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Dispersive Water Waves in One and Two Dimensions

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Abstract

This is the final report of a three-year, Laboratory-Directed Research and Development (LDRD) project at the Los Alamos National Laboratory (LANL). We derived and analyzed new shallow water equations for one-dimensional flows near the critical Froude number as well as related integrable systems of evolutionary nonlinear partial differential equations in one spatial dimension, while developing new directions for the mathematics underlying the integrability of these systems. In particular, we applied the spectrum generating equation method to create and study new integrable systems of nonlinear partial differential equations related to our integrable shallow water equations. We also investigated the solutions of these systems of equations on a periodic spatial domain by using methods from the complex algebraic geometry of Riemann surfaces. We developed certain aspects of the required mathematical tools in the course of this investigation, such as inverse scattering with degenerate potentials, asymptotic reduction of the angle representations, geometric singular perturbation theory, modulation theory and singularity tracking for completely integrable equations. We also studied equations that admit weak solutions, i.e., solutions with discontinuous derivatives in the form of corners or cusps, even though they are solutions of integrable models, a property that is often incorrectly assumed to imply smooth solution behavior. In related work, we derived new shallow water equations in two dimensions for an incompressible fluid with a free surface that is moving under the force of gravity. These equations provide an estimate of the long-time asymptotic effects of slowly varying bottom topography and weak hydrostatic imbalance on the vertically averaged horizontal velocity, and they describe the flow regime in which the Froude number is small -- much smaller even than the small aspect ratio of the shallow domain.

1. Background and Research Objectives

New machine architectures and recent algorithms that numerically compute the time evolution of fluid flows enable today's scientists to simulate events more accurately and more rapidly than ever before. An outstanding example of this is the large-scale, long-time, three-dimensional simulations of global ocean dynamics currently being planned and executed at Los Alamos on the Connection Machine. Beyond the problems of simulation, however, lie the problems of understanding a fluid dynamical system and choosing correct equations to model a given physical situation. Many fluid systems remain too intricate to fully understand, but modern methods of mathematical analysis and approximation theory can sometimes offer insight. Some of this insight is obtained by viewing fluid dynamics in a Hamiltonian

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framework, and in fact several recent advances in fluid dynamics share this perspective. Much of the value of these techniques lies in their applications, and these exist in a broad range of disciplines.

In our work we used multiple-time-scale perturbation theory to derive new model equations for the long-time asymptotic behavior of dispersive shallow water flow over a varying bottom in one and two dimensions. In deriving these new equations we implemented an asymptotic expansion method that preserves the property of being Hamiltonian at each level of approximation. At the same time, this method preserves the (horizontal) particle-relabeling symmetry inherent in the Eulerian description of fluid dynamics. The Hamiltonian structure of the resulting equations was used to analyze their solutions for conservation laws, stability properties, as well as sensitivity to parameters, boundary conditions, and perturbations.

2. Importance to LANL's Science and Technology Base and National R&D Needs

Explicit equations that capture the long-time behavior of fluid flow in shallow domains are important for LANL's science and technology base in assessing the predictability of ocean dynamics in this fluid regime. The model equations we derived captured the long-time behavior of the original barotropic (vertically integrated) ocean dynamics equations, in one and two horizontal dimensions. For these model equations, we addressed questions that may be unanswerable for the full geophysical systems being numerically simulated. Such questions are the following: How do the small scales affect the large ones? What is a sensible level of resolution needed to capture the long-time behavior, after transients have died out? What are the effects of ignoring changes in elevation of the sea surface on the long-time behavior? What are the effects of driving and damping on the solutions of these model equations? Does better physics (e.g., including nonhydrostatic pressure contributions) lead to greater sensitivity of the model to input parameters?

3. Scientific Approach and Accomplishments

Our starting point in making structure-preserving, long-time asymptotic approximations was the three-dimensional Euler equations for a stratified incompressible fluid with a free surface moving under the influence of gravity and an external pressure. (This is also the starting point for the full ocean-dynamics simulations.) Our Hamiltonian approach produced new approximate fluid equations for fluid motion in thin domains and provided a framework for analyzing these equations in (1) determining their conservation laws, (2) studying both the linear and nonlinear stability of their equilibrium solutions, and (3) establishing the time-asymptotic behavior of their nonequilibrium solutions.

Our approach unified previously known approximate theories for shallow water flow and allowed comparisons among them, by placing them into the same Hamiltonian framework. This framework identifies the mathematical properties shared by these theories, while it distinguishes amongst them by the way they each approximate the kinetic energy of vertical motion in the Hamiltonian. (In particular, the standard shallow water theory ignores this energy; the classical Boussinesq theory approximates it by the surface vertical motion; and the recent Green-Naghdi theory obtains it by vertical integration of a solution ansatz). That is, the several previously known approximate theories differ in their Hamiltonians in how they each model the kinetic energy due to vertical motion. The differences among the Hamiltonians result in differences in their corresponding equations, in how they describe wave dispersion and what conservation laws they possess.

These various theories involve non-dominant asymptotics, meaning that they retain terms of different asymptotic order in the same equation. The dominant asymptotics we performed showed that solutions of this entire family of non-dominant equations have the *same* long-time asymptotic behavior, which is described by our unique dominant-asymptotic equation. Thus, we discovered the structural unity of these non-dominant theories and showed that our dominant-asymptotic equations possess a unique universal solution behavior, towards which all of these non-dominant theories approach at long times.

We derived our dominant-asymptotic equations for two-dimensional, long-time, shallow-water dynamics by a multiple time-scale analysis. This analysis led to dominant-asymptotic equations that, remarkably, do possess a Hamiltonian structure that survived the multiple-time-scale approximation process. We then determined stability conditions for equilibrium solutions and analyzed particular vortex solutions for sensitivity to model parameters.

Our Hamiltonian approach showed its power especially well in the one-dimensional asymptotics, by producing a new dispersive shallow water equation with many remarkable properties. For example, in one dimension our multiple time scale equations possess *two* Hamiltonian structures. This biHamiltonian property led to a recursion operator that produced an infinite number of conservation laws which led, in turn, to the exact analytical solution of the initial value problem for the one-dimensional case. So, our new equation has "soliton" solutions and is completely integrable as a Hamiltonian system. The framework for analyzing this equation involved one of the most beautiful mathematical developments in this century -- the theory of integrable, nonlinear, partial differential equations. The "phase space" for our solitons is the space of generalized functions with bounded derivative norm. The phenomenology of these solutions is very rich and includes formation of weak solutions: shocks that form in finite time, then develop into a train of solitons, each of which has a finite jump in derivative at its peak. Subsequently these peaked solitons may collide, and when they do, they pass through each other in a fully nonlinear interaction, from which they emerge again as coherent entities. Because they are weak solutions, our investigation of these peaked solitons required new advances in the mathematics of completely integrable Hamiltonian systems.

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