New mathematical derivations applicable to safety and reliability analysis

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ABSTRACT: Boolean logic expressions are often derived in safety and reliability analysis. Since the values of the operands are rarely exact, accounting for uncertainty with the tightest justifiable bounds is important. Accurate determination of result bounds is difficult when the inputs have “constraints.” One example of a constraint is that an uncertain variable that appears multiple times in a Boolean expression must always have the same value, although the value cannot be exactly specified. A solution for this “repeated variable” problem is demonstrated for two Boolean classes. The classes, termed functions with “unate” variables (including, but not limited to unate functions), and “exclusive-or” functions, frequently appear in Boolean equations for uncertain outcomes portrayed by logic trees (event trees and fault trees).

1 INTRODUCTION

It has recently been recognized [1,2] that using interval-based computations such as interval arithmetic, and fuzzy or possibilistic mathematics in an “unconstrained” mode (applied by sequentially parsing equation solutions) can significantly misrepresent extreme values. It is important to address this phenomenon, its ramifications, and a solution for the problem. Since risk management decisions are commonly based on the results of such analyses, the goal is to derive the most accurate bounds possible, so the resultant decisions can be as good as possible.

2 THE “REPEATED VARIABLE” PROBLEM

Mathematically processing uncertain operands may involve inobvious problems due to constraints. An example of a possible constraint is that repeated appearances of the same uncertain variable must all have the identical value.

This “repeated variable problem” will be addressed in order to show how range-based probabilistic evaluation of Boolean logic expressions, such as those describing the outcomes of fault trees and event trees, can be facilitated. The results are applicable to fuzzy or possibilistic mathematics, interval analysis, Monte Carlo or LHS analysis and other range-portraying techniques. The problem is important, because unconstrained computations result in wider ranges of outputs than those properly obtained with constrained mathematics. Since risk management decisions may be based on this information, the results should be accurate. We illustrate techniques that can be used to transform complex constrained problems into trivial problems in most tree logic expressions, and into tractable problems in most other cases. The approach is based on the Boolean logic characteristics of “unateness” and “minimal compactness,” and differential calculus characteristics related to regions of monotonicity.

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Example problems are used to demonstrate the techniques and the advantages of constrained mathematics. The results obtained are the precise bounds sought for risk management.

3 CONSTRAINED MATHEMATICS

An example illustrating the necessity for constrained mathematics is the union of two independent events for which the probabilities are specified by intervals, \( P(A) = [x_u, x_d] \) and \( P(B) = [y_h, y_d] \). The first member of the ordered pair is, by convention, the lower bound; the second is the upper bound. Also, the notation \( P(A) \) indicates that the uncertain parameter will have a particular value somewhere in the interval, and that value will be the same for every appearance of \( P(A) \) in a probabilistic calculation. Using unconstrained operations of interval analysis, one obtains:

\[
P(A \cup B) = [x_u + y_u - x_d, x_u + y_h - x_d]
\]  

(1)

The results at both ends of the interval can have values outside the range \([0, 1]\), which is erroneous. The problem is that both interval bounds derived from unconstrained operations combine an operand’s lower bound together with its upper bound, which is not possible for uncertainty about an event. Restricting events to have only one value at a time, one obtains the correct answer, which is:

\[
P(A \cup B) = [x_u + y_d - x_d, x_u + y_u - x_d]
\]  

(2)

The simplicity of this result is misleading, however, because most problems cannot be solved correctly by mapping all operand lower bounds to the result lower bound and all operand upper bounds to the result upper bound. However, for most probabilistic evaluation of logic expressions, these types of problems can be worked efficiently. Demonstrating this is a major aim of this paper.

4 RESULTS BASED ON A HUEUSTIC NOTATION

A notation for uncertain variables that helps explain various aspects of the repeated variable problem is:

\[
x_i = X_i + \alpha_i \varepsilon_i
\]  

(3)

where \( x_i = [x_i, x_u] \), \( X_i = \frac{x_i + x_u}{2} \), \( \alpha_i = \frac{x_u - x_i}{2} \), and \( \varepsilon_i = \pm 1 \) (whichever is appropriate for the lower or upper bound sought). Nonlinear operations (e.g., multiplication) produce products of the \( \varepsilon_i \) and multiple appearances of each \( \varepsilon_i \). As a result, the general notation will be \( \varepsilon_y = \pm 1 \) (whichever is appropriate for appearance \( j \) of the variable). This facilitates reasoning for distinguishing between different computations, such as unconstrained mathematics and constrained mathematics.

**Example 1.** Consider the probability range for an “Or” function of two independent interval variables, \( x_1 \) and \( x_2 \). This example illustrates the use of the heuristic notation. It also illustrates why unconstrained mathematics does not produce the correct answer. Converted by the methodology of Ref. [3] into a disjoint set form and expressed in the above notation:

\[
y_1 = X_1 + X_2 - X_1X_2 + \alpha_1 \varepsilon_{11} + \alpha_2 \varepsilon_{21} - X_1 \alpha_2 \varepsilon_{22} - X_2 \alpha_1 \varepsilon_{12} - \alpha_1 \varepsilon_{22} \alpha_2 \varepsilon_{12}
\]  

(4)

where the variables, \( x_1 \) and \( x_2 \), represent probabilities. For \( x_1 = [0.2, 0.4] \), \( x_2 = [0.7, 0.8] \),

\[
y_1 = 0.825 + 0.1 \varepsilon_{11} + 0.05 \varepsilon_{21} - 0.015 \varepsilon_{22} - 0.075 \varepsilon_{12} - 0.005 \varepsilon_{12} \varepsilon_{22}
\]  

(5)

For the unconstrained lower bound, \( \varepsilon_{11} \) and \( \varepsilon_{21} \) are negative, \( \varepsilon_{12} \) and \( \varepsilon_{22} \) are positive, and \( \varepsilon_{12} \varepsilon_{22} \) is identical to the product of \( \varepsilon_{12} \) and \( \varepsilon_{22} \), and therefore positive. This bound is:

\[
y_{1ul} = 0.825 - 0.1 - 0.05 - 0.015 - 0.075 - 0.005 = 0.58
\]  

(6)
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For the unconstrained upper bound, $\varepsilon_{11}$ and $\varepsilon_{21}$ are positive, $\varepsilon_{12}$ and $\varepsilon_{22}$ are negative, and $\varepsilon_{12} \varepsilon_{22}$ is identical to the product of $\varepsilon_{12}$ and $\varepsilon_{22}$, and therefore positive.

$$y_{1uv} = 0.825 + 0.1 + 0.05 + 0.015 + 0.075 - 0.005 = 1.06$$ (7)

For the constrained lower bound, $\varepsilon_{11} = \varepsilon_{12}$ and $\varepsilon_{21} = \varepsilon_{22}$ are negative (the pairs must be identical by the constraint of being the same variable), and $\varepsilon_{11} \varepsilon_{22}$ is therefore positive.

$$y_{1lo} = 0.825 - 0.1 - 0.05 + 0.015 + 0.075 - 0.005 = 0.76$$ (8)

For the constrained upper bound, $\varepsilon_{11} = \varepsilon_{12}$ and $\varepsilon_{21} = \varepsilon_{22}$ are positive (the pairs must be identical by the constraint of being the same variable), and $\varepsilon_{11} \varepsilon_{22}$ is therefore positive.

$$y_{1su} = 0.825 + 0.1 + 0.05 + 0.015 + 0.075 - 0.005 = 0.88$$ (9)

**Example 2.** Consider a general solution for the Exclusive-Or function of two interval independent probability variables:

$$y_2 = x_1 + x_2 - 2x_1 x_2$$

$$= x_1 + \alpha_1 x_1 + x_2 + \alpha_2 x_2 - 2x_1 x_2 - 2x_1 \alpha_1 x_2$$

$$- 2x_2 \alpha_1 x_2 - 2 \alpha_1 x_2 \alpha_2 x_2$$

$$= x_1 + \alpha_1 x_1 + x_2 + \alpha_2 x_2 - 2x_1 x_2 - 2x_1 \alpha_1 x_2$$

$$- 2x_2 \alpha_1 x_2 - 2 \alpha_1 x_2 \alpha_2 x_2$$

(10)

where the variables, $x_1$ and $x_2$, again represent probabilities. For $x_1 = [0.2, 0.4]$, $x_2 = [0.4, 0.8],

$$y_{2} = 0.54 + 0.1 \varepsilon_{11} + 0.2 \varepsilon_{21} - 0.12 \varepsilon_{22} - 0.12 \varepsilon_{12} - 0.04 \varepsilon_{12} \varepsilon_{22}$$

(11)

For the unconstrained lower bound: $\varepsilon_{11} = \varepsilon_{21}$ are negative, and $\varepsilon_{12} = \varepsilon_{22}$ are positive. The bound is:

$$y_{2ul} = 0.54 - 0.1 - 0.2 - 0.12 - 0.12 - 0.04$$

$$= -0.04$$ (12)

Note that the negative probability (impossible) is a byproduct of unconstrained mathematics. For the unconstrained upper bound, $\varepsilon_{11} = \varepsilon_{21}$ are positive, and $\varepsilon_{22} = \varepsilon_{12}$ are negative, so $\varepsilon_{12} \varepsilon_{22}$ is positive. The bound is:

$$y_{2uu} = 0.54 + 0.1 + 0.2 + 0.12 + 0.12 - 0.04$$

$$= 1.04$$ (13)

For the constrained lower bound, $\varepsilon_{11} = \varepsilon_{12}$ are negative, $\varepsilon_{22} = \varepsilon_{21}$ are negative, so $\varepsilon_{12} \varepsilon_{22}$ is negative. The bound is:

$$y_{2ol} = 0.54 - 0.1 - 0.2 + 0.12 + 0.12 - 0.04$$

$$= 0.44$$ (14)

For the constrained upper bound, $\varepsilon_{11} = \varepsilon_{12}$ are negative, $\varepsilon_{22} = \varepsilon_{21}$ are positive, so $\varepsilon_{12} \varepsilon_{22}$ is negative. The bound is:

$$y_{2su} = 0.54 - 0.1 + 0.2 - 0.12 + 0.12 + 0.04$$

$$= 0.68$$ (15)

**Example 3.** General Solution for $y_3 = v_1 v_2 \cup \overline{v_1} v_3 \cup \overline{v_2} v_3$ (Boolean logic variables) expressed in terms of independent interval probabilities ($x_i$ is the probability of $v_i$):

$$y_3 = 1 + 2x_1 x_2 - x_1 - x_2 + x_2 x_3 - x_1 x_2 x_3 = 1 + 2x_1 x_2 + 2x_1 \alpha_2 x_2 + 2x_2 \alpha_1 x_2 - x_1 - x_2 + x_1 - x_2 + x_2 x_3 - x_1 x_2 x_3 +$$

$$+ \alpha_2 x_2 \alpha_3 x_3 - x_1 x_2 x_3 - x_1 x_2 x_3 - x_1 x_2 x_3 - x_1 x_2 x_3 - x_1 x_2 x_3 - x_1 x_2 x_3 - x_1 x_2 x_3 - x_1 x_2 x_3$$

$$+ \alpha_1 x_2 \alpha_3 x_3 - x_1 x_2 x_3 - x_1 x_2 x_3 - x_1 x_2 x_3 - x_1 x_2 x_3 - x_1 x_2 x_3 - x_1 x_2 x_3 - x_1 x_2 x_3 - x_1 x_2 x_3$$

(16)

For $x_1 = [0.2, 0.4]$, $x_2 = [0.7, 0.8]$, and $x_3 = [0.6, 0.8],

$$y_3 = 0.7675 + 0.03 \varepsilon_{21} + 0.15 \varepsilon_{11} + 0.01 \varepsilon_{11} \varepsilon_{21} - 0.1 \varepsilon_{12} - 0.05 \varepsilon_{22} + 0.075 \varepsilon_{31} + 0.035 \varepsilon_{23} +$$

$$+ 0.005 \varepsilon_{23} \varepsilon_{31} - 0.0015 \varepsilon_{24} \varepsilon_{32} - 0.0075 \varepsilon_{13} \varepsilon_{32} - 0.0035 \varepsilon_{13} \varepsilon_{24} - 0.0525 \varepsilon_{13} - 0.0105 \varepsilon_{24} -$$

$$- 0.0225 \varepsilon_{32} - 0.0005 \varepsilon_{13} \varepsilon_{24} \varepsilon_{32}$$

(17)
For the unconstrained lower bound, $\varepsilon_{21}$, $\varepsilon_{11}$, $\varepsilon_{31}$, $\varepsilon_{23}$ are negative, and $\varepsilon_{12}$, $\varepsilon_{22}$, $\varepsilon_{24}$, $\varepsilon_{32}$, $\varepsilon_{13}$ are positive, so $\varepsilon_{11} \varepsilon_{21}$, $\varepsilon_{23} \varepsilon_{31}$, $\varepsilon_{24} \varepsilon_{32}$, $\varepsilon_{13} \varepsilon_{24}$, and $\varepsilon_{13} \varepsilon_{24} \varepsilon_{32}$ are positive.

$$y_{3ul} = 0.7675 - 0.03 - 0.15 + 0.01 - 0.1 - 0.05 - 0.075 - 0.035 + 0.005 - 0.0015 - 0.0075 - 0.0035 - 0.0525 - 0.0105 - 0.0225 - 0.0005 = 0.244$$ (18)

For the unconstrained upper bound, $\varepsilon_{21}$, $\varepsilon_{11}$, $\varepsilon_{31}$, $\varepsilon_{23}$ are positive, and $\varepsilon_{12}$, $\varepsilon_{22}$, $\varepsilon_{24}$, $\varepsilon_{32}$, $\varepsilon_{13}$ are negative, so $\varepsilon_{11} \varepsilon_{21}$, $\varepsilon_{23} \varepsilon_{31}$, $\varepsilon_{24} \varepsilon_{32}$, $\varepsilon_{13} \varepsilon_{24}$ are positive and $\varepsilon_{13} \varepsilon_{24} \varepsilon_{32}$ is negative.

$$y_{3uu} = 0.7675 + 0.03 + 0.15 + 0.01 + 0.1 + 0.05 + 0.075 + 0.035 + 0.005 - 0.0015 - 0.0075 - 0.0035 + 0.0525 + 0.0105 + 0.0225 + 0.0005 = 1.296$$ (19)

For the constrained lower bound, $\varepsilon_{21} = \varepsilon_{22} = \varepsilon_{23} = \varepsilon_{24}$ are positive, $\varepsilon_{11} = \varepsilon_{12} = \varepsilon_{13}$ are negative, $\varepsilon_{31} = \varepsilon_{32}$ are negative, so $\varepsilon_{11} \varepsilon_{21}$ and $\varepsilon_{13} \varepsilon_{24}$ are negative, $\varepsilon_{23} \varepsilon_{31}$, $\varepsilon_{24} \varepsilon_{32}$, $\varepsilon_{13} \varepsilon_{32}$, and $\varepsilon_{13} \varepsilon_{24} \varepsilon_{32}$ are positive.

$$y_{3cl} = 0.7675 + 0.03 - 0.15 - 0.01 + 0.1 - 0.05 - 0.075 + 0.035 - 0.005 + 0.0015 - 0.0075 + 0.0035 + 0.0525 - 0.0105 + 0.0225 - 0.0005 = 0.704$$ (20)

For the constrained upper bound, $\varepsilon_{21} = \varepsilon_{22} = \varepsilon_{23} = \varepsilon_{24}$ are positive, $\varepsilon_{11} = \varepsilon_{12} = \varepsilon_{13}$ are negative, $\varepsilon_{31} = \varepsilon_{32}$ are negative, $\varepsilon_{23} \varepsilon_{31}$ and $\varepsilon_{24} \varepsilon_{32}$ are positive, and $\varepsilon_{13} \varepsilon_{32}$ and $\varepsilon_{13} \varepsilon_{24} \varepsilon_{32}$ are negative.

$$y_{3cu} = 0.7675 + 0.03 - 0.15 - 0.01 + 0.1 - 0.05 + 0.075 + 0.035 + 0.005 - 0.0015 + 0.0075 + 0.0035 + 0.0525 - 0.0105 - 0.0225 + 0.0005 = 0.832$$ (21)

As the examples demonstrate, choice of the signs of the $\varepsilon$ are individually deterministic for unconstrained mathematics. The bounds for constrained mathematics are exact.

It is a more subtle challenge to make the appropriate selection of the $\varepsilon$ to achieve the bounds. This problem will be approached in the following sections.

### 5 UNATE FUNCTIONS

An important situation, which we address with our methodology, involves the probabilistic evaluation of a Boolean function that is unate or has unate variables. Boolean functions are logical descriptions that can be applied to describe the outcomes of fault trees and event trees, among many other safety and reliability analysis applications.

Unateness [4] means that every variable of a Boolean function can be expressed such that each variable appears either complemented or uncomplemented, but both senses are not necessary. Any variable that meets this condition is called a unate variable (positive unate if uncomplemented, negative unate if complemented). Unateness is especially important in logical trees, because an event tree or fault tree having only "ands" and "ors" such that each event affects the probabilistic outcome either positively everywhere it appears or negatively everywhere it appears can be represented by a unate Boolean logic expression. This tree condition is sufficient for unateness. An algorithm that accounts for constrained mathematics in an expression of probability for the Boolean function outcome is:

For a positive unate variable, the lower bound of the result is a function of the lower bound of each appearance of the variable, and the upper bound of the result is a function of the upper bound of each appearance of the variable. For a negative unate variable, the lower bound of the result is a function of the upper bound of each appearance of the variable, and the upper bound of the result is a function of the lower bound of each appearance of the variable.

For the probability, $P(y)$, that the Boolean function is satisfied, all uncertain variables, $x_i$, are constrained to have identical values within their uncertainty range for all appearances of the variable in an algebraic expression for $P(y)$. 
This is true whether the uncertainty is represented by a probability density function, a fuzzy number, an interval, or any other practically meaningful uncertainty measure. Where lower and upper bounds are involved (e.g., interval analysis and fuzzy mathematics) the bound must be the same for each appearance of the variable in the expression for \( P(y) \).

This result naturally extends from independent variables to independent functions, as will be demonstrated in a subsequent example. Once unate variables have been processed, the solution for any non-unate variables can be traditional, but greatly simplified because of the removal of unate variables from the problem.

Parsing for computer solution involves first determining the unate variables and their bounds, and then calculating the bounds for non-unate variables based on the bounds of the unate variables. Finally, all variable bounds are combined to solve for the bounds of the result. The concepts in the theorem, the processing of non-unate variables, and the parsing order will be illustrated through examples.

**Example 3:**

For the function \( y_3 = x_1 \overline{x}_3 \cup x_2 x_4 \cup \overline{x}_2 \overline{x}_3 \), \( x_1 \) and \( x_4 \) are positive unate, \( x_3 \) is negative unate, and \( x_2 \) is not unate (both complemented and uncomplemented forms are necessary). A probability expression (methodology of [3]) is:

\[
P(y_3) = P(x_1)P(x_4) + P(\overline{x}_3)P(\overline{x}_4)
\]

The solution directly implements the bounds for \( x_1, x_3, \) and \( x_4 \) as:

\[
P(y_3)_l = P(x_1)P(x_4)_l + P(\overline{x}_3)_l P(\overline{x}_4)_l
\]

\[
P(y_3)_u = P(x_1)_u P(x_4)_u + P(\overline{x}_3)_u P(\overline{x}_4)_u
\]

and,

\[
P(y_3)_l = P(x_1)P(x_4)_l + P(\overline{x}_3)P(\overline{x}_4)_l
\]

The solution for \( P(x_2) \) depends on the sign of the function of the other variables that are produced with \( P(x_1) \). Taking a partial derivative with respect to \( P(x_1) \):

\[
\frac{\partial P(y_3)}{\partial P(x_2)} = P(x_1) - P(\overline{x}_3)P(x_4)_l P(\overline{x}_4)_l
\]

Then if \( P(x_4)_l - P(\overline{x}_3)_l \geq 0 \), \( P(x_2)_l \) can be used in the computation for \( P(y_3)_l \). If \( P(x_4)_l - P(\overline{x}_3)_l \leq 0 \), \( P(x_2)_u \) can be used in the computation for \( P(y_3)_u \). If \( P(x_4)_u - P(\overline{x}_3)_u \geq 0 \), \( P(x_2)_u \) can be used in the computation for \( P(y_3)_u \). If \( P(x_4)_u - P(\overline{x}_3)_u \leq 0 \), \( P(x_2)_l \) can be used in the computation for \( P(y_3)_u \).

For \( P(x_1) = [0.1, 0.3], P(x_2) = [0.7, 0.9], P(x_3) = [0.4, 0.6], \) and \( P(x_4) = [0.6, 0.8] \),

\[
P(y_3) = [0.5512, 0.8124]
\]

**6 EXCLUSIVE-OR FUNCTIONS**

Many forms of Boolean expressions that do not meet any unateness criteria can be processed almost as efficiently as above. In order to demonstrate this, we will address the “exclusive-or” function (satisfied if an odd number of inputs are satisfied) and its inverse (satisfied if an even number of inputs are satisfied). These have in many respects characteristics completely opposite to unateness. Since these functions are linear and associative, they can be computed iteratively, which simplifies algorithmic implementation. A general result [2] will help illustrate these concepts.

For an exclusive-or of \( n \) Boolean variables (or Boolean functions), consider two of the variables (or functions), \( w \) and \( z \), where \( w \) and \( z \) can each represent either some \( x_i \) or \( \overline{x}_i \). All nine combinations can be treated logically:

**Example 4.** Consider the following Boolean function:

\[
f_4 = v_1 v_2 v_3 v_4 \cup v_1 v_2 v_3 v_5 \cup v_1 v_2 v_4 v_5 \cup v_1 v_2 v_3 v_5 \cup v_1 v_2 v_4 v_5
\]

\[
\cup v_1 v_2 v_3 v_4 \cup v_1 v_2 v_3 v_5 \cup v_1 v_2 v_4 v_5
\]

\[
(26)
\]
where \( x_1 = [0.6, 0.7] \), \( x_2 = [0.4, 0.6] \), \( x_3 = [0.3, 0.5] \), \( x_4 = [0.5, 0.6] \), and \( x_5 = [0.1, 0.3] \). The probability is:

\[
y_4 = x_1 \bar{x}_2 x_3 x_4 + x_1 \bar{x}_2 \bar{x}_4 \bar{x}_5
+ \bar{x}_1 x_2 x_3 x_4 + \bar{x}_1 x_2 \bar{x}_4 \bar{x}_5
\]

(27)

When the Boolean function is simplified to:

\[
f_4 = v_1 \bar{v}_2 v_3 v_4 + v_1 \bar{v}_2 \bar{v}_4 \bar{v}_5
+ \bar{v}_1 v_2 v_3 v_4 \cup \bar{v}_1 v_2 \bar{v}_4 \bar{v}_5
\]

(28)

it is apparent that \( v_3 \) is positive unate, and \( v_4 \) is negative unate. When the methodology of “compacting” is applied to derive \((\oplus \text{ indicates exclusive-or})\):

\[
f_4 = (v_1 \oplus v_2) v_3 v_4 + (v_1 \oplus v_2) \bar{v}_4 \bar{v}_5
\]

(29)

it can be seen that \( v_1 \oplus v_2 \) serves the role of an independent unate subfunction. Using the information in the theorems, we can now determine that \( y_1 \) is a function of \( x_{1u}, x_{2u}, x_{3l} \), and \( x_{5i} \); and \( y_u \) is a function of \( x_{1u}, x_{2l}, x_{3u} \), and \( x_{5l} \). This leaves only the functionality of the bounds of \( x_4 \) to determine. Taking the partial derivative:

\[
\frac{\partial y_4}{\partial x_4} = (x_1 \oplus x_2) (x_3 - \bar{x}_3)
\]

(30)

Since this derivative is everywhere negative, \( y_1 \) is a function of \( x_{4u} \), and \( y_u \) is a function of \( x_{4l} \). The final result is: \( y_4 = [0.2116, 0.378] \).

7 CONCLUDING REMARKS

The methodology outlined in this paper is easily implemented in software, as has been partially done in the PHASER and COSMET fuzzy mathematics routines, and the results obtained are far superior to unconstrained operations.

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7 REFERENCES