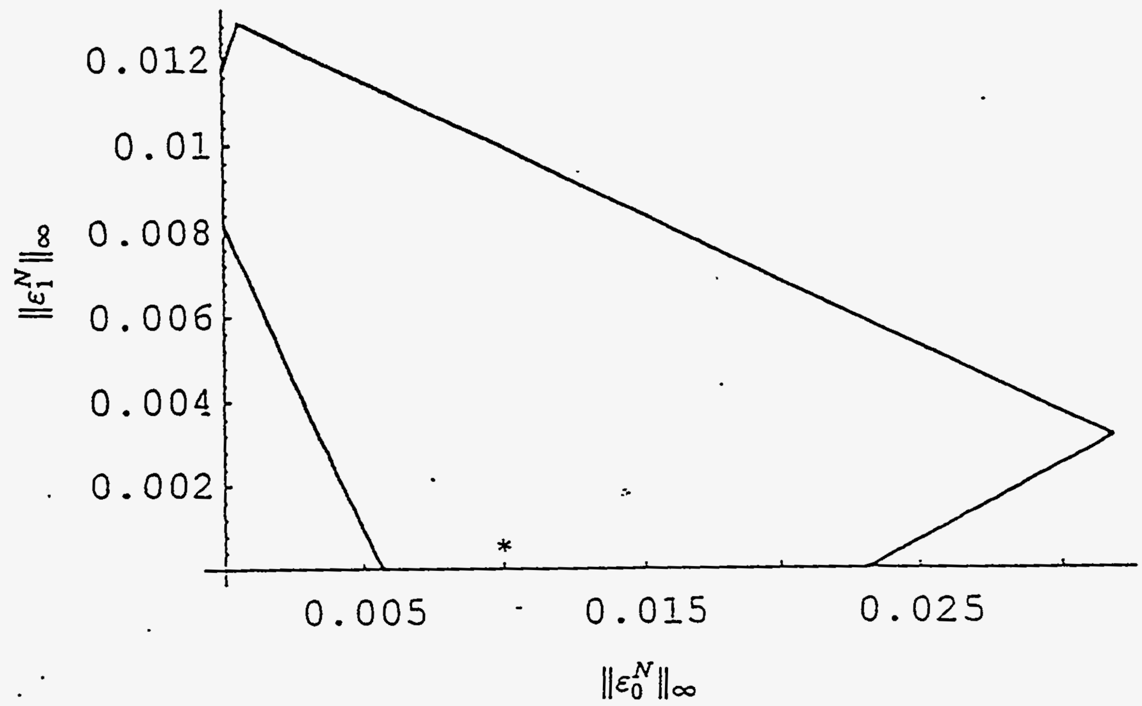
$G_1^N(x)$
0.375
0.35
0.325
0.3
0.275
0.25
x
-1
-0.5
0.5
1
$R_1^N(x)$
0.00006
0.00004
0.00002
-0.00002
-0.00004
-0.00006
$\varepsilon_1^N(x)$
0.0004
0.0002
-0.0002
-0.0004



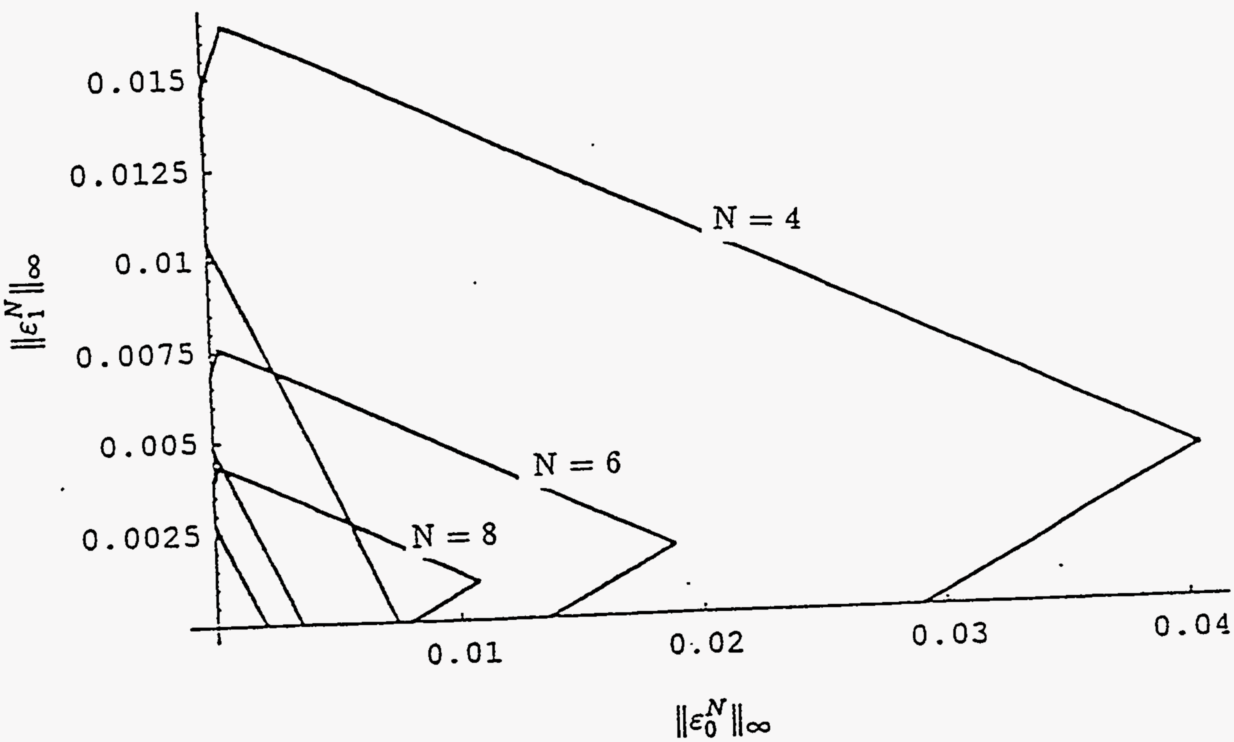
0.375
0.35
0.325
0.3
0.275
0.25
$G_1^N(x)$
x
-1
-0.5
0.5
1
0.0001
0.00008
0.00006
0.00004
0.00002
-0.00002
$R_1^N(x)$
x
-1
-0.5
0.5
1
0.0001
0.00005
-0.00005
$\varepsilon_1^N(x)$
x
-1
-0.5
0.5
1

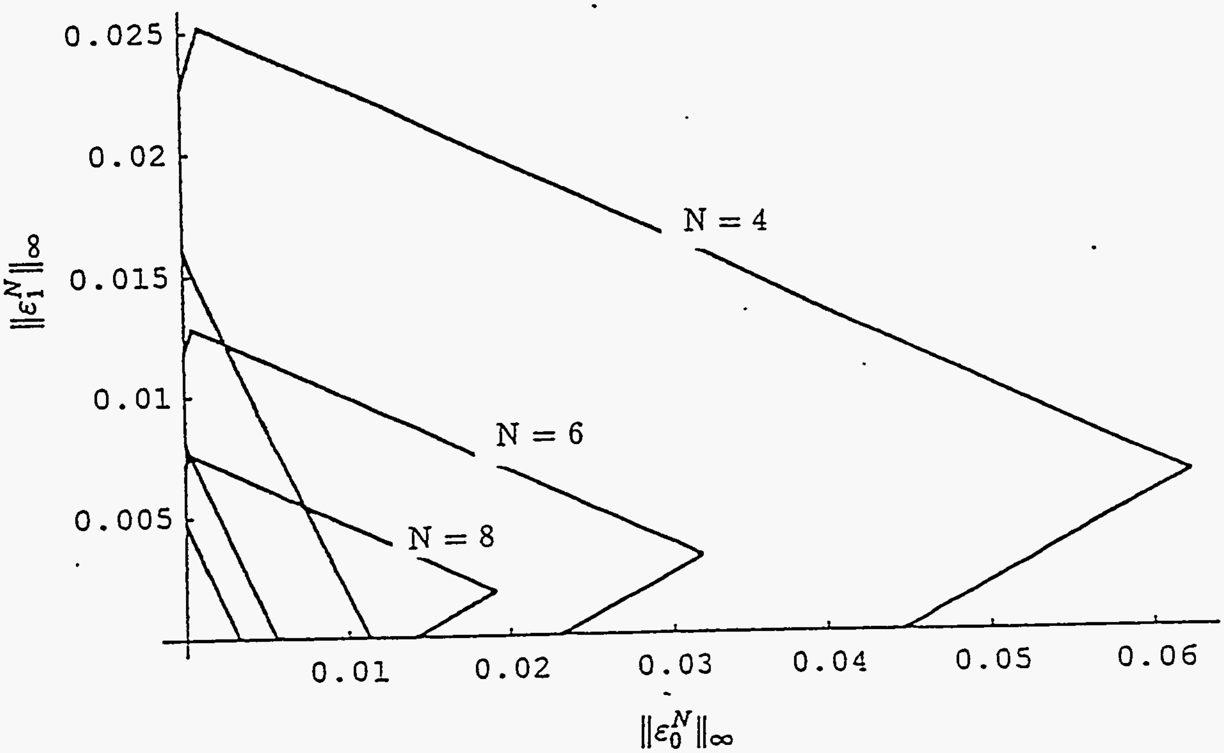


"*" denotes the location of the numerically calculated error norms.



"*" denotes the location of the numerically calculated error norms.


0.015
0.0125
0.01
0.0075
0.005
0.0025
$\|\varepsilon_1^N\|_\infty$
N = 4
N = 6
N = 8
0.01
0.02
0.03
0.04
$\|\varepsilon_0^N\|_\infty$



0.025
0.02
0.015
0.01
0.005
$\|\varepsilon_1^N\|_\infty$
N = 4
N = 6
N = 8
0.01
0.02
0.03
0.04
0.05
0.06
$\|\varepsilon_0^N\|_\infty$

A NOTE ON THE INTEGRAL FORMULATION OF KUMAR AND SLOAN

by

J.I. Frankel, Ph.D.
Associate Professor
Mechanical and Aerospace Engineering Department
University of Tennessee
Knoxville, TN (USA) 37996-2210

Abstract

In this note, we present a uniform approximation and a methodology for developing a posteriori error estimates for the recently proposed method of Kumar and Sloan. Kumar and Sloan proposed a formulation which converts a Hammerstein equation into a conducive form for approximation by a collocation method. Symbolic computation is used in performing the numerous analytic manipulations leading to the establishment of the error estimates. Finally, some remarks on the generalization of the method of Kumar and Sloan to higher-dimensional systems are offered.

Keywords: Hammerstein equation, error analysis, symbolic manipulation, Green's functions.

Running Title: Method of Kumar and Sloan

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1. INTRODUCTION

Kumar and Sloan [1] recently proposed an alternative integral formulation to the conventional Hammerstein form. This new formulation was offered in hopes of reducing the computational effort associated with implementing a collocation method. Thus, the intrinsic merit of this alternative formulation lies in its computational savings. Kumar and Sloan [1] demonstrated (i) conditions guaranteeing convergence to the exact solution, and (ii) the rate of convergence. More recently, Kaneko et al. [2] addressed the issue of superconvergence of the method of Kumar and Sloan in the presence of a weakly-singular kernel.

The purpose of this brief note is twofold; namely to illustrate that a posteriori error estimate may be established, and to illustrate that the method can be extended to higher-dimensional cases. Owing to the large variety of kernels and nonlinearities that occur in practice, an illustrative approach has been opted for from which some generalizations may be inferred.

This note is divided into four sections. In Section 2, we briefly review the method of Kumar and Sloan for contextual purposes. In Section 3, we present two illustrative examples in which error estimates are developed. In Section 4, we present numerical findings for the two examples considered in Section 3. In Section 5, we present an example to illustrate that the method of Kumar and Sloan may be applied to parabolic partial differential equations.

2. METHOD OF KUMAR AND SLOAN [1]

A Hammerstein integral equation may be expressed in the form

$$\Theta(\eta) = f(\eta) - \beta^2\int_{\eta_\circ=-1}^{1} G(\eta,\eta_\circ)S(\eta_\circ,\Theta(\eta_\circ))d\eta_\circ, \quad \eta \in [-1,1], \qquad (2.1)$$

where f, G, and S are known functions and β^2 is a real constant. Here $\Theta(\eta)$ is the unknown function requiring resolution. Often in practice, we transform nonlinear boundary value problems, via Green's functions, into the form displayed in (2.1). Thus, $G(\eta,\eta_\circ)$ would represent an appropriate Green's function [3,4] while $f(\eta)$ would be the conjunct. Kumar and Sloan [1] noted that the direct implementation of a collocation method [5,6] would require that the definite integral displayed in (2.1) be evaluated at each iterate until convergence on the expansion coefficients would be obtained.

Kumar and Sloan [1] proposed that an intermediate function be defined on the basis of the function $S(\eta,\Theta(\eta))$ as shown in (2.1), namely, we let

$$\Psi(\eta) = S(\eta,\Theta(\eta)), \quad \eta \in [-1,1], \qquad (2.2)$$

thus (2.1) can be written as

$$\Theta(\eta) = f(\eta) - \beta^2\int_{\eta_\circ=-1}^{1} G(\eta,\eta_\circ)\Psi(\eta_\circ)d\eta_\circ, \quad \eta \in [-1,1]. \quad (2.3)$$

By substituting (2.3) into (2.2) we obtain the nonlinear integral equation for the intermediate variable $\Psi(\eta)$ as

$$\Psi(\eta) = S(\eta,f(\eta) - \beta^2\int_{\eta_\circ=-1}^{1} G(\eta,\eta_\circ)\Psi(\eta_\circ)d\eta_\circ), \quad \eta \in [-1,1]. \qquad (2.4)$$

This integral form possesses some unique characteristics which have been the topic of study in [1,2,7]. In this form, a collocation method would not require the re-evaluation of the definite integral shown in (2.4) at each iterate. It should be noted that the structure of the new integral equation has become more complicated in appearance than the original form. Once $\Psi(\eta)$ has been resolved satisfactory, $\theta(\eta)$ is reconstructed through the integral transform shown in (2.3). Unlike previous studies, we will use a uniform approximation in developing the approximate solution.

3. TWO EXAMPLES - ERROR ESTIMATES

An illustrative approach is taken in order to demonstrate the development of error estimates for two typical nonlinearities (i.e., algebraic and exponential nonlinearities).

3.1 Example 1: Algebraic Nonlinearity, $S(\eta,\theta(\eta)) = \theta^4(\eta)$

Such a situation arises naturally in the modeling of a fourth-order isothermal, irreversible reaction in a planar geometry [8, p.85] or in steady, one-dimensional heat transfer in a fin placed in a vacuum environment. Let

$$f(\eta) = G_{\eta_0}(\eta,-1) - \theta_2 G_{\eta_0}(\eta,1), \tag{3.1}$$

where the two-point symmetric Green's function is

$$G(\eta,\eta_0) = \begin{cases} \dfrac{(1-\eta)(1+\eta_0)}{2}, & -1 \le \eta_0 \le \eta, \\ \dfrac{(1-\eta_0)(1+\eta)}{2}, & \eta \le \eta_0 \le 1. \end{cases} \tag{3.2}$$

This situation arises from the differential equation

$$\frac{d^2\theta}{d\eta^2}(\eta) - \beta^2\theta^4(\eta) = 0, \quad \eta \in (-1,1), \tag{3.3a}$$

subject to

$$\theta(-1) = 1, \tag{3.3b}$$
$$\theta(1) = \theta_2. \tag{3.3c}$$

Following the method of Kumar and Sloan [1], we define

$$\Psi(\eta) = \theta^4(\eta), \tag{3.4}$$

and upon substituting this into Eq. (2.1), we arrive at

$$\theta(\eta) = f(\eta) - \beta^2 \int_{\eta_o=-1}^{1} G(\eta,\eta_o)\Psi(\eta_o)d\eta_o, \quad \eta \in [-1,1]. \tag{3.5}$$

The explicit form of (2.4) now becomes

$$\Psi(\eta) = [f(\eta) - \beta^2 \int_{\eta_o=-1}^{1} G(\eta,\eta_o)\Psi(\eta_o)d\eta_o]^4, \quad \eta \in [-1,1]. \tag{3.6}$$

Let the unknown function, $\Psi(\eta)$ be represented by the expansion

$$\Psi(\eta) = \sum_{m=0}^{\infty} c_m^* T_m(\eta), \quad \eta \in [-1,1]. \tag{3.7}$$

In general, we seek an approximate solution to $\Psi(\eta)$ by truncating the infinite series shown in (3.7) at a certain order N, namely

$$\Psi_N(\eta) = \sum_{m=0}^{N} c_m^N T_m(\eta), \quad \eta \in [-1,1], \tag{3.8}$$

where c_m^N is an approximation to c_m^* for each fixed m. Thus, the approximate solution is based on solving

$$R_N(\eta) + \Psi_N(\eta) = [f(\eta) - \beta^2 \int_{\eta_o=-1}^{1} G(\eta,\eta_o)\Psi_N(\eta_o)d\eta_o]^4, \quad \eta \in [-1,1], \tag{3.9a}$$

where $R_N(\eta)$ is the local residual function. Upon substituting (3.8) into (3.9a), we arrive at

$$R_N(\eta) = -\sum_{m=0}^{N} c_m^N T_m(\eta) + [f(\eta) - \beta^2 \sum_{m=0}^{N} c_m^N C_m(\eta)]^4, \quad \eta \in [-1,1], \tag{3.9b}$$

where

$$C_m(\eta) = \int_{\eta_o=-1}^{1} G(\eta,\eta_o)T_m(\eta_o)d\eta_o, \quad m = 0,1,\ldots,N. \tag{3.9c}$$

This type of manipulation allows for the direct integration (analytical or numerical) of $C_m(\eta)$, m=0,1,...,N independent of the unknown expansion coefficients (in fact, this is a single time evaluation for each m).

Unless the true solution is a linear combination of the basis functions $\{T_m(\eta)\}_{m=0}^{N}$, we cannot choose $\{c_m^N\}_{m=0}^{N}$ to make $R_N(\eta)$ vanish for all $\eta \in [-1,1]$. However, suitable expansion series representation shown in (3.8). The local residual function, $R_N(\eta)$ is determined from (3.9b). Finally, the approximation to $\Theta(\eta)$, that is $\Theta_N(\eta)$, is obtainable from the transform shown in (3.5), namely

$$\Theta_N(\eta) = f(\eta) - \beta^2 \int_{\eta_o=-1}^{1} G(\eta,\eta_o)\Psi_N(\eta_o)d\eta_o, \quad \eta \in [-1,1]. \qquad (3.12)$$

Let the local error in the intermediate function be defined as

$$\epsilon_N(\eta) = \Psi(\eta) - \Psi_N(\eta), \quad \eta \in [-1,1], \qquad (3.13)$$

and its size may be measured by means of some functional norm. Unfortunately, the error is as inaccessible as the exact solution. Kumar and Sloan [1] did not present or discuss a posteriori error estimates. This type of estimate is highly desirable from the practical point of view. In light of its importance to the evaluation of the numerical solution, we begin by addressing this crucial issue here.

Development of an integral equation for the error in the approximate solution has been illustrated in Delves and Mohamed [5], and used by Frankel [9], and LaClair and Frankel [10] in recent studies involving radiative transport.

It should be noted that our main concern is the original function $\Theta(\eta)$ since it would typically carry the physics of interest. One can arrive at a direct error estimate for $\Theta(\eta) - \Theta_N(\eta)$ through some basic manipulations. To begin, we subtract (3.12) from (3.5) to get

$$\delta_N(\eta) \equiv \Theta(\eta) - \Theta_N(\eta) = -\beta^2 \int_{\eta_o=-1}^{1} G(\eta,\eta_o)\epsilon_N(\eta_o)d\eta_o, \qquad (3.14)$$

where $\delta_N(\eta)$ is the local error introduced by $\Theta_N(\eta)$. Let K denote the integral operator defined by

$$K\gamma \equiv \int_{\eta_o=-1}^{1} G(\eta,\eta_o)\gamma(\eta_o)d\eta_o, \tag{3.15}$$

therefore (3.14) can be expressed as

$$\delta_N = -\beta^2 K\epsilon_N. \tag{3.16}$$

Introducing (3.13) and (3.14) into (3.4) produces

$$\Psi_N + \epsilon_N = [\Theta_N + \delta_N]^4. \tag{3.17}$$

Note that by requiring $\Theta_N(\eta)$ to be defined by the integral transform shown in (3.12), then $\Theta_N^4 \neq \Psi_N$ unless $\epsilon_N(\eta) = 0$ for all $\eta \in [-1,1]$. Clearly, the error δ_N associated with Θ_N is related to ϵ_N through the integral transform shown in (3.14).

In order to obtain an integral equation for the unknown error δ_N, we let $\eta \to \eta_o$ in (3.17), multiply (3.17) by $-\beta^2 G(\eta,\eta_o)$, and integrate over the domain of interest, to arrive at (in operator form)

$$-\beta^2 K\Psi_N + \delta_N = -\beta^2 K[\Theta_N + \delta_N]^4. \tag{3.18}$$

As N becomes sufficiently large, one hopes that δ_N becomes correspondingly small. Expanding (3.18) produces

$$\delta_N = \beta^2 K\Psi_N - \beta^2 K[\Theta_N^4 + 4\Theta_N^3\delta_N] - \beta^2 K[6\Theta_N^2\delta_N^2 + 4\Theta_N\delta_N^3 + \delta_N^4]. \tag{3.19}$$

Assuming

$$|\Theta_N^4 + 4\Theta_N^3\delta_N| >> |6\Theta_N^2\delta_N^2 + 4\Theta_N\delta_N^3 + \delta_N^4|, \tag{3.20}$$

for sufficiently large N, we arrive at the linearized error integral equation (here we denote the linearized error as δ_N^L)

$$\delta_N^L + 4\beta^2 K[\Theta_N^3\delta_N^L] = -\beta^2 KR_N, \tag{3.21}$$

where we have made use of (3.12) and (3.9a), i.e.,

$$R_N + \Psi_N = \Theta_N^4. \tag{3.22}$$

Using the maximum principle [3, p. 64], we can readily establish $|\Theta| \leq 1$. We can now establish some error bounds (assuming $\delta_N^L \approx \delta_N$) after which the assertion shown in (3.20) can be called upon for verification. A correction procedure could be initiated if necessary.

Let the infinity norm of the function g(t) be denoted as $\|g\|_\infty$ and defined by [5, p. 22, 62]

$$\|g\|_\infty \equiv \mathrm{Sup}_{\eta\in[-1,1]} |g(t)|, \qquad (3.23)$$

for any function bounded in $\eta \in [-1,1]$. Using this definition of the norm, we can establish the following error bound from (3.21), namely

$$\|\delta_N^L\|_\infty \le \frac{\beta^2\|KR_N\|_\infty}{1 - 4\beta^2\|K\Theta_N^3\|_\infty}, \qquad (3.24)$$

when $1 - 4\beta^2\|K\Theta_N^3\|_\infty > 0$. With symbolic computation, the norms displayed in (3.24) can be evaluated. This is reminiscent of the error estimates seen in [5].

3.2 Example 2: Exponential Nonlinearity, $S(\eta,\Theta(\eta)) = \exp[\Theta(\eta)]$

Kumar and Sloan [1] considered an example that contained an exponential nonlinearity and known solution. The approach displayed here will use the same basis functions and collocation points as implemented in the previous example.

From [1], we let $\beta^2 = 1/4$, $f(\eta) = 0$ in (2.3), thus arriving at

$$\Theta(\eta) = -\frac{1}{4}\int_{\eta_o=-1}^{1} G(\eta,\eta_o)\exp[\Theta(\eta_o)]d\eta_o, \quad \eta \in [-1,1], \qquad (3.25)$$

where $G(\eta,\eta_o)$ is given in (3.2). This example corresponds to the example considered in [1]. The analytic solution to (3.25) is

$$\Theta(\eta) = -\ln(2) + 2\ln[\frac{c}{\cos(\frac{c\eta}{4})}], \quad \eta \in [-1,1], \qquad (3.26a)$$

where

$$c = \sqrt{2}\cos[\frac{c}{4}]. \qquad (3.26b)$$

Using a similar procedure as outline in Example 1, the upper bound can be established as

$$\|\delta_N^L\|_\infty \leq \frac{4\|\Theta_N\|_\infty + \|Ke^{\theta_N}\|_\infty}{4 - \|Ke^{\theta_N}\|_\infty}, \tag{3.27}$$

when $4 - \|Ke^{\theta_N}\|_\infty > 0$. Rall [16, . 242] indicated that the development of error estimates depends on several factors, including: (1) available information; (2) accuracy requirements; and (3) time and effort to be expelled on obtaining the error estimates. For the simple problems displayed here, a posteriori error estimates [5,17] were sought. In general, a priori error estimates based on projection methods appear to be well suited for Hammerstein integral equations [1,2,7,18]

4. NUMERICAL RESULTS

In this section, we present some illustrative numerical results indicating the merit of the error estimates. Chebyshev polynomials of the first kind [12] $\{T_m(\eta)\}_{m=0}^{N}$ were chosen as the basis functions while the N + 1 distinct collocation points are given by [5]

$$\eta_k = \cos\left(\frac{\pi k}{N}\right), \quad k = 0,1,\ldots,N, \tag{4.1}$$

thus ensuring $R_N(-1) = R_N(1) = 0$. Note that $|T_m(\eta)| \leq 1$, $m = 0,1,\ldots$ and $\eta \in [-1,1]$. The merit in choosing these polynomials is well known [5,11-13].

It should be remarked that implementation of symbolic computation substantially enhanced the analysis process proposed here, especially with regard to the determination of the norms displayed in (3.24) and (3.27). The symbolic software package Mathematica™ [14], Version 2.1, implemented on a NeXT TurboStation with 16 MBytes of memory, was used for developing the solutions and graphics displayed in this paper.

A Newton-Raphson [15] procedure was developed for solving the system of nonlinear algebraic equations for the unknown expansion coefficients. Though Kumar and Sloan [1] used a Brent's method, we noted that convergence could easily be met within four to five iterations where the convergence tolerance on each coefficient was defined as

$$\text{tol} = |(c_k^N)^{p+1} - (c_k^N)^p| < 10^{-15}, \quad k = 0,1,\ldots,N. \qquad (4.2)$$

Here, p represents the p^{th} iterate. Using Mathematica™, an interactive iteration sequence was developed in order to monitor the convergence rate in real time.

The actual CPU time required to determine the approximate solution was less than 3 seconds using a program written in the symbolic language Mathematica™. The computations involving the error estimates and graphics typically took 3 minutes on the NeXT. This is, in part, due to the numerous analytic manipulations required in determining the infinity norms of the linearized error indicated in Eqs. (3.24) and (3.27).

Owing to the availability of an analytic solution to Example 2, we first present some results intended to indicate the merit of the proposed approach. Table 1 indicates typical findings for

N = 3-6 for the chosen Chebyshev basis and closed-rule collocation points. For this example, an accurate depiction appears evident.

Table 2 presents a comparison between a finite element (piecewise linear elements) solution and the orthogonal collocation solution for Example 1. Excellent agreement is evident. Also, the error bound displayed in (3.24) appears quite adequate in the context of (3.20). A posteriori error estimates can be costly in terms of expended CPU time and the amount of required computer code. In most contemporary studies involving Hammerstein integral equations [1,2,18], a priori error estimates are being developed with the aid of interpolatory projections and approximation theory.

Finally, Figure 1 presents two sets of plots where N = 3,5 and $\beta = 1$, $\Theta_2 = 0.2$ illustrating the behaviour of several functions involved in the analysis of Example 1. Theoretically, through successive approximations, Eq. (3.21) has unique solution which uniformly converges to the solution [19]. This can be concluded with the aid of Fig. 1 for both cases N=3,5 using the $\|K\Theta_N^3\|_\infty$ plots. Practically speaking, the development of the resolvent kernel may be computationally prohibitive.

5. FURTHER CONSIDERATIONS

As a final observation, it appears that the method proposed by Kumar and Sloan may be applied to substantial problems of mathematical physics. As an illustration, consider the time varying analog to (3.3a), namely

$$L[\Theta] = \frac{\partial\Theta}{\partial t} + \beta^2\Theta^4, \quad \eta \in (-1,1), \quad t > 0, \tag{5.1a}$$

subject to the time invariant boundary conditions shown in (3.3b,c) and subject to $\Theta(\eta,0) = r(\eta)$, $\eta \in [-1,1]$. Here $\Theta = \Theta(\eta,t)$ and the linear differential operator L is given by

$$L \equiv \frac{\partial^2}{\partial\eta^2} . \tag{5.1b}$$

Inverting the spatial operator produces

$$\Theta(\eta,t) = - \int_{\eta_o=-1}^{1} G(\eta,\eta_o)[\frac{\partial\Theta}{\partial t} + \beta^2\Theta^4]d\eta_o + f(\eta), \; \eta \in [-1,1], \; t > 0, \tag{5.2}$$

where $f(\eta)$ is given in (3.1) and the Green's function is given in (3.2). Next, we assign the intermediate function as

$$\Psi(\eta,t) = \frac{\partial\Theta}{\partial t} + \beta^2\Theta^4, \tag{5.3}$$

thus (5.2) can be written as

$$\Theta(\eta,t) = - \int_{\eta_o=-1}^{1} G(\eta,\eta_o)\Psi(\eta_o,t)d\eta_o + f(\eta), \; \eta \in [-1,1], \; t > 0. \tag{5.4}$$

Substituting (5.4) into (5.3) produces the new formulation

$$\Psi(\eta,t) = \frac{\partial}{\partial t}[- \int_{\eta_o=-1}^{1} G(\eta,\eta_o)\Psi(\eta_o,t)d\eta_o + f(\eta)]$$
$$+ \beta^2[- \int_{\eta_o=-1}^{1} G(\eta,\eta_o)\Psi(\eta_o,t)d\eta_o + f(\eta)]^4, \; \eta \in [-1,1], \; t > 0. \tag{5.5}$$

Next, we can expand the unknown function, $\Psi(\eta,t)$ as

$$\Psi(\eta,t) = \sum_{m=0}^{\infty} c_m(t)T_m(\eta), \quad \eta \in [-1,1], \quad t > 0. \tag{5.6}$$

and follow the procedure previously outlined where we approximate $\Psi(\eta,t)$ by $\Psi_N(\eta,t)$ (other choices for the assumed solution form also exist [8]). Formally proceeding produces

$$\sum_{m=0}^{N} \frac{dc_m^N(t)}{dt} C_m(\eta) = - R_N(\eta,t) - \sum_{m=0}^{N} c_m^N(t)T_m(\eta) \tag{5.7a}$$
$$+ \beta^2[- \sum_{m=0}^{N} c_m^N(t)C_m(\eta) + f(\eta)]^4, \quad \eta \in [-1,1], \; t > 0,$$

where $C_m(\eta)$ is given in (3.9c) and subject to the transformed initial condition

$$r(\eta) = \Theta_N(\eta,0) = -\sum_{m=0}^{N} c_m^N(0)C_m(\eta) + f(\eta). \tag{5.7b}$$

By requiring (5.7a) to be enforced for all instances in time, and upon implementing the previously described collocation method, we arrive at the following system of nonlinear initial value problems

$$\sum_{m=0}^{N} \frac{dc_m^N(t)}{dt} C_m(\eta_k) = -\sum_{m=0}^{N} c_m^N(t)T_m(\eta_k) \tag{5.8}$$
$$+ \beta^2[-\sum_{m=0}^{N} c_m^N(t)C_m(\eta_k) + f(\eta_k)]^4, \quad k = 0,1,\ldots,N, \ t > 0,$$

subject to (5.7b) evaluated at $\eta = \eta_k$, $k = 0,1,\ldots,N$. Carefully note the origin of this expression for the initial condition for $c_m^N(0)$, $m = 0,1,\ldots,N$ and the fact that a matrix inversion of a linear system is necessary for obtaining numerical values for each m. Equation (5.8) subject to the appropriate initial conditions can be resolved numerically as described in the fine book by Finlayson [8, pp. 184-214]. Again, we note that the spatial integrations are carried out only once and are contained in the function $C_m(\eta)$ as shown in (3.9c). This formulation favors the use of well-established initial value algorithms to be used for finding the time varying coefficients, $c_m^N(t)$, $m = 0,1,\ldots,N$.

An alternative implementation using the method of Kumar and Sloan for practical computation involving nonlinear, weakly-singular, integro-partial differential equations has been recently demonstrated [20,21] in the study of radiative and conductive transport in semitransparent materials.

6. CONCLUSIONS

The intent of this note was to illustrate that both a uniform approximation and a posteriori error estimates may be obtained for the method of Kumar and Sloan. Also, there appears to be additional merit to the method with regard to parabolic (and elliptic) partial differential equations.

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N	Actual Error $\|\theta-\theta_N\|_\infty$	Estimated Upper Bound (3.27)
3	1.02×10^{-4}	1.18×10^{-4}
4	1.12×10^{-6}	1.14×10^{-6}
5	4.92×10^{-7}	10.8×10^{-7}
6	4.50×10^{-9}	9.86×10^{-9}

Table 1. Comparison between the actual error and the upper bound estimate shown in (3.27). Results are rounded to three significant figures. Note that the ratio ($N \geq 5$) between the error bound and the true error appears to be approaching a constant.

η	Finite Element Solution N = 20	N = 50	N = 100	Method of Kumar and Sloan [1] N = 3	N = 5	N = 7
-1	1	1	1	1	1	1
-0.8	0.8742	0.8743	0.8743	0.8743	0.8744	0.8743
-0.6	0.7724	0.7725	0.7726	0.7746	0.7728	0.7726
-0.4	0.6852	0.6854	0.6854	0.6905	0.6855	0.6854
-0.2	0.6070	0.6072	0.6072	0.6149	0.6070	0.6072
0	0.5344	0.5346	0.5346	0.5433	0.5343	0.5346
0.2	0.4651	0.4652	0.4653	0.4732	0.4650	0.4653
0.4	0.3978	0.3979	0.3979	0.4038	0.3978	0.3979
0.6	0.3314	0.3315	0.3315	0.3350	0.3316	0.3315
0.8	0.2656	0.2656	0.2657	0.2671	0.2657	0.2657
1	0.2	0.2	0.2	0.2	0.2	0.2
$\|\delta_N^L\|_\infty$ (upper bound)				0.01267	9.571×10^{-4}	7.397×10^{-5}

Table 2. Comparison between a finite element solution and the present formulation for $\theta_N(\eta)$ from (3.12) when $\beta = 1$ and $\theta_2 = 0.2$.

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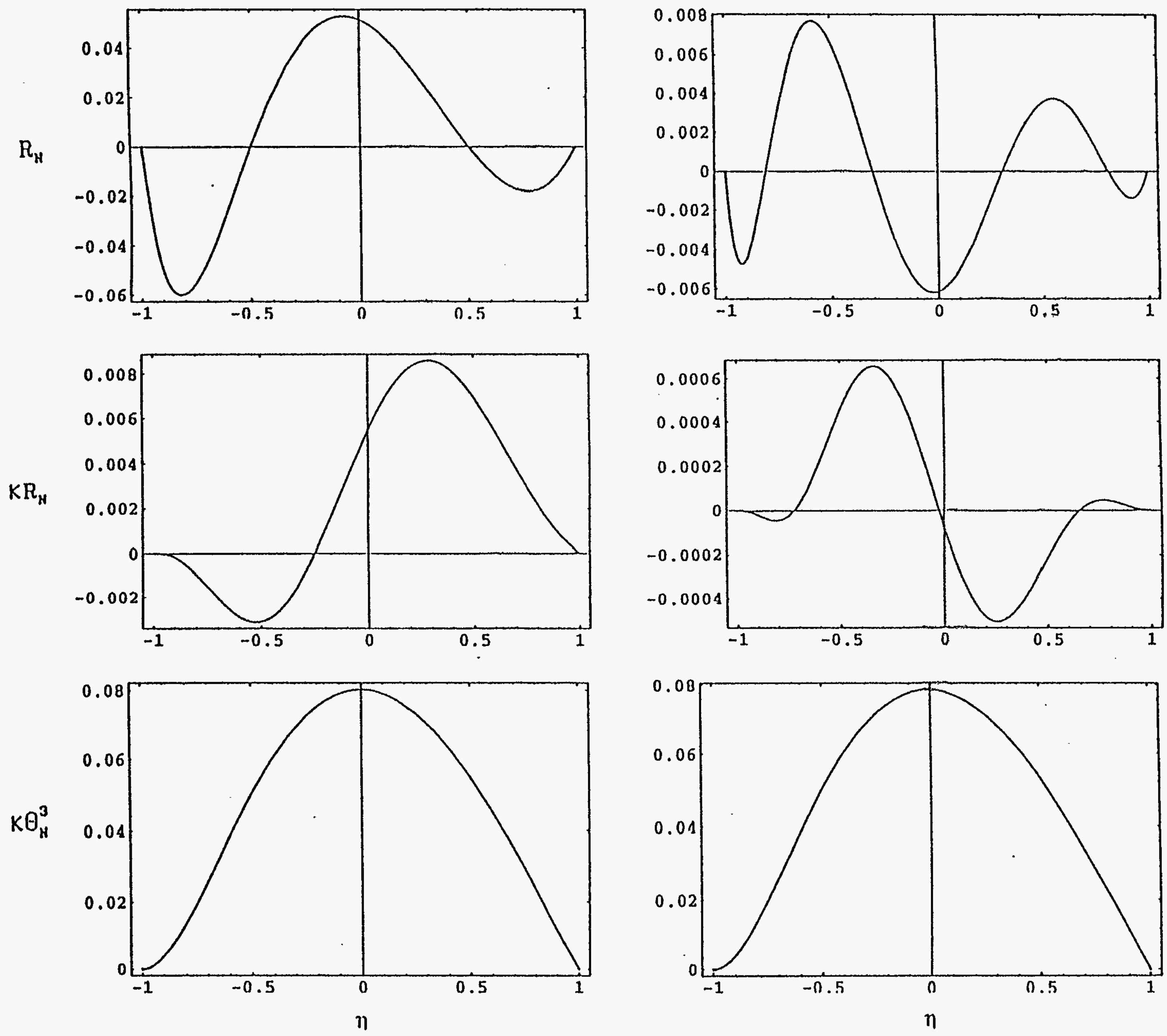

R_N
KR_N
$K\Theta_N^3$
0.04
0.02
0
-0.02
-0.04
-0.06
0.008
0.006
0.004
0.002
-0.002
-0.004
-0.006
0.0006
0.0004
0.0002
-0.0002
-0.0004
0.08
0.06
-1
-0.5
0.5
1
η

An Orthogonal-Collocation Integral Formulation for Transient Radiative Transport

J.I. Frankel

Mechanical and Aerospace Engineering Department, University of Tennessee, Knoxville, Tennessee, 37996-2210

ABSTRACT

A new formulation is offered for transient radiative transport which promotes the use of orthogonal collocation. An intermediate variable is introduced which permits the efficient and rapid development of accurate numerical results. Chebyshev polynomials of the first kind are used as the basis functions for the spatial variable while the temporal variable is resolved by an initial value method. Some a posteriori error estimates are presented illustrating the effectiveness of the approach. This new formulation has potential impact to the boundary element community with regard to nonlinear problems.

INTRODUCTION

The accurate numerical simulation of nonlinear, weakly-singular integro-partial differential equations of mathematical physics often represents a formidible challenge to researchers. Both algebraic nonlinearities in the temperature variable and the appearance of kernels containing logarithmic singularities arise in applications involving transient heat transfer [1] in a participating medium and multidimensional heat transfer in a participating medium.

Recently, Kumar and Sloan [2] proposed a new formulation of one-dimensional Hammerstein integral equations which permits efficient computation by a collocation method. Frankel [3] illustrated that the approach of Kumar and Sloan can be extended to multidimensional and transient studies.

ALTERNATIVE INTEGRO-DIFFERENTIAL FORMULATION

In the present context, we consider [1]

$$\frac{\partial\theta}{\partial t}(\eta,t) = g(\eta) - \theta^4(\eta,t) + \lambda\alpha \int_{\xi=-1}^{1} \theta^4(\xi,t)E_1(\alpha|\xi-\eta|)d\xi, \qquad (1a)$$

$$\eta \epsilon [-1,1], \quad t > 0,$$

where

$$g(\eta) = \lambda \Big[E_2[\alpha(1+\eta)] + \theta_1^4 E_2[\alpha(1-\eta)] \Big], \tag{1b}$$

$$\theta(\eta, 0) = \theta_1, \quad \eta \epsilon [-1,1], \tag{1c}$$

with $\lambda = 1/2$ and $\alpha > 0$.

Here, $\theta(\eta, t)$ is the unknown dependent variable requiring resolution and η, t are the spatial and temporal independent variables, respectively. The n^{th} exponential integral function [4] is denoted by $E_n(z)$ where $E_1(z)$ contains a logarithmic singularity as $z \to 0$.

Let

$$\varphi(\eta, t) = \theta^4(\eta, t), \tag{2}$$

thus Equation (1a) can be written as

$$\frac{\partial \theta}{\partial t}(\eta, t) = g(\eta) - \varphi(\eta, t) + \lambda\alpha \int_{\xi=-1}^{1} \varphi(\xi, t) E_1(\alpha|\xi - \eta|) d\xi, \tag{3}$$

$$\eta \epsilon [-1,1], \quad t > 0.$$

Next, we integrate Equation (3) with respect to t, to get

$$\theta(\eta, t) = h(\eta, t) - \Psi(\eta, t) + \lambda\alpha \int_{\xi=-1}^{1} \Psi(\xi, t) E_1(\alpha|\xi - \eta|) d\xi, \tag{4a}$$

$$\eta \epsilon [-1,1], \quad t \geq 0,$$

where

$$\Psi(\eta, t) = \int_{t_o=0}^{t} \varphi(\eta, t_o) dt_o, \quad t \geq 0, \tag{4b}$$

and

$$h(\eta, t) = g(\eta)t + \theta_1, \quad t \geq 0. \tag{4c}$$

Clearly,

$$\frac{\partial \Psi}{\partial t}(\eta, t) = \varphi(\eta, t) = \theta^4(\eta, t), \tag{5a}$$

and

$$\Psi(\eta,0)=0, \quad \eta\epsilon[-1,1]. \tag{5b}$$

Substituting Equation (4a) into Equation (5a), we obtain the new nonlinear, weakly-singular integro-partial differential equation

$$\frac{\partial\Psi}{\partial t}(\eta,t)=\left[h(\eta,t)-\Psi(\eta,t)+\lambda\alpha\int_{\xi=-1}^{1}\Psi(\xi,t)E_1(\alpha|\xi-\eta|)d\xi\right]^4, \tag{6}$$

$$\eta\epsilon[-1,1], \quad t>0,$$

subject to the initial condition displayed in Equation (5b). Note that the algebraic nonlinearity has been peeled away from within the integral operator shown in Equation (1) to a new position outside the integral operator. This new form permits the implementation of a collocation method in a highly efficient manner in the new variable, $\Psi(\eta,t)$. Once we resolve $\Psi(\eta,t)$, we can reconstruct $\theta(\eta,t)$ through the integral transform shown in Equation (4a).

SOLUTION BY ORTHOGONAL COLLOCATION

Let the unknown function $\Psi(\eta,t)$ be represented by the series expansion

$$\Psi(\eta,t)=\sum_{m=0}^{\infty}c_m^*(t)T_m(\eta), \quad \eta\epsilon[-1,1], \quad t\geq 0, \tag{7}$$

where the basis functions $\{T_m(\eta)\}_{m=0}^{\infty}$ are chosen as the Chebyshev polynomials of the first kind [4]. The unknown time varying expansion coefficients requiring resolution are denoted as $\{c_m^*(t)\}_{m=0}^{\infty}$. In practice, we must truncate this series representation at a finite number of terms, say N. Thus, we express an approximation to $\Psi(\eta,t)$ as $\Psi_N(\eta,t)$, namely

$$\Psi(\eta,t)\approx\Psi_N(\eta,t)=\sum_{m=0}^{N}c_m^N(t)T_m(\eta), \quad \eta\epsilon[-1,1], \quad t\geq 0, \tag{8}$$

where $c_m^N(t)$ is an approximation to $c_m^*(t)$.

Upon substituting Equation (8) into Equation (6), we arrive at

$$R_N(\eta,t)+\sum_{m=0}^{N}\frac{dc_m^N}{dt}(t)T_m(\eta)= \tag{9a}$$

$$\left[h(\eta,t)-\sum_{m=0}^{N}c_m^N(t)\{T_m(\eta)-\lambda\alpha A_m(\eta)\}\right]^4, \quad \eta\epsilon[-1,1], \quad t>0,$$

where

$$A_m(\eta) = \int\limits_{\xi=-1}^{1} T_m(\xi) E_1(\alpha|\eta - \xi|) d\xi, \quad m = 0, 1, ..., N, \quad \eta \epsilon [-1, 1], \quad (9b)$$

which is analytically expressible [5]. The residual function $R_N(\eta, t)$ is introduced in order to maintain the equal sign displayed in Equation (9a). Correspondingly, we find, from Equation (5b), the initial conditions for $c_m^N(t)$, $m = 0, 1, ..., N$, at $t = 0$, namely

$$R_N^{IC}(\eta) + \sum_{m=0}^{N} c_m^N(0) T_m(\eta) = 0, \quad \eta \epsilon [-1, 1]. \quad (9c)$$

Unless the exact solution to $\Psi(\eta, t)$, at any instant in time, $t \geq 0$, is a linear combination of $\{T_m(\eta)\}_{m=0}^N$, we cannot obtain $\{c_m^N(t)\}_{m=0}^N$ which makes both $R_N(\eta, t)$ for $t > 0$ and $R_N^{IC}(\eta)$ at $t = 0$ vanish for all $\eta \epsilon [-1, 1]$. However, we can obtain suitable time varying expansion coefficients by making the residuals indicated in Equation (9a) and Equation (9c) small in some sense. Using the definition of the inner product of two functions shown in Frankel [5], we define the weighted residual method through

$$\langle R_N(\eta, t), \Omega_k(\eta) \rangle_{w_k} = 0, \quad t > 0, \quad (10a)$$

and

$$\langle R_N^{IC}(\eta), \Omega_k(\eta) \rangle_{w_k} = 0, \quad t = 0. \quad (10b)$$

For the collocation method, we have $\Omega_k(\eta) = 1, w_k = \delta(\eta - \eta_k)$, $k = 0, 1, ..., N$. Here the Dirac delta function is denoted by δ while the $N + 1$ collocation points are indicated by η_k, $k = 0, 1, ..., N$ and are defined by the closed rule [5]

$$\eta_k = cos(\frac{\pi k}{N}), \quad k = 0, 1, ..., N. \quad (11)$$

By choosing this set of N+1 collocation points, we ensure that both $R_N(\pm 1, t) = 0$ for $t > 0$ and $R_N^{IC}(\pm 1) = 0$ at $t = 0$.

Applying Equation (10a) on Equation (9a), and Equation (10b) on Equation (9c) formally produces

$$\sum_{m=0}^{N} \frac{dc_m^N}{dt}(t) T_m(\eta_k) = \left[h(\eta_k, t) - \sum_{m=0}^{N} c_m^N(t) \{T_m(\eta_k) - \lambda \alpha A_m(\eta_k)\} \right]^4, \quad (12a)$$

$$k = 0, 1, ..., N, \quad t > 0,$$

and

$$\sum_{m=0}^{N} c_m^N(0) T_m(\eta_k) = 0, \quad k = 0, 1, ..., N, \quad t = 0, \tag{12b}$$

respectively. Since $\{T_m(\eta)\}_{m=0}^N$ forms a set of linearly independent basis functions, we see that Equation (12b) reduces to $c_m^N(0) = 0, \quad m = 0, 1, ..., N$, which now represent the initial conditions necessary for resolving $c_m^N(t)$, $m = 0, 1, ..., N$.

Clearly, once $c_m^N(t)$, $m = 0, 1, ..., N$, $t \geq 0$ are resolved, $\Psi_N(\eta, t)$ is reconstructed through Equation (8). Finally, the approximate solution to $\theta(\eta, t)$, namely $\theta_N(\eta, t)$ is arrived at through Equation (4a) i.e.,

$$\theta_N(\eta, t) = h(\eta, t) - \sum_{m=0}^{N} c_m^N(t)[T_m(\eta) - \lambda\alpha A_m(\eta)], \quad \eta\epsilon[-1, 1], \quad t > 0. \tag{13}$$

STEADY-STATE ANALYSIS

At steady-state conditions, Equation (1) reduces to the linear (in $\hat{\theta}^4(\eta)$) Fredholm integral equation of the second kind

$$\hat{\theta}^4(\eta) = g(\eta) + \lambda\alpha \int_{\xi=-1}^{1} \hat{\theta}^4(\xi) E_1(\alpha|\xi - \eta|) d\xi, \quad \eta\epsilon[-1, 1], \tag{14}$$

where $\hat{\theta}(\eta) = \lim_{t\to\infty} \theta(\eta, t)$.

As before, we can develop a series representation for $\hat{\theta}^4(\eta)$, namely

$$\hat{\theta}^4(\eta) = \sum_{m=0}^{\infty} b_m^* T_m(\eta), \quad \eta\epsilon[-1, 1], \tag{15a}$$

while the N^{th} order approximation is given by

$$\hat{\theta}^4(\eta) \approx \hat{\theta}_N^4(\eta) = \sum_{m=0}^{N} b_m^N T_m(\eta), \quad \eta\epsilon[-1, 1], \tag{15b}$$

where $T_m(\eta)$ was previously defined. Following a similar procedure as described previously, we can obtain the expansion coefficients $\{b_m^N\}_{m=0}^N$ by solving a closed system of linear algebraic equations using conventional means.

TRANSIENT ANALYSIS

From viewing the findings at steady-state conditions, we choose to express the approximate solution for $\theta_N^4(\eta, t_i)$ at discrete time t_i as

$$\theta_N^4(\eta, t_i) = \sum_{m=0}^{N} d_m^N(t_i) T_m(\eta), \quad \eta \epsilon [-1, 1]. \tag{16a}$$

Evaluating Equation (16a) at $\eta = \eta_k$, $k = 0, 1, ..., N$ and substituting Equation (13) into the left-hand side of Equation (16a), we obtain $\{d_m^N(t_i)\}_{m=0}^N$ through matrix inversion at the indicated discrete time, $t = t_i$. These discrete times t_i correspond to the times used in obtaining the numerical results mandated by a initial value method for finding $\{c_m^N(t)\}_{m=0}^N$ as indicated by Equation (12). Doing so produces the linear system of equations for $\{d_m^N(t_i)\}_{m=0}^N$

$$\theta_N^4(\eta_k, t_i) = \sum_{m=0}^{N} d_m^N(t_i) T_m(\eta_k), \quad k = 0, 1, ..., N. \tag{16b}$$

By comparing Equation (15b) to Equation (16b), it is clear that in the limit as $t \to \infty$ (assuming no numerical errors in time)

$$\lim_{t \to \infty} d_m^N(t) = b_m^N, \quad m = 0, 1, ..., N. \tag{16c}$$

RESULTS AND CONCLUSIONS

Let us define the dimensionless dependent variables [1]

$$f^4(\eta, t) = \frac{\theta^4(\eta, t) - \theta_1^4}{1 - \theta_1^4}, \quad \hat{f}^4(\eta) = \frac{\hat{\theta}^4(\eta) - \theta_1^4}{1 - \theta_1^4}, \tag{17a, b}$$

which will be used for transient and steady-state analyses purposes, respectively. At this juncture, we can readily establish a posteriori error bounds for $\hat{f}_N^4(\eta)$. Let the local error of $\hat{f}_N^4(\eta)$ be denoted by

$$\hat{\gamma}_N(\eta) = \hat{f}^4(\eta) - \hat{f}_N^4(\eta), \quad \eta \epsilon [-1, 1], \tag{18}$$

and its size may be measured by means of some functional norm. In general, the error is typically as inaccessible as the exact solution. However, the residual $\hat{R}_N(\eta)$ is a computable measure of how close $\hat{f}_N^4(\eta)$ is to $\hat{f}^4(\eta)$. Following Frankel [5], we arrive at

$$\frac{\|\hat{R}_N\|_\infty/(1 - \theta_1^4)}{1 + |\lambda\alpha| \|\kappa\|_\infty} \leq \|\hat{\gamma}_N\|_\infty \leq \frac{\|\hat{R}_N\|_\infty/(1 - \theta_1^4)}{1 - |\lambda\alpha| \|\kappa\|_\infty}, \tag{19}$$

when $1 - |\lambda\alpha| \|\kappa\|_\infty > 0$. Here, $\|\kappa\|_\infty$ is the infinity norm of the integral operator [5] indicated in Equation (14).

A program was developed using $Mathematica^{TM}$, Version 2.2 on a NeXT TurboStation having 16 MBytes of memory. Table 1 presents a comparision of steady-state results using $\hat{f}_N^4(\eta)$ as defined by Equation (17b) between two previous investigations and the current study when $\alpha = 0.5$ ($L = 1$) for various N. From viewing the upper- and lower-error estimates, the results shown when N=12 appear to be accurate to ± 0.001.

Table 2 presents $f_N^4(\eta, t)$, where $\theta_N^4(\eta, t)$ is arrived at through Equation (16a), when $\alpha = 0.5$ ($L = 1$) and N=12. In this case, the functions $\{c_m^N(t)\}_{m=0}^N$ defined by the differential equations in Equation (12) are resolved numerically using a conventional fully explicit, fifth-order, six-stage Runge-Kutta then the function $\Psi_N(\eta, t)$ is reconstructed through Equation (8). The time step used in presenting this table is $\Delta t = 0.2$, which represents a relatively large time step. It was found that the numbers presented here when compared to smaller time steps ($\Delta t = 0.1$, 0.05) converged to six places of accuracy with exception of a few (rare) occasions. In contrast, Prasad and Hering [1] often required time steps of up to 20 times smaller then used in the present study.

The alternative formulation described in this communication illustrates that an effective and accurate orthogonal collocation method can be conceived and applied to transient radiative transport.

	Heaslet-Warming	Prasad-Hering	Present Investigation, Equation (14).			
η			N = 6	N = 8	N = 10	N = 12
-1	0.756	0.760	0.758186	0.758159	0.758152	0.758149
-0.8	0.698	0.692	0.693707	0.694002	0.694616	0.694771
-0.6	0.646	0.642	0.642139	0.643344	0.642915	0.642706
-0.4	0.590	0.594	0.594817	0.594241	0.593962	0.594316
-0.2	0.551	0.545	0.547736	0.54636	0.547025	0.546715
0	0.500	0.499	0.5	0.5	0.5	0.5
0.2	0.449	0.452	0.452264	0.45364	0.452975	0.453285
0.4	0.410	0.405	0.405183	0.405759	0.406038	0.405684
0.6	0.354	0.355	0.357861	0.356656	0.357085	0.357294
0.8	0.302	0.305	0.306293	0.305998	0.305384	0.305229
1	0.244	0.240	0.241814	0.241841	0.241848	0.241851
Error Bounds for the present study, Equation (19):						
Upper-Error Estimate:			0.004533	0.002587	0.001667	0.001161
Lower-Error Estimate:			0.0008849	0.0005050	0.0003255	0.0002267

Table 1. Steady-state solution $\hat{f}_N^4(\eta)$ when $\alpha = 0.5$ and compared with previous investigations [1]. The error estimates make use of Equation (19).

η	t = 0.2	t = 1	t = 5	t = 15
-1	0.0622082	0.423464	0.749361	0.758148
-0.8	0.0425642	0.297264	0.682926	0.694769
-0.6	0.0326082	0.226019	0.628382	0.642704
-0.4	0.0261886	0.178075	0.577727	0.594314
-0.2	0.021283	0.141842	0.528047	0.546712
0	0.0176481	0.115012	0.479479	0.499997
0.2	0.014785	0.0942173	0.43123	0.453282
0.4	0.0124121	0.0774039	0.382579	0.40568
0.6	0.010571	0.0643619	0.333788	0.35729
0.8	0.00896476	0.0531695	0.282468	0.305225
1	0.00765155	0.0435793	0.222282	0.241848

Table 2. Transient distribution for $f_N^4(\eta, t)$ at equally spaced locations when $N = 12$ and $\alpha = 0.5$.

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A CUMULATIVE VARIABLE FORMULATION FOR TRANSIENT CONDUCTIVE AND RADIATIVE TRANSPORT IN PARTICIPATING MEDIA

by

Jay I. Frankel, Ph.D.
Associate Professor
Mechanical and Aerospace Engineering Department
University of Tennessee
Knoxville, Tennessee 37996-2210

Summary

A new mathematical formulation is proposed for transient conductive and radiative transport in a participating gray, isotropically scattering plane-parallel medium. The methodology can be easily extended to include numerous additional effects. A systematic and unified treatment is presented using cumulative variables which allows for high-order integration using standard initial-value methods in the temporal variable while allowing for an effective orthogonal collocation method to be implemented in the spatial variable. A spectral approach is incorporated in the present context where Chebyshev polynomials of the first kind are used as the basis functions. This paper illustrates the methodology and presents some comparisons with previously reported works.

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1. Introduction

Computational investigations studying transient combined radiative and conductive transport in participating media have recaptured the interest and attention of researchers over the past ten years. This rejuvenation is in part due to new and exciting engineering applications requiring detailed knowledge of temperature and flux distributions inside a medium. The computational prowess of today's machines allow researchers to perform detailed analyses and obtain clear graphical outputs from which interpretation is quickly realized. Today, scientific endeavors requiring dynamical considerations include the study of: heat transfer in ceramic diesel liners [1], transient responses to volumetrically scattering heat shields [2,3], transient studies of high-temperature windows [4], deicing of solids through radiant heating [5], transient combustion of fuel droplets [6], dynamic investigations involving packed-beds, transient responses in active thermal insulation systems [7] and porous thermal insulations [8] as well as numerous other physical applications. Additionally, thermo-mechanical aspects involving semitransparent materials have been investigated for practical assessment in numerous applications [9].

From a review of the literature, the typical setting for most computational investigations begins with the conventional differential form of the heat equation and either the integral or integro-differential form of the linearized Boltzmann transport equation [10-18]. Approximate formulations of the transport equation have also appeared in several studies. Finite difference and finite element methods have been successfully implemented in solving the heat equation while numerous approaches have been applied to the two forms of the radiative equation of transfer. An additional entry to the numerical scene over the past ten years involves the application of Green's functions [19,20]. Boundary integral methods have also made an impact in the investigation of nonlinear heat transfer studies [21]. It is interesting to note that numerous numerical methods have been proposed for solving the transient radiative/conductive heat transfer problem but few investigations have considered alternative formulations which may lead to simplified and unified numerical treatments.

This paper offers a new and unified formulation for mixed-mode, transient radiative and conductive transport in a participating medium where a common computational thread is interwoven into the formulation of the entire system of equations. Generalizations can easily be inferred by the reader through the systematic formulation and discussion offered here. This paper is presented in six major sections which cover the development of the concept to the presentation of some initial findings. Section 2 presents the mathematical formulation of a classical situation. Section 3 presents the new formulation based on the introduction of cumulative variables [22,23]. Section 4 proposes a simple numerical method for solving the system of coupled, transient, integral and differential equations. Section 5 offers some preliminary calculations and comparisons with previously reported works. Finally, Section 6 offers some conclusions and recommendations for future considerations.

2. Mathematical Formulation

In the present context, consider the transient heat equation in the presence of both conductive and volumetric radiative effects in a plane-parallel, isotropically scattering, gray medium [11]

$$\frac{1}{\alpha^2}\frac{\partial^2\theta}{\partial\eta^2}(\eta,\xi)-\frac{1}{N_{cr}}(1-\omega)[\theta^4(\eta,\xi)-G(\eta,\xi)]=\frac{\partial\theta}{\partial\xi}(\eta,\xi),\quad \eta\epsilon(-1,1),\quad \xi>0, \tag{1a}$$

subject to the auxiliary conditions

$$\theta(-1,\xi)=1, \tag{1b}$$

$$\theta(1,\xi)=\theta_2,\quad \xi>0, \tag{1c}$$

and

$$\theta(\eta,0)=\theta_i,\quad \eta\epsilon[-1,1]. \tag{1d}$$

The isothermal boundary conditions are presented in Eqs. $(1b,c)$ while the initial condition is given in Eq. $(1d)$. Here, α is the dimensionless half-width of the slab, N_{cr} is the conventional conduction-radiation parameter, and ω is the single-scattering albedo. The spatial domain is presented in the region $\eta\epsilon[-1,1]$ in anticipation of the choice of orthogonal functions to be introduced in the next section.

The integral form of the radiative equation of transfer in the presence of black surfaces can be written as [11]

$$G(\eta,\xi)=\frac{1}{2}\Big[E_2(\alpha(1+\eta))+\theta_2^4E_2(\alpha(1-\eta)) \tag{2}$$

$$+\alpha\int_{\eta_o=-1}^{1}[(1-\omega)\theta^4(\eta_o,\xi)+\omega G(\eta_o,\xi)]E_1(\alpha|\eta-\eta_o|)d\eta_o\Big].$$

Here, $E_n(z)$ represents the n^{th} exponential integral function [24] where $E_1(z)$ contains a well-known logarithmic (weak) singularity as $z\to 0$.

Before proceeding further, two dimensionless flux relations are presented. The dimensionless net radiative heat flux can be expressed as

$$Q^r(\eta,\xi)=\frac{1}{2}\Big[E_3(\alpha(1+\eta))-\theta_2^4E_3(\alpha(1-\eta))$$

$$+\alpha\int_{\eta_o=-1}^{\eta}[(1-\omega)\theta^4(\eta_o,\xi)+\omega G(\eta_o,\xi)]E_2(\alpha(\eta-\eta_o))d\eta_o \tag{3}$$

$$-\alpha\int_{\eta_o=\eta}^{1}[(1-\omega)\theta^4(\eta_o,\xi)+\omega G(\eta_o,\xi)]E_2(\alpha(\eta_o-\eta))d\eta_o\Big],$$

while the dimensionless net heat flux can be written as

$$Q(\eta,\xi) = -\frac{1}{\alpha}\frac{\partial \theta}{\partial \eta}(\eta,\xi) + \frac{1}{N_{cr}}Q^r(\eta,\xi), \quad \eta\epsilon[-1,1], \quad \xi \geq 0. \tag{4}$$

The cumulative variable approach [23] begins by a partial decomposition of the differential operators displayed in Eq. (1). That is, the approach begins by integrating the temporal variable ξ in both the heat equation displayed in Eq. (1) to get

$$\theta(\eta,\xi) = \theta_i + \frac{1}{\alpha^2}\frac{\partial^2 \Psi_1}{\partial \eta^2}(\eta,\xi) - \frac{(1-\omega)}{N_{cr}}[\Psi_2(\eta,\xi) - \Psi_3(\eta,\xi)], \quad \eta(-1,1), \quad \xi \geq 0, \tag{5a}$$

and in the integral form of the radiative equation of transfer, as displayed in Eq. (2), to arrive at

$$\Psi_3(\eta,\xi) = h(\eta,\xi) + \frac{\alpha}{2}\int_{\eta_o=-1}^{1}[(1-\omega)\Psi_2(\eta_o,\xi) + \omega\Psi_3(\eta_o,\xi)]E_1(\alpha|\eta-\eta_o|)d\eta_o,$$

$$\eta\epsilon[-1,1], \quad \xi \geq 0 \tag{5b}$$

with

$$h(\eta,\xi) = \frac{\xi}{2}[E_2(\alpha(1+\eta)) + \theta_2^4 E_2(\alpha(1-\eta))], \tag{5c}$$

and where the three new cumulative variables $\{\Psi_k(\eta,\xi)\}_{k=1}^3$ are defined as

$$\Psi_1(\eta,\xi) \equiv \int_{\xi_o=0}^{\xi} \theta(\eta,\xi_o)d\xi_o, \tag{6a}$$

$$\Psi_2(\eta,\xi) \equiv \int_{\xi_o=0}^{\xi} \theta^4(\eta,\xi_o)d\xi_o, \tag{6b}$$

$$\Psi_3(\eta,\xi) \equiv \int_{\xi_o=0}^{\xi} G(\eta,\xi_o)d\xi_o. \tag{6c}$$

This formulation is clearly in contrast with other past investigations [1-18] and even those using a Green's function or boundary element approach [21] since only the temporal portion of the linear operator is inverted in the present context. This reformulation blends the concept offered by Frankel and Choudhury [22] when investigating the integro-differential problem of Volterra [26], and the notion offered by Kumar and Sloan [25] when investigating Hammerstein integral equations. Next, Eqs. $(6a-c)$ are differentiated with respect to the temporal variable, ξ to get

$$\frac{\partial \Psi_1}{\partial \xi}(\eta,\xi) = \theta(\eta,\xi), \tag{7a}$$

$$\frac{\partial \Psi_2}{\partial \xi}(\eta, \xi) = \theta^4(\eta, \xi), \tag{7b}$$

$$\frac{\partial \Psi_3}{\partial \xi}(\eta, \xi) = G(\eta, \xi), \tag{7c}$$

respectively. Upon substituting Eq. $(5a)$ into Eqs. $(7a, b)$, we obtain

$$\frac{\partial \Psi_1}{\partial \xi}(\eta, \xi) = \theta_i + \frac{1}{\alpha^2}\frac{\partial^2 \Psi_1}{\partial \eta^2}(\eta, \xi) - \frac{(1-\omega)}{N_{cr}}[\Psi_2(\eta, \xi) - \Psi_3(\eta, \xi)], \quad \eta(-1,1), \quad \xi > 0, \tag{8a}$$

$$\frac{\partial \Psi_2}{\partial \xi}(\eta, \xi) = \left[\theta_i + \frac{1}{\alpha^2}\frac{\partial^2 \Psi_1}{\partial \eta^2}(\eta, \xi) - \frac{(1-\omega)}{N_{cr}}[\Psi_2(\eta, \xi) - \Psi_3(\eta, \xi)]\right]^4, \quad \eta(-1,1), \quad \xi > 0. \tag{8b}$$

Equations $(5b, 8a, 8b)$ represent the new system of equations from which a unified numerical method can be implemented. It is easy to see through viewing Eqs. $(6a-c)$ that the initial conditions for the cumulative variables $\{\Psi_k(\eta, \xi)\}_{k=1}^3$ are

$$\Psi_k(\eta, 0) = 0, \quad k = 1, 2, 3, \quad \eta \epsilon [-1, 1]. \tag{9}$$

The known temperature boundary conditions can be recast into the cumulative variable formulation through Eqs. $(7a, b)$,

$$\frac{\partial \Psi_1}{\partial \xi}(-1, \xi) = 1, \tag{10a}$$

$$\frac{\partial \Psi_2}{\partial \xi}(-1, \xi) = 1, \tag{10b}$$

and

$$\frac{\partial \Psi_1}{\partial \xi}(1, \xi) = \theta_2, \tag{10c}$$

$$\frac{\partial \Psi_2}{\partial \xi}(1, \xi) = \theta_2^4, \quad \xi > 0. \tag{10d}$$

The mathematical formulation in the cumulative variables is now well-posed and ready for further analysis.

Some important observations can be made contrasting the conventional formulation as indicated by Eqs. (1,2) with the new cumulative variable formulation as presented in Eqs. $(5b, 8a, 8b)$. The most apparent differences between the conventional formulation and the present notion lies in the observation that; i) the present system of equations is unified in structure and can be resolved using a single and consistent numerical method, ii) the nonlinearities are rearranged into positions conducive to orthogonal collocation, and iii) three dependent variables are present in the new formulation rather than the usual two dependent variables associated with the conventional formulation.

3. Computational Methodology

In this section, an orthogonal collocation method is presented for finding an approximate solution to $\Psi_k(\eta, t)$, $k = 1, 2, 3$ as shown in Eqs. $(5b, 8a, 8b)$ and subject to the initial conditions shown in (9) and boundary conditions displayed in Eqs. $(10a-d)$. The physical variables can be reconstructed once the cumulative variables are resolved satisfactorily.

Let the unknown functions $\Psi_k(\eta, t)$, $k = 1, 2, 3$ be formally represented by the series expansions

$$\Psi_1(\eta, \xi) = \sum_{m=0}^{\infty} a_m(\xi) T_m(\eta), \tag{11a}$$

$$\Psi_2(\eta, \xi) = \sum_{m=0}^{\infty} b_m(\xi) T_m(\eta), \tag{11b}$$

$$\Psi_3(\eta, \xi) = \sum_{m=0}^{\infty} c_m(\xi) T_m(\eta), \quad \eta \epsilon [-1, 1], \quad \xi \geq 0, \tag{11c}$$

where the basis functions $\{T_m(\eta)\}_{m=0}^{\infty}$ are chosen as the Chebyshev polynomials of the first kind [13,17] and are expressible as

$$T_m(\eta) = cos[m(cos^{-1}\eta)], \quad m = 0, 1, ..., N. \tag{12}$$

Other forms for the expansions of $\{\Psi(\eta, \xi)\}_{k=1}^{3}$ are also possible. These forms may include terms which account for the boundary conditions. Chebyshev polynomials have numerous exploitable features [27,28] and have successfully been used in studies involving fluid mechanics [29], solid mechanics [30], conduction [31] and radiative transport [23,32]. The unknown time-varying expansion coefficients requiring resolution are denoted as $\{a_m(\xi), b_m(\xi), c_m(\xi)\}_{m=0}^{\infty}$. In practice, we must truncate this series representation at a finite number of terms, say at order N. Thus, we denote the N^{th}-order approximation to $\Psi_k(\eta, \xi)$ as $\Psi_k^N(\eta, \xi)$, $k = 1, 2, 3$ namely

$$\Psi_1(\eta, \xi) \approx \Psi_1^N(\eta, \xi) = \sum_{m=0}^{N} a_m^N(\xi) T_m(\eta), \tag{13a}$$

$$\Psi_2(\eta, \xi) \approx \Psi_2^N(\eta, \xi) = \sum_{m=0}^{N} b_m^N(\xi) T_m(\eta), \tag{13b}$$

$$\Psi_3(\eta, \xi) \approx \Psi_3^N(\eta, \xi) = \sum_{m=0}^{N} c_m^N(\xi) T_m(\eta), \quad \eta \epsilon [-1, 1], \quad \xi \geq 0, \tag{13c}$$

where $a_m^N(\xi), b_m^N(\xi), c_m^N(\xi)$ represent approximations to $a_m(\xi), b_m(\xi), c_m(\xi)$, respectively for each fixed m in the finite set.

Upon substituting the series representations shown in Eqs. $(13a-c)$ for $\{\Psi_k^N(\eta,\xi)\}_{k=1}^3$ into Eqs. $(8a, 8b, 5b)$, we arrive at

$$R_1^N(\eta,\xi) + \sum_{m=0}^{N} \frac{da_m^N}{d\xi}(\xi)T_m(\eta) = \theta_i + \frac{1}{\alpha^2}\sum_{m=0}^{N} a_m^N(\xi)T_m''(\eta)$$

$$-\frac{(1-\omega)}{N_{cr}}\sum_{m=0}^{N}[b_m^N(\xi) - c_m^N(\xi)]T_m(\eta), \tag{14a}$$

and

$$R_2^N(\eta,\xi) + \sum_{m=0}^{N} \frac{db_m^N}{d\xi}(\xi)T_m(\eta) \tag{14b}$$

$$= \left[\theta_i + \frac{1}{\alpha^2}\sum_{m=0}^{N} a_m^N(\xi)T_m''(\eta) - \frac{(1-\omega)}{N_{cr}}\sum_{m=0}^{N}[b_m^N(\xi) - c_m^N(\xi)]T_m(\eta)\right]^4, \quad \eta\epsilon(-1,1), \quad \xi > 0,$$

and

$$R_3^N(\eta,\xi) + \sum_{m=0}^{N} c_m^N(\xi)T_m(\eta) = h(\eta,\xi) + \frac{\alpha(1-\omega)}{2}\sum_{m=0}^{N} b_m^N(\xi)A_m^\alpha(\eta)$$

$$+\frac{\alpha\omega}{2}\sum_{m=0}^{N} c_m^N(\xi)A_m^\alpha(\eta), \quad \eta\epsilon[-1,1], \quad \xi \geq 0, \tag{14c}$$

respectively and where $A_m^\alpha(\eta)$, $m = 0, 1, ..., N$ is defined as

$$A_m^\alpha(\eta) \equiv \int_{\eta_o=-1}^{1} T_m(\eta_o)E_1(\alpha|\eta - \eta_o|)d\eta_o, \tag{14d}$$

which can be analytically integrated to yield

$$A_m^\alpha(\eta) = -\sum_{k=0}^{m} \frac{1}{\alpha^{k+1}}\left[(-1)^k T_m^{(k)}(-1)E_{k+2}[\alpha(1+\eta)] + T_m^{(k)}(1)E_{k+2}[\alpha(1-\eta)]\right]$$

$$+ \quad 2\sum_{j=0}^{[\frac{m}{2}]} \frac{1}{\alpha^{2j+1}}T_m^{(2j)}(\eta)E_{2j+2}(0), \quad m = 0, 1, ..., N. \tag{14e}$$

Here, we denote the k^{th} derivative of the m^{th} Chebyshev polynomial of the first kind as $T_m^{(k)}(\eta)$, and where $[\frac{m}{2}]$ is interpreted as the integer result of $(m/2)$. Here, $R_k^N(\eta,\xi)$, $k = 1, 2, 3$ represent the local and instantaneous residual functions as required in order to maintain the equality displayed in Eqs. $(14a-c)$.

Using the finite series representations for $\Psi_k^N(\eta,\xi)$, $k=1,2$, in conjunction with the specified boundary conditions shown by Eqs. $(10a-d)$, we obtain

$$R_1^N(-1,\xi)+\sum_{m=0}^{N}\frac{da_m^N}{d\xi}(\xi)T_m(-1)=1, \tag{15a}$$

$$R_1^N(1,\xi)+\sum_{m=0}^{N}\frac{da_m^N}{d\xi}(\xi)T_m(1)=\theta_2, \tag{15b}$$

and

$$R_2^N(-1,\xi)+\sum_{m=0}^{N}\frac{db_m^N}{d\xi}(\xi)T_m(-1)=1, \tag{15c}$$

$$R_2^N(1,\xi)+\sum_{m=0}^{N}\frac{db_m^N}{d\xi}(\xi)T_m(1)=\theta_2^4, \quad \xi>0, \tag{15d}$$

while the appropriate initial conditions shown in Eq. (9) for $k=1,2$ become

$$R_1(\eta,0)+\sum_{m=0}^{N}a_m^N(0)T_m(\eta)=0, \tag{15e}$$

and

$$R_2(\eta,0)+\sum_{m=0}^{N}b_m^N(0)T_m(\eta)=0, \quad \eta\epsilon[-1,1]. \tag{15f}$$

Unless the exact solution to $\Psi_k(\eta,t)$, $k=1,2,3$ at any instant in time, $\xi\geq 0$, is a linear combination of $\{T_m(\eta)\}_{m=0}^N$, we cannot obtain $\{a_m^N(\xi),b_m^N(\xi),c_m^N(\xi)\}_{m=0}^N$ which makes $R_k^N(\eta,\xi)$ vanish for $k=1,2,3$, $\xi\geq 0$ and $\eta\epsilon[-1,1]$. However, we can obtain suitable time varying expansion coefficients by making the residuals indicated in Eqs. $(14a-c)$ and $(15a-f)$ small in some sense. Let us define the inner product of two real-valued functions $\hat{g}_1(t)$ and $\hat{g}_2(t)$ as

$$\langle\hat{g}_1,\hat{g}_2\rangle_{w_k}=\int_{s=-1}^{1}w_k(s)\hat{g}_1(s)\hat{g}_2(s)ds, \tag{16a}$$

and the corresponding norm as

$$\|\hat{g}_1\|_{w_k}=\left[\int_{s=-1}^{1}w_k(s)\hat{g}_1^2(s)ds\right]^{\frac{1}{2}}, \tag{16b}$$

where $w_k(s)$ is a non-negative, real and integrable weight function.

For the collocation method, the orthogonality relation becomes

$$\left\langle R_k^N(\eta,\xi),\Omega_j(\eta)\right\rangle_{w_j}=0, \quad k=1,2,3, \quad \xi\geq 0, \tag{17a}$$

where $\Omega_j(\eta)=1, w_j=\delta(\eta-\eta_j)$, $j=0,1,...,N$. Here the Dirac delta function is denoted by δ while the $N+1$ collocation points are indicated by η_j, $j=0,1,...,N$ and are defined by the closed rule [28]

$$\eta_j=cos(\frac{\pi j}{N}), \quad j=0,1,...,N. \tag{17b}$$

By choosing this set of $N+1$ collocation points, we ensure that $R_k^N(\pm 1,\xi)=0$, $k=1,2,3$ for $\xi>0$ in Eqs. $(14a-c)$. Note that from Eq. $(17b)$, one interprets that $\eta_0=1$ and $\eta_N=-1$. Error and convergence analyses [33] have been performed illustrating the merit of this choice of basis functions and collocation points in the study of the radiative equation of transfer in an isotropically scattering medium.

Applying the orthogonality condition displayed by Eq. $(17a)$ to Eq. $(14a)$ for $j=1,...,N-1$ produces

$$\sum_{m=0}^{N}\frac{da_m^N}{d\xi}(\xi)T_m(\eta_j)=\theta_i+\frac{1}{\alpha^2}\sum_{m=0}^{N}a_m^N(\xi)T_m''(\eta_j)$$

$$-\frac{(1-\omega)}{N_{cr}}\sum_{m=0}^{N}[b_m^N(\xi)-c_m^N(\xi)]T_m(\eta_j), \quad j=1,2,...,N-1, \tag{18a}$$

along with imposing the boundary constraints

$$\sum_{m=0}^{N}\frac{da_m^N}{d\xi}(\xi)T_m(\eta_0)=\theta_2, \tag{18b}$$

$$\sum_{m=0}^{N}\frac{da_m^N}{d\xi}(\xi)T_m(\eta_N)=1, \quad \xi>0. \tag{18c}$$

Upon similar application of Eq. $(17a)$ on Eq. $(14b)$, we arrive at

$$\sum_{m=0}^{N}\frac{db_m^N}{d\xi}(\xi)T_m(\eta_j) \tag{18d}$$

$$=\left[\theta_i+\frac{1}{\alpha^2}\sum_{m=0}^{N}a_m^N(\xi)T_m''(\eta_j)-\frac{(1-\omega)}{N_{cr}}\sum_{m=0}^{N}[b_m^N(\xi)-c_m^N(\xi)]T_m(\eta_j)\right]^4, \quad j=1,2,...,N-1,$$

along with imposing the boundary constraints

$$\sum_{m=0}^{N}\frac{db_m^N}{d\xi}(\xi)T_m(\eta_0)=\theta_2^4, \tag{18e}$$

$$\sum_{m=0}^{N} \frac{db_m^N}{d\xi}(\xi) T_m(\eta_N) = 1, \quad \xi > 0, \tag{18f}$$

while Eq. (14c) reduces to

$$\sum_{m=0}^{N} c_m^N(\xi) T_m(\eta_j) = h(\eta_j, \xi) + \frac{\alpha(1-\omega)}{2} \sum_{m=0}^{N} b_m^N(\xi) A_m^\alpha(\eta_j)$$

$$+\frac{\alpha\omega}{2} \sum_{m=0}^{N} c_m^N(\xi) A_m^\alpha(\eta_j), \quad \xi \geq 0, \quad j = 0, 1, ..., N. \tag{18g}$$

Under the orthogonality relation defined by Eq. (17a), the initial conditions displayed in Eqs. (15e, f) reduce to

$$a_m^N(0) = b_m^N(0) = 0, \quad m = 0, 1, ..., N, \quad \eta \epsilon [-1, 1]. \tag{18h}$$

Equation (18g) can be alternately expressed as

$$\sum_{m=0}^{N} c_m^N(\xi) [T_m(\eta_j) - \frac{\alpha\omega}{2} A_m^\alpha(\eta_j)] = h(\eta_j, \xi) +$$

$$\frac{\alpha(1-\omega)}{2} \sum_{m=0}^{N} b_m^N(\xi) A_m^\alpha(\eta_j), \quad \xi \geq 0, \quad j = 0, 1, ..., N, \tag{19}$$

which naturally leads to the matrix form $A\overline{c}(\xi) = \overline{f}(\xi)$ where the vector $\overline{c}(\xi)$ contains the unknown time varying coefficients; i.e., $\overline{c}(\xi) = [c_0^N(\xi), c_1^N(\xi), ..., c_N^N(\xi)]^T$. Clearly, we can express $\overline{c}(\xi)$ as

$$\overline{c}(\xi) = A^{-1}\overline{f}(\xi), \quad \xi \geq 0, \tag{20a}$$

when $|A| \neq 0$ and where the coefficient matrix A is given by

$$A = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0N} \\ a_{10} & a_{11} & \cdots & a_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N0} & a_{N1} & \cdots & a_{NN} \end{pmatrix} \tag{20b}$$

while $\overline{f}(\xi) = [f_0^N(\xi), f_1^N(\xi), ..., f_N^N(\xi)]^T$. The components of the coefficient matrix are thus

$$a_{jm} = T_m(\eta_j) - \frac{\alpha\omega}{2} A_m^\alpha(\eta_j), \quad j = 0, 1, ..., N, \quad m = 0, 1, ..., N, \tag{20c}$$

while the unknown time varying components of $\overline{f}(\xi)$ are given as

$$f_j^N(\xi) = h(\eta_j, \xi) + \frac{\alpha(1-\omega)}{2} \sum_{m=0}^{N} b_m^N(\xi) A_m^\alpha(\eta_j), \quad j = 0, 1, ..., N. \tag{20d}$$

This is quite useful since we can analytically eliminate $\overline{c}(\xi) = [c_0^N(\xi), c_1^N(\xi), ..., c_N^N(\xi)]^T$ from $(18a, d)$ which alleviates the need to determine these coefficients explicitly. Thus, we are left with two matrix differential equations requiring resolution for the time varying expansion coefficients $a_m^N(\xi), b_m^N(\xi)$, $m = 0, 1, ..., N$ instead of two matrix differential equations and an algebraic system.

Likewise, Eqs. $(18a - f)$ can be expressed, using compact matrix notation, as

$$B \frac{d\overline{a}(\xi)}{d\xi} = \overline{g}(\xi), \tag{21a}$$

and

$$B \frac{d\overline{b}(\xi)}{d\xi} = \overline{h}(\xi), \quad \xi > 0, \tag{21b}$$

where the common and known $(N+1)$x$(N+1)$ coefficient matrix B is

$$B = \begin{pmatrix} b_{00} & b_{01} & \dots & b_{0N} \\ b_{10} & b_{11} & \dots & b_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N0} & b_{N1} & \dots & b_{NN} \end{pmatrix} \tag{22a}$$

where each entry is given by

$$b_{jm} = T_m(\eta_j), \quad j = 0, 1, ..., N \quad m = 0, 1, ..., N, \tag{22b}$$

while the vectors $\overline{g}(\xi) = [g_0^N(\xi), g_1^N(\xi), ..., g_N^N(\xi)]^T$ and $\overline{h}(\xi) = [h_0^N(\xi), h_1^N(\xi), ..., h_N^N(\xi)]^T$ have components

$$g_0^N(\xi) = \theta_2, \tag{22c}$$

$$g_j^N(\xi) = \theta_i + \frac{1}{\alpha^2} \sum_{m=0}^{N} a_m^N(\xi) T_m''(\eta_j) - \frac{(1-\omega)}{N_{cr}} \sum_{m=0}^{N} [b_m^N(\xi) - c_m^N(\xi)] T_m(\eta_j),$$
$$j = 1, 2, ..., N-1, \tag{22d}$$

$$g_N^N(\xi) = 1, \tag{22e}$$

while

$$h_j^N(\xi) = [g_j^N(\xi)]^4, \quad j = 0, 1, ..., N. \tag{22f}$$

The vectors defined as $\overline{a}(\xi)$ and $\overline{b}(\xi)$ are expressible as $\overline{a}(\xi) = [a_0^N(\xi), a_1^N(\xi), ..., a_N^N(\xi)]^T$ and $\overline{b}(\xi) = [b_0^N(\xi), b_1^N(\xi), ..., b_N^N(\xi)]^T$. Inverting the known coefficient matrix B in Eqs.

$(21a, b)$ produces the system of nonlinear initial value problems requiring numerical approximation. Once the time varying coefficients $\{a_m^N(\xi)\}_{m=0}^N$ and $\{b_m^N(\xi)\}_{m=0}^N$ for $\xi \geq 0$ are determined then the approximation of the necessary field variables $\theta_N(\eta, \xi)$ and $Q_N^r(\eta, \xi)$ can be obtained.

Construction of the Physical Variables

Three physical variables can now be recovered without major effort. Indeed, the majority of the computational effort lies in calculating the unknown expansion coefficients, $\{a_m^N(\xi)\}_{m=0}^N$ and $\{b_m^N(\xi)\}_{m=0}^N$. The physical variables are constructed during a post-processing procedure and require minimal computational effort. It is advantageous to reconstruct the approximate solution for the dimensionless temperature variable $\theta_N(\eta, \xi)$ using Eq. $(7a)$, namely

$$\theta_N(\eta, \xi) = \frac{\partial \Psi_1^N}{\partial \xi}(\eta, \xi) = \sum_{m=0}^{N} \frac{da_m^N}{d\xi}(\xi) T_m(\eta). \tag{23}$$

The inversion formula shown by Eq. $(5a)$ also will produce results identical to Eq. (23) at the interior collocation points. However, at the endpoints Eq. $(5a)$ will not reproduce the imposed boundary conditions. This makes sense since the boundary conditions are manually imposed (overlayed) into Eq. $(21a, b)$. The right-hand side of Eq. $(21a)$, which has been precalculated, can be used in determining $da_m^N(\xi)/d\xi$, $m = 0, 1, ..., N$ at the discrete times corresponding with the numerical integrator. Inverting the coefficient matrix B shown in Eq. $(21a)$ allows us to obtain numerical values for $da_m^N(\xi)/d\xi$, $m = 0, 1, ..., N$. Note that B^{-1} has already been used and prevously stored.

As indicated in Eq.(3), both $\theta^4(\eta, \xi)$ and $G(\eta, \xi)$ are required in establishing the local radiative heat flux $Q^r(\eta, \xi)$. Following a similar procedure as outlined previously for determining $\theta(\eta, \xi)$, one expresses the approximation of $\theta^4(\eta, \xi)$ as

$$\theta_N^4(\eta, \xi) = \frac{\partial \Psi_2^N}{\partial \xi}(\eta, \xi) = \sum_{m=0}^{N} \frac{db_m^N}{d\xi}(\xi) T_m(\eta). \tag{24}$$

An alternative approach is now offered for determining $G(\eta, \xi)$. This is broached since the approximation displayed by Eq. $(13c)$ would require us to differentiate $\Psi_3^N(\eta, \xi)$ with respect to ξ. This is clearly undesirable since an additional approximation would be incurred in obtaining $dc_m^N(\xi)/d\xi$, $m = 0, 1, ..., N$. Let the approximation for $G(\eta, \xi)$ be written as $G_N(\eta, \xi)$ and expressed by

$$G_N(\eta, \xi) = \sum_{m=0}^{N} \gamma_m^N(\xi) T_m(\eta). \tag{25}$$

Substituting Eqs. (24,25) into Eq. (2), produces

$$R_4^N(\eta, \xi) + \sum_{m=0}^{N} \gamma_m^N(\xi) T_m(\eta) = \frac{1}{2}\Big[E_2(\alpha(1+\eta)) + \theta_2^4 E_2(\alpha(1-\eta))$$

$$+\alpha(1-\omega)\sum_{m=0}^{N}\frac{db_m^N}{d\xi}(\xi)A_m^\alpha(\eta)+\alpha\omega\sum_{m=0}^{N}\gamma_m^N(\xi)A_m^\alpha(\eta)\Big]. \tag{26a}$$

Clearly, $\gamma_m^N(\xi) = dc_m^N(\xi)/d\xi$, $m = 0, 1, ..., N$ and can be explicitly calculated using previously determined results. Applying the orthogonality relation shown in Eq. (17a) on Eq. (26a) produces

$$\sum_{m=0}^{N}\gamma_m^N(\xi)[T_m(\eta_j)-\frac{\alpha\omega}{2}A_m^\alpha(\eta_j)]=\frac{1}{2}\Big[E_2(\alpha(1+\eta_j))+\theta_2^4E_2(\alpha(1-\eta_j))$$

$$+\alpha(1-\omega)\sum_{m=0}^{N}\frac{db_m^N}{d\xi}(\xi)A_m^\alpha(\eta_j)\Big],\quad j=0,1,...,N, \tag{26b}$$

which can be written in the compact matrix form as

$$A\overline{\gamma}(\xi)=\overline{p}(\xi),\quad \xi\geq 0, \tag{26c}$$

where $\overline{\gamma}(\xi)$ denotes the solution vector defined as $\overline{\gamma}(\xi) = [\gamma_0^N(\xi), \gamma_1^N(\xi), ..., \gamma_N^N(\xi)]^T$. The constant coefficient matrix A is identical to that previously expressed by Eq. (20b). At this point, the vector $\overline{p}(\xi) = [p_0^N(\xi), p_1^N(\xi), ..., p_N^N(\xi)]^T$ is completely known at a finite set of discrete times. Each component in the vector $\overline{p}(\xi)$ is given by

$$p_j^N(\xi)=\frac{1}{2}\Big[E_2(\alpha(1+\eta_j))+\theta_2^4E_2(\alpha(1-\eta_j))+\alpha(1-\omega)\sum_{m=0}^{N}\frac{db_m^N}{d\xi}(\xi)A_m^\alpha(\eta_j)\Big], \tag{26d}$$

$$j=0,1,...,N.$$

Clearly, $\overline{\gamma}(\xi)$ is easily recoverable through matrix inversion at the discrete times corresponding the numerical integrator, namely

$$\overline{\gamma}(\xi)=A^{-1}\overline{p}(\xi). \tag{26e}$$

Using these variables, the local radiative heat flux, as shown by Eq. (3), can be written as

$$Q^r(\eta,\xi)\approx Q_N^r(\eta,\xi)=\frac{1}{2}\Big[E_3(\alpha(1+\eta))-\theta_2^4E_3(\alpha(1-\eta))$$

$$+\alpha\sum_{m=0}^{N}[(1-\omega)\frac{db_m^N}{d\xi}(\xi)+\omega\gamma_m^N(\xi)]W_{1,m}^\alpha(\eta)$$

$$-\alpha\sum_{m=0}^{N}[(1-\omega)\frac{db_m^N}{d\xi}(\xi)+\omega\gamma_m^N(\xi)]W_{2,m}^\alpha(\eta)\Big],\quad \eta\epsilon[-1,1],\quad \xi\geq 0, \tag{27a}$$

where the functions $W_{1,m}^\alpha(\eta)$, $W_{2,m}^\alpha(\eta)$ for $m = 0, 1, ..., N$ are defined as

$$W^{\alpha}_{1,m}(\eta) \equiv \int_{\eta_o=-1}^{\eta} T_m(\eta_o)E_2(\alpha(\eta-\eta_o))d\eta_o, \tag{27b}$$

and

$$W^{\alpha}_{2,m}(\eta) \equiv \int_{\eta_o=\eta}^{1} T_m(\eta_o)E_2(\alpha(\eta_o-\eta))d\eta_o, \tag{27c}$$

which analytically integrate to

$$W^{\alpha}_{1,m}(\eta) = \sum_{i=0}^{m} \frac{(-1)^i}{\alpha^{i+1}}[T^{(i)}_m(\eta)E_{i+3}(0) - T^{(i)}_m(-1)E_{i+3}(\alpha(1+\eta))], \tag{27d}$$

and

$$W^{\alpha}_{2,m}(\eta) = \sum_{i=0}^{m} \frac{1}{\alpha^{i+1}}[T^{(i)}_m(1)E_{i+3}(\alpha(1-\eta)) - T^{(i)}_m(\eta)E_{i+3}(0)], \quad m=0,1,...,N. \tag{27e}$$

At this point, all the ingredients are available to obtain numerical results for the temperature distribution and radiative, conductive, and total heat flux distributions.

5. Preliminary Results

Preliminary findings are presented illustrating some computational features produced by this formulation. A program was written implementing the expressions and algorithm discussed in Section 4 using the symbolic software package $Mathematica^{TM}$ [34] as implemented on a NeXT computer (NeXTstation Turbo). The computations required in the matrix manipulations were performed symbolically and left in analytic form. A fully explicit fifth-order, six-stage Runge-Kutta [35] was written in the $Mathematica^{TM}$ language. This initial value method merely serves to produce preliminary results. This method will be replaced by an implicit method in order to remove the stability constraints associated with an explicit method.

Some unique features of the new formulation are presented in order to elaborate on the computational attributes of the scheme. Figure 1 illustrates the nature of the time-varying expansion coefficients $a^N_m(\xi)$, $m = 0,1,2,3,4,5$ when $N = 8$, $N_{cr} = 0.1$, $\alpha = 0.5$, $\omega = 0.5$, $\theta_i = \theta_2 = 0$. Accurate representations of these coefficients are critical in reconstructing the cumulative variable $\Psi^N_1(\eta,\xi)$. The well-behaved nature of these functions under the imposed constraints is clearly illuminated by this figure. This type of behavior lends itself to a good approximation. As $\xi \to \infty$, it appears that these coefficients grow no worse than linearly with respect to time. This observation has theoretical basis [23]. It should be noted that the fully explicit numerical integrator's stability constraint becomes evident as N increases thus requiring smaller time steps to be used in order for the numerical method to remain stable. This minor inconvenience can be easily rectified by changing the initial value numerical method [35].

The development of rigorous error estimates for this formulation is presently under consideration. As a preliminary indicator, comparison with other published works is offered. Frankel has developed rigorous error estimates [23,33] and convergence rates [33] using this set of basis functions when investigating an integral form of the transport equation for steady-state radiative transport in a plane-parallel geometry. In transient studies, it is possible to develop estimates based on truncation errors associated with Runge-Kutta methods [35] in the temporal variable. Unfortunately, in complicated physical problems it is often difficult to obtain realistic error estimates. Thus, without rigorous error estimates, only qualitative assessments are available.

As an indicator of numerical accuracy, Tables 1 and 2 are presented comparing several reported works to the present analysis. The set of system parameters chosen corresponds to an example with a wealth of tabular results [16]. Comparison of the present method to other investigations are quite favorable, as shown in Tables 1 and 2, for the case where $N = 8$, $N_{cr} = 0.1$, $\alpha = 0.5$, $\omega = 0.5$, $\theta_i = \theta_2 = 0$ and $\xi = 0.05$. Table 1 presents temperature values at three interior locations. Agreement is apparent when compared with the reported results of Sutton, Barker and Sutton, and Tsai and Lin, as taken from Ref. [16]. Note that the underlined entries in these tables indicate locations which coincide with the collocation points. Correspondingly good results for the net radiative heat flux are displayed in Table 2. The present conclusion is that the proposed methodology produces comparable numerical results to that reported by other studies. Other cases have been considered and compared to tabulated results when available in order to validate the proposed methodology. Finally, letting $N = 6$ in the present formulation produces graphically identical results to $N = 8$.

The second example considers the situation where $N_{cr} = 0.01$, $\omega = 0.2$, $\alpha = 0.5$, $\theta_i = \theta_2 = 0.25$, which was previously reported by Sutton [15]. Figure 2 displays the time-varying expansion coefficients $a_m^N(\xi)$, $m = 0, 1, 2, 3$ when $N = 8$. Similar qualitative features to that of the previous case are indicated in this figure. Figure 3 presents the constructed temperature distribution $\theta_N(\eta, \xi)$ at four distinct times $\xi = 0.005, 0.01, 0.025, 0.05$. Meanwhile, Figure 4 shows the corresponding radiative heat flux distribution $Q_N^r(\eta, \xi)$ at the identical times. This figure produces identical graphical results to that illustrated by Sutton [15].

Some additional remarks are offered concerning the development of the spatial temperature and flux distributions. The numerical solution at the collocation points $\{\eta_k\}_{k=0}^{N}$ will typically be more accurate than at other spatial locations [23,33] owing in part to the oscillatory behavior of the residual function which is well documented [23,33]. Numerical values for the temperatures and radiative heat fluxes at noncollocation positions were obtained through Eqs. (23) and ($27a$), respectively. An alternative procedure which has yet to be explored involves the use of a least-squares method for developing an appropriate polynomial approximation based on minimizing the deviations from the proposed curve fit to the predefined values of the function at the discrete collocation points. An orthogonal basis would be proposed since this choice could alleviate the well-known potential for ill-conditioning associated with the coefficient matrix in the linear system [36, p. 641]. This method would certainly produce fast numerical results when desiring numerical values for the radiative heat flux. Also, careful monitoring of the condition number and the determination of the optimal order of the curve fit based on the variance could be incorporated into a simulation package.

6. Conclusions

The objective of this paper was to present a unified formulation which renders a systematic and unified numerical treatment to transient combined conductive/radiative transport in a participating medium. The numerical procedure postulated and then demonstrated uses a common computational theme for solving the heat equation and equation of radiative transport. This first communication is meant to present the concept and to illuminate some initial findings. Generalization of the present concept to include anisotropic scattering [37], variable thermal properties and nongray optical properties will be addressed in an upcoming investigation. Finally, the choice of a uniform approximation using Chebyshev polynomials could be modified to reflect a different set of orthogonal basis functions or even a set of piecewise continuous basis functions. As an aside, the cumulative variable formulation used in concert with the proposed computational methodology has been successfully demonstrated with regard to a one-dimensional Burger's equation. Thus, this approach possesses broad appeal to a wide range of engineering problems.

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Nomenclature

A = Coefficient matrix, Eq. (20b)

$A_m^\alpha(\eta)$ = Function defined in Eq. (14d)

$\overline{a}(\xi)$ = Vector containing $a_m^N(\xi)$, $m = 0, 1, ..., N$

a_{jm} = Matrix entries for A, Eq. (20c)

$a_m(\xi)$ = Exact expansion coefficients, Eq. (11a)

$a_m^N(\xi)$= Approximate expansion coefficients, Eq. (13a)

B= Coefficient matrix, Eq. (22a)

$\overline{b}(\xi)$ = Vector containing $b_m^N(\xi)$, $m = 0, 1, ..., N$

b_{jm} = Matrix entries for B, Eq. (22b)

$b_m(\xi)$ = Exact expansion coefficients, Eq. (11b)

$b_m^N(\xi)$= Approximate expansion coefficients, Eq. (13b)

$\overline{c}(\xi)$ = Vector containing $c_m^N(\xi)$, $m = 0, 1, ..., N$

$c_m(\xi)$ = Exact expansion coefficients, Eq. (11c)

$c_m^N(\xi)$= Approximate expansion coefficients, Eq. (13c)

c_p = Specific heat

$E_n(z)$= n^{th} exponential integral function

$\overline{f}(\xi)$ = Vector containing $f_j^N(\xi)$, $j = 0, 1, ..., N$

$f_j^N(\xi)$ = Entries for vector $\overline{f}(\xi)$, Eq. (20d)

$G(\eta, \xi)$ = $G^*(\alpha(1 + \eta), \xi)$, Dimensionless Chebyshev incident radiation function

$G^*(\tau, \xi)$= $\frac{\overline{G}(\tau,\xi)}{4n^2\sigma T_r^4}$, Dimensionless incident radiation function, Ref. [11]

$\overline{G}(\tau, \xi)$ = Incident radiation function defined in Ref. [11]

$\overline{g}(\xi)$ = Vector containing $g_j^N(\xi)$, $j = 0, 1, ..., N$

$g_j^N(\xi)$ = Entries for vector $\overline{g}(\xi)$, Eq. (22$c - e$)

$\overline{h}(\xi)$ = Vector containing $h_j^N(\xi)$, $j = 0, 1, ..., N$

$h_j^N(\xi)$ = Entries for vector $\overline{h}(\xi)$, Eq. (22f)

$h(\eta, \xi)$ = Function defined by Eq. (5c)

k = Thermal conductivity

l = Dimensional length of plate

N = N^{th}- order approximation

$N_{cr} = \frac{k\beta}{4n^2\sigma T_r^3}$, Conduction-radiation number, Ref. [11]

n = Index of refraction

$\overline{p}(\xi)$ = Vector containing $p_j^N(\xi)$, $j = 0, 1, ..., N$

$p_j^N(\xi)$ = Entries for vector $\overline{p}(\xi)$, Eq. (26d)

$\overline{Q}^r(\tau,\xi) = \frac{q_r''}{4n^2\sigma T_r^4}$, Dimensionless radiative heat flux, Ref. [11]

$Q^r(\eta,\xi) = \overline{Q}^r(\alpha(1+\eta),\xi)$, Dimensionless radiative heat flux in Chebyshev domain

$Q_N^r(\eta,\xi) = N^{th}$-order approximation to $Q^r(\eta,\xi)$

q_r'' = Dimensional radiative heat flux

$R_k^N(\eta,\xi)$ = Local residual function, $k = 1, 2, 3, 4$

s = Dummy variable

$T_m(\eta) = m^{th}$ Chebyshev polynomial of the first kind

T_r = Reference temperature

t = Time

$W_{1,m}^{\alpha}(\eta)$ = Function defined by Eq. (27b)

$W_{2,m}^{\alpha}(\eta)$ = Function defined by Eq. (27c)

y = Dimensional spatial coordinate

α = Dimensionless half depth, $\frac{\tau_D}{2}$

β = Extinction coefficient, Ref. [11]

δ = Dirac delta function

$\overline{\gamma}(\xi)$ = Vector containing $\gamma_j^N(\xi)$, $j = 0, 1, ..., N$

$\gamma_j^N(\xi)$ = Approximate Expansion coefficients, Eq. (25)

$\eta = \frac{\tau}{\alpha} - 1$, Dimensionless (Chebyshev domain) spatial variable

$\eta_j = j^{th}$ collocation point defined by Eq. (17b)

η_o = Dummy variable

$\theta(\eta,\xi) = \overline{\theta}(\alpha(1+\eta),\xi)$, Dimensionless temperature in Chebyshev domain

$\overline{\theta}(\tau,\xi) = \frac{T}{T_r}$, Dimensionless temperature, Ref. [11]

$\theta_N(\eta,\xi) = N^{th}$ order approximation to $\theta(\eta,\xi)$

θ_i = Initial temperature at $\xi = 0$

θ_2 = Imposed boundary condition at $\eta = 1$

ρ = density

σ = Stefan-Boltzmann constant

$\tau = \beta y$, Optical variable, Ref. [11]

$\tau_D = \beta l$, Optical depth, Ref [11]

ω = Single-scattering albedo

$\xi = \left(\frac{k}{\rho c_p}\right)\beta^2 t$, Dimensionless time, Ref [11]

ξ_o = Dummy variable

$\Psi_k(\eta, \xi)$ = Cumulative variables, $k = 1, 2, 3$

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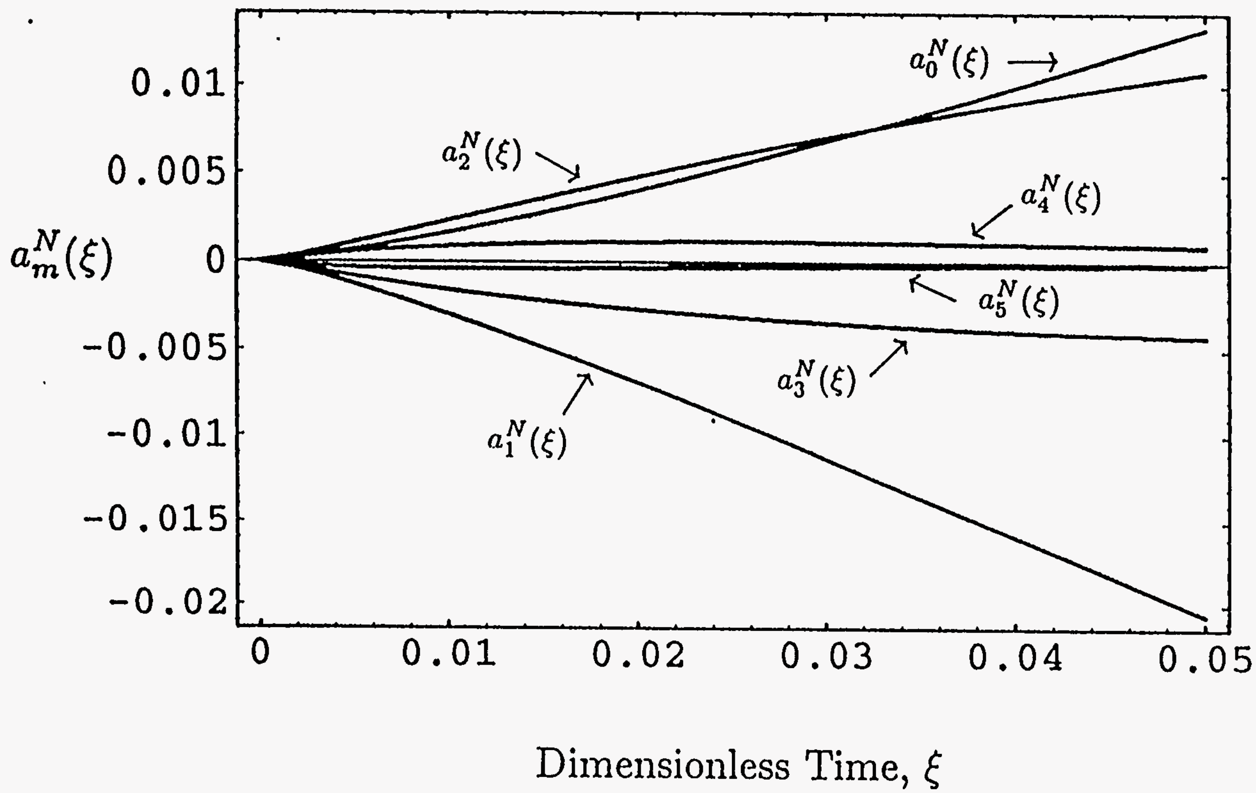

$a_m^N(\xi)$
0.01
0.005
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-0.01
-0.015
-0.02
$a_0^N(\xi)$
$a_2^N(\xi)$
$a_4^N(\xi)$
$a_5^N(\xi)$
$a_3^N(\xi)$
$a_1^N(\xi)$
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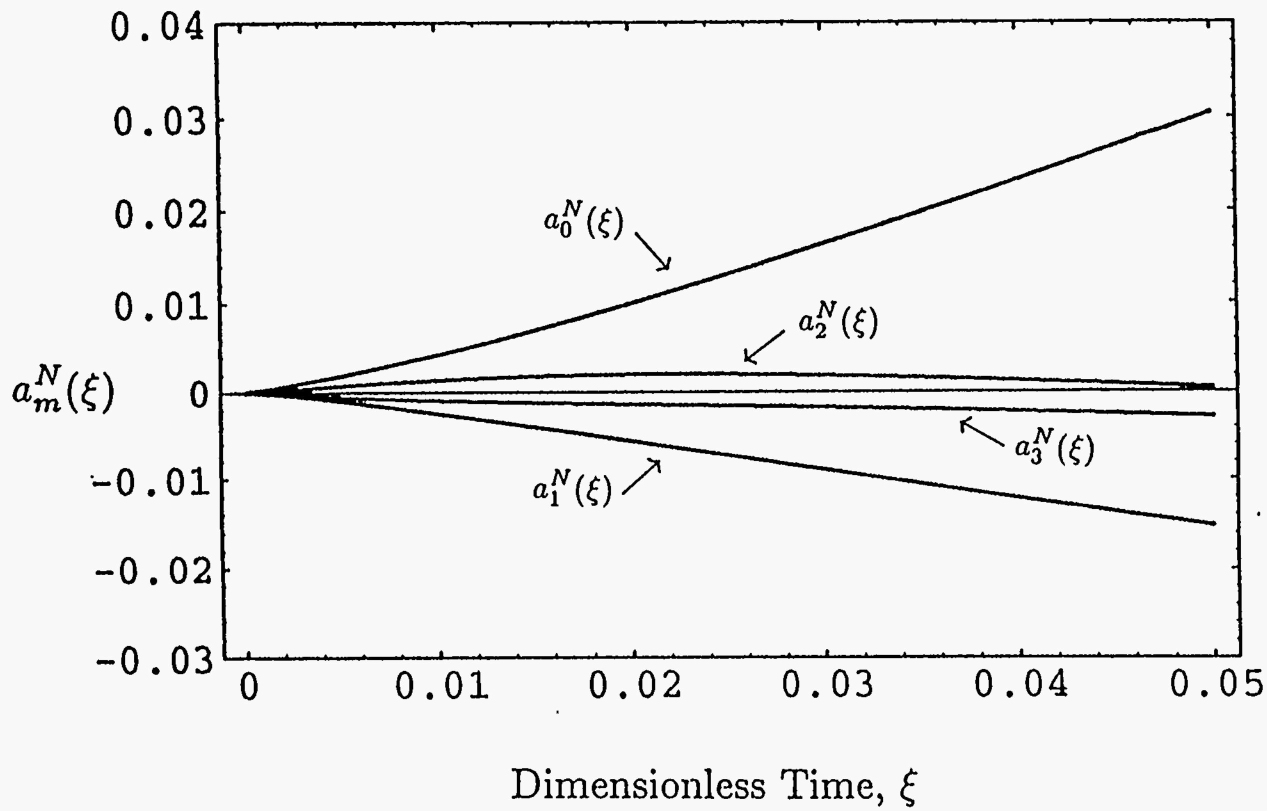
0.03
0.04
0.05
Dimensionless Time, ξ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0.02
0.03
0.04
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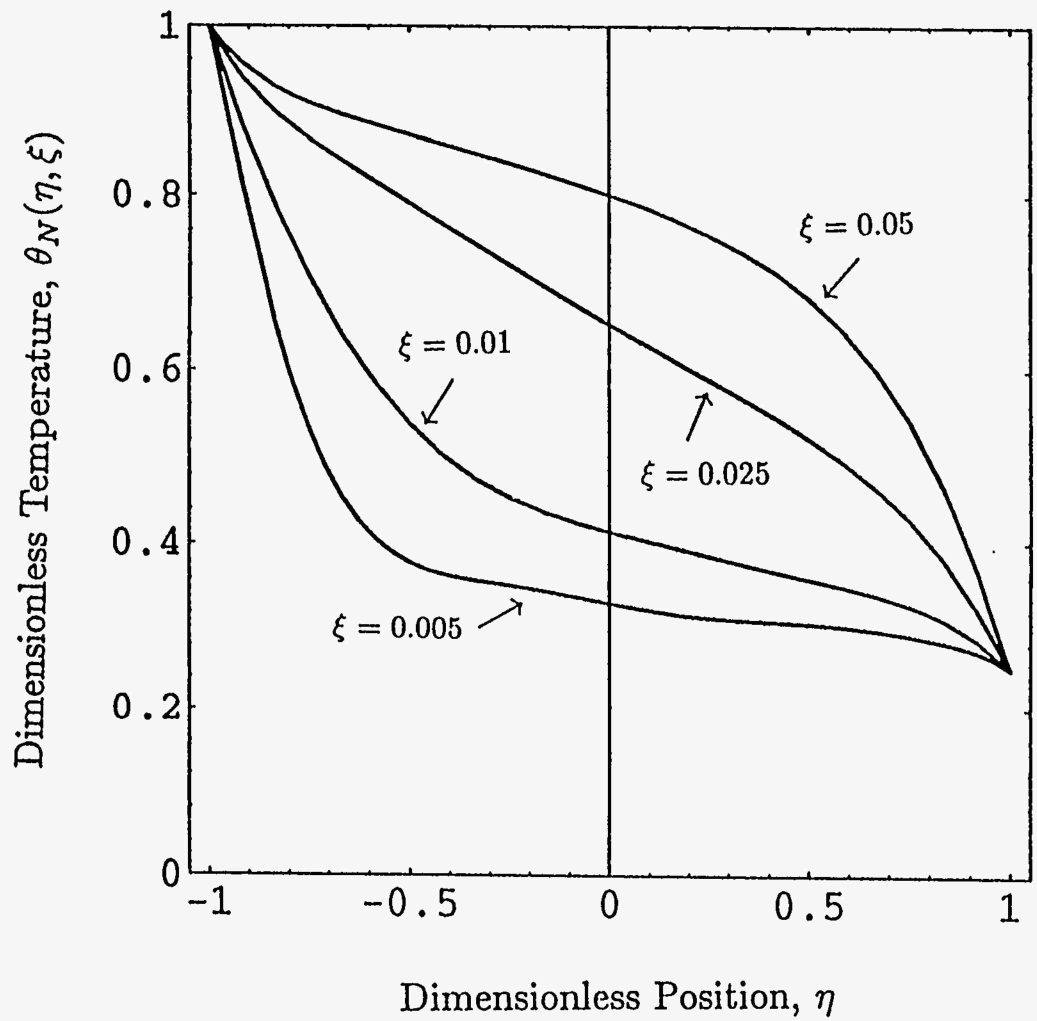
$a_0^N(\xi)$
$a_1^N(\xi)$
$a_2^N(\xi)$
$a_3^N(\xi)$
Dimensionless Time, ξ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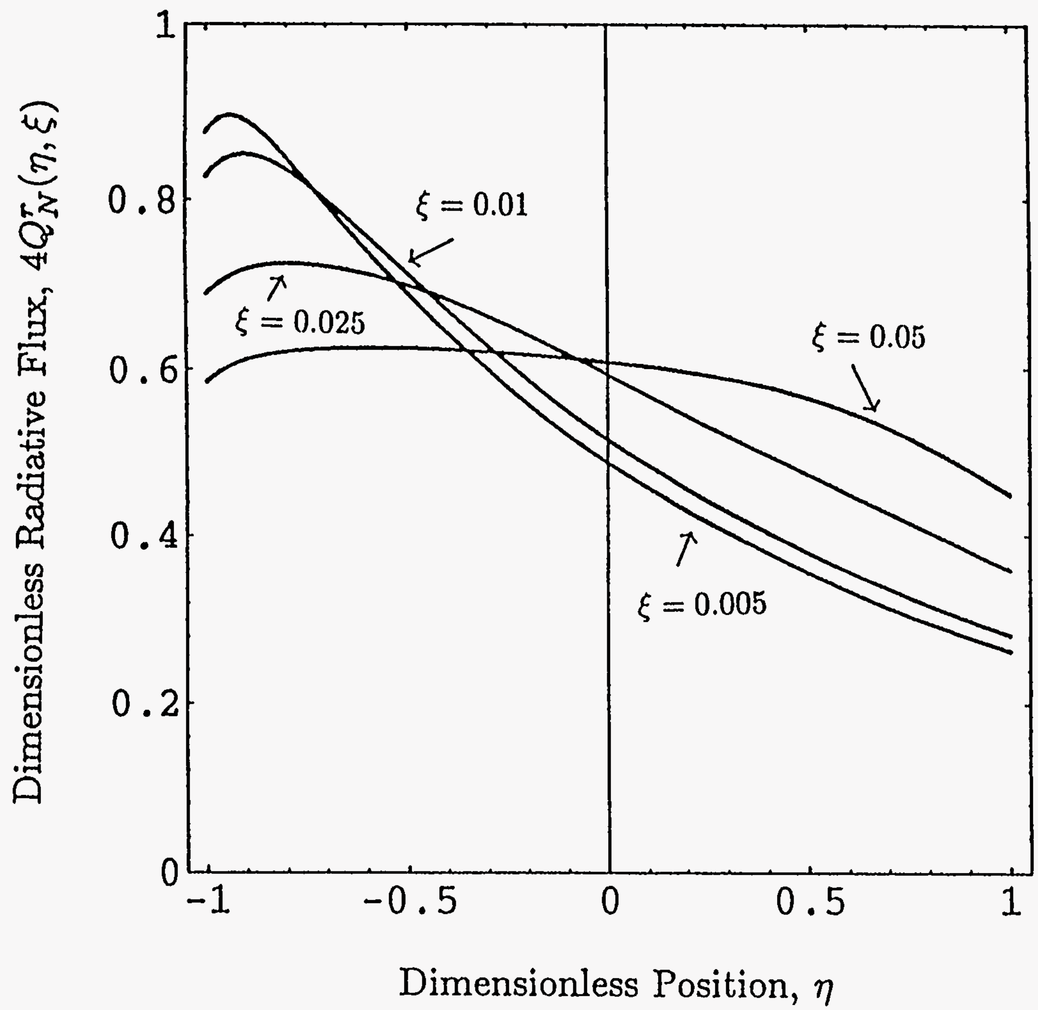
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
$\xi = 0.05$
$\xi = 0.025$
$\xi = 0.01$
$\xi = 0.005$
Dimensionless Position, η



Dimensionless Radiative Flux, $4Q_N^r(\eta,\xi)$
$\xi = 0.01$
$\xi = 0.025$
$\xi = 0.05$
$\xi = 0.005$
Dimensionless Position, η

Table 1.

Dimensionless Temperatures

Investigators [16]	$\eta = -0.5$	$\eta = 0$	$\eta = 0.5$
Lii and Ozisik	0.4617	0.1474	0.0277
Sutton	0.4888	0.1778	0.0591
Barker and Sutton	0.4893	0.1775	0.0588
Tsai and Lin	0.4889	0.1773	0.0588
Present Study: $\theta_N(\eta, 0.05)$			
N = 4	0.4996	<u>0.1797</u>	0.0504
N = 6	<u>0.4888</u>	<u>0.1777</u>	<u>0.0584</u>
N = 8	0.4893	<u>0.1773</u>	0.0587

Table 2.

Dimensionless Radiative Heat Fluxes

Investigators [16]	$\eta = -1$	$\eta = 0$	$\eta = 1$
Lii and Ozisik	1.6436	1.2529	0.9746
Sutton	1.9304	1.3305	0.8332
Barker and Sutton	1.9300	1.3314	0.8335
Tsai and Lin	1.9328	1.3292	0.8321
Present Study: $Q_N^r(\eta, 0.05)/N_{cr}$			
N = 4	1.9355	<u>1.3025</u>	0.8339
N = 6	<u>1.9348</u>	<u>1.3284</u>	<u>0.8317</u>
N = 8	1.9342	<u>1.3289</u>	0.8319

GENERALIZATION OF THE METHOD OF PETERS TO CAUCHY SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS

J.I. Frankel, Ph.D.
Associate Professor
Mechanical and Aerospace Engineering Department
Florida Institute of Technology
Melbourne, Florida (USA) 32901

Abstract

An analytic methodology is presented for solving linear Cauchy singular integro-differential equations. A representative equation is studied detailing the approach. Peters' notion, conceived when studying Cauchy singular integral equations of the airfoil type, is generalized to include Cauchy singular integro-differential equations. The final outcome from the analytic preconditioning suggests the use of Chebychev polynomials as the basis functions for developing an approximate analytic solution. The proposed analytic procedure is augmented with symbolic computation for performing algebraic manipulations. Results indicate that the approach has merit and deserves additional consideration.

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Nomenclature

a	= constant
a_m	= m^{th} expansion coefficent corresponding to the m^{th} Chebychev polynomial of the first kind
$A(x)$	= prescribed functions, Eq. (1.1a)
$A_1(x)$	= prescribed function, Eq. (1.1a)
A_{mk}	= constants defined by Eq. (3.24c)
B_k	= constants defined by Eq. (3.24d)
$B(x)$	= prescribed function, Eq. (1.1a)
$B_1(x)$	= prescribed function, Eq. (1.1a)
c_k	= coefficients, Eq. (3.12c)
C_o, C_1	= zeroth and first moments of $f(\eta)$, Eq. (2.2b,c)
$\tilde{C}_o$, $\tilde{C}_1$	= zeroth and first moments of $f(x)$, Eq. (3.7b,c)
C_{mk}	= constants defined by Eq. (3.24e)
$f(x)$	= unknown function, $x \in [-1,1]$
$f'(x)$	= derivative of $f(x)$, $x \in [-1,1]$
f_o	= prescribed boundary condition, Eq. (1.1b)
f_1	= prescribed boundary condition, Eq. (1.1b)
$g(x)$	= prescribed function, Eq. (2.1c)
$G(x)$	= prescribed function, Eq. (1.1a)
$h(\eta)$	= function given in Eq. (2.5b)
$H(\eta;f)$	= function given in Eq. (2.8b)
$K_o(x,t)$	= kernel function, Eq. (1.1a)
$K_1(x,t)$	= kernel function, Eq. (1.1a)
N_{mk}	= normalization integral, Eq. (3.24b)
r	= independent variable, $r \in [-1,1]$

t	=	independent variable, $t \in [-1,1]$
$T_m(x)$	=	m^{th} Chebychev polynomial of the first kind
$U_m(x)$	=	m^{th} Chebychev polynomial of the second kind
x	=	independent variable, $x \in [-1,1]$
y	=	independent variable, $y \in [-1,1]$
z	=	independent variable, $z \in [-1,1]$

Greek

α	=	constant, a/2
β_k	=	coefficients in Eq. (3.12b)
δ_{jk}	=	Kronecker delta function
γ_k	=	coefficients in Eq. (3.13)
η	=	independent variable, $\eta \in [0,1]$
η_o	=	dummy variable, $\eta_o \in [0,1]$
ν	=	dummy variable, $\nu \in [0,1]$
ρ_k	=	constants defined in Eq. (3.23b)
σ	=	dummy variable, $\sigma \in [0,1]$
ω_k	=	constants defined in Eq. (3.15e)
ξ	=	dummy variable, $\xi \in [0,1]$
Ψ	=	intermediate function, Eq. (2.7)

1. INTRODUCTION

Cauchy singular integral equations appear in inviscid airfoil theory (Bland 1970), radiative transport theory (Sparrow and Cess 1978), and elasticity (Erdogan 1969), three areas which cover a broad spectrum of scientific endeavor. Cauchy singular integro-differential equations make their appearance in molecular conduction and infrared radiative transport (Cess and Tiwari 1969), and elastic contact (Sankar et al. 1982) studies to name only a few. The development of accurate approximate analytic solutions to the Cauchy singular integral and integro-differential equations of mathematical physics is a formidable challenge requiring additional study.

Such equations may at times be expressed as (Sankar et al. 1982)

$$A(x)f(x) + A_1(x)f'(x) + B(x)⨍_{t=-1}^{1} \frac{f(t)}{t-x}\,dt + B_1(x)⨍_{t=-1}^{1} \frac{f'(t)}{t-x}\,dt + \int_{t=-1}^{1} K_o(x,t)f(t)dt + \int_{t=-1}^{1} K_1(x,t)f'(t)dt = G(x), \quad x \in (-1,1), \tag{1.1a}$$

subject to (Sankar et al. 1982)

$$f(-1) = f_o \quad \text{and} \quad f(1) = f_1. \tag{1.1b}$$

Here the functions $A(x)$, $A_1(x)$, $B(x)$, $B_1(x)$, and $G(x)$ are prescribed functions which are continuous and differentiable in the interval $x \in (-1,1)$. The kernels $K_o(x,t)$ and $K_1(x,t)$ are assumed to be at worst weakly singular. Also, we denote integration in the Cauchy principal value sense with the symbol $⨍$. We suppose that if $f(x)$, $x \in (-1,1)$ is continuous and that its derivative $f'(x)$ exists and is continuous in the interval

$x \in (-1,1)$ then $⨍_{t=-1}^{1} f(y)\,(x - y)^{-1}dy$ exists in the principal value sense.

Numerous approaches for solving linear integro-differential equations containing Cauchy kernels have been described in the literature (e.g. Ioakimidis and Theocaris 1979). Some methods are based on purely numeric approaches while others make use of approximate analytic solutions. The appearance of symbolic manipulation on the computational scene, such as available in software packages like Mathematica™, Derive, and Maples, supports the development of approximate analytic solutions. Symbolic manipulation allows for copious amounts of algebra to performed in an automated fashion. Another application worthy of symbolic manipulation involves the determination of the unknown expansion coefficients as they arise when changing from one basis to another basis. The introduction and acceptance of symbolic algebraic manipulation may change the approach taken by analysts since tedious manipulations can now be performed by software.

This paper is divided into three sections. In §2, we generalize Peters' (1963, 1968) concept to include Cauchy singular integro-differential equations. In §3, we develop an approximate analytic solution based on a Chebychev series expansion for the unknown function. Finally, in §4, we present some results and draw some conclusions on the importance of the proposed approach.

2. FORMULATION

In this section, we consider a specialized form of Eq. (1.1a). The reduced integro-differential equation under consideration becomes (Sankar et al. 1982)

$$f(x) + af'(x) + \int_{t=-1}^{1} \frac{f(t)}{t - x}\,dt + \int_{t=-1}^{1} \frac{f'(t)}{t - x}\,dt + \int_{t=-1}^{1} xtf(t)dt = g(x), \quad x \in (-1,1), \tag{2.1a}$$

subject to the auxiliary conditions specified in Sankar et al. (1982), namely

$$f(-1) = 0 \quad \text{and} \quad f(1) = 2, \tag{2.1b}$$

where

$$g(x) = \frac{14}{3} + 2008.4x + 3003x^2 + x^3 + (2x + 4x^2 + x^3)\log\left|\frac{1 - x}{1 + x}\right|, \tag{2.1c}$$

and $a = 1000$. The rationale for studying this equation is twofold; namely, 1) a simple closed form solution exists, i.e., $f(x) = x^2 + x^3$, and 2) this equation has been previously considered by Sankar et al. (1982) where some numerical results have been presented. Sankar et al. (1982) developed an approximate analytic solution to the above equation for benchmark purposes. This type of equation appears in elastic contact, and combined molecular conduction and infrared radiation studies.

The solution approach proposed here begins by using a notion offered by Peters (1963, 1968). His notion of manipulation was developed while studying equations of the classical airfoil type. Frankel (1992) recently extended Peters concept to a simple integro-differential equation which arises in combined molecular conduction and infrared radiation in a plane-parallel region. (The case where $A(x) = B_1(x) = K_o(x,t) = K_1(x,t) = 0$ in Eq. (1.1a) was considered by Frankel (1992).) Since no exact solution existed, a purely numeric solution was developed for comparison purposes. The developed approximate analytic solution appeared to yield benchmark results. Thus, the present work is intended to illustrate that the approach can be extended to a more general class of equation then originally considered Frankel (1992).

Before proceding, it is convenient to map the physical domain from $x \in [-1,1]$ to $\eta \in [0,1]$. Doing so transforms Eq. (2.1a) to

$$f(\eta) + \alpha f'(\eta) + ⨍_{\xi=0}^{1} [f(\xi) + \frac{f'(\xi)}{2}]\frac{d\xi}{\xi - \eta}$$
$$+ 2(2\eta - 1)(2C_1 - C_o) = g(2\eta - 1), \quad \eta \in (0,1), \tag{2.2a}$$

where

$$C_o \equiv \int_{\xi=0}^{1} f(\xi)d\xi, \tag{2.2b}$$

$$C_1 \equiv \int_{\xi=0}^{1} \xi f(\xi)d\xi, \tag{2.2c}$$

where $\alpha = a/2$ and subject to the auxiliary conditions

$$f(0) = 0 \quad \text{and} \quad f(1) = 2. \tag{2.2d}$$

To illustrate the method of Peters, we multiple Eq. (2.2a) by η, then add and subtract the integral quantity

$$\pm ⨍_{\xi=0}^{1} [f(\xi) + \frac{f'(\xi)}{2}] \frac{\xi d\xi}{\xi - \eta}, \tag{2.3}$$

to the left-hand side of Eq. (2.2a), make use of the auxiliary conditions shown in Eq. (2.2d), and then divide the resultant by $\sqrt{\eta}$ to arrive at

$$\sqrt{\eta}f(\eta) + \alpha \frac{d}{d\eta}[\sqrt{\eta}f(\eta)] - \frac{\alpha f(\eta)}{2\sqrt{\eta}} + ⨍_{\xi=0}^{1} [f(\xi) + \frac{f'(\xi)}{2}]\frac{\xi d\xi}{\sqrt{\eta}(\xi - \eta)} =$$
$$\frac{(1 + C_o)}{\sqrt{\eta}} - 2\sqrt{\eta}(2\eta - 1)(2C_1 - C_o) + \sqrt{\eta}g(2\eta - 1), \quad \eta \in (0,1). \tag{2.4}$$

Next, we integrate Eq. (2.4) from $\eta_o = 0$ to $\eta_o = \eta$ (after renaming η by η_o in Eq. (2.4)). After a straightforward set of manipulations, we find

$$\int_{\eta_o=0}^{\eta} f(\eta_o)\left[\frac{2\eta_o - \alpha}{2\sqrt{\eta_o}}\right]d\eta_o + \alpha\sqrt{\eta}f(\eta) -$$

$$\int_{\eta_o=0}^{1} \sqrt{\eta_o}\left[f(\eta_o) + \frac{f'(\eta_o)}{2}\right]\log\left|\frac{\sqrt{\eta} - \sqrt{\eta_o}}{\sqrt{\eta} + \sqrt{\eta_o}}\right| d\eta_o = h(\eta), \qquad \eta \in (0,1), \tag{2.5a}$$

where

$$h(\eta)=\int_{\eta_o=0}^{\eta} \left\{\frac{(1 + C_o)}{\sqrt{\eta_o}} - \sqrt{\eta_o}[2(2\eta_o - 1)(2C_1 - C_o) - g(2\eta_o-1)]\right\}d\eta_o, \tag{2.5b}$$

or

$$\int_{\eta_o=0}^{\eta} f(\eta_o)\left[\frac{2\eta_o - \alpha}{2\sqrt{\eta_o}}\right]d\eta_o + \alpha\sqrt{\eta}f(\eta) +$$

$$\int_{\xi=0}^{\eta} \frac{d\xi}{\sqrt{\eta - \xi}} \int_{\eta_o=\xi}^{1} \frac{\sqrt{\eta_o}\left[f(\eta_o) + \frac{f'(\eta_o)}{2}\right]}{\sqrt{\eta_o - \xi}}d\eta_o = h(\eta), \qquad \eta \in (0,1), \tag{2.5c}$$

where we have made use of (Peters 1963)

$$-\log\left|\frac{\sqrt{\eta} - \sqrt{\eta_o}}{\sqrt{\eta} + \sqrt{\eta_o}}\right| = \begin{cases} \int_{\sigma=0}^{\eta_o} \dfrac{d\sigma}{\sqrt{\eta_o - \sigma}\sqrt{\eta - \sigma}}, & \eta > \eta_o, \\ \\ \int_{\sigma=0}^{\eta} \dfrac{d\sigma}{\sqrt{\eta_o - \sigma}\sqrt{\eta - \sigma}}, & \eta_o > \eta. \end{cases} \tag{2.6}$$

Next, we define the intermediate function, $\Psi(\eta)$ as

$$\Psi(\xi) \equiv \int_{\eta_o=\xi}^{1} \frac{\sqrt{\eta_o}\left[f(\eta_o) + \frac{f'(\eta_o)}{2}\right]}{\sqrt{\eta_o - \xi}}d\eta_o. \tag{2.7}$$

Thus, Eq. (2.5c) becomes

$$\int_{\xi=0}^{\eta} \frac{\Psi(\xi)d\xi}{\sqrt{\eta - \xi}} = H(\eta;f), \qquad \eta \in (0,1), \tag{2.8a}$$

where

$$H(\eta;f) = -\int_{\eta_o=0}^{\eta} f(\eta_o)\left[\frac{2\eta_o - \alpha}{2\sqrt{\eta_o}}\right] d\eta_o - \alpha\sqrt{\eta}f(\eta) + h(\eta). \tag{2.8b}$$

If one treats the right-hand side of Eq. (2.8a) as an effective nonhomogeneity, then one can interpret Eq. (2.8a) as an Abel integral equation for the unknown function $\Psi(\eta)$. One can easily find the solution to $\Psi(\eta)$ by classical means (Tricomi 1985). We operate on Eq. (2.8a) with (after renaming η by ν)

$$\int_{\nu=0}^{\eta} \frac{d\nu}{\sqrt{\eta - \nu}}$$

to get

$$\int_{\nu=0}^{\eta} \frac{d\nu}{\sqrt{\eta - \nu}} \int_{\xi=0}^{\nu} \frac{\Psi(\xi)d\xi}{\sqrt{\nu - \xi}} = \int_{\nu=0}^{\eta} \frac{H(\nu;f)d\nu}{\sqrt{\eta - \nu}}, \qquad \eta \in (0,1). \tag{2.9}$$

Interchanging orders of integration in Eq. (2.9), leads to

$$\int_{\xi=0}^{\eta} \Psi(\xi)d\xi = \frac{1}{\pi} \int_{\nu=0}^{\eta} \frac{H(\nu;f)d\nu}{\sqrt{\eta - \nu}}, \qquad \eta \in (0,1). \tag{2.10}$$

Integrating the right-hand side of Eq. (2.10) by parts and incorporating the condition that $H(0;f(0)) = 0$ produces

$$\int_{\xi=0}^{\eta} \Psi(\xi)d\xi = \frac{2}{\pi} \int_{\nu=0}^{\eta} \sqrt{\eta - \nu}\, \frac{dH}{d\nu}(\nu;f)d\nu, \quad \eta \in (0,1). \tag{2.11}$$

Next, we differentiate Eq. (2.11) with respect to η to produce

$$\Psi(\eta) = \frac{1}{\pi} \int_{\nu=0}^{\eta} \frac{H'(\nu;f)}{\sqrt{\eta - \nu}}\, d\nu, \qquad \eta \in (0,1), \tag{2.12a}$$

where from Eq. (2.8b) we find

$$H'(\eta;f) = - f(\eta)[\frac{2\eta - \alpha}{2\sqrt{\eta}}] - \frac{\alpha f(\eta)}{2\sqrt{\eta}} - \alpha\sqrt{\eta}f'(\eta) + h'(\eta). \tag{2.12b}$$

Using our original definition of the intermediate function, $\Psi(\eta)$, as shown in Eq. (2.7), we can express Eq. (2.12a) as

$$\int_{\nu=\eta}^{1} \frac{\sqrt{\nu}[f(\nu) + \frac{f'(\nu)}{2}]}{\sqrt{\nu - \eta}}\, d\nu = \frac{1}{\pi} \int_{\nu=0}^{\eta} \frac{H'(\nu;f)}{\sqrt{\eta - \nu}}\, d\nu, \qquad \eta \in (0,1), \tag{2.13}$$

which may be interpreted as an Abel integral equation for the unknown function $\sqrt{\nu}[f(\nu) + \frac{f'(\nu)}{2}]$. Next, we operate on Eq. (2.13) with (after renaming η by ξ)

$$\int_{\xi=\eta}^{1} \frac{d\xi}{\sqrt{\xi - \eta}}$$

to get

$$\int_{\xi=\eta}^{1} \frac{d\xi}{\sqrt{\xi - \eta}} \int_{\nu=\xi}^{1} \frac{\sqrt{\nu}[f(\nu) + \frac{f'(\nu)}{2}]}{\sqrt{\nu - \xi}} d\nu = \frac{1}{\pi} \int_{\xi=\eta}^{1} \frac{d\xi}{\sqrt{\xi - \eta}} \int_{\nu=0}^{\xi} \frac{H'(\nu;f)}{\sqrt{\xi - \nu}} d\nu,$$

$$\eta \in (0,1), \qquad (2.14)$$

or after a lengthy but straightforward set of manipulations, we arrive at

$$\int_{\nu=\eta}^{1} \sqrt{\nu}[f(\nu) + \frac{f'(\nu)}{2}]d\nu = \frac{1}{\pi^2} \int_{\nu=0}^{1} H'(\nu;f) \log\left|\frac{\sqrt{1 - \nu} + \sqrt{1 - \eta}}{\sqrt{1 - \nu} - \sqrt{1 - \eta}}\right| d\nu,$$

$$\eta \in (0,1). \qquad (2.15)$$

Carefully differentiating Eq. (2.15) with respect to η leads to

$$f(\eta) + \frac{f'(\eta)}{2} = - \frac{1}{\pi^2\sqrt{\eta}\sqrt{1 - \eta}} ⨍_{\nu=0}^{1} \frac{H'(\nu;f)\sqrt{1 - \nu}}{\nu - \eta} d\nu, \quad \eta \in (0,1), \quad (2.16)$$

and upon substituting the definition of $H'(\eta;f)$, from Eq. (2.12b), into Eq. (2.16) we get

$$f(\eta) + \frac{f'(\eta)}{2} = - \frac{1}{\pi^2} ⨍_{\nu=0}^{1} \frac{\sqrt{\nu}\sqrt{1 - \nu}}{\sqrt{\eta}\sqrt{1 - \eta}} \frac{d\nu}{\nu - \eta}[- f(\nu) - \alpha f'(\nu) + \qquad (2.17)$$

$$\frac{(1 + C_0)}{\nu} - 2(2\nu - 1)(2C_1 - C_0) + g(2\nu - 1)]d\nu, \qquad \eta \in (0,1),$$

subject to the auxiliary conditions displayed in Eq. (2.2d). It is clear from viewing Eq. (2.17) that a function which can be interpreted as a Chebychev weight has been generated by these manipulations.

In the next section, we develop an approximate analytic solution based on a series representation for the unknown function $f(\eta)$ in terms of a Chebychev series. Though operationally complex, the unknown coefficients in the expansion are obtained by a simple Fourier approach which makes use of the known orthogonality relation associated with the Chebychev polynomials of the first kind. In practice, it may be more prudent to introduce a collocation approach (Baker 1978, Delves and Mohamed 1988).

3. SOLUTION METHOD

In this section, we develop an approximate analytic solution to Eq. (2.17) subject to the auxiliary conditions previously defined. Before proceding further, it is now convenient to map the physical domain from $\eta \in [0,1]$ back to $x \in [-1,1]$, thereby arriving at

$$- [f(x) + \frac{df}{dx}(x)] = \frac{1}{\pi^2} ⨍_{t=-1}^{1} \frac{\sqrt{1 - t^2}}{\sqrt{1 - x^2}} \frac{dt}{t - x}[- f(t) - 2\alpha\frac{df}{dt}(t) +$$

$$\frac{2}{1 + t} + g(t) + \frac{1}{1 + t} \int_{r=-1}^{1} f(r)dr - t\int_{r=-1}^{1} rf(r)dr], \quad x \in (-1,1). \quad (3.1)$$

Equation (3.1) is subject to auxiliary conditions (Sankar et al. 1982)

$$f(-1) = 0 \quad \text{and} \quad f(1) = 2, \quad (3.2)$$

where $2\alpha = a$ and $g(x)$ is displayed in Eq. (2.1c).

It is now clear that the analytic preconditioning lead us to a form which suggests the use of a Chebychev series representation for $f(x)$. Let $T_m(x) = \cos[m(\cos^{-1}x)]$, $x \in [-1,1]$, $m = 0,1,\ldots$, denote the Chebychev polynomials of the first kind while $U_m(x) = \sin[(m + 1)(\cos^{-1}x)]/\sin(\cos^{-1}x)$, $m = 0,1,\ldots$, denote Chebychev polynomials of the second kind. It is well known that $\{T_m(x)\}$, $m = 0,1,\ldots$, form an orthogonal sequence of functions with respect to the weight function $1/\sqrt{(1 - x^2)}$ while $\{U_m(x)\}$, $m = 0,1,\ldots$, form an orthogonal sequence of functions with respect to the weight function $\sqrt{(1 - x^2)}$.

For this particular case, we choose our series representation for the unknown function $f(x)$ as

$$f(x) = \sum_{m=0}^{\infty} a_m T_m(x) + 1, \qquad x \in (-1,1), \quad (3.3)$$

where $T_m(x)$ represents the m^{th} Chebychev polynomial of the first kind. The unknown expansion coefficients a_m for $m = 0,1,\ldots,$ are to be obtained by some means, typically by either a collocation procedure or with the assistance of some known integral relations such as orthogonality. Directly applying the boundary condition at x = 1 as shown in Eq. (3.2) provides a constraint on the unknown constants in our approximate analytic solution. Thus, from evaluating Eq. (3.3) at x = 1 and making use of Eq. (3.2), we find

$$\sum_{m=0}^{\infty} a_m = 1. \tag{3.4}$$

Appending this constraint to Eq. (3.2) produces

$$f(x) = \sum_{m=0}^{\infty} a_m[T_m(x) + 1], \quad x \in (-1,1], \tag{3.5}$$

which now automatically satisfies the known boundary condition at x = 1 upon imposing the constraint relation shown in Eq. (3.4).

From Andrews (1992), we also know

$$\frac{df}{dx}(x) = \sum_{m=1}^{\infty} m a_m U_{m-1}(x), \quad x \in (-1,1], \tag{3.6}$$

where $U_{m-1}(x)$ represents the $(m - 1)^{th}$ Chebychev polynomial of the second kind. Upon substituting Eqs. (3.5) and (3.6) into Eq. (3.1), we find

$$- \{\sum_{m=0}^{\infty} a_m[T_m(x) + T_o(x)], + \sum_{m=1}^{\infty} m a_m U_{m-1}(x)\} = \tag{3.7a}$$

$$- \frac{1}{\pi^2} \int_{t=-1}^{1} \frac{\sqrt{1 - t^2}}{\sqrt{1 - x^2}} \frac{dt}{t - x}\{ \sum_{m=0}^{\infty} a_m[T_m(t) + T_o(t)] + a\sum_{m=1}^{\infty} m a_m U_{m-1}(t)\} +$$

$$\frac{1}{\pi^2} \int_{t=-1}^{1} \frac{\sqrt{1 - t^2}}{\sqrt{1 - x^2}} \frac{dt}{t - x} g(t) + \frac{1}{\pi^2} \int_{t=-1}^{1} \frac{\sqrt{1 - t^2}}{\sqrt{1 - x^2}} \frac{dt}{t - x}[\frac{2 + \tilde{C}_o}{1 + t} - t\tilde{C}_1],$$

$$x \in (-1,1],$$

where

$$\tilde{C}_o \equiv \int_{r=-1}^{1} f(r)dr = 2[1 - \sum_{m=0,2,\ldots}^{\infty} \frac{a_m}{m^2 - 1}], \tag{3.7b}$$

$$\tilde{C}_1 \equiv \int_{r=-1}^{1} rf(r)dr = -\sum_{m=1,3,\ldots}^{\infty} \frac{2a_m}{m^2-4}, \qquad (3.7c)$$

and where $T_0(x) = 1 = U_0(x)$.

Before proceding further, we state several well-known integral relations associated with Chebychev polynomials. These relations are useful in the subsequent analysis.

ORTHOGONALITY PROPERTY (Andrews 1992):

$$\int_{t=-1}^{1} \frac{T_m(t)T_n(t)}{\sqrt{1-t^2}}dt = \begin{cases} 0, & m \neq n, \\ \pi, & m = n = 0, \\ \frac{\pi}{2}, & m = n > 0 \end{cases} \qquad (3.8a)$$

CLOSED FORM INTEGRAL EXPRESSIONS (Kaya and Erdogan 1987):

$$⨍_{t=-1}^{1} \frac{U_m(t)\sqrt{1-t^2}}{t-x}\,dt = -\pi T_{m+1}(x), \quad m = 0,1,\ldots \qquad (3.8b)$$

$$\int_{t=-1}^{1} T_m(t)dt = \begin{cases} 0, & m = 1,3,\ldots \\ \frac{2}{1-m^2}, & m = 0,2,\ldots \end{cases} \qquad (3.8c)$$

$$⨍_{t=-1}^{1} \frac{T_m(t)}{\sqrt{1-t^2}\,(t-x)}\,dt = \begin{cases} \pi U_{m-1}(x), & m = 1,2,\ldots \\ 0, & m = 0. \end{cases} \qquad (3.8d)$$

Equally important in our analysis is the use of several known algebraic relations associated with Chebychev polynomials. Again, as a convenience to the reader, we state them:

ALGEBRAIC RELATIONS (Abramowitz and Stegun 1972):

$$T_m(x)T_n(x) = \frac{1}{2}[T_{m+n}(x) + T_{|m-n|}(x)], \quad m=0,1,\ldots \quad n=0,1,\ldots \qquad (3.9a)$$

$$xU_m(x) = \frac{1}{2}[U_{m+1}(x) + U_{m-1}(x)], \; m=1,2,\ldots \qquad (3.9b)$$

$$T_m(x) = U_m(x) - xU_{m-1}(x), \quad m=1,2,\ldots \qquad (3.9c)$$

Another interesting observation can be made concerning the logarithmic term displayed in g(x), namely

$$⨍_{t=-1}^{1} \frac{dt}{t - x} = \log\left|\frac{1 - x}{1 + x}\right|. \qquad (3.10)$$

This relation will also be quite useful.

Let us rewrite Eq. (3.7a) in the compact form

$$- [\sum_{m=0}^{\infty} a_m[T_m(x) + T_o(x)], + \sum_{m=1}^{\infty} ma_mU_{m-1}(x)] = I(x) + J(x), \qquad (3.11a)$$

with

$$I(x) = I_1(x) + I_2(x), \qquad (3.11b)$$

$$J(x) = J_1(x) + J_2(x), \qquad (3.11c)$$

where

$$I_1(x) = \frac{1}{\pi^2} ⨍_{t=-1}^{1} \frac{\sqrt{1 - t^2}}{\sqrt{1 - x^2}} \frac{dt}{t - x} g(t), \qquad (3.11d)$$

$$I_2(x) = \frac{1}{\pi^2} ⨍_{t=-1}^{1} \frac{\sqrt{1 - t^2}}{\sqrt{1 - x^2}} \frac{dt}{t - x}[\frac{2 + \tilde{C}_o}{1 + t} - t\tilde{C}_1], \qquad (3.11e)$$

and

$$J_1(x) = - \frac{1}{\pi^2} ⨍_{t=-1}^{1} \frac{\sqrt{1 - t^2}}{\sqrt{1 - x^2}} \frac{dt}{t - x} \sum_{m=0}^{\infty} a_m[T_m(t) + T_o(t)], \qquad (3.11f)$$

$$J_2(x) = - \frac{a}{\pi^2} ⨍_{t=-1}^{1} \frac{\sqrt{1 - t^2}}{\sqrt{1 - x^2}} \frac{dt}{t - x} \sum_{m=1}^{\infty} ma_mU_{m-1}(t). \qquad (3.11g)$$

We can now find analytic expressions for J(x) and I(x). First, we further decompose $I_1(x)$ into two components, namely

$$I_1(x) = I_{1a}(x) + I_{1b}(x), \qquad (3.12a)$$

where

$$I_{1a}(x) = \frac{1}{\pi^2} ⨍_{t=-1}^{1} \frac{\sqrt{1 - t^2}}{\sqrt{1 - x^2}} \frac{dt}{t - x} [\sum_{k=0}^{3} \beta_k t^k], \qquad (3.12b)$$

and

$$I_{1b}(x) = \frac{1}{\pi^2} ⨍_{t=-1}^{1} \frac{\sqrt{1-t^2}}{\sqrt{1-x^2}} \frac{dt}{t-x} \left[\sum_{k=1}^{3} c_k t^k\right] \log\left|\frac{1-t}{1+t}\right|, \qquad (3.12c)$$

where we have made explicit use of the functional form for $g(x)$ as defined in Eq. (2.1c). Here $\beta_0 = 14/3$, $\beta_1 = 2008.4$, $\beta_2 = 3003$, and $\beta_3 = 1$ while $c_1 = 2$, $c_2 = 4$, and $c_3 = 1$.

Initial attention is directed toward evaluating $I_{1a}(x)$ in an exact manner. From viewing Eq.(3.8b), we expand the partial sum shown in Eq. (3.12b) in terms of an equivalent partial sum of Chebychev polynomials of the second kind, namely

$$\sum_{k=0}^{3} \beta_k t^k = \sum_{k=0}^{3} \gamma_k U_k(t). \qquad (3.13a)$$

Symbolic software makes the task of determining the unknown coefficients γ_k, k=0,1,2,3 a straightforward procedure. Doing so produces the following explicit analytic expressions

$$\gamma_0 = \beta_0 + \frac{\beta_2}{4}, \qquad \gamma_1 = \frac{\beta_1}{2} + \frac{\beta_3}{4},$$

$$\gamma_2 = \frac{\beta_2}{4}, \qquad \gamma_3 = \frac{\beta_3}{8}. \qquad (3.13b)$$

Substituting Eq. (3.13a) into Eq. (3.12b) and making use of Eq. (3.8b), we find

$$I_{1a}(x) = -\frac{1}{\pi} \sum_{k=0}^{3} \frac{\gamma_k T_{k+1}(x)}{\sqrt{1-x^2}}. \qquad (3.14)$$

Second, to obtain an explicit analytic evaluation for $I_{1b}(x)$, as shown in Eq. (3.12c), we make use of Eq. (3.10) to get

$$I_{1b}(x) = -\frac{1}{\pi^2\sqrt{1-x^2}} ⨍_{t=-1}^{1} \left\{\frac{\sqrt{1-t^2}}{t-x} \left[\sum_{k=1}^{3} c_k t^k\right] ⨍_{z=-1}^{1} \frac{dz}{t-z}\right\}dt. \qquad (3.15a)$$

Upon carefully using the Hardy-Poincaré-Bertrand formula (Tricomi 1985, Muskhelishvili 1992), Eq. (3.15a) becomes

$$I_{1b}(x) = -\frac{1}{\pi^2\sqrt{1-x^2}} \fint_{z=-1}^{1} \bar{I}_B(x,z)dz - \sum_{k=1}^{3} c_k t^k, \qquad (3.15b)$$

where

$$\bar{I}_B(x,z) = \fint_{t=-1}^{1} \frac{\sqrt{1-t^2}}{(t-x)(t-z)} \sum_{k=1}^{3} c_k t^k \, dt. \qquad (3.15c)$$

Again expanding the partial sum shown in Eq. (3.15c) in terms of a finite sum of Chebychev polynomials of the second kind, namely

$$\sum_{k=1}^{3} c_k t^k = \sum_{k=0}^{3} \omega_k U_k(t), \qquad (3.15d)$$

where we obtain

$$\omega_o = \frac{c_2}{4}, \qquad \omega_1 = \frac{c_2}{2} + \frac{c_3}{4},$$

$$\omega_2 = \frac{c_2}{4}, \qquad \omega_3 = \frac{c_3}{8}, \qquad (3.15e)$$

and making use of a partial fraction maneuver in Eq. (3.15c) and upon substituting Eq. (3.15d) into Eq. (3.15c), we arrive at

$$\bar{I}_B(x,z) = -\pi \sum_{k=0}^{3} \omega_k \frac{T_{k+1}(x) - T_{k+1}(z)}{x - z}. \qquad (3.15f)$$

Substituting Eq. (3.15f) into Eq. (3.15b) for $\bar{I}_b(x,z)$, and performing the integration analytically, we arrive at

$$I_{1b}(x) = \frac{2}{\pi\sqrt{1-x^2}}[\, x^3 + 4x^2 + \frac{11}{6}x - \frac{2}{3}] - \sum_{k=1}^{3} c_k t^k, \qquad (3.16)$$

where for simplicity we have made use of the actual numeric values for the constants of ω_k, $k = 0,1,2,3$.

Next, our attention is directed toward evaluating $I_2(x)$ as defined in Eq. (3.11e). Let us define

$$I_2(x) = I_{2a}(x) + I_{2b}(x), \qquad (3.17a)$$

such that

$$I_{2a}(x) = \frac{\tilde{C}_o + 2}{\pi^2\sqrt{1-x^2}} \fint_{t=-1}^{1} \frac{\sqrt{1-t^2}}{1+t} \frac{dt}{t-x}, \qquad (3.17b)$$

and

$$I_{2b}(x) = -\frac{\tilde{C}_1}{\pi^2\sqrt{1-x^2}} ⨍_{t=-1}^{1} t\sqrt{1-t^2}\,\frac{dt}{t-x}\,. \tag{3.17c}$$

After performing some elementary manipulations and noting that $1 = T_o(t)$, $t = T_1(t)$, and making use of Eq. (3.8d), we find

$$I_{2a}(x) = -\frac{2+\tilde{C}_o}{\pi\sqrt{1-x^2}}. \tag{3.18a}$$

Next, once noting that $t = U_1(t)/2$ and making use of Eq. (3.8b), we find

$$I_{2b}(x) = \frac{\tilde{C}_1 T_2(x)}{2\pi\sqrt{1-x^2}}\,, \tag{3.18b}$$

or finally after reconstruction, we obtain $I_2(x)$ as

$$I_2(x) = \frac{\left[-(2+\tilde{C}_o) + \frac{\tilde{C}_1}{2}T_2(x)\right]}{\pi\sqrt{1-x^2}}\,. \tag{3.19}$$

Next, we evaluate the integrals defined as $J_1(x)$ and $J_2(x)$, as shown in Eqs. (3.11f) and (3.11g), respectively. The functional form shown in the integrand displayed in Eq. (3.11f) permits some interpretation leading us to express the Chebychev function of the first kind as a Chebychev function of the second kind. This conversion should be made in light of viewing Eq. (3.8b). Noting that $T_o(t) = U_o(t)$ and using Eq. (3.9c) in conjunction with Eq. (3.9b), we can express $J_1(x)$ as

$$J_1(x) = -\frac{1}{\pi^2\sqrt{1-x^2}}\Big\{ (1+a_o)⨍_{t=-1}^{1} U_o(t)\sqrt{1-t^2}\,\frac{dt}{t-x} + \tag{3.20a}$$

$$a_1 ⨍_{t=-1}^{1}\frac{U_1(t)\sqrt{1-t^2}}{2}\,\frac{dt}{t-x} + \sum_{m=2}^{\infty} a_m ⨍_{t=-1}^{1}\frac{[U_m(t) - U_{m-2}(t)]}{2}\sqrt{1-t^2}\,\frac{dt}{t-x}\Big\}.$$

Using Eq. (3.8b), we find

$$J_1(x) = \frac{1}{\pi\sqrt{1-x^2}}\Big\{ (1+a_o)T_1(x) + \frac{a_1}{2}T_2(x) +$$

$$\sum_{m=2}^{\infty} a_m\frac{[T_{m+1}(x) - T_{m-1}(x)]}{2}\Big\}. \tag{3.20b}$$

The evaluation of $J_2(x)$ is rather straightforward and thus we merely present the final result, namely

$$J_2(x) = \frac{a}{\pi\sqrt{1 - x^2}} \sum_{m=1}^{\infty} m a_m T_m(x). \tag{3.21}$$

Upon reconstructing the right-hand side of Eq. (3.11a) using Eqs. (3.12a), (3.14), (3.16), (3.19), (3.20b), and (3.21), and recasting the polynomic expressions

$$x^3 + 4x^2 + \frac{11}{6}x - \frac{2}{3} = \sum_{k=0}^{3} h_k T_k(x), \tag{3.22a}$$

$$x^3 + 4x^2 + 2x = \sum_{k=0}^{3} p_k T_k(x), \tag{3.22b}$$

into equivalent Chebychev forms of the first kind, where $h_o = 4/3$, $h_1 = 31/12$, $h_2 = 2$, and $h_3 = 1/4$ while $p_o = 2$, $p_1 = 11/4$, $p_2 = 2$, and $p_3 = 1/4$, we find

$$- \{\sum_{m=0}^{\infty} a_m[T_m(x) + T_o(x)] + \sum_{m=1}^{\infty} m a_m U_{m-1}\} + \sum_{m=0}^{3} p_m T_m(x) =$$

$$\frac{1}{\pi\sqrt{1 - x^2}}\{ \sum_{m=1}^{\infty} a_m \frac{T_{m+1}(x)}{2} - \sum_{m=2}^{\infty} a_m \frac{T_{m-1}(x)}{2} + a\sum_{m=1}^{\infty} m a_m T_m(x) -$$

$$\sum_{m=0}^{3} \gamma_m T_{m+1}(x) + \sum_{m=0}^{3} \rho_m T_m(x)\}, \tag{3.23a}$$

where

$$\rho_o = - (2 + \tilde{C}_o) + 2h_o, \qquad \rho_1 = (1 + a_o) + 2h_1,$$

$$\rho_2 = \frac{\tilde{C}_1}{2} + 2h_2, \qquad \rho_3 = 2h_3. \tag{3.23b}$$

At this point, the use of a collocation method for determining the unknown expansion coefficients, a_m, $m = 0,1,...$ appears to be the judicious choice. As an alternative approach which requires additional analytic consideration, we proceed to find the unknown expansion coefficients using a Fourier series approach based on the orthogonality relation displayed in Eq. (3.8a). We operate on Eq. (3.23a) with

$$\int_{x=-1}^{1} T_k(x)dx, \quad k=0,1,...$$

and upon making use of the orthogonality property shown in Eq.

(3.8a), we arrive at

$$-\left\{\sum_{m=0}^{\infty} a_m A_{mk} + \sum_{m=1}^{\infty} m a_m C_{mk} + \frac{N_{kk}}{2\pi}[a_{k-1}(1 - \delta_{k1}) - a_{k+1} + 2aka_k]\right\} = \tag{3.24a}$$

$$B_k - \sum_{m=0}^{3} p_m A_{mk} + \frac{N_{kk}}{\pi}\left\{\rho_k \prod_{j=4}^{\infty}(1 - \delta_{kj}) - \gamma_{k-1} \prod_{j=5}^{\infty}(1 - \delta_{kj})\right\}, \quad k = 1,2,...$$

where the normalization integral (see Eq. (3.8a)) is given as

$$N_{mk} = \int_{t=-1}^{1} \frac{T_m(t)T_k(t)}{\sqrt{1 - t^2}}dt = \begin{cases} 0, & m \neq k, \\ \pi, & m = k = 0, \\ \frac{\pi}{2}, & m = k > 0, \end{cases} \tag{3.24b}$$

and where the constants A_{mk}, B_k, and C_{mk} are defined as

$$A_{mk} \equiv \int_{x=-1}^{1} T_k(x)T_m(x)dx = \begin{cases} 0, & m,k \text{ mixed odd/even} \\ \frac{1}{1 - (m + k)^2} + \frac{1}{1 - |m - k|^2}, & \text{both } m,k \text{ even or odd} \end{cases} \tag{3.24c}$$

$$B_k \equiv \int_{x=-1}^{1} T_k(x)dx = \begin{cases} 0, & k \text{ odd } (k = 1,3,5,....) \\ \frac{2}{1 - k^2}, & k \text{ even } (k = 0,2,4,...) \end{cases} \tag{3.24d}$$

$$C_{mk} \equiv \int_{x=-1}^{1} T_k(x)U_{m-1}(x)dx = \begin{cases} 0, & m = k \geq 1 \\ \frac{1}{2}\left[\frac{1 - (-1)^{m+k}}{m + k} + \frac{1 - (-1)^{|m-k|}}{m - k}\right], & m \neq k,\ m = 1,2,...,\ \ k=0,1,2,... \end{cases} \tag{3.24e}$$

Here we introduce δ_{jk} (i.e., the Kronecker delta) for notational convenience. We note that the index k begins at unity in Eq. (3.24a) since we impose the constraint condition

$$\sum_{m=0}^{\infty} a_m = 1, \tag{3.24f}$$

in lieu of the k = 0 case.

All the ingredients for determining the unknown expansion coefficients are now available. From the practical point of view, the infinite number of expansion coefficients to be simultaneously obtained requires us to truncate at a finite number of terms, say N. Thus, we actually find an approximate solution based on a finite series expansion of Chebychev polynomials, namely (see Eq. (3.5))

$$f(x) \cong f^N(x) = \sum_{m=0}^{N} a_m \, [T_m(x) + 1], \quad x \in [-1,1]. \qquad (3.25)$$

We refer to $f^N(x)$ as the approximate analytic solution to f(x) in terms of a finite Chebychev series. The system of N + 1 unknown linear algebraic coefficients can now be resolved using matrix algebra.

4. RESULTS AND CONCLUSIONS

Due to the inherent coupling among the expansion coefficients, a simple Gauss-Seidel (Atkinson 1992) iterative scheme was developed for determining the expansion coefficients, a_m, m = 0,1,...,N as required when viewing Eq. (3.24a) and Eq. (3.24f). When choosing an iterative scheme, one requires convergence to be established with respect to some error criteria. Here, an absolute error criteria has been chosen as defined by:

$$\text{ERROR} = |a_m^{r+1} - a_m^r| < \text{tolerance}, \quad m = 0,1,\ldots,N, \qquad (3.26)$$

where 'r' represents the r^{th} iterate. The tolerance was set at 10^{-16}. All calculation were performed on an Ardent Titan II Graphics Supercomputer in double precision using a single processor. Table 1 presents numeric results for the expansion coefficients based on naive choices of N. It is apparent from viewing Table 1 that the present approximate analytic solution produces highly accurate results. Recalling that the exact solution is $x^3 + x^2$ (or in terms of Chebychev polynomials of the first kind, we would find $a_0 = -1/2$, $a_1 = 3/4$, $a_2 = 1/2$, $a_3 = 1/4$, and $a_k = 0$, $k > 3$), one concludes that with a proper choice of N, benchmark accuracy can be achieved for the problem under consideration by the present scheme.

The unknown function f(x) can be reconstructed from Eq. (3.25) once the expansion coefficients have been resolved. The approximate analytic solution displayed in Eq. (3.25) also permits a massively parallel computation for f(x) for preassigned values of x.

In closing, the extension of Peters' method to linear, Cauchy singular integro-differential equations appears viable and fruitful. The analytic preconditioning offered by Peters' method suggests a natural set of basis functions to be used in the approximation process. In light of the numerous analytic manipulations required in the present study, it appears that the inclusion of symbolic computation reduces that portion of the effort. Also, future studies should consider the use of a collocation method for determining the unknown Chebychev expansion coefficients since this approach should greatly reduce the operation count.

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m	a_m (N = 1)	a_m (N = 3)	a_m (N = 6)
0	0.2483339	-0.5000000	-0.5000000
1	0.7516661	0.7500000	0.7500000
2		0.5000000	0.5000000
3		0.2500000	0.2500000
4			$O(10^{-19})$
5			$O(10^{-19})$
6			$O(10^{-19})$

Table 1. Illustrative table showing numeric values of the expansion coefficients as a function of the parameter N. CPU times were always less than 0.07 seconds using a standard Gauss-Seidel method. For N > 2, numerically "exact" results for f(x) occur.