DETERMINATION OF THE MULTIPLICATION FACTOR AND ITS BIAS
BY THE $^{252}$Cf-SOURCE TECHNIQUE: A METHOD FOR CODE
BENCHMARKING WITH SUBCRITICAL CONFIGURATIONS

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DETERMINATION OF THE MULTIPLICATION FACTOR AND ITS BIAS
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ABSTRACT

A brief discussion of the \(^{252}\text{Cf}\) source driven method for subcritical measurements serves as an introduction to the concept and use of the spectral ratio, \(\Gamma\). It has also been shown that the Monte Carlo calculation of spectral densities and effective multiplication factors have as a common denominator the transport propagator. This commonality follows from the fact that the Neumann series expansion of the propagator lends itself to the Monte Carlo method. On this basis a linear relationship between the spectral ratio and the effective multiplication factor has been shown. This relationship demonstrates the ability of subcritical measurements of the ratio of spectral densities to validate transport theory methods and cross sections.

I. INTRODUCTION

Unfortunately, the effective neutron multiplication factor, \(k\), can only be inferred easily at critical. To infer the \(k\) at subcritical has been a busy field of endeavor since the advent of nuclear technology. To this end both neutron die-away and stochastic measurements were used. However, the measured quantity (within experimental constraints) in this type of measurements is a time-eigenvalue, the reactor period, not the reactivity eigenvalue that has to be indirectly inferred by calibration at delayed critical and by the use of spectral and spatial correction factors if point kinetics methods are used. With the availability of sophisticated Monte Carlo and deterministic codes, the criticality safety community adopted the method of calculating the effective multiplication of systems at critical to determine the bias by comparison of the calculation with the corresponding critical experiments. The same bias is then applied at the subcritical level for a similar configuration. Although there are several reasons to object to the use of the same bias at all subcritical levels, this method is helpful, and it is widely applied.

Now consider the case of a spent fuel cooling pool where the subcritical margin of the pool is needed, before more spent fuel can be stored. Clearly, the previously described procedure cannot be used because at critical experiments for the pool are not possible. What is needed is a parameter that would allow benchmarking the calculational method (including cross sections) at all subcritical levels. For this purpose the measured parameter should have the following properties: a) it must be accessible to measurement at any level of subcriticality, b) it must be reliably calculated by the same code used for the calculation of the effective multiplication factor, and c) it must be related in a univocally fashion to the effective neutron multiplication factor. Such a parameter would then provide the necessary tool for benchmarking calculational method and would also play the role of a surrogate measurement for the inference of the neutron multiplication factor. Obviously, this parameter must be independent from the driving experimental input, such as the strength of external source, as well as from the detector efficiency and the response of the electronics setup. A parameter, the spectral-ratio that satisfied all of the above conditions, was introduced by Mihalczo, et al. The spectral ratio is defined as the ratio of spectra between signals acquired from a driving \(^{252}\text{Cf}\)-source contained in an ionization chamber which provides an electrical pulse for each spontaneous fission and then serves as a timed source of fission neutrons and two or more neutron detectors.
Although an extensive bibliography exists on the Cf-source driven method (see reference 2 and its references), it is still pertinent to give a brief description of its main features. A sketch of the measurement configuration is shown in Figure 1.

![Diagram of measurement configuration](image)

Fig. 1. $^{252}$Cf-Driven Frequency-Analysis Measurement Configuration

A Fourier analyzer is employed to digitally sample the source and detector signals and estimate frequency-dependent, or spectral, signatures between signals. These signatures essentially measure the amount of correlated information between the two signals analyzed. For example, the cross-spectrum between detector signals #2 and #3, denoted $\Phi_{23}$, indicates the number of events in detector #3 that are correlated with events in detector #2. Uncorrelated events do not contribute to the cross-spectrum signature, but they do affect the uncertainty in the magnitude of the signature $\Phi_{23}$. The signatures obtained from the described measurement configuration involve all two-channel combinations (i.e., 1&2, 1&3, 2&3, etc.) including signatures that correlate an individual signal with itself (i.e., 1&1, 2&2, 3&3, etc.). The signatures between different signals are termed cross-spectra, while those involving the same signal twice are termed autospectra. The measured autospectra and cross-spectra have the following properties. The source autospectra $\Phi_{11}$ is simply a measure of the fission source strength. The source-detector cross-spectra ($\Phi_{12}, \Phi_{13}$) measure counting events in the detectors correlated with fission events in the source. Consequently, the source-detector cross-spectra indicate the amount of source-induced fission occurring in the system analyzed. The detector cross-spectra ($\Phi_{23}$) correlate events in one detector with events in another detector, so their magnitudes indicate the amount of both source-induced and inherent fission occurring in the system analyzed. Finally, the detector autospectra ($\Phi_{22}, \Phi_{33}$) are a measure of the source-induced and inherent fission rate of the system and the background rate. The particular frequency spectral ratio, $\Gamma(\omega)$, that satisfied the above requirements is given by

$$
\Gamma(\omega) = \frac{\Phi_{12}(\omega) \Phi_{13}(\omega)}{\Phi_{11}(\omega) \Phi_{23}(\omega)}
$$

Both Monte Carlo calculations and measurements have shown a linear relationship over a wide range of values, between the spectral ratio and $\Delta k/k$ where $k$ is the effective neutron multiplication factor of the system and $\Delta k/k$ is one quantification of the shutdown margin. The motivation for this work is to investigate the reasons for the existence of this relationship. To this end, it is shown that the Monte Carlo calculation of both the spectral ratio and the multiplication factor have a common denominator, the Neumann series expansion of the transport propagator. To illustrate this point, a formulation of the variation of the propagator due to system parameter changes is developed. The above formalism is then used for the calculation of the subsequent changes in both the spectral ratio and the neutron multiplication factor.

II. SYMBOLS, NOTATIONS, AND GENERAL BACKGROUND

This section contains information of the symbols, and notations used in the present work, as well as a summary of expressions for the various spectral densities, included in the definition (1.1) of the spectral ratio $\Gamma(\omega)$.

Symbols and Notations are as follows:

$\bar{r}_s, \bar{r}_d$ = source and detector locations

(phase space) $\bar{x}$ (r, v, $\Omega$)

(i) = (phase space and time) ($\bar{x}$, t)
\[ d^1 = \text{volume element in time and phase space} \]
\[ < f | g > = \text{time-phase space inner product} = \int d^1 f g \]
\[ (f | g) = \text{phase space inner product} = \int dx f g \]
\[ k_q(\omega) = \text{response function of the qth detector system} \]
\[ (\text{dimensionless}) \]
\[ \bar{\gamma}_q = \text{qth detector efficiency} \]
\[ V_D = \text{detector volume} \]
\[ k_q(\omega), \gamma_q = \text{source containing chamber response function and efficiency} \]

\[ F_c = \text{average source strength (fission/sec)} \]
\[ \bar{\chi}, \bar{\chi}_s = \text{average neutron multiplicity for the fissile system and source, respectively} \]
\[ \chi, \chi_s = \text{normalized fission neutron spectrum for the fuel neutrons and source neutron respectively.} \]
\[ G(\Omega | 2) = \text{Neutron transport propagator - (response to a unit input} \delta(1-2) \]
\[ (\text{a time-displacement kernel}) \]
\[ G(\bar{x} | \bar{x}' \omega) = \text{The Fourier transform of the propagator.} \]
\[ G(\bar{x} | \bar{x}') = \text{The time independent (static) propagator.} \]
\[ \Sigma_{\alpha q} = \text{capture cross section for the qth detector (standard value)} \]

The neutron transport operator, \( L_0 \), is written in the form
\[ L_0 = L_{oo} - (S + F) \]  
(2.1)

where, in standard notation:
\[ L_{oo} = \bar{\chi} \bar{\chi} + \Sigma_T \]  
(2.2)
\[ S = \int dv' d\Omega' \Sigma_S(v' | \bar{\Omega} | v \bar{\Omega}) \]  
(2.3)
\[ F = \frac{1}{4\pi} \int dv' d\Omega' \chi(v) \Sigma_F(v' | v') \nu_T(v') \]  
(2.4)

and where the adjoint operator, \( L^\dagger \), is obtained in the usual fashion. The forward and adjoint propagators satisfy the transport equations.
\[ \left[ \frac{1}{v_1} \frac{\partial}{\partial t_1} + L_0 \right] G(1|2) = \delta(1-2) \]  
(2.5)
\[ \left[ \frac{1}{v_1} \frac{\partial}{\partial t_1} + L^\dagger \right] G^+(1|2) = \delta(1-2) \]  
(2.6)

The boundary conditions on a convex domain are:
\[ G(12) = 0 \quad (\bar{\Omega} \cdot \vec{n} < 0); \quad G^+(12) = 0 \quad (\bar{\Omega} \cdot \vec{n} > 0) \] (2.7)

and the following causality relations apply

\[ G(12) = 0 \quad (t_1 < t_2) \] (2.8)

\[ G^+(12) = 0 \quad (t_1 > t_2) \] (2.9)

as well as the reciprocity condition

\[ G^+(12) = G(21). \] (2.10)

An issue of some interest is to determine the variation of the propagator upon a change, \( \Delta L \), in the transport operator, that is now written in the form:

\[ L'_o = L_o + \varepsilon \Delta L \] (2.11)

where \( \varepsilon \) (\( 0 \leq \varepsilon \leq 1 \)) is a parameter that continuously varies with the change from zero up to maximum strength (\( \varepsilon = 1 \)). With the assumption that the propagator becomes an analytical function of the change parameter, \( \varepsilon \), it can be shown that the rate of change of the propagator is given by \(^3\)

\[ \frac{\partial}{\partial \varepsilon} G(12) = -(G(13, \varepsilon) - \frac{\partial}{\partial \varepsilon} L'_o(\bar{\Omega}_3, \varepsilon) G(3|2, \varepsilon)). \] (2.12)

Since the propagators are time displacement kernels, Fourier transformation of equation (2.12) yields

\[ \frac{\partial}{\partial \varepsilon} G(\bar{\Omega}_1, \bar{\Omega}_2, \omega) = -(G(\bar{\Omega}_1, \bar{\Omega}_2, \omega) - \frac{\partial}{\partial \varepsilon} L'_o(\bar{\Omega}_3) G(\bar{\Omega}_3, \bar{\Omega}_2, \omega)). \] (2.13)

The result (2.12) is a nonlinear integral equation. The knowledge of the propagator at \( \varepsilon = 0 \), (the free propagator) provides the setting up of an initial value problem marching the solution from \( \varepsilon = 0 \) up to \( \varepsilon = 1 \). This procedure generates a Neumann series solution for the propagator.

\[ G(12) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \left[ \frac{\partial}{\partial \varepsilon}^n G_o(12) \right] \] (2.14)

where the derivatives involved in (2.14) are calculated by successive use of (2.12). In what follows, two versions of the changed operator are given. In the first version, it is written as the sum

\[ L'_o = L_o - \varepsilon (S + F) \] (2.15)

so that for \( \varepsilon = 0 \), it becomes the collisionless operator. In the second version:

\[ L'_o = L_o + \varepsilon \sum_{P=1}^{3} (\alpha_p - 1) L_{sp} \] (2.16)

where the subindex \( s \) symbolizes the standard value of the system cross sections and concentrations, and the \( \alpha_p \) quantities are equal to unity for the standard case, and:

\[ L_{st} = \sum_{ST} : L_{st} = -S_{SS}; L_{st} = -F_S. \] (2.17)
This version is convenient for the evaluation of the effects of changing system parameters.

Given generalized functions, X(1), X '(1) that satisfy boundary, initial and "final" conditions congruent with the boundary and causality conditions (2.7), (2.8), and (2.9) and the transport equations:

\[
\left[-\frac{1}{\upsilon_1} \frac{\partial}{\partial \upsilon_1} + L_0^*\right] X^+(1) = Q^+(1) \\
\left[\frac{1}{\upsilon_1} \frac{\partial}{\partial \upsilon_1} + L_0\right] X(1) = Q (1)
\]  
(2.18)

(2.19)

it can easily be shown that

\[
X^+(1) = \langle G (2|1) | Q^+(2) \rangle
\]  
(2.20)

\[
X (1) = \langle G (1|2) | Q (2) \rangle
\]  
(2.21)

where the time and phase space integration goes over the repeated variable symbols, and Q, Q *, are generalized sources. Similarly, one writes in the frequency domain

\[
X^+ (\bar{x}, \omega) = \langle G (\bar{x}^* | \bar{x}, \omega) | Q^* (\bar{x}', \omega) \rangle
\]  
(2.22)

with a similar result for X (\bar{x}, \omega).

Next, summarize the expressions for the \(\Phi_{11}, \Phi_{12}, \Phi_{13}, \) and \(\Phi_{23}\), spectral densities.\(^5,6\)

\[
\Phi_{11}(\omega) = F \gamma_1^2 k_1(\omega) k_1^*(\omega)
\]  
(2.23)

\[
\Phi_{12}(\omega) = \gamma_1 \bar{\gamma}_q \gamma_q^* k_1(\omega) k_q(\omega) h_{q*}(\omega) / (2\pi)^2
\]  
(2.24)

\[
\Phi_{23}(\omega) = M_{12}[\gamma_1(\omega) \gamma_1(\omega) / (2\pi)]
\]  
(2.25)

where the following spectrum and angle weighted quantities are introduced

\[
F_R(\bar{\tau}, \bar{r}_s) = \bar{v}_s \int d\Omega H_F(\bar{r}_s | \bar{r}) \sum_F(\bar{r})
\]  
(2.26)

\[
h_{q*}(\omega) = T_R(G(\bar{x}^* | \bar{x}, \omega) | \sum_{q*}(\bar{x}'))
\]  
(2.27)

\[
h_{q*}(\omega) = T_S(G(\bar{x}^* | \bar{x}, \omega) | \sum_{q*}(\bar{x}'))
\]  
(2.28)

with

\[
M_{12}(\omega) = \gamma_1 \gamma_2 k_1(\omega) k_1^*(\omega) / (2\pi)^2
\]  
(2.29)

and

\[
H_F(\bar{x} | \bar{r}_s) = T_S G_S(\bar{r}_s \bar{v} \gamma_1^* | \bar{x})
\]  
(2.30)
III. THE PROPAGATOR AND THE MONTE CARLO CALCULATION OF STOCHASTIC OBSERVABLES AND OF THE EFFECTIVE MULTIPLICATION FACTOR

The goal of this section is to formalize the relationship between the propagator and the Monte Carlo method. The reason behind this exercise is to illustrate the commonality between the calculation of measured stochastic observables and the determination of the effective multiplication factor. To this end, we use the expansion (2.14) for the propagator and (2.15) for the $L_\omega$ operator to obtain the series expansion

$$G(1|2) = G_0(1|2) + \langle G_0(1|3)S_s(\bar{x}_3)G_0(3|2) \rangle + \langle G_0(1|3)|F(\bar{x}_3)G_0(3|2) \rangle$$
$$+ \langle G_0(1|3)S_s(\bar{x}_3)G_0(3|4)|F(\bar{x}_4)G_0(4|2) \rangle$$
$$+ \langle G_0(1|3)|F(\bar{x}_3)G_0(3|4)|F(\bar{x}_4)G_0(4|2) \rangle$$
$$+ \cdots$$

(3.1)

The laborious task of writing the various terms in the series (3.1) is made easier by the use of diagrams. To this end we establish the following conventions:

(a) The free propagator, $G_0(1|2)$, is represented by an arrow going from the phase space point, $\bar{x}_2$ to $\bar{x}_1$ and $t_2$ to $t_1$ ($t_1 > t_2$).

(b) A scattering event is represented by a dot, $\bullet$.

(c) A fission event is represented by an square, $\square$.

(d) The full propagator, $G(1|2)$, is represented by the graph

Following the above conventions, equation (3.1) can be pictured as:

Note that the nth term in the expansion contains $2n$ different diagrams, each diagram being repeated, $n!$ times. The higher order terms in the expansion (3.1) are combinations of the elementary scattering ($\rightarrow \bullet \rightarrow$) and fission diagrams ($\rightarrow \square \rightarrow$). The latter may adopt several structures because a progeny of prompt neutrons can be generated in a fission event. Then fission diagrams like the following exist

where the bubble indicates neutron absorption by fission without generation of fission neutrons (The exit propagator $G_0(3|2)$, has been replaced by $\delta(3-2)$).
To illustrate the relationship between the diagrammatic and Monte Carlo methods, start at the zeroth order approximation, \( G_o(1|2) \) that obeys the equation

\[
\left[ \frac{1}{v_1} \frac{\partial}{\partial t_1} + \Omega \cdot \mathbf{v} + \Sigma_f (\mathbf{x}_1) \right] G_o(1|2) = \delta(1 - 2).
\]  

(3.3)

A closed form solution is obtained by making the change of variables

\[
\mathbf{r}_1 = \mathbf{r} + s \mathbf{x}_2; t_1 = t' + s \, / \, v_2
\]

(3.4)

where \( s \) is a parameter with the dimensions of a length along the direction, \( \mathbf{x}_2 \). Now

\[
G_o(1|2) = \delta(\mathbf{x}_0 - \mathbf{x}_2) \, \delta(v_1 - v_2) \, \exp(I(s))
\]

(3.5)

where

\[
I(s) = \int_0^s ds' \Sigma_f (s', v_2)
\]

(3.6)

and where the following relations hold between the input and output legs of the free propagator:

\[
\mathbf{r}_1 = \mathbf{r}_2 + s \mathbf{x}_2; t_1 = t_2 + s \, / \, v_2
\]

(3.7)

Clearly, the zeroth diagram, analytically described by the result (3.5), corresponds to the Monte Carlo determination of the distance to the next collision and the time elapsed while in free flight.

The matrix element that corresponds to the scattering diagram is

\[
\int d\mathbf{x}_3 \, dt_3 \, dv_2 \, d\Omega_2 \, G_o(1|3) \Sigma_f (v_2 \mathbf{x}_2; \mathbf{x}_3) \, G_o(v_2 \mathbf{x}_2; \mathbf{r}_3 t_3|2)
\]

(3.8)

and has the following implementation by the Monte Carlo method: at source location, select, \( v_2 \) and \( \mathbf{x}_2 \), the particle then flies freely until it reaches a collision point at phase-space \( \mathbf{x}_3 \) at \( t_3 = t_2 + s \, / \, v_2 \). The exit velocity angle pair \( (v_3, \mathbf{x}_3) \) is determined by sampling of the scattering kernel and the energy-angle correlation. Afterwards, the propagator \( G_o(3|2) \) controls the particle evolution until the occurrence of the next collision.

The diagram describing fission with the production of a single neutron corresponds to the matrix element:

\[
\frac{1}{4\pi} \int d\mathbf{x}_3 \, dt_3 \, dv_2 \, d\Omega_2 \, G_o(1|3) \chi(v_3) v(v_2) \Sigma_f (v_2) \, G_o(v_2 \mathbf{x}_2; \mathbf{r}_3 t_3|2)
\]

(3.9)

that has the following Monte Carlo interpretation: at source location select the velocity-angle pair \( (v_2, \mathbf{x}_2) \). Use \( \Sigma_f (v_2) \) to determine if a fission event took place. Use the neutron multiplicity probability, \( P(v) \), and the normalized spectrum, \( \chi(v) \) to determine the exit speed of the generated neutron.

When more than one fission neutron is produced, the fission diagrams split into as many of them as neutrons in the progeny. In the Monte Carlo method the extensive effort to calculate the many diagrams in (3.1) is replaced by averaging over many particle histories. The contribution of a given history to the full propagator may take different forms determined by the ordering of events. Each particle history partial diagrammatic equivalent is labeled by the particle followed and by the time from its birth to extinction. Thus the full propagator going from the source location, 2, to the observation point, 1, can be pictured as:
where, \( I_N \), is the number of histories, and the \( l \)-label, keeps track of the time elapsed following the \( t \)th particle history. In practice, the matrix elements of the propagator such as the count rate, \( C_q \), in a detector placed at, \( \vec{r}_q \), in the presence of a neutron source at \( \vec{r}_s \) are of interest. In terms of the propagator

\[
C_q(t) = (D_q(\vec{x}_q) | G(\vec{x}_q; t_q | \vec{x}_s, t_s) | S(\vec{x}_s, t_s))
\]

with

\[
D_q(\vec{x}_q) = \sum_{\text{spec}} (v) V \delta (\vec{r} - \vec{r}_q).
\]

Equivalently, the Monte-Carlo method calculation will be represented by the diagram

\[
C_q(t) = \frac{\Delta}{I_N} \sum_{id} [S(\vec{x}_s, t_s)] D_q(\vec{x}_q)
\]

that can be interpreted as the sum of matrix elements of each partial diagram, whereby its input leg is plugged into the source region, and its output leg to the detector region.

The Monte Carlo calculation of the effective multiplication factor, \( k \), may start (there are various options) by setting up, \( F_0(\vec{x}) \), a general distribution of fission sites serving as the initial neutron source distribution. An initial cycle of Monte Carlo calculations yields a new fission sites distribution serving as input for the next cycle. The multiplication factor is calculated at each cycle as the ratio \( m=0 \)

\[
k_m = \frac{P_m}{P_{m-1}}
\]

where, \( P_m \), the total neutron production is given by

\[
P_m = (\nu_T | F_m(\vec{x}))
\]

and the fission site density by

\[
F_m(\vec{x}) = \sum_{\vec{x}} \Phi_m(\vec{x})
\]

with the neutron flux at the end \( (t=t_m) \), of the \( m \)th cycle given by

\[
\Phi_m(\vec{x}) = (G(\vec{x}, t_m | \vec{x}', t_{m-1}) \bar{V}_T \chi(\nu') F_{m-1}(\vec{x}')) / 4\pi
\]

because on the average, each fission site will contribute, \( \bar{V}_T \), neutrons with an energy distribution given by the fission neutron spectrum. Iteration of the result (3.17), yields the following relationship between the equilibrium and initial fission site densities

\[
F_M(\vec{x}) = \sum_{\vec{x}} \Omega_M(\vec{x}) F_0(\vec{x})
\]

where the operator
\[ \Omega_M = \prod_{j=1}^{M} \left( dx_j \right) G(x_{j-1}, t_{M+1-j}, x_j, t_{M-j}) a(\bar{x}_j) \Sigma_R(\bar{x}_j) \]  
\[ \text{with} \quad a(\bar{x}_j) = \frac{\nu_T}{x(\nu_j)} / 4\pi. \]  
Thus at equilibrium
\[ k = \frac{P_M}{P_{M-1}} \]  
with
\[ P_M = (\nu_T | F_M(\bar{x})|. \]  
The results above were derived for two reasons. First, to show how the Monte Carlo calculation of the multiplication factor can be related to the propagator, and secondly, to their use as the basis to investigate the linear behavior of both the spectral ratio, \( \Gamma \), and the multiplication factor, \( k \).

IV. THE RELATIONSHIP BETWEEN THE SPECTRAL RATIO AND THE EFFECTIVE MULTIPLICATION FACTOR.

All the ingredients needed to investigate the relationship between the spectral ratio and the multiplication factor now have been formulated. To this end consider the change experienced by the two quantities upon a small change from the standard system parameters to a new set, \( \alpha_p \). The perturbed transport operator will take the form in equation (2.16). First operate with \( \partial / \partial \alpha_p \) on both sides of equations (1.1) for the spectral ratio and (3.21) for the multiplication factor. The result of this operation consists in each instance of the sum of two contributions: the direct contribution arising from the change of the parameters themselves and an indirect contribution due to the variation of the Propagator upon parameter changes. For the case of the spectral ratio, the pertinent expression for the variation of the propagator is the frequency dependent equation (2.13) and for the multiplication factor is equation (2.12). After some algebra for the changes from the standard values, \( \Gamma_s \), \( k_s \), to the new values \( \Gamma_p \), \( k_p \), the following expressions are obtained
\[ \Gamma_p(\omega) - \Gamma_s(\omega) = A_s(\omega)(\alpha_p - 1) \]  
\[ k_p - k_s = Z_s(\alpha_p - 1) \]  
where
\[ A_s(\omega) = \Gamma_s(\omega) \left[ \frac{1}{\Phi_{12R}} (F_{12D} + F_{12T}) - \sum_{q=1}^{2} \frac{1}{\Phi_{qFR}} T_s M_q \right] \]  
\[ Z_s = \frac{1}{P_{SM}} (\nu_T (Z_{MD} + Z_{MB})) - \frac{1}{P_{SM-1}} (\nu_T (Z_{M-1,D} + Z_{M-1,B})) \]  
with
\[ F_{12D} = \nu_T(\nu_s - 1) \int d\nu d\xi \mathbb{H}_F(\bar{\nu}_s \bar{\xi}) h_{1R}^* (\bar{\nu}, \bar{\xi}) h_{2R} (\bar{\nu}, \bar{\xi}) \]  
\[ F_{12T} = -\nu_T(\nu_s - 1) \int d\nu d\xi \mathbb{H}_F(\bar{\nu}_s \bar{\xi}) \left( L_{3F} (\bar{\xi}_s) G(\bar{\xi}_s) \right) \Sigma_{3F} (l) + \left\{ \frac{1}{\nu_T} \left[ S_R(\bar{\nu}_s M_{1} + h_{1R}^* M_{2}) \right] \right\} \]  
\[ \Phi_{12R} = \Phi_{12}/M_{12}; \Phi_{qFR} = \Phi_{qF}/\nu_{qF} \]
Note that all the terms entering in the expressions (4.1) and (4.2) for the slopes $A_n(\omega)$ and $Z_n$ are calculated at the standard conditions and can therefore be obtained from the unperturbed Monte Carlo calculation. Next we use equations (4.1) and (4.2) to arrive at the desired relationship between the spectral density, extrapolated to zero frequency, $T(\omega)$, and the multiplication factor. Then let $T_m$ be the measured spectral ratio in a subcritical assembly, and $T_m$, the ratio calculated with the standard parameters. Upon modifying the $p$th parameter from its standard value of unity to say, $\% m$, let $T_m$ be the new value calculated for the spectral ratio. Similarly, let $k_m$ and $k_n$ be the standard and “perturbed” values of the multiplication factor, obtained by the same programs and parameters used in the spectral ratio calculation. Repeated use of equations (4.1) and (4.2) yields

$$\Gamma_m - \Gamma_s = \frac{\alpha_p(m) - 1}{1 - \alpha_p(c)} \frac{k_m - k_c}{k_m - k_c} = \frac{\alpha_p(m) - 1}{1 - \alpha_p(c)}$$

(4.15)

where, $\alpha_p(m)$, $\alpha_p(c)$, are the values of the $p$th parameter corresponding to the measured and perturbed cases. The above results can be used in two ways: either to solve for $\alpha_p(m)$, and use its value to calculate, $k_m$, or combine the equations (4.15) to arrive at the relation, applicable for the changes of any of the system parameters:

$$\Gamma_m - \Gamma_s = \frac{k_m - k_c}{k_m - k_c}$$

(4.16)

that provides an equation for the determination of the bias $(k_m - k_c)$ in the calculation of the multiplication factor from subcritical experiments.

V. SUMMARY AND DISCUSSION

It has been shown that the Monte Carlo calculations of spectral densities and multiplication factors have as a common denominator the transport propagator as defined in Eq (2.5). Once this commonality was established, first order perturbation theory approximations for the change of both quantities can be derived. What this means is that as these quantities ride along their defining trajectories in parameter space, one can consider they do so in small linear steps within which the results (4.15) are valid. However, there is substantial evidence, both calculational and experimental,$^{2,8}$ showing that the linear approximation is in fact a valid one over comfortably large ranges of parameter changes. The result (4.16) serves for code benchmarking at any subcritical level, for systems where the benchmarking at critical is either unsafe or unreachable. Code benchmarking at subcritical system configurations is needed, even when the critical configuration is available, because of the changes in spatial and spectral neutron distributions. This methodology demonstrate the validity of using subcritical measurements of the ratio of spectral densities to validate transport theory methods and cross sections.
An interesting spin-off, coming from the developments in Section 4, is that the first order perturbation theory developed there may replace direct, brute force Monte Carlo calculations for small system changes. Indeed, the effect of small parameter changes can be masked by the statistical fluctuations in the Monte Carlo algorithm. However, to prove this point more work is needed for exactly describe in some detail how to obtain the slopes, $A$, and $Z$, from a single Monte Carlo calculation.

VI. References


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