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in Shear Flows

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Asymptotic persistence of collective modes in shear flows

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Abstract

A new nonasymptotic method is presented that reveals an unexpected richness in the spectrum of fluctuations sustained by a shear flow with nontrivial *arbitrary* mean kinematics. The vigor of the method is illustrated by analyzing a two-dimensional, compressible hydrodynamic shear flow. The temporal evolution of perturbations spans a wide range of nonexponential behavior from growth-cum oscillations to monotonic growth. The principal characteristic of the revealed exotic collective modes is their *asymptotic persistence*. “Echoing” as well as unstable (including parametrically-driven) solutions are displayed. Further areas of application, for both the method and the new physics, are outlined.

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It is well known that the predictions of the classical asymptotic stability theory (normal mode approach) often fail to match the relevant experimental results for a variety of shear flows (e.g., the plane Couette, the Poiseuille and the pipe Poiseuille flows). This failure, generally attributed to the non self-adjoint character of the governing equations [1,2], strongly suggests that alternative nonasymptotic routes to solve the initial value problem must be explored. In fact, a method originally proposed by Lord Kelvin [3] to study the nonexponential temporal evolution of fluctuations has recently been exploited to unearth fascinating aspects of fluctuation dynamics both in neutral fluids [4-8] and in plasmas [9-13].

In this *letter*, we follow the spirit of the Kelvin approach, i.e. carry out a nonasymptotic evolutionary analysis, but extend the domain of investigation to shear flows with arbitrary mean kinematics. We propose a simple method which reduces the multidimensional initial value problem to a set of manageable ordinary differential equations in time. The increase in the spatial dimensionality of the velocity inhomogeneity imparts an immense richness to the temporal behavior of the perturbations. The transient growths, seen in simple parallel shear flows, can either become asymptotic growths or can appear in a periodic pattern in a multidimensionally sheared flow. A totally unexpected phenomenon of parametric instabilities (present in equations with periodic coefficients) is brought out; the origin of periodicity lies in the periodic motion executed by a fluid element in an appropriately sheared flow.

After giving a brief synopsis of the general approach, we will discuss in some detail the relatively simple, two-dimensional hydrodynamic system. An analysis of this example illustrates the salient features of new physics without much algebraic complication.

In the close neighborhood of a point $A(x_0, y_0, z_0)$ ($|x - x_0|/|x_0| \ll 1$, etc.), the spatial variation of a general mean velocity field $\mathbf{U}(x, y, z)$ could be approximated by the linear terms in its Taylor expansion. Then a set of nine constants $U_{i,k}(x_0, y_0, z_0)$ ($i, k = x, y, z$)

forming the *Shear Matrix* $\mathcal{S}$:

$$\mathcal{S} \equiv \begin{pmatrix} U_{x,x} & U_{x,y} & U_{x,z} \\ U_{y,x} & U_{y,y} & U_{y,z} \\ U_{z,x} & U_{z,y} & U_{z,z} \end{pmatrix} \equiv \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad (1)$$

will fully characterize the shear flow in the region. For flows with *homogeneous* equilibrium density, the velocity field is divergence-free $\nabla \cdot \mathbf{U} = 0$ which translates into the constraint that the Shear Matrix is traceless: $a_{11} + a_{22} + a_{33} = 0$. The components of the mean velocity may, now, be expressed as $U_i(x, y, z) \simeq U_{0i} + a_{ik}x_k$ where $U_{0i} = U_i(x_0, y_0, z_0)$ and $x_i \equiv x_i - x_{0i}$. This will convert the linearized convective derivative operating on the perturbations to $\mathcal{D}u_i + a_{ik}u_k$, where $\mathcal{D}$ stands for the spatially inhomogeneous operator $\mathcal{D} \equiv \partial_t + U_i(x, y, z)\partial_i$. Our first task is to find, if possible, a transformation that will annihilate the spatial dependence in the operator $\mathcal{D}$. For any fluctuation $F(x, y, z; t)$, the ansatz

$$F(x, y, z; t) \equiv \hat{F}(\mathbf{k}(t), t) \exp[i\varphi], \quad \varphi \equiv k_i(t)x_i - U_{0i} \int k_i(t)dt \quad (2)$$

seems to do the job provided the wave vector $\mathbf{k}$ acquires the time dependence given by

$$\partial_t \mathbf{k} + \mathcal{S}^T \cdot \mathbf{k} = 0, \quad (3)$$

where $\mathcal{S}^T$ is the transposed shear matrix. The net effect is that the convective derivative has become an ordinary derivative in time: $\mathcal{D}F = \exp[i\varphi]\partial_t \hat{F}$.

Equations (3), even in their most general form, have simple analytic solutions that embrace all possible kinds of background flows. For example, the plane Couette flow, corresponding to $a_{12} = A$ and all other $a_{ik} = 0$, leads to the obvious and well-known solutions: $k_x(t) = k_x(0)$, $k_z(t) = k_z(0)$, and $k_y(t) = k_y(0) - Atk_x(0)$ [2].

For shear flows with more sophisticated kinematics, the temporal dependence of $\mathbf{k}(t)$ would be much more complex encompassing both exponentially evolving and periodic behavior. Notice that the “orbits” in the wave number space are closely related to the orbits of the fluid elements (moving with the the given sheared velocity) in the physical space. In

the two dimensional case to be discussed soon, the closed (diverging) orbits in physical space will lead to closed (diverging) orbits in $\mathbf{k}$ -space; the former will lead to oscillatory $\mathbf{k}$'s while the latter yield exponentially increasing $\mathbf{k}$'s.

Thus the ansatz (2) manages to reduce the entire burden of the multidimensional spatial inhomogeneity to the time variation of the $\mathbf{k}$'s. The resulting nonautonomous ordinary differential equations (ODE's) can be analyzed to 'smoke out' some overlooked modes of collective behavior, excited by nontrivial velocity fields.

The scope of this method is best demonstrated by examining the fluctuation dynamics of a two-dimensional hydrodynamic system with a homogeneous mean density. With the ambient flow velocity given by $\mathbf{U}(x, y) \equiv U_x(x, y)\mathbf{e}_x + U_y(x, y)\mathbf{e}_y$ with $a_{11} = -a_{22} \equiv \sigma$, $a_{12} \equiv a$ and $a_{21} \equiv b$, we can readily derive the set of linearized equations

$$\mathcal{D}\rho' + \rho_0(\partial_x u_x + \partial_y u_y) = 0, \quad (4a)$$

$$\mathcal{D}u_x + \sigma u_x + a u_y = -c_s^2 \partial_x(\rho'/\rho_0), \quad (4b)$$

$$\mathcal{D}u_y + b u_x - \sigma u_y = -c_s^2 \partial_y(\rho'/\rho_0), \quad (4c)$$

governing the evolution of small-scale perturbations in this flow. On applying ansatz (2), we convert the system to the set of first order ODE's:

$$\partial_\tau \varrho = \mathcal{K}_x v_x + \mathcal{K}_y v_y, \quad (5a)$$

$$\partial_\tau v_x + \varepsilon v_x + R_1 v_y = -\mathcal{K}_x \varrho, \quad (5b)$$

$$\partial_\tau v_y + R_2 v_y - \varepsilon v_y = -\mathcal{K}_y \varrho. \quad (5c)$$

with the components of the variable wave vector obeying

$$\partial_\tau \mathcal{K}_x + \varepsilon \mathcal{K}_x + R_2 \mathcal{K}_y = 0, \quad (6a)$$

$$\partial_\tau \mathcal{K}_y + R_1 \mathcal{K}_x - \varepsilon \mathcal{K}_y = 0, \quad (6b)$$

where we have used the dimensionless notation: $\tau \equiv c_s k_x(0)t$, $\varepsilon \equiv \sigma/c_s k_x(0)$, $R_1 \equiv a/c_s k_x(0)$, $R_2 \equiv b/c_s k_x(0)$, $\mathcal{K}_{x,y} \equiv k_{x,y}/k_x(0)$, $\varrho \equiv i(\rho'/\rho_0)$, and $v_{x,y} \equiv \hat{u}_{x,y}/c_s$.

Equations (5) and (6) contain *two* conserved quantities: the first one $\Delta \equiv \mathcal{K}_y \partial_\tau \mathcal{K}_x - \mathcal{K}_x \partial_\tau \mathcal{K}_y$ is associated with (6), while the second $\mathcal{C} \equiv \mathcal{K}_y v_x - \mathcal{K}_x v_y + (R_1 - R_2)\varrho$ follows from (5) and (6) and links the physical variables of the system with one another.

Defining $\mathcal{K}^2 \equiv \mathcal{K}_x^2 + \mathcal{K}_y^2$, and $\varrho \equiv \mathcal{K}\Psi$ we can derive from (5)–(6), the following explicit second-order ODE [$\Lambda^2 \equiv R_1 R_2 + \sigma^2$]:

$$\partial_\tau^2 \Psi + \left[\mathcal{K}^2 - \Lambda^2 + \frac{3\Delta^2}{\mathcal{K}^4} \right] \Psi = \frac{2\mathcal{C}\Delta}{\mathcal{K}^3}, \quad (7)$$

which determines the time behavior of the perturbations.

The constant of integration $\mathcal{C}$, appearing in the inhomogeneous term, can be interpreted as the source of vortical fluctuations. If one considers (7) as the oscillator equation then the inhomogeneity term represents time-dependent external force, while the function in square brackets stands for the square of the oscillator's time-dependent effective frequency. The entire time dependence of both coefficients in Eq. (7) comes through $\mathcal{K}^2$. The actual form of $\mathcal{K}^2$ crucially depends on the value and sign of the parameter Λ^2 . In particular, here are the following three cases:

(a) $\Lambda^2 = 0$. In this simple case $\mathcal{K}^2 = \mathcal{K}^2(0) + \partial_\tau \mathcal{K}^2(0)\tau + (R_1 - R_2)\Delta\tau^2$. This form of $\mathcal{K}$ pertains to the standard plane Couette flows, extensively studied in the recent literature both in the hydrodynamical [6,8] and in the plasma [12,13] contexts.

(b) $\Lambda^2 > 0$. Here $\mathcal{K}^2 = \delta + \mathcal{A} \cosh(2\Lambda\tau + \psi_0)$ with $\delta \equiv -(R_1 - R_2)\Delta/2\Lambda^2$ and $\mathcal{A}, \psi_0$, determined by initial values of the wave numbers. This case allows a simple asymptotic analysis. For $\Lambda\tau \rightarrow \infty$, $\mathcal{K}^2 \approx ae^{2\Lambda\tau}$, and (7) may be approximated by a *Bessel* equation

$$\partial_\tau^2 \Psi + \left[ae^{2\Lambda\tau} - \Lambda^2 \right] \Psi = 0, \quad (8)$$

with the solution

$$\Psi = J_1 \left(\frac{a^{1/2}}{\Lambda} e^{\Lambda\tau} \right) \rightarrow \left(\frac{2\Lambda}{\pi a^{1/2}} \right)^{1/2} e^{-\Lambda\tau/2} \cos \left[\frac{a^{1/2}}{\Lambda} e^{\Lambda\tau} - \frac{3\pi}{4} \right],$$

leading to an exponential growth for the physical density perturbation:

$$\varrho = \mathcal{K}\Psi \approx e^{\Lambda\tau/2} \cos \left[\frac{a^{1/2}}{\Lambda} e^{\Lambda\tau} - \frac{3\pi}{4} \right]. \quad (9)$$

Thus the diverging orbits with $\mathcal{K}^2 \rightarrow e^{2\Lambda\tau}$ imply an acoustic wave with exponentially growing amplitude and ‘frequency.’ The numerical solution displayed in Fig. 1, where we have plotted ϱ as a function of time, clearly confirms the predictions of Eq. (9). It must be stressed, however, that for this class of flows ($\mathcal{K}^2 \rightarrow e^{2\Lambda\tau}$), the viscous damping will tend to kick in due time and will eventually damp the mode. This is, indeed, found to be the case, when viscosity is incorporated into the original setup.

(c) $-\omega^2 \equiv \Lambda^2 < 0$. The solution for $\mathcal{K}^2$ is, now, periodic: $\mathcal{K}^2 = \delta + \mathcal{B} \cos(2\omega\tau + \phi_0)$, with $\mathcal{B}$, ϕ_0 , depending, again, on the initial values of the wave numbers. Equation (7) acquires the form of an inhomogeneous Hill equation because $\mathcal{K}^2$ is periodic in time. For this class of flows, the fluid elements (therefore the $\mathbf{k}$ vector) have closed elliptic orbits. Simple numerical analysis reveals the following three subclasses of solutions:

1. *Vortical solutions* interacting with the Acoustic wave. These solutions are expected whenever $\mathcal{C}$ is large, in some sense, and the frequency $\omega \equiv [-(R_1 R_2 + \varepsilon^2)]^{1/2}$ is small. These are new, very peculiar versions of the Kelvin modes [6,8,12,13] with the difference that the new vortex (Fig. 2) is not a transient, it does have a transient growth, but it repeats with the periodicity 2ω . The vortex forms at a given time, gives up its energy back to the flow and the acoustic oscillations, and echoes back after a time $T = \pi/\omega$. We call this picturesque phenomenon *Asymptotic Persistence*, and we believe it is a hallmark of the multidimensional shear flows.

2. *Unstable Acoustic Waves:* It is well known that equations with periodic coefficients allow unstable solutions in certain ranges of parameters. Notice that for $\Delta = 0$, (7) becomes a *Mathieu* equation whose solutions are very well known. It is reasonable to expect that even for $\Delta \neq 0$, the solutions will retain the peculiarities of the Mathieu solution, e.g. the regions of instability. For Mathieu equation, the regions of instability are tabulated, but for more general cases, one has to determine them by numerical methods.

Numerical solutions show that (7) has a number of unstable regions in the parameter space defined by $[\omega, R_2 \text{ and } \mathcal{K}_y(0)]$. One such parametrically unstable solution is displayed in Fig. 3, where the perturbation is exponentially growing in time. We do not yet have approximate analytic formulas for either the growth rates or the regions of instability.

3. *Stable Acoustic Waves:* the basic solutions of Eq. (7) are some combinations of the vortical and acoustic type of perturbations. For a variety of initial conditions, acoustic type will dominate the vortical type. In this case the frequency as well as the amplitude of the mode changes periodically. The periodicity of frequency change is, naturally, the periodicity of $\mathcal{K}^2$. The amplitude waxes and wanes as the acoustic mode exchanges energy with the flow. A pair of typical plots is shown on Figures 4a and 4b. These figures are drawn for the same set of parameters as Fig. 3. but with the difference that $\mathcal{K}_y(0) = 10$ for Fig. 4a, and $\mathcal{K}_y(0) = 6$ for Fig. 4b. Comparison of these solutions with the unstable one shows that the acoustic waves become unstable for relatively narrow and restricted ranges of the system parameters. This feature clearly reflects on the parametric nature of the solution displayed in Fig. 3.

For all the cases discussed, the fluctuations do not go away; they are not *transient*, they amplify and persist.

It seems reasonable and tempting to argue that the phenomenon of asymptotic persistence of fluctuations is quite general; it will manifest itself in all systems (both in neutral fluids and in plasmas) with multidimensionally sheared background mean flows. Note that the ambient flow must vary at least in two directions, otherwise the fluctuations will be mere transients like in the standard Couette flow.

In order to demonstrate that our considerations are not limited to hydrodynamics, we carried out a similar analysis for the electrostatic Langmuir waves in a sheared plasma. The mathematical similarity of these two systems translates to quite an equivalence in their time evolution. We find echoing Langmuir vortices, a weak parametric instability, persisting Langmuir fluctuations etc.

We expect that the notions introduced in this letter will have very widespread applications. Since the sheared flows are being recognized as major determinants of the fate of fusion plasmas, we believe that this shear generated and maintained fluctuation spectrum could be a crucial new element in understanding the anomalous transport in magnetic-confinement experiments. Astrophysical systems stand out as yet another class of examples of shear flows with complex kinematics. A rather incomplete list will consist of: solar plasma flows, pulsar magnetospheric plasmas, accretion columns in X-ray bursters, bipolar outflows (jets) in Active Galactic Nuclei and Quasars. It is hard to imagine that the shear-induced, asymptotically persisting fluctuations will not lead to the delineation of processes with direct or indirect observational consequences.

We end this paper with a word of caution. In order to determine the appropriate set of new modes of collective behavior pertaining to concrete cases of hydrodynamical and plasma (laboratory, geophysical or astrophysical) shear flows, one must be careful. It should be remembered that not *all* $\mathbf{U}(x, y, z)$'s are possible; the relevant flow must satisfy the stationary state equations for a given physical system. In this *letter*, for the sake of generality, we consciously *did not* circumscribe the range of possible mean velocity fields. We simply con-

centrated on developing an approach for studying the initial value problem valid for arbitrary ambient shear flows. The flows discussed in this paper are consistent with the equilibrium solutions. The general question of the sources of the ambient flows, along with many other details of the phenomena described in this letter, will be dealt with in a forthcoming larger publication.

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Figure Captions

Fig. 1 Normalized density perturbation $\varrho(\tau) = \mathcal{K}(\tau)\Psi(\tau)$ versus time for $R_1 = 0.1$, $R_2 = 0.05$, $\varepsilon = 0$, $\mathcal{K}_y(0) = 0.1$, $\mathcal{C} = 0$, $\Psi(0) = 10^{-2}$, and $\partial_\tau \Psi(0) = 0$.

Fig. 2 Asymptotic persistence of “echoing” Kelvin vortices. Normalized density perturbation $\varrho(\tau)$ versus time when $R_1 = 0.1$, $R_2 = -0.01$, $\varepsilon = 0$, $\mathcal{K}_y(0) = 1$, $\mathcal{C} = 1$, $\Psi(0) = 3.87 \times 10^{-2}$, and $\partial_\tau \Psi(0) = 8.66 \times 10^{-3}$.

Fig. 3 Parametrically unstable acoustic wave solution plotted for $\varrho(\tau)$ when $R_1 = 2.0$, $R_2 = -2 \times 10^{-2}$, $\varepsilon = 0$, $\mathcal{K}_y(0) = 8$, $\mathcal{C} = 1$, $\Psi(0) = 10^{-4}$, and $\partial_\tau \Psi(0) = 0$.

Fig. 4 Stable acoustic wave solutions presented by $\varrho(\tau)$ versus time graphs, for the same set of system parameters as in Fig. 3, except that here $\mathcal{K}_y(0) = 10$ (on Fig. 4a) and $\mathcal{K}_y(0) = 6$ (on Fig. 4b).

Figure 1

Figure 2

Figure 3