Final Report for Symmetric Truncations of the Shallow Water Equations and Modelling of Coherent Structures in a Boundary Layer

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Grant Number DE-FG-03-91ER61222

DOE/ER/61222--T2

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Contents

1 Introduction

In this final report we will briefly describe three achievements in the funding period, related to the original goals of the research, and attach some representative papers for each achievement.

The original goal of the project was to implement a numerical scheme to numerically solve the shallow water equations (in periodic boundary conditions and on the sphere), in such a way as to preserve not just the enstrophy, but all other invariants consistent with the level of the numerical truncation. The theoretical framework for this work was set up successfully and simple numerical runs were carried out. However because the scheme was set up in Lagrangian coordinates, and even though preserved the invariants in the desired way, it was inefficient. Therefore this idea was put aside.

The second goal achieved arose indirectly from the work on the original project however. The Hamiltonian formalism used in the constructions of the original method, were used successfully in treating a problem in boundary layer theory. This probelem is of general fluid dynamical interest and therefore applies to the atmospheric boundary layer as well.

The third goal acheived, is much closer in spirit to the original project. Unfortunately at the time the proposal for renewal was written, this was still not developed well, and could not be included in the renewal request. However, now the project to be described is nearly complete. This project led ultimately to an interesting method for solving the shallow water equations on the sphere. This method was implemented, and led to successful results, and in our opinion, an improvement over the shallow water solver supplied by NCAR.

2 Symmetric Truncations

In climate dynamics, the hydrodynamical equations of motion for the coupled ocean-atmosphere system must be integrated over very long time periods. This integration can only be carried out numerically, and it is important to have some assurance that the numerically computed approximate solution is close the actual solution of the dynamical equations governing the ocean-atmosphere system. Current coupled ocean-atmosphere general circulation models are extremely complex, therefore the issue of the numerical *accuracy* of computed solutions is a very difficult one to resolve. Of course, one can by now have some degree of confidence in numerically computed solutions of general circulation models, as evidenced for example, by the recent successes in the prediction of the El Niño southern oscillation events, see for example [1] and references therein. However, these successes are still for integrations over fairly short periods of time. There are certain important questions, such as the effects of increased greenhouse gases in the atmosphere, in which one would like to have confidence in numerical simulations for a period of 100 years or more. An observational test of the numerical predictions is clearly problematic. On the one hand, we cannot wait 100 years to see if the prediction comes true and on the other hand, we cannot really test these models by "predicting" the past climate from the present one, as accurate climatic data exist only from the relatively recent past.

Here we proposed to address this issue of *accuracy* of numerically computed solutions in a new manner. At first, we will look at a fairly limited, albeit important, aspect of this problem. Specifically, we will consider the *shallow water equations.* The majority of general circulation models are *layer* models of the atmosphere and oceans [1, 2]. The fluid in each layer is assumed to behave in a way that is basically two dimensional. This is a good approximation to the actual three dimensional motions because of the thinness of both the oceanic and atmospheric fluid layers compared to the large horizontal length scales of interest. The layers are coupled together in complicated ways in order to account for the vertical stratification. Additionally, for a realistic model, complex thermodynamical equations must be included, bringing in and coupling the pressure, temperature and salinity fields with the velocity field in each layer. Other important phenomena, such as cloud cover and precipitation must also be modeled into the equations.

Nonetheless, it is true that the basic building block of these models is the set of shallow water equations governing the evolution of the velocity field and fluid depth in each layer. It seems reasonable, therefore, to look first at the above mentioned issue of accuracy in the context of the shallow water equations governing the evolution of a single thin layer of fluid.

In our investigation of the long time dynamics of the shallow water equations, we will take the matters of *symmetry* and *conservation principles* to be our guiding light. Evolution equations of physical interest frequently exhibit symmetries. These symmetries imply the existence of conserved quantities, if our dynamical equations of interest constitute a *Hamiltonian* or *friction free system.* This is always an idealization, since some dissipation is present in all physical systems. Nevertheless, ignoring dissipation in the shallow water equations is a good first approximation, given that geophysical length scales of interest are greater than viscous length scales by many orders of magnitude. Dissipation is a very important factor in many situations, and later, we will have something to say about the implications of our investigation when it is taken into account.

The importance of conserved quantities in long term dynamics is not difficult to see. Even though the conservation of all conserved quantities for a particular dynamical system does not guarantee the accuracy of a numerical code, the *nonconservation* of some of the conserved quantities by a numerical code certainly suggests that one has wandered off the true solution. This is the case since this nonconservation implies that the numerical solution is in a region of phase space that is *never* visited by the true solution.

A problematic issue in the numerical simulation of many continuum inviscid fluid systems, including the shallow water equations is that they have *infinitely many* conserved quantities besides the obvious ones such as energy. Till now, numerical truncations have at best been able to conserve energy and enstrophy [3]. It is of course impossible to ask that a truncation, which is inherently finite dimensional, conserve an infinite number of conserved quantities. It is however, reasonable to demand of a truncation that it *have a set of conserved quantities, the number of which increase without bound and whose values approach that of the conserved quantities of the original infinite dimensional system, as the number of degrees of freedom in the truncation is increased.*

In the first attached paper, just such a scheme is described [4]. Preliminary numerical investigations of this model are also described in that document. In one sense these numerical tests were a great success, since by the techniques of symplectic numerical integration [5, 6, 7, 8], we conserved potential vorticity in our runs to within roundoff error. In another sense though, they were not satisfactory from a practical point of view. Our discretization employed an awkward Lagrangian formulation, with all the well known difficulties of such formulations. Since the whole purpose of this investigation was to investigate very long time dynamics of the shallow water equations, inevitably many "remeshings" of the Lagrangian grid would be necessary. A naive remeshing will destroy the potential vorticity conserving aspect of our scheme. Therefore a very subtle sort of remeshing is necessary. This is possible, but would render an already awkward numerical problem even more awkward and inefficient. For this reason, I spent the greater part of my time during the last funding period, attempting to find an equivalent Eulerian potential vorticity conserving discretization scheme. This involved finding answers to very intricate mathematical questions, indeed it was not *a priori* obvious whether an Eulerian formulation was even theoretically possible. Despite help from mathematicians expert in the field of "reduction" (notably Tudor Ratiu at U.C. Santa Cruz), which addresses how one proceeds from Lagarangian to Eulerian formulations of fluid mechanics using symmetry, [9, 10], I got close but was not quite successful at finding an Eulerian formulation. Such a scheme was finally devised however (based partly on my work) by Zhong Ge (Fields Research institute, Waterloo, Ontario, Cananda), and Clint Scovel at Los Alamos National Laboratory have been successful at formulating an Eulerian symmetric truncation [11]. The main ideas they employed, were similar to the ones we used in our Lagrangian truncation. This scheme however seems extremely awkward also, and it did not seem worthwhile to pursue a numerical integration of the Ge Scovel truncation. From a practical point of view numerical integration of these equations appeared quite difficult.

3 Coherent Structures in Boundary Layers

The second project the we investigated a rather different problem. This was not part of the original proposal. In short, the applicant and his collaborator, Jon Wright, constructed a unified Hamiltonian formalism for the investigation of the interaction of singular vorticity distributions with free and rigid boundaries [12]. This formalism was then applied to a model for instabilities in a boundary layer [13] and in a very simple manner yielded results in striking agreement with experimental observations of coherent structures in a boundary layer.

The formalism and analysis have so far been restricted to the simpler situation of classical hydrodynamics, i.e., the case of incompressible fluids. However, this research was begun by the applicant with an eye to future geophysical applications, in particular to the modelling of the atmospheric boundary layer. The idea here is to model complicated coherent structures arising in a turbulent boundar layer, using singular vorticity distributions, such as vortex filaments, and vortex layers (with constant vorticity). This is simpler than a full numerical modelling of a boundary layer, thus insights into physical processes are easier to gain. Since funding was not continued on the project this project is currently on hold. The papers cited above have been attached to this report however.

4 Split Step Methods

Through familiarity with geometrical methods in differential equation, the researcher devised a new method for solving partial differential equations. This methods was implemented for the case of the shallow water equations on the sphere and led to very satisfactory results.

We recently introduced a method of operator splitting for a wide class of partial differential equations [14, 15], which permit one to utilize high order time stepping methods which have appeared in the recent past in the literature [16, 17, 18, 19, 20]. In addition to higher accuracy in time, these methods have the advantage that the higher accuracy does not come at the expense of computer memory. In investigating this operator splitting method we have discovered further advantanges that it presents.

In the attached preprint we applied our method to the solution of the shallow water equations in spherical geometry. The initial conditions we have used are the ones proposed by Williamson et. al. [21]. In this paper we compared our scheme against the the one provided by NCAR not for all the initial conditions suggested by Williamson et. al. [21], but rather ones that bring out the differences between the NCAR scheme and ours. In particular, since our scheme uses spherical harmonic spectral discretization on the sphere, for the simpler initial conditions, the results obtained are nearly identical to those obtained by running the NCAR code. [22]

The main advantages of our method are better treatment of gravity waves, use of less storage and better treatment of artificial viscosity. This material is described in the attached preprint.

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