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A Finite Element Simulation System in Reservoir Engineering

A Project

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1. RESERVOIR SIMULATION TOOLS

1.1 BOAST

Reservoir simulation is performed to predict the future performance of a reservoir based on its current state and past performance and to explore other methods for increasing the recovery of hydrocarbons from a reservoir. Reservoir simulators are routinely used for these purposes. A reservoir simulator is a sophisticated computer program which solves a system of partial differential equations describing multiphase fluid flow (oil, water and gas) in a porous reservoir rock.

Reservoir simulators are usually based on:

- (1) reservoir engineering principles,
- (2) a set of partial differential equations that describe the flow of fluids through porous media,
- (3) a numerical technique (Finite Difference or Finite Element Method) to solve the set of differential equations,
- (4) and a computer program which implements the computing model. It usually requires a high performance computer and a graphics workstation to display the computed results.

BOAST is one of many current reservoir simulators in use today. The

well region. Relatively little information has appeared in the literature about detailed representations of wells in numerical simulators. Some discussions of well regions was published by Peaceman(1977). These state that when we want to represent an oil production rate:

$$Q_o = PI \left(\frac{\lambda_o}{B_o} \right) (p_e - p_{wf}) \text{ where } p_e > p_{wf}$$

where p_{wf} is the oil pressure in the wellbore and Q_o is the amount of oil produced from the well. PI is a constant, representing the rock and wellbore properties. This well model assumes the fluid flow is at steady-state and the oil is incompressible. These assumptions are not accurate. The fluid flow is not at steady state and the fluids are compressible in every stage of a simulation.

(3) BOAST uses the IMPES method, which solves the pressure equation first, then it solves for the phase saturations. This method is only conditionally stable. The stability limit for any explicit method is directly proportional to the grid size that is used. This is a major concern when small grid cells and very high flux rates are used that produce rapid saturation changes, which is the environment in the well region. Thus, to get reasonable solutions, either a large grid or a small time step must be used.

(4) Most commercial reservoir simulators today are fully implicit and BOAST is not.

tween neighbouring blocks in BOAST is determined by the transmissibilities. Assuming that the conditions needed for Darcy flow are satisfied, then an average phase volumetric flow-rate can be determined at each face of a block.

After the transmissibilities are computed in BOAST, they are used to determine the phase volumetric flow at each face of all of the blocks. This volumetric data is used in our well block after it is computed by BOAST.

The flow in BOAST is given in the direction of west, east, south, north, top and bottom of a block (Figure 2). Figure 2 presents these directions and what we named each phase variable for our well region model.

Figure 2

OW, OE, OS, ON, OT and OB arrays are the variable names for the phase

transmissibilities for oil phase. WW,WE,WS,WN,WT and WB are the phase transmissibilities for water. GW,GE,GS,GN,GT and GB are the phase transmissibilities for gas. DPi (i=1,6) are the pressure differences in the direction of two neighbouring blocks and these are needed to compute the volumes of each phase that cross each boundary of the well region block.

OIL:

$$QOW(k)=DP1*OW(i,j,k)$$

$$QOE(k)=DP2*OE(i,j,k)$$

$$QOS(k)=DP3*OS(i,j,k)$$

$$QON(k)=DP4*ON(i,j,k)$$

$$QOT(K)=DP5*OT(i,j,k)$$

$$QOB(K)=DP6*OB(i,j,k)$$

Water:

$$QWW(K)=DP1*WW(i,j,k)$$

$$QWE(k)=DP2*WE(i,j,k)$$

$$QWS(K)=DP3*WS(i,j,k)$$

$$QWN(k)=DP4*WN(i,j,k)$$

$$QWT(k)=DP5*WT(i,j,k)$$

$$QWB(k)=DP6*WB(i,j,k)$$

Free Gas:

$$QDW(k)=DP1*GW(i,j,k)$$

$$QGE(k)=DP2*GE(i,j,k)$$

$$QGS(k)=DP3*GS(i,j,k)$$

$$QGN(k)=DP4*GN(i,j,k)$$

$$QGT(k)=DP5*GT(i,j,k)$$

$$QGB(k)=DP6*GB(i,j,k)$$

The inner boundary condition in the wellbore is implemented by the total fluid flow (oil, water and gas) being equally distributed in each node at the wellbore. That is, the total fluid withdrawn during a single time step is divided by the number of nodes in the wellbore and this quantity is then removed from each node. The total fluid flow is specified in the BOAST input file. It is controlled by the well production conditions.

The flowchart of FEM model with the interface to BOAST that obtains these volumetric amounts is presented :

The reason that we reduced the time step in the FEM part was because of the highly non-linear effect of fluid flow in the well region.

3. A REAL TIME DISTRIBUTED GRAPHICS SYSTEM

3.1 SYSTEM DESCRIPTION

Traditionally, the output of a simulator is saved in a disk file for further analysis. It is both tedious and time intensive to read and understand the output text file. Another limitation of the existing BOAST reservoir simulator is that the results are available after the entire simulation is over. But the whole simulation may take a few hours or even a few days, depending on the size and accuracy of the real problem.

To better understand and subsequently use the simulation results, we propose a real time distributed graphics system. This system will visualize the output data even while the simulation is running. Using the new computing technology of X-Windows, Motif and a TCP/IP network, a real time graphics system has been implemented using both an IBM AIX RS 6000 and a DEC Alpha workstation (UNIX). The UNIX 4.3 BSD socket was used to transfer data between the two computer systems, that are used in the simulation, one for the simulation and the other for the real time graphics system.

This graphics system provides a great flexibility for the petroleum engineer to visualize the simulation results. This graphics interface is easy to

understand. The user can use as many of the window widgets as desired, such as pulldown menus and pushdown buttons to make selections, and plot the desired graph in the real time.

The main software used in the design and implementation of this real time graphics system were:

1. Graphics routines from X-Windows (Xlib),
2. User interface design (Motif 1.2),
3. and Network Communications (4.3 BSD Sockets).

3.2. WINDOW INTERFACE/NETWORK

3.2.1 X Window

The X Window System is a hardware and operating system-independent windowing system. It was developed jointly by MIT and Digital Equipment Corporation, and has been adopted by the computer industry as a standard for graphics applications.

Like other windowing systems, the X-Window System divides the screen into multiple input and output areas called windows. The X-Window System takes user input from a pointer, usually by clicking a mouse. This method of using programs is often easier to learn than traditional keyboard control because it is more intuitive.

Programming a graphically-based window system is fundamentally different from standard procedural programming. In traditional character-based interfaces, once the application starts execution, it is always in control. The executing application knows what kind of input will it allow, and may define exclusive modes to limit that input. Only "keyboard" input is expected. In a windowing system, by contrast, multiple graphic applications may be running simultaneously. In addition to the "keyboard" input, the user can use the pointer to select data. Window applications are almost modeless, that

3.2.2 Xt/Motif

The Xt Intrinsics are based upon Xlib, the lowest-level C-Language interface to X windows. Xlib provides full access to the capabilities of the X Protocol, but does little to make programming easier.

It is quite difficult to build applications that have a graphics user interface using a low level programming library such as Xlib. To simplify development, Xt Intrinsics is designed to contain each of the user-interface elements of a graphical application such as scrollbars, command buttons, dialog boxes, popup or pulldown menus. The purpose of Xt Intrinsics is to provide an object-oriented layer that supports the user-interface abstraction called widget. A widget is a reusable, configurable piece of code that operates independently from the application.

Motif is based on Xt Intrinsics. Motif is a set of guidelines that specify how a user interface of graphical computers could “look and feel.” This term, “look and feel”, describes how an application appears on the screen (the look) and how the user interacts with it (the feel).

3.2.3 TCP/IP and Socket

3.2.3.1 TCP/IP - the Internet Protocols

The Advanced Research Projects Agency (APRA) of the Department of Defense (DoD) sponsored the development of the APRANET during the late 1960s and the early 1970s. The ARPANET included military, university, and research sites, and was used to support computer science and military research projects. Although the accurate name for this family of protocols is the "DARPA Internet protocol suite," it is commonly referred to as TCP/IP protocol suite, or just TCP/IP.

TCP (Transmission Control Protocol) is a connection-oriented protocol that provides a reliable, full-duplex, byte stream for the user process. It is used by most Internet application programs such FTP and TELNET protocols. IP (Internet Protocol) is the protocol that provides the packet delivery service for TCP. Since TCP uses IP, the entire Internet protocol suite is often called the TCP/IP protocol family.

There are several interesting properties for TCP/IP:

1. It is not vendor-specific.
2. It has been implemented on almost every computer from personal computers to the largest supercomputers.

3. It is used for both LANs and WANs.

4. It is widely used by many different government agencies, Universities and commercial sites today and not just DARPA-funded research projects.

One reason for the increased use of the TCP/IP protocols during the 1980s was their inclusion in the BSD Unix system around 1982. This, along with the use of BSD Unix in technical workstations, allowed many organizations and university departments to establish their own LANs.

3.2.3.2 Berkeley Sockets

The API (Application Program Interface) is the interface available to an application programmer. The availability of an API depends on both the operating system and the programming language. The two most prevalent communication APIs for Unix systems are Berkeley sockets and the System V Transport Layer Interface (TLI). Both of these interfaces were developed for the C language.

The new release of 4.3 BSD Unix socket supports communication protocols in the Unix domain and Internet domain (TCP/IP). Unix domain sockets provide a data communication between two processes in the same Unix machine, while Internet domain sockets provide data communication between two processes on physically different machines.

3.3. SYSTEM REQUIREMENTS

Before the design and implementation of a system, one needs to study the requirements of the user. Since the user probably doesn't know much about our programming language, they are probably not interested in how the functions associated with a window are implemented. What they usually want is a simple and working graphics interface.

The user may specify many of the function requirements. In this system they are :

1. The interface system should be easily understood. It should provide a certain kind of on-line help.
2. The system should provide a set of window widgets such as the pull-down menu and the pushdown button to assist the user's selection.
3. The system should be able to plot 2D graphics and 3D color images to visualize the data.
4. The system should be real time and support network connection.

3.4. DESIGN

There were three main issues related to the design of this system. Xlib is used to provide the basic draw routines. An internal data structure is built to retrieve data from an input file or socket. If the input is from a socket, the window system should receive the data as soon as it arrives from the network and then do the required computing.

3.4.1 Xlib plotting routines

Xlib provides a set of plotting functions useful in implementing this system, the functions we used from Xlib are XDrawRectangle, XDrawLine, XDrawImageString and XDrawPoint.

The descriptions of these functions follow:

1. XDrawRectangle(Display *display, Drawable drawable, GC gc, int x, int y, unsigned width, unsigned height). Display is associated with the physical device to display the plots. Drawable is an object to make plots on. GC is defined as graphics context, which is a data structure specifying the setting up of a window object, for example foreground color and background color. XDrawRectangle draws the outline of a rectangle by using ordinate and abscissa as coordinates.

2. XDrawLine(Display *display, Drawable drawable, GC gc, int x, int

y, unsigned width, unsigned height). XDrawLine draws a line between two points on the screen.

3. XDrawString(Display *display, Drawable drawable, GC gc, int x, int y, char *string, int length). XDrawString draws a string on the screen.

4. XDrawPoint(Display *display, Drawable drawable, GC gc, int x, int y). XDrawPoint draws a pixel on the screen. The color of the pixel can be specified by the value of gc.

It is interesting that any complicated computer presentation graphics can be developed based on these four basic routines.

3.4.2 System Input

This system accepts input data from three different sources. These input data sources are: 1) a disk file, 2) the Unix domain socket, 3) and the TCP/IP socket. Each of these input sources are virtual files since Unix treats its input and output devices as files. All of these input sources need to be opened as a file descriptor before reading or writing to them. The code to handle the reading from or writing to the file descriptor is the same.

In order to capture the event that the data from socket is ready, the concept of event handler in X windows is used. `XtAppAddInput()` can be used to add an event handler to a window object. It registers a procedure to be called when input is available from a file or socket. The registered procedure will perform the reading and plot in the window.

The input file records the information about the three variable distributions in the real space associated with coordinates x, y, z over time. In the C program, a data structure is written in a four dimension array and has the following format:

```
typedef struct {  
float variable[nθ][nr][nz][3];  
float day;
```

... other variables ...

} DATA

A dynamic array of the type DATA is allocated whenever the window receives data from the input. The C function call for a dynamic allocation in Unix is `realloc()` and this is used in this system.

3.4.3 Network Communications

This system uses a client/server model to transfer data between two processes in the network. This graphics system acts as a server on a Dec Alpha Station 200 (233MHZ). The reservoir simulator runs as a client on an IBM RS 6000 workstation located in another building. The Dec Alpha Station 200 and the IBM RS6000 communicates through the local Ethernet. 4.3 BSD socket is used for the data and communication control.

The scenario of the system call of the socket is presented in in Figure 4:

Figure 4

3.5. PROTOTYPE USER'S MANUAL

There are three main windows in this system. This section describes each.

3.5.1 Main control window

Figure 5

This Real Time Graphics System has File, View Data, Control and Help in its pulldown menu. The explanation for each of these follows:

1. File has "quit" submenu that can terminate the program.
2. View Data has "First Step", "Next Step" and "Prev Step" in its submenu. It can be used to control the time of the plots.
3. Control has the "Step Size" submenu. It controls the time interval for the real time plots.
4. Help has the "Help" submenu. It pops up a dialog window which contains a help manual and a graphics illustration of the problem.

3.5.2 Variable's space distribution window

Figure 6

In Figure 6, a sample window plot is presented and the variables that are plotted are pressure vs distance. This was plotted at a simulation time of

0.8 day. It is named the pressure distribution. This window has File, Graph, Image and Data for pulldown menus.

Each of these icons for the pressure distribution has additional features.

These are:

1. File has a "save" submenu. This saves the graphics in the window as a GIF file for future reference as shown in Figure 7.

Figure 7

2. Graph has "by Radical " and "Direction" submenus. They provide two

3.5.3 Variable's change over time

Figure 9

The above window is similar to the previous window except that in this plot the abscissa is time. The variable's real time behavior is displayed in this window.

4. TEST RESULTS

Two sets of test data for our FEM well block model were selected by Dr. Ming-Ming Chang, senior research engineer from NIPER. Each of these two sets of data has been used in testing our simulation system. Both sets of test results obtained were consistent with the fundamental principles of Petroleum Engineering and exhibited expected fluid flow characteristics.

4.1 Test Problem 1

Test Problem 1 is shown in Figure 10. It has a single oil production well located at block (6,5,1) in a reservoir. Initially, there were two phases (oil and water) present in the reservoir. All the gas was dissolved in the oil phase. The water phase was immobile since its concentration was always below the critical saturation of water present to cause mobility. When the oil pressure decreased as oil was being produced, the gas came out of the oil phase (this occurred at the bubble point) and this produces free gas. We expected the oil pressure around the well bore in the well block to drop below bubble point and to observe an increase in gas saturation.

This was in fact, what our calculation using the output data for test problem 1, produced.

Figure 10

Our reservoir simulation calculated the oil pressure, water saturation and gas saturation implicitly in our FEM grid over the "6,5,1" well block over time. Some example results from this calculation are shown in the following figures.

Figure 12 shows the gas saturation's distribution also at day 20. The upper line is the gas saturation at the top of the well block. The lower line is the gas saturation at the bottom of the well block. Since gas is less dense, with respect to oil, it tends to gather at the higher plane. The distance is measured out from the wellbore. As seen in Figure 12, the gas saturations decreases as the distance from the wellbore increases. This is the expected result since the reservoir pressure is increasing as the distance from the wellbore increases.

the top plane. The results became unstable at day 24 and the pressure was slightly different vs distance from the wellbore depending upon which grids were traversed from the wellbore.

Figure 15 shows the gas saturation versus distance from the wellbore at the top plane at day 24. The unstable result is very apparent here. All of the gas saturations should be congruent but they are not. The gas saturation was not well behaved and this will be discussed on the next page.

Figure 16 shows the gas saturation distribution vs. distance from the wellbore at the bottom plane at day 24. These vary depending upon the radius traversed from the wellbore.

As shown in Figure 15 and 16, the gas saturation in our solution has an instability. This may be due to two possible reasons (1) the FEM stiffness matrix may be ill-conditioned and not stable or (2) the boundary condition at the wellbore may not be correctly stated and implemented. The boundary condition being addressed in another research project. The final discretized FEM equation is a first order difference equation with respect to time. For first order linear system, the solution is stable if all of the eigenvalues are less than 1. The eigenvalues of the final FEM stiffness matrix are less than

1. So we believe that the boundary condition at the wellbore is the problem because we only know the total fluid flow being removed from the well and not how it is distributed among the nodes at the wellbore.

4.2 Test Problem 2

Figure 17

The reservoir mesh for this problem is $10 \times 10 \times 3$. It has 3 layers, and a grid that has 100 blocks that are 1000 feet on each side. The model has a gas injection well on one corner and an oil production well at the opposite corner as depicted in figure 17 (A). The gas is injected into the top layer of the 3 layers and the oil is to be produced from the lowest layer of the 3 layers as shown in figure 17 (B).

The injected gas drives the oil from the reservoir into the oil production well. The oil well in Figure 17 (A) is located in block (10,10,3). This oil well was completed in only one layer and this was the lowest of the 3 layers.

Figure 18 shows the change of oil pressure in the wellbore over time. It also shows a graphics presentation that was developed to present these results as a part of this research.

Figure 18

Figure 19 shows the oil pressure distribution vs distance from the wellbore at 0.8 day for test problem 2.

Figure 19

The instability that occurred with test data 1 also occurred with test data 2. The reason for this is believed to be the same as discussed in Test Problem One.

5. CONCLUSIONS

This finite element reservoir simulation system provides an accurate solution in the well block region for general three dimension and three phase problems in reservoir engineering. Its formulation and coding have been extensively tested on an IBM RS 6000 Unix workstation. This simulation is consistent with the fundamental principles of Petroleum Engineering and fluid flow in a porous media.

The limitation of this system is that it exhibits an instability during the simulation. We believe that this instability is caused after the breakthrough of a second phase and how this is to be distributed at the inner wellbore boundary. Therefore additional knowledge is required from experimental reservoir engineering and research before this system can be completed and used in actual field conditions. For certain well test problems known, this system is a working tool .

Although this system has limitations, it can still provide an accurate solution to many reservoir well simulations in their initial stages. Most reservoir simulators are not accurate in their initial solution stages.

A real time distributed graphics system has been developed to be used

with this simulation system. This graphics system is able to visualize the simulation results while the simulation is running. It is intended to reduce the cycle of development in a well block region simulator.

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Three Phase 3-D Finite Element Method in Reservoir Simulation

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The flow of fluid into and out of a single reservoir cube:

Assume fluid flows into this cube at $x(\tau_x)$ and out of the cube at $x+\Delta x (\tau_{x+\Delta x})$. τ denotes the fluid flux and is

defined as the rate of flow of mass per unit cross-sectional area normal to the direction of flow. Conservation of mass yields:

1.1 Conservation of mass

mass entering - mass exiting = accumulation of mass in cube

$$\text{mass in} = [(\tau_x)_x \Delta y \Delta z + (\tau_y)_y \Delta x \Delta z + (\tau_z)_z \Delta x \Delta y] \Delta t \quad (1.1.1)$$

$$\text{mass out} = [(\tau_x)_{x+\Delta x} \Delta y \Delta z + (\tau_y)_{y+\Delta y} \Delta x \Delta z + (\tau_z)_{z+\Delta z} \Delta x \Delta y] \Delta t + q \Delta x \Delta y \Delta z \Delta t \quad (1.1.2)$$

where a source/sink term q which represents mass flow into (source) or out of (sink) a "well."

Accumulation of mass in cube is the change of concentration of phase p (C_p) in the time interval Δt . If the concentration C_p is defined as the total mass of phase p (oil, water, or gas) in the whole cube divided by its volume, then the accumulation term is:

$$[(C_p)_{t+\Delta t} - (C_p)_t] \Delta x \Delta y \Delta z \quad (1.1.3)$$

$$\text{Mass IN} - \text{Mass OUT} = \text{Mass Accumulation} \quad (1.1.4)$$

and

$$\begin{aligned} & [(\tau_x)_x \Delta y \Delta z + (\tau_y)_y \Delta x \Delta z + (\tau_z)_z \Delta x \Delta y] \Delta t - [(\tau_x)_{x+\Delta x} \Delta y \Delta z + (\tau_y)_{y+\Delta y} \Delta x \Delta z + (\tau_z)_{z+\Delta z} \Delta x \Delta y] \Delta t + q \Delta x \Delta y \Delta z \Delta t \\ &= [(C_p)_{t+\Delta t} - (C_p)_t] \Delta x \Delta y \Delta z - \left[\frac{(\tau_x)_{x+\Delta x} - (\tau_x)_x}{\Delta x} \right] \Delta y \Delta z \Delta t - \left[\frac{(\tau_y)_{y+\Delta y} - (\tau_y)_y}{\Delta y} \right] \Delta x \Delta z \Delta t - \left[\frac{(\tau_z)_{z+\Delta z} - (\tau_z)_z}{\Delta z} \right] \Delta x \Delta y \Delta t - q \Delta x \Delta y \Delta z \Delta t \end{aligned} \quad (1.1.5)$$

$$= \frac{[(C_p)_{t+\Delta t} - (C_p)_t]}{\Delta t} \Delta x \Delta y \Delta z \quad (1.1.6)$$

and in the limit as $\Delta x, \Delta y, \Delta z, \Delta t \rightarrow 0$

$$-\frac{\partial \tau_x}{\partial x} - \frac{\partial \tau_y}{\partial y} - \frac{\partial \tau_z}{\partial z} - q = \frac{\partial C_p}{\partial t} \quad (1.1.7)$$

Each phase (oil, water, gas) satisfies the mass conservation equation.

1.2 Flow Equations

A flux in a given direction can be written as the density of the fluid times its velocity in that given direction:

$$(t)_o = \frac{\rho_o \dot{V}_o}{B_o} \quad (1.2.1)$$

$$(t)_w = \frac{\rho_w \dot{V}_w}{B_w} \quad (1.2.2)$$

$$(t)_g = \frac{\rho_g \dot{V}_g}{B_g} + \frac{R_{so} \rho_g \dot{V}_o}{B_o} + \frac{R_{sw} \rho_g \dot{V}_w}{B_w} \quad (1.2.3)$$

where R_{so} and R_{sw} are gas solubilities in SCF/STB. B_o , B_w and B_g are formation volume factors in reservoir volume/standard volume and the ρ 's are fluid densities at standard conditions (60°F and 14.7 psia).

The velocities are assumed to be Darcy velocities and their X-components are:

$$(v_o)_x = -K_x \lambda_o \frac{\partial}{\partial x} \left[P_o - \frac{\rho_o g z}{144 g_c} \right] \quad (1.2.4)$$

$$(v_w)_x = -K_x \lambda_w \frac{\partial}{\partial x} \left[P_w - \frac{\rho_w g z}{144 g_c} \right] \quad (1.2.5)$$

$$(v_g)_x = -K_x \lambda_g \frac{\partial}{\partial x} \left[P_g - \frac{\rho_g g z}{144 g_c} \right] \quad (1.2.6)$$

Assume $g = g_c$

Similar expressions can be written for the y and z components. Each phase mobility λ_p is defined as the ratio of the

relative permeability to flow of the phase divided by its viscosity, $\lambda_p = \frac{K_{rp}}{\mu_p}$.

The phase densities are related to the formation volume factors and gas solubilities by

$$\rho'_o = \frac{1}{B_o} [P_o + R_{so} \rho_g] \quad (1.2.7)$$

$$\rho'_w = \frac{1}{B_w} [\rho_w + R_{sw} \rho_g] \quad (1.2.8)$$

$$\rho'_g = \frac{\rho_g}{B_g} \quad (1.2.9)$$

and the concentrations are given by

$$C_o = \frac{\phi \rho_o S_o}{B_o} \quad (1.2.10)$$

$$C_w = \frac{\phi \rho_w S_w}{B_w} \quad (1.2.11)$$

$$C_g = \phi \rho_g \left[\frac{S_g}{B_g} + \frac{R_{so} S_o}{B_o} + \frac{R_{sw} S_w}{B_w} \right] \quad (1.2.12)$$

where ϕ is the porosity and S_p is the saturation of phase p . The saturations satisfy:

$$S_o + S_w + S_g = 1 \quad (\text{and } 0 \leq S_o, S_w, S_g \leq 1) \quad (1.2.13)$$

combining these equations¹:

Oil

$$-\left[\frac{\partial}{\partial x} \left(\frac{\rho_o}{B_o} V_{xo} \right) + \frac{\partial}{\partial y} \left(\frac{\rho_o}{B_o} V_{yo} \right) + \frac{\partial}{\partial z} \left(\frac{\rho_o}{B_o} V_{zo} \right) \right] - q_o = \frac{\partial}{\partial t} \left(\frac{\phi \rho_o S_o}{B_o} \right) \quad (1.2.14)$$

Water

$$-\left[\frac{\partial}{\partial x} \left(\frac{\rho_w}{B_w} V_{xw} \right) + \frac{\partial}{\partial y} \left(\frac{\rho_w}{B_w} V_{yw} \right) + \frac{\partial}{\partial z} \left(\frac{\rho_w}{B_w} V_{zw} \right) \right] - q_w = \frac{\partial}{\partial t} \left(\frac{\phi \rho_w S_w}{B_w} \right) \quad (1.2.15)$$

Gas

1. All densities are at standard conditions.

$$\begin{aligned}
& -\frac{\partial}{\partial x} \left[\frac{\rho_g}{B_g} V_{xg} + \frac{R_{so}\rho_g}{B_o} V_{xo} + \frac{R_{sw}\rho_g}{B_w} V_{xw} \right] - \frac{\partial}{\partial y} \left[\frac{\rho_g}{B_g} V_{yg} + \frac{R_{so}\rho_g}{B_o} V_{yo} + \frac{R_{sw}\rho_g}{B_w} V_{yw} \right] \\
& - \frac{\partial}{\partial z} \left[\frac{\rho_g}{B_g} V_{zg} + \frac{R_{so}\rho_g}{B_o} V_{zo} + \frac{R_{sw}\rho_g}{B_w} V_{zw} \right] - q_g = -\frac{\partial}{\partial t} \left[\phi \rho_g \left(\frac{S_g}{B_g} + \frac{R_{so}S_o}{B_o} + \frac{R_{sw}S_w}{B_w} \right) \right]
\end{aligned} \tag{1.2.16}$$

The densities at standard conditions are constants, and these equations can be reduced as shown below:

Oil:

$$-\left[\frac{\partial}{\partial x} \left(\frac{1}{B_o} V_{xo} \right) + \frac{\partial}{\partial y} \left(\frac{1}{B_o} V_{yo} \right) + \frac{\partial}{\partial z} \left(\frac{1}{B_o} V_{zo} \right) \right] - \frac{q_o}{\rho_o} = \frac{\partial}{\partial t} \left(\frac{\phi S_o}{B_o} \right) \tag{1.2.17}$$

Water:

$$-\left[\frac{\partial}{\partial x} \left(\frac{1}{B_w} V_{xw} \right) + \frac{\partial}{\partial y} \left(\frac{1}{B_w} V_{yw} \right) + \frac{\partial}{\partial z} \left(\frac{1}{B_w} V_{zw} \right) \right] - \frac{q_w}{\rho_w} = \frac{\partial}{\partial t} \left(\frac{\phi S_w}{B_w} \right) \tag{1.2.18}$$

Gas:

$$\begin{aligned}
& -\frac{\partial}{\partial x} \left(\frac{V_{xg}}{B_g} + \frac{R_{so}V_{xo}}{B_o} + \frac{R_{sw}V_{xw}}{B_w} \right) - \frac{\partial}{\partial y} \left(\frac{V_{yg}}{B_g} + \frac{R_{so}V_{yo}}{B_o} + \frac{R_{sw}V_{yw}}{B_w} \right) - \frac{\partial}{\partial z} \left(\frac{V_{zg}}{B_g} + \frac{R_{so}V_{zo}}{B_o} + \frac{R_{sw}V_{zw}}{B_w} \right) - \frac{q_g}{\rho_g} \\
& = \frac{\partial}{\partial t} \left[\phi \left(\frac{S_g}{B_g} + \frac{R_{so}S_o}{B_o} + \frac{R_{sw}S_w}{B_w} \right) \right]
\end{aligned} \tag{1.2.19}$$

These equations can be written as:

$$-\nabla \cdot \frac{\vec{v}_o}{B_o} - \frac{q_o}{\rho_o} = \frac{\partial}{\partial t} \left(\frac{\phi S_o}{B_o} \right) \tag{1.2.20}$$

$$-\nabla \cdot \frac{\vec{v}_w}{B_w} - \frac{q_w}{\rho_w} = \frac{\partial}{\partial t} \left(\frac{\phi S_w}{B_w} \right) \quad (1.2.21)$$

$$-\nabla \cdot \left\{ \frac{\vec{v}_g}{B_g} + \frac{R_{so}}{B_o} \vec{v}_o + \frac{R_{sw}}{B_w} \vec{v}_w \right\} - \frac{q_g}{\rho_g} = \frac{\partial}{\partial t} \left(\phi \left[\frac{S_g}{B_g} + \frac{R_{so} S_o}{B_o} + \frac{R_{sw} S_w}{B_w} \right] \right) \quad (1.2.22)$$

where

$$\nabla \cdot \vec{v} = \frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y + \frac{\partial}{\partial z} v_z \quad (1.2.23)$$

Define the potential¹ Ψ_p of phase p as:

$$\Psi_p = P_p - \frac{\rho_p Z}{144} \quad (1.2.24)$$

Also introduce the dyadic notation $\hat{M}_o, \hat{M}_w, \text{ and } \hat{M}_g$ to signify that mobility is a tensor of rank two.

$\hat{M}_o = \frac{1}{\mu_o} (\vec{k} \vec{k}_{ro})$ where $(\vec{k} \vec{k}_{ro})$ is the permeability matrix whose terms are given individually by the product $k_{ijk} k_{ro}$, with k_{ijk} the local permeability and k_{ro} the permeability to oil, and μ_o is the oil viscosity.

1.3 Finite Elements

A finite element method will be used to obtain simultaneous solution of the partial differential equations for the flow of oil, water, and gas. The following fluids were considered to be compressible and immiscible which justifies the treatment of relative permeability differences, complex fluid properties, and the effects of capillary pressure.

The continuity equation for each flowing phase can be found in "Modern Reservoir Engineering - A Simulation Approach" by H.B. Crichlow, Prentice Hall, Englewood Cliffs, NJ (1977) or "Petroleum Reservoir Simulation" by K. Aziz and A. Settari, Applied Science Publications, New York (1979). Using Darcy's flow law with these continuity

1. In BOAST, the density is in $lb(mass)/ft^3$ and the division by 144 converts this into psi. A density of

$1 gm/cm^3 = 62.4 lbs/ft^3$.

equations the following expressions can be applied for each flowing phase at standard conditions, the formula that follows was taken from "A Finite Element Solution of a Fully Coupled Implicit Formulation for Reservoir Simulation," by Y. Sukirman and R.W. Lewis, International Journal for Numerical and Analytical Methods in Geomechanics, Vol. 17, 677-698 (1993).

$$-\Delta(\dot{M}_l \Delta(P_l + \rho_l g h)) + (\text{rate of fluid accumulation}) = Q_{\text{net}} \quad (1.3.1)$$

$$\dot{M}_l = \frac{K k_{rl}}{\mu_l B_l} \quad (1.3.2)$$

$$l = o, g, w \quad (1.3.3)$$

where P_l is the pressure, ρ_l is the fluid density, B_l is the formation volume factor, K the absolute permeability matrix, μ_l is the dynamic fluid viscosity and h is the height above an arbitrary datum. The subscripts o , g , and w refer to oil, gas, and water phases, respectively.

$$\dot{M}_w = \frac{1}{\mu_w} (\vec{K} K_{rw}) \quad (1.3.4)$$

$$\dot{M}_g = \frac{1}{\mu_g} (\vec{K} K_{rg}) \quad (1.3.5)$$

let

$$\hat{\lambda}_o = \frac{\dot{M}_o}{B_o} = \frac{K_o}{B_o \mu_o} \quad (1.3.6)$$

$$\hat{\lambda}_w = \frac{\dot{M}_w}{B_w} = \frac{K_w}{B_w \mu_w} \quad (1.3.7)$$

$$\hat{\lambda}_g = \frac{\dot{M}_g}{B_g} = \frac{K_g}{B_g \mu_g} \quad (1.3.8)$$

then

$$\dot{b}_o = -\dot{M}_o \cdot \nabla \Psi_o \quad (1.3.9)$$

$$\dot{b}_w = -\dot{M}_w \cdot \nabla \Psi_w \quad (1.3.10)$$

$$\vec{V}_g = -M_g \cdot \nabla \Psi_g \quad (1.3.11)$$

and,

$$\nabla \cdot (\hat{\lambda}_o \cdot \nabla \Psi_o) - \frac{q_o}{\rho_o} = \frac{\partial}{\partial t} \left(\frac{\phi S_o}{B_o} \right) \quad (1.3.12)$$

$$\nabla \cdot (\hat{\lambda}_w \cdot \nabla \Psi_w) - \frac{q_w}{\rho_w} = \frac{\partial}{\partial t} \left(\frac{\phi S_w}{B_w} \right) \quad (1.3.13)$$

$$\nabla \cdot [\hat{\lambda}_g \cdot \nabla \Psi_g + R_{so} \hat{\lambda}_o \cdot \nabla \Psi_o + R_{sw} \hat{\lambda}_w \cdot \nabla \Psi_w] - \frac{q_g}{\rho_g} = \frac{\partial}{\partial t} \left[\phi \left(\frac{S_g}{B_g} + \frac{R_{so} S_o}{B_o} + \frac{R_{sw} S_w}{B_w} \right) \right] \quad (1.3.14)$$

can be found in Peaceman (1977) equations (1-121) through (1-123). Equivalent Ψ_o, Ψ_w, Ψ_g are the phase potentials, to simplify the handling of these the phase pressures (P_o, P_w, P_g) can be related as capillary pressures:

$$P_{cow} = P_o - P_w \quad (1.3.15)$$

$$P_{cgo} = P_g - P_o \quad (1.3.16)$$

These differences P_{cow} (capillary pressure of oil-to-water) and P_{cgo} (capillary pressure of gas-to-oil). Experimentally P_{cow} and P_{cgo} can be measured and are assumed to be functions of water and gas saturations only. Using these equations, the water and gas phase potentials are:

$$\Psi_w = P_o - P_{cow} - \frac{\rho_w' Z}{144} \quad (1.3.17)$$

$$\Psi_g = P_o - P_{cgo} - \frac{\rho_g' Z}{144} \quad (1.3.18)$$

combining these equations with our previous equations:

Oil:

$$\nabla \cdot (\hat{\lambda}_o \cdot \nabla P_o) - \nabla \cdot \left(\hat{\lambda}_o \cdot \nabla \left(\frac{\rho_o' Z}{144} \right) \right) - \frac{q_o}{\rho_o} = \frac{\partial}{\partial t} \left(\frac{S_o}{B_o} \right) \quad (1.3.19)$$

Water:

$$\nabla \cdot (\hat{\lambda}_w \cdot \nabla P_o) - \nabla \cdot \left(\hat{\lambda}_w \cdot \nabla \left(\frac{\rho'_w Z}{144} + P_{cow} \right) \right) - \frac{q_w}{\rho_w} = \frac{\partial}{\partial t} \left(\varphi \frac{S_w}{B_w} \right) \quad (1.3.20)$$

Gas:

$$\begin{aligned} \nabla \cdot \left[\hat{\lambda}_g \cdot \nabla \left(P_o + P_{cgo} - \frac{\rho'_g Z}{144} \right) + R_{so} \hat{\lambda}_o \cdot \nabla \left(P_o - \frac{\rho'_o Z}{144} \right) \right. \\ \left. + R_{sw} \hat{\lambda}_w \cdot \nabla \left(P_o - P_{cow} - \frac{\rho'_w Z}{144} \right) \right] - \frac{q_g}{\rho_g} = \frac{\partial}{\partial t} \left[\varphi \left(\frac{S_g}{B_g} + \frac{R_{so} S_o}{B_o} + \frac{R_{sw} S_w}{B} \right) \right] \end{aligned} \quad (1.3.21)$$

$$S_o = 1 - S_w - S_g \quad (1.3.22)$$

$$\frac{dS_o}{dt} = -\frac{dS_w}{dt} - \frac{dS_g}{dt} \quad (1.3.23)$$

$$\frac{\partial}{\partial t} \left(\varphi \frac{S_o}{B_o} \right) = \frac{\varphi}{B_o} \cdot \frac{\partial S_o}{\partial t} + \frac{S_o}{B_o} \cdot \frac{\partial \varphi}{\partial t} + \varphi S_o \left(-\frac{1}{B_o^2} \cdot \frac{\partial B_o}{\partial t} \right) \quad (1.3.24)$$

$$\frac{\partial \varphi}{\partial t} = \frac{d\varphi}{dP_o} \cdot \frac{\partial P_o}{\partial t} \quad (1.3.25)$$

$$\frac{\partial B_o}{\partial t} = \frac{dB_o}{dP_o} \cdot \frac{\partial P_o}{\partial t} \quad (1.3.26)$$

so that,

$$\frac{\partial}{\partial t} \left(\varphi \frac{S_o}{B_o} \right) = \frac{\varphi}{B_o} \cdot \frac{\partial S_o}{\partial t} + \frac{S_o}{B_o} \left(\frac{d\varphi}{dP_o} \right) \frac{\partial P_o}{\partial t} - \frac{\varphi S_o}{B_o^2} \left(\frac{dB_o}{dP_o} \right) \frac{\partial P_o}{\partial t} = \frac{\varphi}{B_o} \cdot \frac{\partial S_o}{\partial t} + \frac{\varphi S_o}{B_o} \left[\frac{1}{\varphi} \frac{d\varphi}{dP_o} - \frac{1}{B_o} \frac{dB_o}{dP_o} \right] \frac{\partial P_o}{\partial t} \quad (1.3.27)$$

$$\frac{\partial}{\partial p} \left[\frac{\partial}{\partial p} \left[\frac{\partial}{\partial p} \right] \right] - \frac{\partial}{\partial p} \left[\frac{\partial}{\partial p} \right] + \frac{\partial}{\partial p} \left[\frac{\partial}{\partial p} \right] = 17$$

$$\frac{\partial}{\partial p} \frac{\partial p}{\partial q} = \frac{\partial}{\partial q}$$

$$\frac{\partial}{\partial \theta} \frac{\partial p}{\partial p} = \frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial s} \left(\frac{g}{\phi_s} \right) = \frac{g}{\phi_s^2} \frac{\partial \phi_s}{\partial s} - \frac{1}{\phi_s} \frac{\partial g}{\partial s} + \frac{g}{\phi_s^2} \frac{\partial \phi_s}{\partial s} = \frac{g}{\phi_s^2} \frac{\partial \phi_s}{\partial s} = L_7$$

$$\epsilon_7 + \epsilon_2 + \epsilon_1 = \left[\left(\frac{B^w}{S^w} + \frac{B^o}{S^o} + \frac{B^g}{S^g} \right) \phi \right] \frac{\partial \epsilon}{\partial \epsilon}$$

[illegible]

$$\frac{\partial \rho}{\partial p} \cdot \frac{\partial p}{\partial \rho} = \frac{\partial \rho}{\partial \rho}$$

$$\frac{\partial \rho}{\partial p} \cdot \frac{\partial p}{\partial \rho} = \frac{\partial \rho}{\partial \rho}$$

$$\frac{\partial}{\partial \mathbf{B}} \frac{\partial}{\partial \mathbf{w}} \cdot \frac{\mathbf{w}}{2} \cdot \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \mathbf{w}} \cdot \frac{\mathbf{w}}{\phi} \cdot \frac{\partial}{\partial \mathbf{B}} = \left(\frac{\partial}{\partial \phi} \frac{\partial}{\partial \mathbf{B}} \right) \frac{\mathbf{w}}{2}$$

$$\left[\frac{\partial}{\partial p} \left(\frac{\partial p}{\partial p} \frac{\partial \phi}{\partial p} \right) - \frac{\partial}{\partial p} \left(\frac{\partial p}{\partial p} \frac{\partial \phi}{\partial p} \right) \right] \frac{\partial \phi}{\partial p} =$$

(1.3.37)

(1.3.36)

(1.3.35)

(1.3.34)

(1.3.33)

(1.3.32)

(1.3.31)

(1.3.30)

(1.3.29)

(1.3.28)

$$L_1 = \frac{\phi}{B_g} \frac{\partial S_g}{\partial t} + \frac{\phi S_g}{B_g} \left[\frac{1}{\phi} \frac{d\phi}{dP_o} - \frac{1}{B_g} \frac{dB_g}{dP_o} \right] \frac{\partial P_o}{\partial t} \quad (1.3.38)$$

$$L_2 = \frac{\partial}{\partial t} \left[\frac{\phi R_{so} S_o}{B_o} \right] \quad (1.3.39)$$

$$L_2 = \frac{\partial R_{so}}{\partial t} \frac{\phi S_o}{B_o} + R_{so} \frac{\partial}{\partial t} \left(\frac{\phi S_o}{B_o} \right) \quad (1.3.40)$$

$$= \frac{dR_{so}}{dP_o} \left(\frac{\phi S_o}{B_o} \right) \frac{dP_o}{dt} + R_{so} \frac{\partial}{\partial t} \left(\frac{\phi S_o}{B_o} \right) \quad (1.3.41)$$

$$L_3 = \frac{\partial}{\partial t} \left[\frac{\phi R_{sw} S_w}{B_w} \right] = \frac{dR_{sw}}{dP_o} \frac{\phi S_w}{B_w} \frac{dP_o}{dt} + R_{sw} \frac{\partial}{\partial t} \left(\frac{\phi S_w}{B_w} \right) \quad (1.3.42)$$

$$\frac{\partial \phi}{\partial t} = \frac{d\phi}{dP_o} \cdot \frac{\partial P_o}{\partial t} \quad (1.3.43)$$

$$L_1 + L_2 + L_3 = \frac{\phi}{B_g} \frac{\partial S_g}{\partial t} + \frac{\phi S_g}{B_g} \left(\frac{1}{\phi} \frac{d\phi}{dP_o} - \frac{1}{B_g} \frac{dB_g}{dP_o} \right) \frac{\partial P_o}{\partial t} + \frac{dR_{so}}{dP_o} \frac{\phi S_o}{B_o} \frac{\partial P_o}{\partial t} + \frac{dR_{sw}}{dP_o} \frac{\phi S_w}{B_w} \frac{\partial P_o}{\partial t} + R_{so} \frac{\partial}{\partial t} \left(\frac{\phi S_o}{B_o} \right) + R_{sw} \frac{\partial}{\partial t} \left(\frac{\phi S_w}{B_w} \right) \quad (1.3.44)$$

Compressibilities of oil, water, gas, and rock:

$$C_o = -\frac{1}{B_o} \frac{\partial B_o}{\partial P_o} + \frac{B_g}{B_o} \frac{\partial R_{so}}{\partial P_o} \quad (1.3.45)$$

$$C_w = -\frac{1}{B_w} \frac{\partial B_w}{\partial P_o} + \frac{B_g}{B_w} \frac{\partial R_{sw}}{\partial P_o} \quad (1.3.46)$$

$$C_g = -\frac{1}{B_g} \frac{\partial B_g}{\partial P_o} \quad (1.3.47)$$

$$C_r = \frac{1}{\phi} \frac{\partial \phi}{\partial P_o} \quad (1.3.48)$$

and the compressibility total is:

$$C_t = C_r + S_o C_o + S_w C_w + S_g C_g \quad (1.3.49)$$

Equation (1):

$$\frac{\partial}{\partial t} \left(\frac{\varphi S_o}{B_o} \right) = \frac{\varphi}{B_o} \left[\frac{\partial S_o}{\partial t} + S_o \left(\frac{1}{\varphi} \frac{\partial \varphi}{\partial P_o} - \frac{1}{B_o} \frac{dB_o}{dP_o} \right) \frac{\partial P_o}{\partial t} \right] \quad (1.3.50)$$

$$= \frac{\varphi}{B_o} \left[\frac{\partial S_o}{\partial t} \right] + \frac{\varphi S_o}{B_o} \left(C_r - \frac{1}{B_o} \frac{dB_o}{dP_o} \right) \frac{\partial P_o}{\partial t} \quad (1.3.51)$$

Equation (2):

$$\frac{\partial}{\partial t} \left(\frac{\varphi S_w}{B_w} \right) = \frac{\varphi}{B_w} \left[\frac{\partial S_w}{\partial t} + S_w \left(\frac{1}{\varphi} \frac{\partial \varphi}{\partial P_o} - \frac{1}{B_w} \frac{dB_w}{dP_o} \right) \frac{\partial P_o}{\partial t} \right] \quad (1.3.52)$$

$$= \frac{\varphi}{B_w} \left[\frac{\partial S_w}{\partial t} \right] + \frac{\varphi S_w}{B_w} \left(C_r - \frac{1}{B_w} \frac{dB_w}{dP_o} \right) \frac{\partial P_o}{\partial t} \quad (1.3.53)$$

Equation (3):

$$L_1 + L_2 + L_3 = \frac{\varphi}{B_g} \cdot \frac{\partial S_g}{\partial t} + \varphi \left[\frac{1}{B_g} \left(S_g C_r + S_g C_g \right) + \frac{S_o}{B_o} \frac{dR_{so}}{dP_o} + \frac{S_w}{B_w} \frac{dR_{sw}}{dP_o} \right] \cdot \frac{\partial P_o}{\partial t} + R_{so} \frac{\partial}{\partial t} \left(\frac{\varphi S_o}{B_o} \right) + R_{sw} \frac{\partial}{\partial t} \left(\frac{\varphi S_w}{B_w} \right) \quad (1.3.54)$$

Let,

$$C_\alpha = \frac{B_g}{B_o} \frac{dR_{so}}{dP_o} \quad (1.3.55)$$

Oil:

$$\nabla \cdot (\hat{\lambda}_o \cdot \nabla P_o) - \nabla \cdot \left(\hat{\lambda}_o \cdot \nabla \left[\frac{\rho'_o z}{144} \right] \right) - \frac{q_o}{\rho_o} = -\frac{\varphi}{B_o} \left[\frac{\partial S_w}{\partial t} + \frac{\partial S_g}{\partial t} \right] + \frac{\varphi}{B_o} \left[C_r - \frac{1}{B_o} \frac{dB_o}{dP_o} \right] \left[\frac{\partial P_o}{\partial t} - S_w \frac{\partial P_o}{\partial t} - S_g \frac{\partial P_o}{\partial t} \right] \quad (1.3.56)$$

Water:

$$\nabla \cdot (\hat{\lambda}_w \cdot \nabla P_o) - \nabla \cdot \left(\hat{\lambda}_w \cdot \nabla \left[\frac{\rho'_w z}{144} + P_{cow} \right] \right) - \frac{q_w}{\rho_w} = \frac{\varphi}{B_w} \frac{\partial S_w}{\partial t} + \frac{\varphi}{B_w} \left[C_r - \frac{1}{B_w} \frac{dB_w}{dP_o} \right] S_w \frac{\partial P_o}{\partial t} \quad (1.3.57)$$

Gas:

$$\begin{aligned} \nabla \cdot \left(\hat{\lambda}_g \cdot \nabla \left[P_o + P_{cgo} - \frac{\rho'_g Z}{144} \right] + R_{so} \hat{\lambda}_o \cdot \nabla \left[P_o - \frac{\rho'_o Z}{144} \right] + R_{sw} \hat{\lambda}_w \cdot \nabla \left[P_o - P_{cow} - \frac{\rho'_w Z}{144} \right] \right) - \frac{q_g}{\rho_g} = \\ = \frac{\phi}{B_g} \cdot \frac{\partial S}{\partial t} + \phi \left[\frac{1}{B_g} (S_g C_r + S_g C_g) + \frac{S_o dR_{so}}{B_o dP_o} + \frac{S_w dR_{sw}}{B_w dP_o} \right] \cdot \frac{\partial P_o}{\partial t} + R_{so} \frac{\partial}{\partial t} \left(\frac{\phi S_o}{B_o} \right) + R_{sw} \frac{\partial}{\partial t} \left(\frac{\phi S_w}{B_w} \right) \end{aligned} \quad (1.3.58)$$

Now perform the following operation: Equation (1.3.58) - R_{so} times Equation (1.3.19) - R_{sw} times Equation (1.3.20)

Then we would have:

Gas:

$$\begin{aligned} \nabla \cdot \left(\hat{\lambda}_g \cdot \nabla \left[P_o + P_{cgo} - \frac{\rho'_g Z}{144} \right] \right) + \nabla \cdot \left(R_{so} \hat{\lambda}_o \cdot \nabla \left[P_o - \frac{\rho'_o Z}{144} \right] \right) - R_{so} \nabla \cdot \left(\hat{\lambda}_o \cdot \nabla \left[P_o - \frac{\rho'_o Z}{144} \right] \right) \\ + \nabla \cdot \left(R_{sw} \hat{\lambda}_w \cdot \nabla \left[P_o - P_{cow} - \frac{\rho'_w Z}{144} \right] \right) - R_{sw} \nabla \cdot \left(\hat{\lambda}_w \cdot \nabla \left[P_o - P_{cow} - \frac{\rho'_w Z}{144} \right] \right) - \left(\frac{q_g}{\rho_g} - \frac{R_{so} q_o}{\rho_o} - \frac{R_{sw} q_w}{\rho_w} \right) \end{aligned} \quad (1.3.59)$$

$$= \frac{\phi}{B_g} \cdot \frac{\partial S}{\partial t} + \phi \left[\frac{1}{B_g} (S_g C_r + S_g C_g) + \frac{S_o dR_{so}}{B_o dP_o} + \frac{S_w dR_{sw}}{B_w dP_o} \right] \cdot \frac{\partial P_o}{\partial t} \quad (1.3.60)$$

An approach to this problem is to find an average value in each element for R_{sw} and R_{so} . Then treat these as constants in the elemental equation for gas in each element. That is to find the average pressure in the element and find the value of R_{sw} and R_{so} for this average element pressure. For 8 nodes this average pressure would be:

$$P_{ave} = \sum_{i=1}^8 P_i N_i$$

We use $\bar{R}_{so}$ as the average for R_{so} and $\bar{R}_{sw}$ as the average for R_{sw} in each element. Then the left side of the Gas Equation becomes:

$$\begin{aligned} \nabla \cdot \left(\hat{\lambda}_g \cdot \nabla P_o \right) + \nabla \cdot \left(\hat{\lambda}_g \cdot \nabla P_{cgo} \right) - \nabla \cdot \left(\hat{\lambda}_g \cdot \nabla \left(\frac{\rho'_g Z}{144} \right) \right) + \nabla \cdot \left(\bar{R}_{so} \hat{\lambda}_o \cdot \nabla P_o \right) - \nabla \cdot \left(\bar{R}_{so} \hat{\lambda}_o \cdot \nabla \left(\frac{\rho'_o Z}{144} \right) \right) + \nabla \cdot \left(\bar{R}_{sw} \hat{\lambda}_w \cdot \nabla P_o \right) \\ - \nabla \cdot \left(\bar{R}_{sw} \hat{\lambda}_w \cdot \nabla P_{cow} \right) - \nabla \cdot \left(\bar{R}_{sw} \hat{\lambda}_w \cdot \nabla \left(\frac{\rho'_w Z}{144} \right) \right) - \bar{R}_{so} \left(\nabla \cdot \left(\hat{\lambda}_o \cdot \nabla P_o \right) - \nabla \cdot \left(\hat{\lambda}_o \cdot \nabla \left(\frac{\rho'_o Z}{144} \right) \right) \right) \\ - \bar{R}_{sw} \left(\nabla \cdot \left(\hat{\lambda}_w \cdot \nabla P_o \right) - \nabla \cdot \left(\hat{\lambda}_w \cdot \nabla \left(\frac{\rho'_w Z}{144} + P_{cow} \right) \right) \right) - \left(\frac{q_g}{\rho_g} - \frac{R_{so} q_o}{\rho_o} - \frac{R_{sw} q_w}{\rho_w} \right) \\ = \nabla \cdot \left(\hat{\lambda}_g \cdot \nabla P_o \right) + \nabla \cdot \left(\hat{\lambda}_g \cdot \nabla P_{cgo} \right) - \left(\nabla \cdot \hat{\lambda}_g \cdot \nabla \left(\frac{\rho'_g Z}{144} \right) \right) - \left(\frac{q_g}{\rho_g} - \frac{R_{so} q_o}{\rho_o} - \frac{R_{sw} q_w}{\rho_w} \right) \end{aligned} \quad (1.3.61)$$

So the Gas equation becomes:

$$\begin{aligned} \nabla \cdot \left(\hat{\lambda}_g \cdot \nabla P_o \right) + \nabla \cdot \left(\hat{\lambda}_g \cdot \nabla P_{cgo} \right) - \nabla \cdot \left(\hat{\lambda}_g \cdot \nabla \left(\frac{\rho'_g Z}{144} \right) \right) - \left(\frac{q_g}{\rho_g} - \frac{R_{so} q_o}{\rho_o} - \frac{R_{sw} q_w}{\rho_w} \right) \\ = \frac{\phi}{B_g} \cdot \frac{\partial S}{\partial t} + \phi \left[\frac{1}{B_g} (S_g C_r + S_g C_g) + \frac{S_o dR_{so}}{B_o dP_o} + \frac{S_w dR_{sw}}{B_w dP_o} \right] \cdot \frac{\partial P_o}{\partial t} \end{aligned}$$

Define,

$$\frac{q_g}{\rho_g} = \frac{q_g}{\rho_g} - R_{so} \frac{q_o}{\rho_o} - R_{sw} \frac{q_w}{\rho_w} \quad (1.3.62)$$

which is the free gas production rate at surface conditions from the source (it is zero if the closed region does not contain a well).

(1.3.63)

We assume that the capillary pressure for oil-to-water (P_{cow}) is a function of the water saturation only and the capillary for gas-to-oil (P_{cgo}) is a function of the gas saturation only. Then

$$\nabla P_{cow} = i \frac{\partial P_{cow}}{\partial x} + j \frac{\partial P_{cow}}{\partial y} + k \frac{\partial P_{cow}}{\partial z} = \left(i \frac{\partial S_w}{\partial x} + j \frac{\partial S_w}{\partial y} + k \frac{\partial S_w}{\partial z} \right) \frac{dP_{cow}}{dS_w} \quad (1.3.64)$$

$$\nabla P_{cow} = \frac{dP_{cow}}{dS_w} \nabla S_w \quad (1.3.65)$$

and,

$$\nabla P_{cgo} = i \frac{\partial P_{cgo}}{\partial x} + j \frac{\partial P_{cgo}}{\partial y} + k \frac{\partial P_{cgo}}{\partial z} = \left(i \frac{\partial S_g}{\partial x} + j \frac{\partial S_g}{\partial y} + k \frac{\partial S_g}{\partial z} \right) \frac{dP_{cgo}}{dS_g} \quad (1.3.66)$$

$$\nabla P_{cgo} = \frac{dP_{cgo}}{dS_g} \nabla S_g \quad (1.3.67)$$

Using these two equations:

Oil:

$$\nabla \cdot (\hat{\lambda}_o \cdot \nabla P_o) - \nabla \cdot \left(\hat{\lambda}_o \cdot \nabla \left(\frac{\rho_o' z}{144} \right) \right) - \frac{q_o}{\rho_o} = -\frac{\Phi}{B_o} \left[\frac{\partial S_w}{\partial t} + \frac{\partial S_g}{\partial t} \right] + \frac{\Phi}{B_o} \left[C_r - \frac{1}{B_o} \frac{\partial B_o}{\partial P_o} \right] \left[\frac{\partial P_o}{\partial t} - S_w \frac{\partial P_o}{\partial t} - S_g \frac{\partial P_o}{\partial t} \right] \quad (1.3.68)$$

Water:

$$\nabla \cdot (\hat{\lambda}_w \cdot \nabla P_o) - \nabla \cdot \left(\hat{\lambda}_w \cdot \nabla \left(\frac{\rho'_w z}{144} \right) \right) - \frac{dP_{cow}}{dS_w} \nabla \cdot (\hat{\lambda}_w \cdot \nabla S_w) - \frac{q_w}{\rho_w} = \frac{\phi}{B_w} \frac{\partial S_w}{\partial t} + \frac{\phi}{B_w} \left[C_r - \frac{1}{B_w} \frac{\partial B_w}{\partial P_o} \right] S_w \frac{\partial P_o}{\partial t} \quad (1.3.69)$$

Gas:

$$\nabla \cdot \hat{\lambda}_g \cdot \nabla \left[P_o - \frac{\rho'_g z}{144} \right] + \frac{dP_{cga}}{dP_o} \nabla \cdot (\hat{\lambda}_g \cdot \nabla S_g) - \frac{q_g}{\rho_g} = \frac{\phi}{B_g} \frac{\partial S_g}{\partial t} + \phi \left[\frac{1}{B_g} (S_g C_r + S_g C_g) + \frac{S_o}{B_o} \frac{dR_{so}}{dP_o} + \frac{S_w}{B_w} \frac{dR_{sw}}{dP_o} \right] \cdot \frac{\partial P_o}{\partial t} \quad (1.3.70)$$

The "mobility" tensor for phase l ,

$$\hat{\lambda}_l = \frac{1}{B_l \mu_l} \begin{bmatrix} k_x & k_y & k_z \\ k_{rlx} & k_{rly} & k_{rlz} \\ k_{rlx} & k_{rly} & k_{rlz} \\ k_z & k_{rlx} & k_{rly} & k_{rlz} \end{bmatrix} \quad (1.3.71)$$

which, for problems in which we have permeabilities that are assumed to be parallel to the coordinates in the x, y, and z directions; that is, K_x, K_y, K_z are non-zero and all other permeabilities are zero:

$$\hat{\lambda}_l = \frac{k_{rl}}{B_l \mu_l} \begin{bmatrix} K_x & 0 & 0 \\ 0 & K_y & 0 \\ 0 & 0 & K_z \end{bmatrix} \quad (1.3.72)$$

We will use these assumptions in later development to simplify the discretized equations, letting ϵ denote the transmissibility of phase l :

$$\hat{\epsilon}_l = \begin{bmatrix} \epsilon_{zl} \\ \epsilon_{yl} \\ \epsilon_{xl} \end{bmatrix} = \frac{1}{\mu_l} \begin{bmatrix} K_x k_{rlx} \\ K_y k_{rly} \\ K_z k_{rlz} \end{bmatrix}$$

1.4 Finite Element Formulation:

Finite Element formulation of the 3 phase, 3-D reservoir problem follows:

The element trial solution for the oil pressure:

$$P_o^{(e)} = \sum_{j=1}^n P_j^{(e)}(t) N_j^{(e)}(x, y, z) = \sum_{j=1}^n P_j N_j \quad (1.4.1)$$

The element trial solution for the water saturation:

$$S_w^{(e)} = \sum_{j=1}^n S_{wj}^{(e)}(t) N_j^{(e)}(x, y, z) = \sum_{j=1}^n S_{wj} N_j \quad (1.4.2)$$

The element trial solution for the gas saturation:

$$S_g^{(e)} = \sum_{j=1}^n S_{gj}^{(e)}(t) N_j^{(e)}(x, y, z) = \sum_{j=1}^n S_{gj} N_j \quad (1.4.3)$$

The element trial solution for the density term:

$$\rho_o = \sum_{j=1}^n (\rho_o)_j N_j^{(e)}(x, y, z)$$

where the $N_j^{(e)}(x, y, z)$ are the shape functions and n is the number of degrees of freedom (DOF) in the element.

Applying Galerkin's Method to the equations that have been developed yields the following set of weighted residuals equations over each finite element e :

Oil:

$$\int_V \left(\nabla \cdot (\hat{\lambda}_o \cdot \nabla P_o) - \nabla \cdot \hat{\lambda}_o \cdot \nabla \left(\frac{\rho_o' z}{144} \right) - \frac{q_o}{\rho_o} + \frac{\Phi}{B_o} \left[\frac{\partial S_w}{\partial t} + \frac{\partial S_g}{\partial t} + \frac{\partial S_o}{\partial t} \right] \right. \\ \left. - \frac{\Phi}{B_o} \left[C_r - \frac{1}{B_o} \frac{\partial B_o}{\partial P_o} \right] \left[\frac{\partial P_o}{\partial t} - S_w \frac{\partial P_o}{\partial t} - S_g \frac{\partial P_o}{\partial t} \right] \right) N_i dV = 0 \quad (1.4.4)$$

Water:

$$\int_V^{(e)} \left\{ \nabla \cdot (\hat{\lambda}_w \cdot \nabla P_o) - \nabla \cdot \left(\hat{\lambda}_w \cdot \nabla \left(\frac{\rho'_w z}{144} \right) \right) - \frac{dP_{cow}}{dS_w} \nabla \cdot (\hat{\lambda}_w \cdot \nabla S_w) - \frac{q}{\rho_w} \right. \\ \left. - \frac{\phi}{B_w} \frac{\partial S_w}{\partial t} - \frac{\phi}{B_w} \left[C_r - \frac{1}{B_w} \frac{\partial B_w}{\partial P_o} \right] S_w \frac{\partial P_o}{\partial t} \right\} N_i dV = 0 \quad (1.4.5)$$

Gas:

$$\int_V^{(e)} \left(\nabla \cdot \hat{\lambda}_g \cdot \nabla \left(P_o - \frac{\rho'_g z}{144} \right) + \frac{dP_{cgo}}{dS_g} \nabla \cdot (\hat{\lambda}_g \cdot \nabla S_g) - \frac{q_{fg}}{\rho_g} - \frac{\phi}{B_g} \frac{\partial S_g}{\partial t} - \phi \left[\frac{1}{B_g} (S_g C_r + S_g C_g) + \left(\frac{S_o}{B_o} \frac{dR_{so}}{dP_o} + \frac{S_w}{B_w} \frac{dR_{sw}}{dP_o} \right) \right] \right) \frac{\partial P_o}{\partial t} N_i dV = 0 \quad (1.4.6)$$

1.5 Oil FEM:

We now proceed to produce the formulation, one term at a time.

Conductance Term

$$\int_V^{(e)} \left(\nabla \cdot (\hat{\lambda}_o \cdot \nabla P_o) \right) N_i dV \quad (1.5.1)$$

where

$$\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \quad (1.5.2)$$

$$\nabla \cdot P_o = \sum_{j=1}^n P_j^{(e)} \nabla \cdot N_j \quad (1.5.3)$$

Using Green's Theorem:

$$\iiint_V (\nabla^2 g + \nabla f \cdot \nabla g) dV = \iint_S f \frac{\partial g}{\partial n} dA \quad (1.5.4)$$

$$\iiint_V (\nabla^2 g) dV = -\iiint_V (\nabla f \cdot \nabla g) dV + \iint_S f \frac{\partial g}{\partial n} dA \quad (1.5.5)$$

Green's Theorem becomes:

$$\int_V \nabla \cdot (\hat{\lambda}_o \cdot \nabla P_o) N_i dV = - \int_V (\hat{\lambda}_o \cdot \nabla P_o) \cdot \nabla N_i dV + \int_S (\hat{\lambda}_o \cdot \nabla P_o) \cdot \hat{n} N_i ds \quad (1.5.6)$$

where $\hat{n}$ is the unit outer normal vector to the surface S. We can observe that $\hat{\lambda}_o \cdot \nabla P_o$ represents the oil flux, so we may represent this oil flux normal to the surface S as:

$$\tau_o^{(e)} = (\hat{\lambda}_o \cdot \nabla P_o) \cdot \hat{n} \quad (1.5.7)$$

$$\int_S \tau_o^{(e)} N_i ds = \tau_o^{(e)} \int_S N_i ds \quad (1.5.8)$$

for sufficiently small time steps and elements, we may assume that the flux on each face in an element e is a constant.

For a quadrilateral face:

$$\tau_o^{(e)} \int_S N_i ds = \frac{\tau_o^{(e)} A}{4} \quad (1.5.9)$$

(1/4 of the flux is assigned to each node).

Assuming a medium with only K_x, K_y and K_z , then :

$$-\int_V (\hat{\lambda}_o \cdot \nabla P_o) \cdot \nabla N_i dV \approx -\int_V \sum_{j=1}^n (\hat{\lambda}_o P_j \cdot \nabla N_j) \cdot \nabla N_i dV \quad (1.5.10)$$

$$\approx -\sum_{j=1}^n \left\{ \int_V [(\hat{\lambda}_o \cdot \nabla N_j) \cdot \nabla N_i] dV \right\} P_j \quad (1.5.11)$$

$$\approx -\frac{1}{B_o \mu_o} \sum_{j=1}^n P_j \int_V \left[\left((K_{ro})_x \frac{\partial N_j}{\partial x} + (K_{ro})_y \frac{\partial N_j}{\partial y} + (K_{ro})_z \frac{\partial N_j}{\partial z} \right) \cdot \left(\frac{\partial N_i}{\partial x} + \frac{\partial N_i}{\partial y} + \frac{\partial N_i}{\partial z} \right) \right] dV P_j \quad (1.5.12)$$

$$-[(\hat{\lambda}_o \cdot \nabla P_o) \cdot \nabla N_i] dV \approx -\frac{1}{B_o \mu_o} \sum_{j=1}^n P_j \int_V \left[\left((K_{ro})_x \frac{\partial N_j}{\partial x} \frac{\partial N_i}{\partial x} + (K_{ro})_y \frac{\partial N_j}{\partial y} \frac{\partial N_i}{\partial y} + (K_{ro})_z \frac{\partial N_j}{\partial z} \frac{\partial N_i}{\partial z} \right) \right] dV \quad (1.5.13)$$

$$\approx -\frac{1}{B_o \mu_o} \sum_{j=1}^n P_j \int_V \left[\left((K_{ro})_x \frac{\partial N_j}{\partial x} \frac{\partial N_i}{\partial x} + (K_{ro})_y \frac{\partial N_j}{\partial y} \frac{\partial N_i}{\partial y} + (K_{ro})_z \frac{\partial N_j}{\partial z} \frac{\partial N_i}{\partial z} \right) \right] dV \quad (1.5.14)$$

let,

$$A_{ij}^{(e)} = \int_V \frac{1}{B_o \mu_o} \left[\left((K_{ro})_x \frac{\partial N_j}{\partial x} \frac{\partial N_i}{\partial x} + (K_{ro})_y \frac{\partial N_j}{\partial y} \frac{\partial N_i}{\partial y} + (K_{ro})_z \frac{\partial N_j}{\partial z} \frac{\partial N_i}{\partial z} \right) \right] dV \quad (1.5.15)$$

Gravity Term

$$-\int_V \left[\nabla \cdot \left(\hat{\lambda}_o \cdot \nabla \left(\sum_{j=1}^n \frac{(\rho'_o z_j) N_j}{144} \right) \right) \right] N_i dV = -\sum_{j=1}^n A_{ij}^{(e)} \left[\frac{\rho'_o z_j}{144} \right]_j = D_{oi} \quad (1.5.16)$$

$$D_{oi} = -\sum_{j=1}^n \frac{A_{ij}^{(e)} [\rho'_o z_j]_j}{144}$$

Injection/Production Term

$$-\int_V \frac{q_o}{\rho_o} N_i dV = Q_{oi}^{(e)} \quad (1.5.17)$$

We can probably ignore this term and handle all injection/production with the flux terms. We could also choose to use this term and then integrate it as shown.

Saturation Capacity Terms

$$\int_v \frac{\Phi}{B_o} \left(\frac{\partial S_w}{\partial t} \right) N_i dV = \frac{\Phi}{B_o} \int_v \sum_{j=1}^n \left(\frac{\partial S_{wj}}{\partial t} N_j \right) N_i dV = \frac{\Phi}{B_o} \sum_{j=1}^n \left[\left(\frac{\partial S_{wj}}{\partial t} \right) \int_v N_j N_i dV \right] \quad (1.5.18)$$

$$C_{ij}^o = \frac{\Phi}{B_o} \int_v N_i N_j dV \quad (1.4.4.1)$$

$$\int_v \frac{\Phi}{B_o} \left(\frac{\partial S_g}{\partial t} \right) N_i dV = \frac{\Phi}{B_o} \sum_{j=1}^n \left[\left(\frac{\partial S_{gj}}{\partial t} \right) \int_v N_j N_i dV \right] \quad (1.5.5)$$

Compressibility Term

$$\int_v -\frac{\Phi}{B_o} \left[C_r - \frac{1}{B_o} \frac{\partial B_o}{\partial P_o} \right] \left[\frac{\partial P_o}{\partial t} - S_w \frac{\partial P_o}{\partial t} - S_g \frac{\partial P_o}{\partial t} \right] N_i dV \quad (1.5.6)$$

let

$$\alpha_o = \left[C_r - \frac{1}{B_o} \frac{\partial B_o}{\partial P_o} \right] \quad (1.5.7)$$

(these saturations are fixed at the k-timestep or the last timestep represented by the k superscript)

$$S_w^{(e)} = \sum_{l=1}^n (S_w^k)_l N_l \quad (1.5.8)$$

$$S_g^{(e)} = \sum_{l=1}^n (S_g^k)_l N_l \quad (1.5.9)$$

$$\frac{\partial P_o}{\partial t} = \sum_{j=1}^n \left(\frac{\partial P}{\partial t} \right)_j N_j \quad (1.5.10)$$

$$\begin{aligned} & -\frac{\varphi \alpha_o}{B_o} \int_v \left[\frac{\partial P_o}{\partial t} - S_w \frac{\partial P_o}{\partial t} - S_g \frac{\partial P_o}{\partial t} \right] N_i dV = -\frac{\varphi \alpha_o}{B_o} \left[\int_v \left\{ \sum_{j=1}^n \left(\frac{\partial P}{\partial t} \right)_j N_j N_i dV \right\} - \int_v \left\{ \sum_{l=1}^n \left(S_w^k \right)_l N_l \right\} \left\{ \sum_{j=1}^n \left(\frac{\partial P}{\partial t} \right)_j N_j \right\} N_i dV \right. \\ & \left. - \int_v \left\{ \sum_{l=1}^n \left(S_g^k \right)_l N_l \right\} \left\{ \sum_{j=1}^n \left(\frac{\partial P}{\partial t} \right)_j N_j \right\} N_i dV \right] \end{aligned} \quad (1.5.11)$$

$$= -\frac{\varphi \alpha_o}{B_o} \left\{ \sum_{j=1}^n \left(\frac{\partial P}{\partial t} \right)_j \int_v N_i N_j dV - \sum_{j=1}^n \left(\frac{\partial P}{\partial t} \right)_j \left[\sum_{l=1}^n \left[\left(S_w^k \right)_l \int_v N_i N_j N_l dV \right] - \sum_{l=1}^n \left(\frac{\partial P}{\partial t} \right)_l \left[\sum_{j=1}^n \left[\left(S_g^k \right)_l \int_v N_i N_j N_l dV \right] \right] \right] \right\} \quad (1.5.12)$$

Define:

$$f_2(i, l) = \int_v N_i N_l dV \quad (1.5.13)$$

and

$$f_3(i, j, l) = \int_v N_i N_j N_l dV \quad (1.5.14)$$

then:

$$\begin{aligned} & -\frac{\varphi \alpha_o}{B_o} \int_v \left[\frac{\partial P_o}{\partial t} - S_w \frac{\partial P_o}{\partial t} - S_g \frac{\partial P_o}{\partial t} \right] N_i dV = \frac{\varphi \alpha_o}{B_o} \left\{ \sum_{j=1}^n \left(\frac{\partial P}{\partial t} \right)_j \left[\sum_{l=1}^n \left[\left(S_w^k \right)_l + \left(S_g^k \right)_l \right] f_3(i, j, l) \right] - \sum_{j=1}^n \left(\frac{\partial P}{\partial t} \right)_j f_2(i, j) \right\} \\ & = \frac{\varphi \alpha_o}{B_o} \sum_{j=1}^n \left[\sum_{l=1}^n \left\{ \left[\left(S_w^k \right)_l + \left(S_g^k \right)_l \right] f_3(i, j, l) - f_2(i, j) \right\} \left(\frac{\partial P}{\partial t} \right)_j \right] \end{aligned} \quad (1.5.15)$$

$$CO_{ij}^{(\epsilon)} = \frac{\varphi \alpha_o}{B_o} \left[\sum_{l=1}^n \left\{ \left[\left(S_w^k \right)_l + \left(S_g^k \right)_l \right] f_3(i, j, l) - f_2(i, j) \right\} \right] \quad (1.5.16)$$

The matrix representation for oil becomes:

$$A_{ij}^{(\epsilon)} P_j^{k+1} + CO_{ij}^{(\epsilon)} P_j + C_{ij}^{(\epsilon)} (S_w + S_g)_j + Q_{oi}^{(\epsilon)} + D_{oi}^{(\epsilon)} = 0 \quad (1.5.17)$$

1.6 Water FEM

Conductance Term

$$\int_V \nabla \cdot (\hat{\lambda}_w \cdot \nabla P_o) N_i dV = - \int_V (\hat{\lambda}_w \cdot \nabla P_o) \cdot \nabla N_i dV + \int_S (\hat{\lambda}_w \cdot \nabla P_o) \cdot \hat{n} N_i ds \quad (1.6.1)$$

$$\tau_w^{(e)} = (\hat{\lambda}_w \cdot \nabla P_o) \cdot \hat{n} \quad (1.6.2)$$

$$\int_S \tau_w^{(e)} N_i ds \approx \tau_w^{(e)} \int_S N_i ds \approx \frac{\tau_w^{(e)} A}{4} \quad (1.6.3)$$

$$- \int_V (\hat{\lambda}_w \cdot \nabla P_o) \cdot \nabla N_i dV = - \frac{1}{B_w \mu_{wj}} \sum_{j=1}^n P_j \int_V \left[(K_{rw})_x \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + (K_{rw})_y \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} + (K_{rw})_z \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial z} \right] dV \quad (1.6.4)$$

$$H_{ij}^{(e)} = - \frac{1}{B_w \mu_w} \int_V \left[(K_{rw})_x \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + (K_{rw})_y \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} + (K_{rw})_z \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial z} \right] dV \quad (1.6.5)$$

Gravity Term

$$- \int_V \nabla \cdot \left(\hat{\lambda}_w \cdot \nabla \left(\frac{\rho'_w z}{144} \right) \right) N_i dV = \sum_{j=1}^n H_{ij}^{(e)} \left[\frac{\rho'_w z}{144} \right]_j \quad (1.6.6)$$

$$\approx D_{wi}^{(e)} \quad (1.6.7)$$

Injection/Production Term

$$- \int_V \frac{q_w}{\rho_w} N_i dV = Q_{wi}^{(e)} \quad (1.6.8)$$

Capillary Term

$$-\int_V \frac{dP_{cow}^{(e)}}{dS_w} [\nabla \cdot (\hat{\lambda}_w \cdot \nabla S_w)] N_i dV = -\frac{dP_{cow}^{(e)}}{dS_w} \int_V [\nabla \cdot (\hat{\lambda}_w \cdot \nabla S_w)] N_i dV \quad (1.6.9)$$

$$= \frac{dP_{cow}^{(e)}}{dS_w} \int_V (\hat{\lambda}_w \cdot \nabla S_w) \cdot \nabla N_i dV - \frac{dP_{cow}^{(e)}}{dS_w} \int_S (\hat{\lambda}_w \cdot \nabla S_w) \cdot \lambda N_i dS \quad (1.6.10)$$

The value of $\frac{dP_{cow}}{dS_w}$ is determined empirically by observing a table of relationships between P_{cow} and S_w , and may be considered a constant within a given time step. The flux term as before, "fluxes" at interior boundaries will cancel. For external boundaries, if there is no change in saturation then $(\hat{\lambda}_w \cdot \nabla S_w) \cdot \lambda = 0$, this will be modified as required at the production nodes.

$$S_w^{(e)} = \sum_{j=1}^n (S_w)_j N_j \quad (1.6.11)$$

and,

$$\nabla S_w^{(e)} = \sum_{j=1}^n (S_w)_j \nabla N_j \quad (1.6.12)$$

$$\frac{dP_{cow}^{(e)}}{dS_w} \int_V (\hat{\lambda}_w \cdot \nabla S_w) \cdot \nabla N_i dV = \frac{dP_{cow}^{(e)}}{dS_w} \int_V \left(\hat{\lambda}_w \cdot \left(\sum_{j=1}^n (S_w)_j \nabla N_j \right) \cdot \nabla N_i \right) dV \quad (1.6.13)$$

$$= \frac{dP_{cow}^{(e)}}{dS_w} \left(\sum_{j=1}^n (S_w)_j \left\{ \int_V (\hat{\lambda}_w \cdot \nabla N_j) \cdot \nabla N_i dV \right\} \right) \quad (1.6.14)$$

$$= \frac{dP_{cow}^{(e)}}{dS_w} \frac{1}{B_w \mu_w} \left(\sum_{j=1}^n (S_w)_j \int_V \left[(K_{rw})_x \frac{\partial N_j}{\partial x} \frac{\partial N_i}{\partial x} + (K_{rw})_y \frac{\partial N_j}{\partial y} \frac{\partial N_i}{\partial y} + (K_{rw})_z \frac{\partial N_j}{\partial z} \frac{\partial N_i}{\partial z} \right] dV \right) \quad (1.6.15)$$

$$B_{ij}^{(e)} = \frac{dP_{cow}}{dS_w} \frac{1}{B_w \mu_w} \left\{ \int_v \left[(K_{rw})_x \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + (K_{rw})_y \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} + (K_{rw})_z \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial z} \right] dV \right\} \quad (1.6.16)$$

Saturation Capacity Term

$$- \int_v \left(\frac{\Phi}{B_w} \frac{\partial S_w}{\partial t} \right) N_i dV = - \frac{\Phi}{B_w} \int_v \frac{\partial S_w}{\partial t} N_i dV = - \frac{\Phi}{B_w} \sum_{j=1}^n \left\{ \frac{\partial S_w}{\partial t} \int_v N_j N_i dV \right\} \quad (1.6.17)$$

$$C_{ij}^w = - \frac{\Phi}{B_w} \int_v N_j N_i dV \quad (1.6.18)$$

Compressibility Term

$$\int_v - \frac{\Phi}{B_w} \left[C_r - \frac{1}{B_w} \frac{\partial B_w}{\partial P_o} \right] S_w \frac{\partial P_o}{\partial t} N_i dV = - \frac{\Phi}{B_w} \left[C_r - \frac{1}{B_w} \frac{\partial B_w}{\partial P_o} \right] \int_v \left(S_w \frac{\partial P_o}{\partial t} \right) N_i dV \quad (1.6.19)$$

$$= - \frac{\Phi}{B_w} \left[C_r - \frac{1}{B_w} \frac{\partial B_w}{\partial P_o} \right] \left\{ \int_v \left[\sum_{l=1}^n (S_w^k)_l N_l \right] \left[\sum_{j=1}^n \left(\frac{\partial P_o}{\partial t} \right)_j N_j \right] N_i dV \right\} \quad (1.6.20)$$

$$= - \frac{\Phi}{B_w} \left[C_r - \frac{1}{B_w} \frac{\partial B_w}{\partial P_o} \right] \left\{ \sum_{j=1}^n \left(\frac{\partial P_o}{\partial t} \right)_j \left[\sum_{l=1}^n (S_w^k)_l \int_v N_l N_j N_i dV \right] \right\} \quad (1.6.21)$$

$$f_3(i, j, l) = \int_v N_l N_j N_i dV \quad (1.6.22)$$

$$= - \frac{\Phi}{B_w} \left[C_r - \frac{1}{B_w} \frac{\partial B_w}{\partial P_o} \right] \left\{ \sum_{j=1}^n \left(\frac{\partial P_o}{\partial t} \right)_j \left[\sum_{l=1}^n (S_w^k)_l f_3(i, j, l) \right] \right\} \quad (1.6.23)$$

$$C_{ij}^{w(e)} = - \frac{\Phi}{B_w} \left[C_r - \frac{1}{B_w} \frac{\partial B_w}{\partial P_o} \right] \left\{ \sum_{l=1}^n (S_w^k)_l f_3(i, j, l) \right\} \quad (1.6.24)$$

The matrix representation for water becomes:

$$H_{ij}^{(e)} P_j^{k+1} + CW_{ij}^{(e)} \frac{\partial P_j}{\partial t} + Cw_{ij}^{(e)} \frac{\partial S_{wj}}{\partial t} + B_{ij}^{(e)} S_{wj}^{k+1} + Q_{wi}^{(e)} + D_{wi}^{(e)} = 0 \quad (1.6.25)$$

1.7 Gas FEM

$$\int_v^{(e)} \nabla \cdot (\hat{\lambda}_g \cdot \nabla P_o) N_i dV = - \int_v^{(e)} (\hat{\lambda}_g \cdot \nabla P_o) \cdot \nabla N_i dV + \int_s^{(e)} (\hat{\lambda}_g \cdot \nabla P_o) \cdot \hat{n} N_i dS \quad (1.7.1)$$

$$\tau_g^{(e)} = (\hat{\lambda}_g \cdot \nabla P_o) \cdot \hat{n} \quad (1.7.2)$$

$$\tau_g^{(e)} \int_s N_i dS = \frac{\tau_g^{(e)} A}{4} \quad (1.7.3)$$

$$- \int_v^{(e)} (\hat{\lambda}_g \cdot \nabla P_o) \cdot \nabla N_i dV = - \sum_{j=1}^n \left\{ \int_v^{(e)} [(\hat{\lambda}_g \cdot \nabla N_j) \cdot \nabla N_i] dV \right\} P_j \quad (1.7.4)$$

$$= - \frac{1}{B_g \mu} \sum_{j=1}^n P_j \int_v^{(e)} \left[(K_{rg})_x \frac{\partial N_j}{\partial x} \frac{\partial N_i}{\partial x} + (K_{rg})_y \frac{\partial N_j}{\partial y} \frac{\partial N_i}{\partial y} + (K_{rg})_z \frac{\partial N_j}{\partial z} \frac{\partial N_i}{\partial z} \right] dV \quad (1.7.5)$$

let

$$E_{ij}^{(e)} = \int_v^{(e)} \left[- \frac{1}{B_g \mu} \left((K_{rg})_x \frac{\partial N_j}{\partial x} \frac{\partial N_i}{\partial x} + (K_{rg})_y \frac{\partial N_j}{\partial y} \frac{\partial N_i}{\partial y} + (K_{rg})_z \frac{\partial N_j}{\partial z} \frac{\partial N_i}{\partial z} \right) \right] dV \quad (1.7.6)$$

Capillary Term

$$\int_v^{(e)} \frac{dP}{dS_g} \left[\nabla \cdot (\hat{\lambda}_g \cdot \nabla S_g) \right] N_i dV = \frac{dP}{dS_g} \int_v^{(e)} \left(\nabla \cdot (\hat{\lambda}_g \cdot \nabla S_g) \right) N_i dV \quad (1.7.7)$$

$$= \frac{dP}{dS_g} \int_v^{(e)} (\hat{\lambda}_g \cdot \nabla S_g) \cdot \nabla N_i dV + \frac{dP}{dS_g} \int_s^{(e)} (\hat{\lambda}_g \cdot \nabla S_g) \cdot \hat{n} dS \quad (1.7.8)$$

as before,

$$\frac{dP_{cgo}}{dS_g} \int_V (\hat{\lambda}_g \cdot \nabla S_g) \cdot \nabla N_i dV = \frac{dP_{cgo}}{dS_g} \sum_{j=1}^n \frac{(S_g)_j}{B_g \mu_g} \int_V \left[(K_{rg})_x \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + (K_{rg})_y \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} + (K_{rg})_z \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial z} \right] dV \quad (1.7.9)$$

$$G_{ij}^{(e)} = \int_V \frac{dP_{cgo}}{dS_g} \frac{\mu_g}{B_g} \left[(K_{rg})_x \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + (K_{rg})_y \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} + (K_{rg})_z \frac{\partial N_i}{\partial z} \frac{\partial N_j}{\partial z} \right] dV \quad (1.7.10)$$

Saturation Capacity Term

$$-f_v^{(e)} \left(\frac{\Phi}{B_g} \frac{\partial S_g}{\partial t} \right) N_i dV = - \sum_{j=1}^n \int_V \frac{\Phi}{B_g} \left(\frac{\partial S_g}{\partial t} \right)_j N_j N_i dV \quad (1.7.11)$$

Let,

$$f_g^{(e)} = - \int_V \frac{\Phi}{B_g} N_j N_i dV \quad (1.7.12)$$

The Gravity Term

$$- \int_V \nabla \cdot \left(\hat{\lambda}_g \cdot \nabla \left(\frac{\rho'_g z}{144} \right) \right) N_i dV = - \sum_{j=1}^n E_{ij}^{(e)} \left[\frac{\rho'_g z}{144} \right]_j = D_{gi}^{(e)} \quad (1.7.13)$$

$$(1.7.14)$$

Injection/Production Term

$$- \int_V \frac{q_{fg}}{\rho_g} N_i dV = Q_{gi}^{(e)} \quad (1.7.15)$$

let

$$\int_V \frac{\Phi}{B_g} \left(S_g C_r + S_g C_g + S_o C_\alpha \right) \frac{\partial P_o}{\partial t} N_i dV = \int_V \frac{\Phi}{B_g} \left(S_g C_r + S_g C_g + (1 - S_w - S_g) C_\alpha \right) \frac{\partial P_o}{\partial t} N_i dV \quad (1.7.16)$$

$$= \int_V \frac{\Phi}{B_g} \left(S_g C_r + S_g C_g + (1 - S_w - S_g) C_\alpha \right) \sum_{j=1}^n \left(\frac{\partial P_o}{\partial t} \right)_j N_j N_i dV \quad (1.7.17)$$

Let,

$$GX_{ij} = - \int_V \frac{\Phi}{B_g} \left(S_g C_r + S_g C_g + (1 - S_w - S_g) C_\alpha \right) N_j N_i dV \quad (1.7.18)$$

$$(1.7.19)$$

$$\int_V \frac{\Phi}{B_w} \frac{\partial R_{sw}}{\partial P_o} S_w \frac{\partial P_o}{\partial t} N_i dV = \sum_{j=1}^n \int_V \frac{\Phi}{B_w} \frac{\partial R_{sw}}{\partial P_o} S_w \left(\frac{\partial P_o}{\partial t} \right)_j N_j N_i dV$$

let

$$WX_{ij} = \int_V \frac{\Phi}{B_w} \frac{\partial R_{sw}}{\partial P_o} S_w N_j N_i dV \quad (1.7.20)$$

$$[E_{ij}^{(\epsilon)}] P_j^{k+1} + [G_{ij}^{(\epsilon)}] S_{gj}^{k+1} + D_{gi}^{(\epsilon)} + Q_{gi}^{(\epsilon)} + f_g \frac{\partial S_{gj}}{\partial t} + [GX_{ij}] \frac{\partial P_j}{\partial t} + [WX_{ij}] \frac{\partial P_j}{\partial t} = 0 \quad (1.7.21)$$

Using the finite element equations that we have developed for 3-D 3-phase flow in a porous media in matrix-vector form and replacing the time derivatives with a finite-difference approximations and by indicating with superscripts k and $k+1$ quantities at the beginning and at the end of a generic iteration step, then:

(1)

$$[A] P^{k+1} + [CO] \frac{\partial P}{\partial t} + [C^o] \left(\frac{\partial S_w}{\partial t} + \frac{\partial S_g}{\partial t} \right) + Q_o + D_o = 0 \quad (1.7.22)$$

(2)

$$[H] P^{k+1} + [CW] \frac{\partial P}{\partial t} + [C^w] \frac{\partial S_w}{\partial t} + [B] S_w^{k+1} + Q_w + D_w = 0 \quad (1.7.23)$$

(3)

$$[E] p^{k+1} + [G] s_g^{k+1} + D_g + Q_g + f_g \frac{\partial S}{\partial t} + [GX] \frac{\partial P}{\partial t} + [WX] \frac{\partial P}{\partial t} = 0 \quad (1.7.24)$$

With a time step of Δt and doing an "implicit" solution:

(1)

$$[A] p^{k+1} + [CO] \left(\frac{p^{k+1} - p^k}{\Delta t} \right) + [C^o] \left(\frac{s_w^{k+1} - s_w^k}{\Delta t} + \frac{s_g^{k+1} - s_g^k}{\Delta t} \right) + Q_o + D_o = 0 \quad (1.7.25)$$

$$\left([A] + \frac{1}{\Delta t} [CO] \right) p^{k+1} + \frac{1}{\Delta t} [C^o] (s_w^{k+1} + s_g^{k+1}) = -Q_o - D_o + \frac{1}{\Delta t} [CO] p^k + \frac{1}{\Delta t} [C^o] (s_w^k + s_g^k) \quad (1.7.26)$$

(2)

$$[H] p^{k+1} + [CW] \left(\frac{p^{k+1} - p^k}{\Delta t} \right) + [C^w] \left(\frac{s_w^{k+1} - s_w^k}{\Delta t} \right) + [B] s_w^{k+1} + Q_w + D_w = 0 \quad (1.7.27)$$

$$\left([H] + \frac{1}{\Delta t} [CW] \right) p^{k+1} + \left([B] + \frac{1}{\Delta t} [C^w] \right) s_w^{k+1} = -Q_w - D_w + \frac{1}{\Delta t} [CW] p^k + \frac{1}{\Delta t} [C^w] s_w^k \quad (1.7.28)$$

(3)

$$[E] p^{k+1} + [G] s_g^{k+1} + D_g + Q_g + f_g \frac{\partial S}{\partial t} + [GX] \frac{\partial P}{\partial t} + [WX] \frac{\partial P}{\partial t} = 0 \quad (1.7.29)$$

and,

$$\begin{aligned} [E] p^{k+1} + [G] s_g^{k+1} + [GX] \left(\frac{p^{k+1} - p^k}{\Delta t} \right) + [WX] \left(\frac{p^{k+1} - p^k}{\Delta t} \right) + f_g \left(\frac{s_g^{k+1} - s_g^k}{\Delta t} \right) \\ = -D_g - Q_g \end{aligned} \quad (1.7.30)$$

and,

$$\left([E] + \frac{1}{\Delta t} [GX] + \frac{1}{\Delta t} [WX] \right) p^{k+1} + \left([G] + \frac{1}{\Delta t} f_g \right) s_g^{k+1} = -D_g - Q_g + \frac{1}{\Delta t} ([C^w] + [GX] + [WX]) p^k + \frac{[f_g]}{\Delta t} s_g^k \quad (1.7.31)$$

To write this in a more compact form, let:

$$[Y] = [GX] + [WX] \quad (1.7.32)$$

$$F_i = -D_g - Q_g + \frac{1}{\Delta t} [Y] p^k + \frac{1}{\Delta t} [f_g] s_g^k \quad (1.7.33)$$

then:

$$\left([E] + \frac{1}{\Delta t} [Y] \right) p^{k+1} + \left([G] + \frac{1}{\Delta t} [F_g] \right) s_g^{k+1} = -D_g - Q_g + \frac{1}{\Delta t} [Y] p^k + \frac{1}{\Delta t} [f_g] s_g^k \quad (1.7.34)$$

These three equations can be solved using a fully implicit scheme:

$$\begin{bmatrix} [A] + \frac{1}{\Delta t} [CO] & \frac{1}{\Delta t} [C^o] \\ [H] + \frac{1}{\Delta t} [C_w] & [B] + \frac{1}{\Delta t} [C^w] \\ [E] + \frac{1}{\Delta t} [Y] & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\Delta t} [C^o] \\ 0 \\ [G] + \frac{1}{\Delta t} [F_g] \end{bmatrix} \begin{bmatrix} p^{k+1} \\ s_w^{k+1} \\ s_g^{k+1} \end{bmatrix} = \begin{bmatrix} -Q_o - D_o + \frac{1}{\Delta t} [CO] p^k + \frac{1}{\Delta t} [C^o] (s_w^k + s_g^k) \\ -Q_w - D_w + \frac{1}{\Delta t} [CW] p^k + \frac{1}{\Delta t} [C^w] s_w^k \\ F_i \end{bmatrix} \quad (1.7.35)$$

The simulation of 3-phase, 3-D flow in a porous medium requires solving this coupled system of nonlinear approximation to the partial differential equations. These approximations arise from the conservation of mass principle for an oil-water-gas system.