SUSPNDRS: A Numerical Simulation Tool for the Nonlinear Transient Analysis of Cable Supported Bridge Structures

Part I: Theoretical Development

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Oakland-San Francisco Bay Bridge Western Crossing

June 1997

This work was performed for the *Advanced Earthquake Hazards Research Project*, a Campus-Laboratory Collaboration between the Lawrence Livermore National Laboratory and the University of California. Funding for this study was provided by the University of California Directed Research and Development Fund.

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ABSTRACT

Classical suspension bridges and modern cable-stayed bridges dominate the class of bridges which are often referred to simply as "long-span bridges". Long-span bridges constitute some of the largest, architecturally pleasing and essential structures for human activity. The present day worth of many of these structures is enormous, for example consider the cost for replacement or the annual cost-benefit to tourism of San Francisco's Golden Gate Bridge. It is essential that these structures be adequately protected from attack by natural forces.

There are numerous long-span bridges in the United States and throughout the world which were designed and built prior to the development of computer based analysis procedures and modern structural design codes and standards. It is widely recognized that many of these bridges have serious structural deficiencies when considering extreme environmental loads such as seismic excitation from major earthquakes. Recent earthquakes have brought to the forefront the seismic deficiencies of many older bridge structures and have demonstrated the potential for major loss of life and adverse economic impact, and many state, local and federal transportation entities are currently evaluating the feasibility of seismic upgrades for major bridges under their jurisdictions.

Construction details of many older bridges lead to significant numerical modeling challenges. For example, to accommodate thermal expansion and contraction, the hanging decks of suspension bridges are often weakly coupled to the bridge towers, leading to the possibility of sudden impact between the suspended deck system and the main supporting towers when the deck system sways during transient earthquake response. Another example of an area which is often problematic in older suspension bridges is the main towers and their support system where rocking and up-lift of the tower bases or rocking of the supporting caissons may lead to significant nonlinear response behavior. Adding to the complexity of these types of local nonlinearities is the fact that suspension structures are inherently nonlinear in a global sense due to the geometric dependence of the system stiffness on the global geometry and the displaced shape of the suspension cable systems.

Cable stayed bridges represent a relatively new structural configuration and have enjoyed widespread acceptance and application over the past few years. Similar to suspension structures, cable-stayed bridges rely on the tensile load carrying capacity of a tensioned

cable system to provide support and stability of the vehicle roadway, and many of the issues prevalent in modeling suspension bridges, i.e. the importance of geometry change and geometric nonlinearities, are relevant in cable-stayed bridge modeling.

Even with modern computational capabilities the analysis of a major suspension or cable-stayed bridge is a daunting task. Aside from the fact that the sheer size of these structures results in large, nonlinear computational models, there are many fundamental loading issues which are not currently well understood. For example, the length of these structures leads to the potential for significantly different seismic input motions at the support towers, and the potential for spatial variability of seismic motions has not been well quantified nor is its effect on long-span structures completely understood.

The work reported on herein was aimed at developing methodologies and tools for efficient and accurate numerical simulation of the seismic response of suspension and cable-stayed structures. A special purpose finite element program has been constructed and the underlying theory and demonstration example problems are presented. A companion report [Ref 1] discusses the application of this technology for a major suspension bridge structure.

1.0 Background

Engineers and Earth Scientists from the Lawrence Livermore National Laboratory, the University of California at Berkeley and the University of California at Los Angeles have combined efforts in a UC funded Campus-Laboratory-Collaboration (CLC) research project aimed at developing a better understanding of the physical processes associated with the generation of earthquake ground motions and the response of important major structures to large earthquakes.

One component of the *Advanced Earthquake Hazard Research* project includes a seismic response study of the western span of the San Francisco-Oakland Bay Bridge. The Bay Bridge was selected for study because it embodies many of the complicating issues which are currently of great interest to engineers evaluating the response of long-span bridges. These issues include the effect of spatially varying ground motion and the degree to which system global and local nonlinearities influence the system response. Another motivation for investigating the Bay Bridge western span is the fact that this structure is critical to the transportation infrastructure and economy of the San Francisco Bay Area. The hardships caused by partial collapse of a segment of the Eastern Bay Bridge span during the 1989 Loma Prieta earthquake have driven home the necessity of seismic integrity for critical transportation systems. The California Department of Transportation (Caltrans) is currently developing a seismic retrofit strategy for the western span suspension structures, and the Caltrans' retrofit study and retrofit construction will be performed over the next three to five years. In light of this, a corollary reason for studying this particular structure is that any information generated on the earthquake response can provide timely input for Caltrans' retrofit design effort and thereby allow UC to make a contribution to California's ongoing seismic hazard mitigation efforts.

The Bay Bridge western crossing consists of classic steel suspension structures which were completed in November 1936. The twin suspension structures of the western crossing are shown in Figure 1 and the entire length of the structure is approximately 3.14 Km (1.95 miles). Both anchorages and the supporting piers extend vertically through soil layers and are founded on Franciscan sandstone as shown in Figure 1.

The work reported on herein is one component of the Bay Bridge study, and provides an overview of the development and implementation of a nonlinear finite element program created specifically for the analysis of long-span suspension bridges. Heretofore, computer analyses of this class of structure have been performed sparingly because of the computationally intensive nature of the numerical simulations. The motivation for the current work was to provide a problem specific, tailored analysis tool with enough computational efficiency to allow extensive parametric evaluations of the response of long-span suspension bridges.

The basic element technologies employed in the program are shown in Figure 2. The focus of the current report is on the development of the model of the bridge superstructure system (i.e. the towers, cable system and deck system), and the theoretical development of the various elements is outlined herein along with example problems which illustrate the

FIGURE 1. Profile and dimensions of the western crossing of the Oakland-San Francisco Bay Bridge

application and utility of the modeling methodology. A companion report (Ref 1), pro-

FIGURE 2. Element technologies employed in the suspension bridge model

vides the details of application of this technology for an actual bridge structure.

2.0 Global nonlinear solution framework

The foundation of any nonlinear analysis tool is the nonlinear solution algorithm utilized to solve the nonlinear equations of equilibrium. The global nonlinear solution framework employed in the SUSPNDRS program was developed to allow accommodation of both geometric and material nonlinearities in the characterization of the bridge system. The procedure followed is based on a Newton-Raphson technique with a number of special procedures to enhance the stability and convergence of the nonlinear equilibrium iterations. The basic approach follows that outlined in McCallen and Romstad [Ref 2, Ref 3, Ref 4].

To simply illustrate the development of the nonlinear solution algorithm a static analysis problem will be addressed. The extension to dynamic analysis is relatively straight forward and consists of insertion of a time stepping loop. In a computational structural model, the deformation of the structure is defined by the active global displacement components $\{D\} \in R^n$ where n = the number of active degrees of freedom required to adequately characterize the deformations of the structure. For a given set of external loads on the structure, $\{P\} \in R^n$, an equilibrium configuration of the structure is achieved if the external load component at each degree of freedom i , denoted P_i , is in equilibrium with the sum of the internal resisting forces of the structure in the direction of degree of freedom i , denoted Q_i , and any nodal force in the direction of degree of freedom i which is generated by contact of disjoint model parts, denoted C_i , for all active degrees of freedom.

The contact forces can simply be treated as an additional nodal load vector in the same manner as the externally applied loads $\{P\}$. The internal resisting forces $\{Q\}$ are the result of deformations of the model elements and represent the generalized nodal forces for the elements.

In a nonlinear problem the internal resisting forces and the contact forces are nonlinear functions of the global displacements. If a residual vector $\{R\langle\{D\}\rangle\} \in R^n$ is defined as the difference between all of the various force components in the direction of each degree of freedom,

$$\{R\langle\{D\}\rangle\} = \{\{Q\langle\{D\}\rangle\} - \{P\} - \{C\langle\{D\}\rangle\}\} \quad (\text{EQ } 1)$$

then an equilibrium configuration of the structure $\{D^\star\}$ results in a null residual vector, i.e.

$$\{R\langle\{D^\star\}\rangle\} = \{0\} \quad (\text{EQ } 2)$$

If $\{D^k\}$ is the k th approximation of $\{D^\star\}$ then a Taylor series expansion of the residual vector about $\{D^k\}$ yields

$$\{R\langle\{D^\star\}\rangle\} = \{R\langle\{D^k\}\rangle\} + \left[\frac{\partial\{R\langle\{D^k\}\rangle\}}{\partial D} \right] \{\Delta D\} + o(\{\Delta D\}^2) \quad (\text{EQ 3})$$

where,

$$\left[\frac{\partial R}{\partial D} \right] = \begin{bmatrix} \left(\frac{\partial R_1}{\partial D_1} \right) & \cdots & \left(\frac{\partial R_1}{\partial D_n} \right) \\ \cdots & \cdots & \cdots \\ \left(\frac{\partial R_n}{\partial D_1} \right) & \cdots & \left(\frac{\partial R_n}{\partial D_n} \right) \end{bmatrix} \quad (\text{EQ 4})$$

and from EQ. 1,

$$\left[\frac{\partial R}{\partial D} \right] = \left[\frac{\partial Q}{\partial D} \right] - \left[\frac{\partial C}{\partial D} \right] \quad (\text{EQ 5})$$

$$= \begin{bmatrix} \left(\frac{\partial Q_1}{\partial D_1} \right) & \cdots & \left(\frac{\partial Q_1}{\partial D_n} \right) \\ \cdots & \cdots & \cdots \\ \left(\frac{\partial Q_n}{\partial D_1} \right) & \cdots & \left(\frac{\partial Q_n}{\partial D_n} \right) \end{bmatrix} - \begin{bmatrix} \left(\frac{\partial C_1}{\partial D_1} \right) & \cdots & \left(\frac{\partial C_1}{\partial D_n} \right) \\ \cdots & \cdots & \cdots \\ \left(\frac{\partial C_n}{\partial D_1} \right) & \cdots & \left(\frac{\partial C_n}{\partial D_n} \right) \end{bmatrix} \quad (\text{EQ 6})$$

Disarding the higher order terms in EQ. 3 and invoking the fact that $\{D^\star\} = \{0\}$, the incremental displacements are given by

$$\left[\frac{\partial\{R\langle\{D^k\}\rangle\}}{\partial D} \right] \{\Delta D\} = -\{R\langle\{D^k\}\rangle\} \quad (\text{EQ 7})$$

The instantaneous stiffness matrix of the structure is defined as the instantaneous rate of change of the internal resisting forces and contact forces with respect to displacements, thus

$$[K_I\langle\{D^k\}\rangle] = \left[\frac{\partial\{R\langle\{D^k\}\rangle\}}{\partial D} \right] \quad (\text{EQ } 8)$$

The individual terms of the instantaneous stiffness matrix are given by EQ. 6.

EQ. 8 provides the basis for the Newton-Raphson solution algorithm in which the external loads are applied incrementally in a load step, and within the given load step equilibrium iterations are performed to drive the residual to zero (or more accurately, to an acceptably small value). Quasi-Newton methods can also be readily incorporated within this framework and these methods will be investigated in the full scale bridge study (Ref 1).

3.0 Local-global coordinate transformation for a one dimensional element

In order to develop the system coordinate transformations for a one dimensional element with a convected local coordinate system, the direction cosines and transformation matrices must be derived. The transformation matrices, the components of which are the direction cosines of the axes, define the transformation of vector quantities between coordinate systems. Consider the one dimensional element shown in Figure 3.

FIGURE 3. Global and local coordinate systems

Axis x' is defined in terms of the unit vector $\hat{i}'$,

$$\hat{i}' = l_x \hat{i} + m_x \hat{j} + n_x \hat{k} \quad (\text{EQ 9})$$

where,

$$l_x = \frac{x_J - x_I}{\sqrt{(x_J - x_I)^2 + (y_J - y_I)^2 + (z_J - z_I)^2}} \quad (\text{EQ 10})$$

$$m_x = \frac{y_J - y_I}{\sqrt{(x_J - x_I)^2 + (y_J - y_I)^2 + (z_J - z_I)^2}} \quad (\text{EQ 11})$$

$$n_x = \frac{z_J - z_I}{\sqrt{(x_J - x_I)^2 + (y_J - y_I)^2 + (z_J - z_I)^2}} \quad (\text{EQ 12})$$

Based on vectors from I to J and from I to K, the unit vector in the z' direction can be obtained,

$$\vec{IK} = (x_K - x_I)\hat{i} + (y_K - y_I)\hat{j} + (z_K - z_I)\hat{k} \quad (\text{EQ 13})$$

$$\vec{IJ} = (x_J - x_I)\hat{i} + (y_J - y_I)\hat{j} + (z_J - z_I)\hat{k} \quad (\text{EQ 14})$$

the cross product of IK and IJ is given by,

$$\vec{IJ} \otimes \vec{IK} = (x_J - x_I)\hat{i} + (y_J - y_I)\hat{j} + (z_J - z_I)\hat{k} \otimes (x_K - x_I)\hat{i} + (y_K - y_I)\hat{j} + (z_K - z_I)\hat{k} \quad (\text{EQ 15})$$

thus,

$$\begin{aligned} \vec{IJ} \otimes \vec{IK} = & [(y_J - y_I)(z_K - z_I) - (z_J - z_I)(y_K - y_I)]\hat{i} + \\ & [-(x_J - x_I)(z_K - z_I) + (z_J - z_I)(x_K - x_I)]\hat{j} + \\ & [(x_J - x_I)(y_K - y_I) - (y_J - y_I)(x_K - x_I)]\hat{k} \end{aligned} \quad (\text{EQ 16})$$

Now, unitize the product to get the unit vector in the z' direction, if

$$\vec{IJ} \otimes \vec{IK} = C_1\hat{i} + C_2\hat{j} + C_3\hat{k} \quad (\text{EQ 17})$$

then

$$\hat{k}' = \frac{C_1}{\sqrt{C_1^2 + C_2^2 + C_3^2}}\hat{i} + \frac{C_2}{\sqrt{C_1^2 + C_2^2 + C_3^2}}\hat{j} + \frac{C_3}{\sqrt{C_1^2 + C_2^2 + C_3^2}}\hat{k} \quad (\text{EQ 18})$$

or

$$\hat{k}' = l_z\hat{i} + m_z\hat{j} + n_z\hat{k} \quad (\text{EQ 19})$$

where

$$l_z = \frac{C_1}{\sqrt{C_1^2 + C_2^2 + C_3^2}} \quad (\text{EQ 20})$$

$$m_z = \frac{C_2}{\sqrt{C_1^2 + C_2^2 + C_3^2}} \quad (\text{EQ 21})$$

$$n_z = \frac{C_3}{\sqrt{C_1^2 + C_2^2 + C_3^2}} \quad (\text{EQ 22})$$

To get the unit vector in the y' direction, construct the cross product $\hat{k}' \otimes \hat{i}$,

$$\begin{aligned} \hat{j}' = \hat{k}' \otimes \hat{i} = & l_z\hat{i} + m_z\hat{j} + n_z\hat{k} \otimes l_x\hat{i} + m_x\hat{j} + n_x\hat{k} = \\ & (m_zn_x - n_zm_x)\hat{i} + (-l_zn_x + n_zl_x)\hat{j} + (l_zm_x - m_zl_x)\hat{k} \end{aligned} \quad (\text{EQ 23})$$

or

$$\hat{j}' = l_y\hat{i} + m_y\hat{j} + n_y\hat{k} \quad (\text{EQ 24})$$

where

$$l_y = (m_z n_x - n_z m_x) \quad (\text{EQ 25})$$

$$m_y = (-l_z n_x + n_z l_x) \quad (\text{EQ 26})$$

$$n_y = (l_z m_x - m_z l_x) \quad (\text{EQ 27})$$

A vector quantity $\vec{D}$ can now be described in either coordinate system shown in Figure 4,

FIGURE 4. Vector quantity in global and local coordinates

given $\vec{D} = d_x \hat{i} + d_y \hat{j} + d_z \hat{k}$, then

$$d_x = \vec{D} \cdot \hat{i} \quad (\text{EQ 28})$$

thus,

$$d_x = (d_x \hat{i} + d_y \hat{j} + d_z \hat{k}) \cdot \hat{i} \quad (\text{EQ 29})$$

$$\therefore d_x = d_x (l_x \hat{i} + m_x \hat{j} + n_x \hat{k}) \cdot \hat{i} \quad (\text{EQ 30})$$

$$+ d_y (l_y \hat{i} + m_y \hat{j} + n_y \hat{k}) \cdot \hat{i}$$

$$+ d_z (l_z \hat{i} + m_z \hat{j} + n_z \hat{k}) \cdot \hat{i}$$

$$\therefore d_x = d_x l_x + d_y l_y + d_z l_z \quad (\text{EQ 31})$$

similarly,

$$d_y = d_x m_x + d_y m_y + d_z m_z \quad (\text{EQ 32})$$

$$d_z = d_x n_x + d_y n_y + d_z n_z \quad (\text{EQ 33})$$

In matrix form these relationships become

$$\begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = \begin{bmatrix} l_{x'} & l_{y'} & l_{z'} \\ m_{x'} & m_{y'} & m_{z'} \\ n_{x'} & n_{y'} & n_{z'} \end{bmatrix} \begin{bmatrix} d_{x'} \\ d_{y'} \\ d_{z'} \end{bmatrix} \quad (\text{EQ 34})$$

or

$$\{d\} = [T]^T \{d'\} \quad (\text{EQ 35})$$

The inverse transformation is given by,

$$\begin{aligned} d_{x'} &= \vec{D} \cdot \hat{i}' = (d_x \hat{i} + d_y \hat{j} + d_z \hat{k}) \cdot (l_{x'} \hat{i} + m_{x'} \hat{j} + n_{x'} \hat{k}) \\ &= d_x l_{x'} + d_y m_{x'} + d_z n_{x'} \end{aligned} \quad (\text{EQ 36})$$

similarly,

$$d_{y'} = d_x l_{y'} + d_y m_{y'} + d_z n_{y'} \quad (\text{EQ 37})$$

$$d_{z'} = d_x l_{z'} + d_y m_{z'} + d_z n_{z'} \quad (\text{EQ 38})$$

or in matrix form

$$\begin{bmatrix} d_{x'} \\ d_{y'} \\ d_{z'} \end{bmatrix} = \begin{bmatrix} l_{x'} & m_{x'} & n_{x'} \\ l_{y'} & m_{y'} & n_{y'} \\ l_{z'} & m_{z'} & n_{z'} \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} \quad (\text{EQ 39})$$

or

$$\{d'\} = [T] \{d\} \quad (\text{EQ 40})$$

EQ. 34 and EQ. 40 provide the transformation between global and local coordinates for any vector quantity. For example, these relationships provide the transformation between local and global coordinates for the end displacement and force vectors for a one dimensional truss element as shown in Figure 5.

FIGURE 5. Application of coordinate transformation relationships to a one dimensional truss member

4.0 Measurement of element deformations

The class of problems considered in the SUSPNDRS program includes problems in which global displacements and rotations can take on arbitrarily large values. However, the deformations of the elements are restricted to be small, and in the nonlinear solution algorithm the incremental displacements and rotations (i.e. the increments obtained from the current equilibrium iteration) are assumed to be small.

In large displacement problems, total rotations cannot be treated as vector quantities - i.e. the rotations will not generally obey the parallelogram laws of vector addition, and the addition of large rotations does not satisfy the commutative properties of vector addition. In planar problems, where only one rotation component is present, this does not create any particular difficulties. Element rotations are easily determined in an updated system by obtaining the difference between the global nodal rotation and the updated coordinate system orientation as shown in Figure 6. The element rotational deformation at node J is simply obtained from,

$$\gamma_j = (\theta_j - \alpha) \quad (\text{EQ 41})$$

where α is the angle measured from the global axis to the convected coordinate system.

FIGURE 6. Deformation measurement for a beam element undergoing planar displacement

It should be noted that the updated coordinate system to be utilized is not unique. Both updated coordinate systems shown in Figure 7 could be used remove rigid body translations and define element deformations for a beam undergoing planar deformation.

FIGURE 7. Alternatives for element updated coordinate systems

In three dimensional problems, the element rotational deformations are not immediately obtainable from the global rotations. The large global rotations are not vector quantities and cannot be resolved vectorially into element coordinates as in a linear, small displacement problem

4.0.1 Element and nodal coordinate systems

To allow for determination of element rotations in the SUSPNDRS beam element, two element nodal coordinate systems are established in addition to the element convected coordinate system (Figure 8). The element nodal coordinates are used to track the position and orientation of the element principal axes as the element deforms in space, and to allow determination of the element rotational deformations at the respective nodes of the element. In Figure 8, the element convected coordinate system is denoted by the x', y', z' axes, the nodal coordinate system at element node I is denoted by the x'', y'', z'' axes and the nodal coordinate system at element node J is denoted by the x''', y''', z''' axes.

The element nodal coordinate system at node I of the element is defined by the triad of unit vectors $\hat{i}'', \hat{j}'', \hat{k}''$, and the element nodal coordinate system at node J is defined by the unit vectors $\hat{i}''', \hat{j}''', \hat{k}'''$. The initial orientations of the nodal coordinate systems are defined by the locations of the element nodes I and J and the initial K node location (i.e. node K_0 in Figure 8).

The x'', y'', z'' coordinate system tracks with the principal axes of the beam element at node I as the beam deforms, and the x''', y''', z''' coordinate system tracks with the principal axes of the beam element at node J as the beam deforms. The incremental displacement and incremental rotation vectors in global coordinates, determined from an

FIGURE 8. Convected element coordinate system, element nodal coordinates and element deformations

incremental equation solution in the finite element analysis, for node I in global coordinates are given by

$$\overrightarrow{\Delta D_I} = \Delta d_{x_I} \hat{i} + \Delta d_{y_I} \hat{j} + \Delta d_{z_I} \hat{k} \quad (\text{EQ 42})$$

$$\overrightarrow{\Delta \theta_I} = \Delta \theta_{x_I} \hat{i} + \Delta \theta_{y_I} \hat{j} + \Delta \theta_{z_I} \hat{k} \quad (\text{EQ 43})$$

and the incremental displacement and incremental rotations at node J are given by

$$\overrightarrow{\Delta D_J} = \Delta d_{x_J} \hat{i} + \Delta d_{y_J} \hat{j} + \Delta d_{z_J} \hat{k} \quad (\text{EQ 44})$$

$$\overrightarrow{\Delta \theta_J} = \Delta \theta_{x_J} \hat{i} + \Delta \theta_{y_J} \hat{j} + \Delta \theta_{z_J} \hat{k} \quad (\text{EQ 45})$$

In order to determine the element deformations after incremental displacements take place, it is first necessary to determine the new location and orientation of the nodal coordinate systems after the incremental displacements occur. The new locations of nodes I and J are immediately found from the incremental displacements of element node I and element node J. Thus

$$x_I^{k+1} = x_I^k + \Delta d_{x_I}^k \quad (\text{EQ 46})$$

$$y_I^{k+1} = y_I^k + \Delta d_{y_I}^k \quad (\text{EQ 47})$$

$$z_I^{k+1} = z_I^k + \Delta d_{z_I}^k \quad (\text{EQ 48})$$

$$x_J^{k+1} = x_J^k + \Delta d_{x_J}^k \quad (\text{EQ 49})$$

$$y_J^{k+1} = y_J^k + \Delta d_{y_J}^k \quad (\text{EQ 50})$$

$$z_J^{k+1} = z_J^k + \Delta d_{z_J}^k \quad (\text{EQ 51})$$

The new orientation of the nodal coordinate systems can be obtained from the incremental nodal rotations. Consider the nodal coordinate system at node I, i.e. the x'' , y'' , z'' system, as the beam element deforms from iteration k to iteration $k+1$ (Figure 9). Assuming the incremental rotations are small and can therefore be transformed vectorially, the incremental rotations at node I can be transformed into the element nodal coordinate system (see Figure 9).

$$\begin{bmatrix} \Delta \theta_{x''}^k \\ \Delta \theta_{y''}^k \\ \Delta \theta_{z''}^k \end{bmatrix} = \begin{bmatrix} l_{x''}^k & m_{x''}^k & n_{x''}^k \\ l_{y''}^k & m_{y''}^k & n_{y''}^k \\ l_{z''}^k & m_{z''}^k & n_{z''}^k \end{bmatrix} \begin{bmatrix} \Delta \theta_{x_I} \\ \Delta \theta_{y_I} \\ \Delta \theta_{z_I} \end{bmatrix} \quad (\text{EQ 52})$$

The current direction cosines of the x'' , y'' , z'' axes provide the unit vector components for the nodal coordinate system,

FIGURE 9. Incremental rotations of nodal coordinate system at node I from iteration k to iteration k+1

$$\hat{i}_k'' = l_{x''}^k \hat{i} + m_{x''}^k \hat{j} + n_{x''}^k \hat{k} \quad (\text{EQ 53})$$

$$\hat{j}_k'' = l_{y''}^k \hat{i} + m_{y''}^k \hat{j} + n_{y''}^k \hat{k} \quad (\text{EQ 54})$$

$$\hat{k}_k'' = l_{z''}^k \hat{i} + m_{z''}^k \hat{j} + n_{z''}^k \hat{k} \quad (\text{EQ 55})$$

The magnitude of the cross product of the unit vectors in the x direction is given by

$$|\hat{i}_k'' \otimes \hat{i}_{k+1}''| = |\hat{i}_k''| \cdot |\hat{i}_{k+1}''| \cdot \sin \gamma_i \quad (\text{EQ 56})$$

For small rotations $\sin \gamma_i \approx \gamma_i$ and the cross product is given by

$$|\hat{i}_k'' \otimes \hat{i}_{k+1}''| = \gamma_i \quad (\text{EQ 57})$$

Thus for small rotations the cross product defines the rotation about the direction defined by the cross product $\hat{i}_k'' \otimes \hat{i}_{k+1}''$. Based on small rotation assumptions, the rotation about the cross product direction can be written as the vector sum of the incremental nodal coordinate rotations (note that the cross product must always lie in the plane defined by the $\hat{j}''$ and $\hat{k}''$ vectors because the cross product vector must be orthogonal to the $\hat{i}''$ vector),

$$\hat{i}_k'' \otimes \hat{i}_{k+1}'' = \Delta \theta_{y''}^k \hat{j}_k'' + \Delta \theta_{z''}^k \hat{k}_k'' \quad (\text{EQ 58})$$

In the x'' , y'' , z'' coordinate system, let the vector $\hat{i}_{k+1}''$ be denoted by,

$$\hat{i}_{k+1}'' = \phi_1 \hat{i}_k'' + \phi_2 \hat{j}_k'' + \phi_3 \hat{k}_k'' \quad (\text{EQ 59})$$

Substituting EQ. 59 into EQ. 58 yields,

$$\hat{i}_k'' \otimes \phi_1 \hat{i}_k'' + \phi_2 \hat{j}_k'' + \phi_3 \hat{k}_k'' = \Delta\theta_{y''}^k \hat{j}_k'' + \Delta\theta_{z''}^k \hat{k}_k'' \quad (\text{EQ 60})$$

or

$$\phi_2 \hat{k}_k'' - \phi_3 \hat{j}_k'' = \Delta\theta_{y''}^k \hat{j}_k'' + \Delta\theta_{z''}^k \hat{k}_k'' \quad (\text{EQ 61})$$

thus

$$\phi_2 = \Delta\theta_{z''}^k \quad (\text{EQ 62})$$

$$\phi_3 = -\Delta\theta_{y''}^k \quad (\text{EQ 63})$$

The third component of the unit vector can be found based on the fact that the vector has magnitude of unity, i.e.,

$$\phi_1 = \sqrt{1 - (\phi_2)^2 - (\phi_3)^2} \quad (\text{EQ 64})$$

The sign of ϕ_1 is always positive since $\hat{i}_{k+1}''$ is nearly in the x'' direction for any small angular deformations.

The updated direction cosines of the $\hat{i}''$ vector in global coordinates are finally given by,

$$\begin{bmatrix} l_{x''}^{k+1} \\ m_{x''}^{k+1} \\ n_{x''}^{k+1} \end{bmatrix} = \begin{bmatrix} l_{x''}^k & l_{y''}^k & l_{z''}^k \\ m_{x''}^k & m_{y''}^k & m_{z''}^k \\ n_{x''}^k & n_{y''}^k & n_{z''}^k \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \quad (\text{EQ 65})$$

Similarly,

$$|\hat{j}_k'' \otimes \hat{j}_{k+1}''| = \gamma_j \quad (\text{EQ 66})$$

and

$$\hat{j}_k'' \otimes \hat{j}_{k+1}'' = \Delta\theta_{x''}^k \hat{i}_k'' + \Delta\theta_{z''}^k \hat{k}_k'' \quad (\text{EQ 67})$$

In the x'' , y'' , z'' coordinate system let the vector $\hat{j}_{k+1}''$ be denoted by

$$\hat{j}_{k+1}'' = \lambda_1 \hat{i}_k'' + \lambda_2 \hat{j}_k'' + \lambda_3 \hat{k}_k'' \quad (\text{EQ 68})$$

Substituting EQ. 68 into EQ. 67 yields,

$$\hat{j}_k'' \otimes \lambda_1 \hat{i}_k'' + \lambda_2 \hat{j}_k'' + \lambda_3 \hat{k}_k'' = \Delta\theta_{x''}^k \hat{i}_k'' + \Delta\theta_{z''}^k \hat{k}_k'' \quad (\text{EQ 69})$$

or

$$-\lambda_1 \hat{k}_k'' + \lambda_3 \hat{i}_k'' = \Delta\theta_{x''}^k \hat{i}_k'' + \Delta\theta_{z''}^k \hat{k}_k'' \quad (\text{EQ 70})$$

thus,

$$\lambda_1 = -\Delta\theta_{z''} \quad (\text{EQ 71})$$

$$\lambda_3 = \Delta\theta_{x''} \quad (\text{EQ 72})$$

$$\lambda_2 = \sqrt{1 - (\lambda_1)^2 - (\lambda_3)^2} \quad (\text{EQ 73})$$

The updated global coordinates of the $\hat{j}''$ vector is finally given by

$$\begin{bmatrix} l_{y''}^{k+1} \\ m_{y''}^{k+1} \\ n_{y''}^{k+1} \end{bmatrix} = \begin{bmatrix} l_{x''}^k & l_{y''}^k & l_{z''}^k \\ m_{x''}^k & m_{y''}^k & m_{z''}^k \\ n_{x''}^k & n_{y''}^k & n_{z''}^k \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \quad (\text{EQ 74})$$

Similarly,

$$|\hat{k}_k'' \otimes \hat{k}_{k+1}''| = \gamma_k \quad (\text{EQ 75})$$

and

$$\hat{k}_k'' \otimes \hat{k}_{k+1}'' = \Delta\theta_{x''}^k \hat{i}_k'' + \Delta\theta_{y''}^k \hat{j}_k'' \quad (\text{EQ 76})$$

In the x'' , y'' , z'' coordinate system let the vector $\hat{k}_{k+1}''$ be denoted by

$$\hat{k}_{k+1}'' = \mu_1 \hat{i}_k'' + \mu_2 \hat{j}_k'' + \mu_3 \hat{k}_k'' \quad (\text{EQ 77})$$

Substituting EQ. 77 into EQ. 76 yields,

$$\hat{k}_k'' \otimes \mu_1 \hat{i}_k'' + \mu_2 \hat{j}_k'' + \mu_3 \hat{k}_k'' = \Delta\theta_{x''}^k \hat{i}_k'' + \Delta\theta_{y''}^k \hat{j}_k'' \quad (\text{EQ 78})$$

or

$$\mu_1 \hat{j}_k'' - \mu_2 \hat{i}_k'' = \Delta\theta_{x''}^k \hat{i}_k'' + \Delta\theta_{y''}^k \hat{j}_k'' \quad (\text{EQ 79})$$

thus,

$$\mu_1 = \Delta\theta_{y''} \quad (\text{EQ 80})$$

$$\mu_2 = -\Delta\theta_{x''} \quad (\text{EQ 81})$$

$$\mu_3 = \sqrt{1 - (\mu_1)^2 - (\mu_2)^2} \quad (\text{EQ 82})$$

The updated global coordinates of the $\hat{k}''$ vector is finally given by

$$\begin{bmatrix} l_{z''}^{k+1} \\ m_{z''}^{k+1} \\ n_{z''}^{k+1} \end{bmatrix} = \begin{bmatrix} l_{x''}^k & l_{y''}^k & l_{z''}^k \\ m_{x''}^k & m_{y''}^k & m_{z''}^k \\ n_{x''}^k & n_{y''}^k & n_{z''}^k \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \quad (\text{EQ 83})$$

The element convected coordinate system defined by the x' , y' , z' coordinate axes must also be updated for the new element configuration. Based on the new locations of the element I and J nodes, as given by EQ. 46 to EQ. 51, the direction cosines of the x' axes are immediately found as outlined in Section 3.0. The direction cosines of the x' axis are given by the formulas of EQ. 10 to EQ. 12. There is some flexibility in the definition of the element y' and z' axes as discussed for a planar problem in Section 3 (Figure 7). These axes are used to measure the amount of rotational deformation at each end of the element (note as shown in Figure 8, the displacement at each node will be zero since the element nodal and convected coordinate systems have the same origin). In light of the fact that the theory developed here is based on the assumption of small deformations, it is advantageous to define the convected coordinate system in a manner which minimizes the angular deformation measurement. The element y' axis is defined by a new K node location, where the K node is based on an average of the nodal $\hat{j}''$ and $\hat{j}'''$ vectors. Specifically, a directional vector $\hat{\eta}$ is defined where,

$$\hat{\eta} = xl \times \frac{(\hat{j}'' + \hat{j}''')}{|\hat{j}'' + \hat{j}'''|} \quad (\text{EQ 84})$$

and xl is the element length. The new K node position is established from element node I in the direction of vector $\hat{\eta}$ as indicated in Figure 10. Thus if the $\hat{\eta}$ vector is written,

$$\hat{\eta} = \eta_x \hat{i} + \eta_y \hat{j} + \eta_z \hat{k} \quad (\text{EQ 85})$$

the coordinates of the new K node are given by,

$$x_K = x_I + \eta_x \quad (\text{EQ 86})$$

$$y_K = y_I + \eta_y \quad (\text{EQ 87})$$

$$z_K = z_I + \eta_z \quad (\text{EQ 88})$$

The direction cosines of the element x' , y' , z' axes are then found exactly as outlined in section 3.0.

FIGURE 10. Establishment of a new element coordinate system

4.0.2 Element rotational deformations

Once the updated locations and orientations of the element convected and nodal coordinate systems are established, it is necessary to determine the element deformations as measured in the convected element coordinate system. The element deformation can be obtained by considering the small angles between the nodal coordinate systems and the convected element coordinate system. Consider node I and the x', y', z' and x'', y'', z'' coordinate systems as indicated in Figure 11.

The angular rotations which the x', y', z' axes must go through to arrive at the x'', y'', z'' axes are denoted by (Figure 11) $\theta_{x'_I}, \theta_{y'_I}, \theta_{z'_I}$. These rotations provide element rotational deformations for three dimensional flexure in the same manner as the angular deformation γ provides the element rotational deformation in the planar bending problem of Figure 6.

The magnitudes of these angular deformations can be obtained from the updated locations of the x', y', z' and x'', y'' and z'' axes. The magnitude of the cross product of the unit vectors in the x' and x'' directions is given by,

$$|\hat{i}' \otimes \hat{i}''| = |\hat{i}'| \cdot |\hat{i}''| \cdot \sin \alpha_i \quad (\text{EQ 89})$$

For small angular deformations $\sin \alpha_i \approx \alpha_i$ and the magnitude of the cross product is given by,

$$|\hat{i}' \otimes \hat{i}''| = \alpha_i \quad (\text{EQ 90})$$

FIGURE 11. Determination of angular deformation between the x', y', z' and the x'', y'', z'' coordinate axes

The cross product vector lies in the plane defined by the $\hat{j}', \hat{k}'$ vectors and provides the rotation about the axis defined by the $\hat{i}' \otimes \hat{i}''$ direction. For small angular deformations, this rotation can be written as the vector sum of the rotations about the y' and z' axes, i.e.

$$\hat{i}' \otimes \hat{i}'' = \theta_{y'} \hat{j}' + \theta_{z'} \hat{k}' \quad (\text{EQ 91})$$

To evaluate the cross product in EQ. 91 the coordinates of the $\hat{i}''$ vector must be translated from global coordinates to the element convected coordinate system. Denoting the $\hat{i}''$ vector in the convected coordinate system by,

$$\hat{i}'' = \Omega_1 \hat{i}' + \Omega_2 \hat{j}' + \Omega_3 \hat{k}' \quad (\text{EQ 92})$$

where from a coordinate transformation between the convected and global coordinate system,

$$\begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} = \begin{bmatrix} l_{x'} & m_{x'} & n_{x'} \\ l_{y'} & m_{y'} & n_{y'} \\ l_{z'} & m_{z'} & n_{z'} \end{bmatrix} \begin{bmatrix} l_{x''} \\ m_{x''} \\ n_{x''} \end{bmatrix} \quad (\text{EQ 93})$$

Substituting EQ. 92 into EQ. 91 gives

$$\hat{i} \otimes \Omega_1 \hat{i} + \Omega_2 \hat{j} + \Omega_3 \hat{k} = \theta_{y'} \hat{j} + \theta_{z'} \hat{k} \quad (\text{EQ 94})$$

or

$$\Omega_2 \hat{k} - \Omega_3 \hat{j} = \theta_{y'} \hat{j} + \theta_{z'} \hat{k} \quad (\text{EQ 95})$$

thus

$$\theta_{y'} = -\Omega_3 \quad (\text{EQ 96})$$

$$\theta_{z'} = \Omega_2 \quad (\text{EQ 97})$$

Similarly

$$|\hat{j} \otimes \hat{j}''| = |\hat{j}| \cdot |\hat{j}''| \cdot \sin \alpha_j \quad (\text{EQ 98})$$

and for small angular deformations $\sin \alpha_j \approx \alpha_j$ and the magnitude of the cross product is given by,

$$|\hat{j} \otimes \hat{j}''| = \alpha_j \quad (\text{EQ 99})$$

The cross product of the vectors lies in the plane defined by the $\hat{i}$ and $\hat{k}$ vectors and provides the rotation about the axis defined by the $\hat{j} \otimes \hat{j}''$ direction. For small angular deformations, this rotation can be written as the vector sum of the rotations about the x' and z' axes, i.e.

$$\hat{j} \otimes \hat{j}'' = \theta_{x'} \hat{i} + \theta_{z'} \hat{k} \quad (\text{EQ 100})$$

To evaluate EQ. 100 the coordinates of the $\hat{j}''$ vector must be translated from global to convected coordinates. Denoting the $\hat{j}''$ vector in the convected coordinate system by

$$\hat{j}'' = \tau_1 \hat{i} + \tau_2 \hat{j} + \tau_3 \hat{k} \quad (\text{EQ 101})$$

where from the coordinate transformation between the convected and global coordinate system,

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \begin{bmatrix} l_{x'} & m_{x'} & n_{x'} \\ l_{y'} & m_{y'} & n_{y'} \\ l_{z'} & m_{z'} & n_{z'} \end{bmatrix} \begin{bmatrix} l_{y''} \\ m_{y''} \\ n_{y''} \end{bmatrix} \quad (\text{EQ 102})$$

Substituting EQ. 101 into EQ. 100 gives,

$$\hat{j} \otimes \tau_1 \hat{i} + \tau_2 \hat{j} + \tau_3 \hat{k} = \theta_{x'} \hat{i} + \theta_{z'} \hat{k} \quad (\text{EQ 103})$$

or

$$-\tau_1 \hat{k}' + \tau_3 \hat{i}' = \theta_{x'} \hat{i}' + \theta_{z'} \hat{k}' \quad (\text{EQ 104})$$

thus

$$\theta_{x'} = \tau_3 \quad (\text{EQ 105})$$

5.0 Representation of steel structure plasticity

A simple elasto-plastic, linear kinematic hardening model is utilized for a number of elements in the SUSPNDRS program. A general stress evaluation and stress-strain relationship can be developed for the uniaxial plasticity model which can be utilized in all of the elasto-plastic elements.

The classical bilinear plasticity model with kinematic hardening is shown in Figure 12. In the evaluation of element nodal force vectors and instantaneous stiffness matrices, it is necessary to carry out state determinations in which the element stresses at a given equilibrium iteration are evaluated for the element force and stiffness calculations and the instantaneous stress-strain relationship is obtained for the element stiffness determination.

FIGURE 12. Elasto-plastic model with linear kinematic hardening

To evaluate the current state of stress in the incremental-iterative solution procedure, a path independent stress integration procedure will be employed. In the most general form, this relationship can be stated

$$\sigma_k^n = \sigma^{n-1} + \int_{\epsilon^{n-1}}^{\epsilon_k^n} \frac{d\sigma}{d\epsilon} d\epsilon \quad (\text{EQ 106})$$

In EQ. 106, n refers to the current load step number, k refers to the equilibrium iteration number in the current load step n , ϵ^{n-1} is the converged strain at the end of the previous

load step and ϵ_k^n is the current strain obtained from the displacement field for iteration k . EQ. 106 provides the stress value for the current configuration based on an integration of the stress-strain relationship between the last fully converged state and the current displacement configuration. Performing the integration from the last fully converged state rather than the last iteration state¹ ensures that the algorithm will be path independent.

In the bilinear model, the integration of EQ. 106 is readily accomplished if a number of simple rules are adopted to integrate around yield points. In Figure 12, (ϵ_c, σ_c) denotes the current location of the center of the yield region. When performing the integration of EQ. 106 a number of possible cases must be considered as shown in Figure 13. In case I, the last converged state resides within the elastic region and the current configuration for equilibrium iteration k also lies within the elastic region. In Case II the last converged state is within the yield region and the current configuration lies outside the yield limits. In Case III the last equilibrium configuration lies on the upper or lower yield surface and after load reversal, the new configuration lies either in the elastic range or past the opposite yield point (i.e reverse yielding in a very large load step). In Case IV, yielding continues in the same direction with growth of the yield surface.

To facilitate the development of a simple formula for the stress integration, the parameters λ and η will be defined (Figure 14). These parameters are defined by

$$\lambda = \frac{\epsilon_y}{2} \quad (\text{EQ 107})$$

$$\eta = \frac{E}{E_y} \quad (\text{EQ 108})$$

In EQ. 107 ϵ_y is the initial yield strain and λ represents the half width of the yield region. The parameter η is the ratio of the stress-strain slope in the linear region to the stress-strain slope in the inelastic regime.

Consider first the stress integration from point 1 to point 2 in Figure 15. Letting the total strain increment from point 1 to point 2 be denoted by $\Delta\epsilon$ where

$$\Delta\epsilon = \epsilon_k^n - \epsilon^{n-1} \quad (\text{EQ 109})$$

and this strain increment would be obtained from the displacement field in a finite element analysis. The stress increment is given by,

$$\Delta\sigma = \sigma_k^n - \sigma^{n-1} = E_y[\Delta\epsilon - (\epsilon_c + \lambda - \epsilon^{n-1})] + E(\epsilon_c + \lambda - \epsilon^{n-1}) \quad (\text{EQ 110})$$

or

1. For example, performing the integration $\sigma_k^n = \sigma_{k-1}^n + \int_{\epsilon_{k-1}^n}^{\epsilon_k^n} \frac{d\sigma}{d\epsilon} d\epsilon$ would not preserve path independence

since the integration from converged state to converged state would rely on intermediate, unconverged configurations and would introduce path dependency along a nonequilibrium path

FIGURE 13. Cases for consideration in stress determination

$$\Delta\sigma = E_y\Delta\epsilon + (E - E_y)(\epsilon_c + \lambda - \epsilon^{n-1}) \quad (\text{EQ 111})$$

Defining

$$\Delta\epsilon_y \equiv \epsilon_c + \lambda - \epsilon^{n-1} \quad (\text{EQ 112})$$

which as shown in Figure 15 represents the strain increment from the last fully converged point to the yield point, the stress increment given in EQ. 111 can be written,

FIGURE 14. Stress-strain curve and parameters λ and η .

$$\Delta\sigma = E_y\Delta\epsilon + (E - E_y)\Delta\epsilon_y \quad (\text{EQ 113})$$

or

$$\Delta\sigma = E_y\Delta\epsilon + (\eta E_y - E_y)\Delta\epsilon_y \quad (\text{EQ 114})$$

EQ. 114 then reduces to,

$$\Delta\sigma = E_y(\Delta\epsilon + (\eta - 1)\Delta\epsilon_y) \quad (\text{EQ 115})$$

Defining the additional parameter,

$$\rho \equiv \frac{\Delta\epsilon_y}{\Delta\epsilon} \quad (\text{EQ 116})$$

The stress increment of EQ. 115 can finally be written,

$$\Delta\sigma = E_y[1 + \rho(\eta - 1)]\Delta\epsilon \quad (\text{EQ 117})$$

For a negative strain increment, e.g. moving from point 1 to point 3 in Figure 15, a similar expression can be developed and it is found that EQ. 117 still holds if the appropriate strain to yield expression is employed, i.e.

$$\Delta\epsilon_y = \epsilon_c - \lambda - \epsilon^{n-1} \quad (\text{EQ 118})$$

With the appropriate definition of the strain to yield term as given in EQ. 112 and EQ. 118, the stress increment given in EQ. 117 is applicable for Case II in Figure 13 and in Case III when the current equilibrium configuration results in reverse direction yielding.

FIGURE 15. Integration of stress from the last fully converged state to the new current equilibrium configuration

In a Newton-Raphson incremental-iterative scheme, the stresses must be updated in a state determination for the current equilibrium configuration and the instantaneous stiffness of the structural system must be formed. For the state determination the total stress is found by obtaining the stress increment from EQ. 117 and adding the stress increment to the total stress at the last fully converged state. The instantaneous stiffness is found from the

current rate of change of stress with respect to strain $\frac{d\sigma}{d\epsilon}$. For the simple elasto-plastic model, the logic for accomplishing this, which will adequately capture all of the cases shown in Figure 13, is indicated in Figure 16.

When applied to the bilinear elasto-plastic model, the Newton-Raphson solution algorithm can experience serious divergence problems. In the well known paper by Matthies and Strang [Ref 5] for example, the authors discuss some simple pathological cases in which the classical Newton-Raphson procedure diverges rapidly. A general observation, based on the author's numerical experimentation, is that since the Newton-Raphson procedure is a second order method, if the iterations do converge, convergence is quite rapid. However, divergence is a distinct possibility unless special controls are implemented. In the current work, a special control was implemented to assist in convergence of the Newton-Raphson procedure at the global finite element level. The procedure which was implemented con-

Define yield indicator variables

$y_{indint} \equiv$ yield indicator for stress integration

$y_{indstf} \equiv$ yield indicator for stiffness

Perform preloadstep loop initialization

set $y_{indint} = 0$

set $y_{indstf} = 0$

set $(\epsilon_c, \sigma_c) = (0, 0)$

for each stress-strain location

Load step loop

Equilibrium iteration loop

Form finite element residual vector and stiffness matrix

Residual vector:

update stress from last converged state

if $y_{indint} = 0 \dots$

if $\epsilon_c - \lambda \leq \epsilon_k^n \leq \epsilon_c + \lambda$ then

$$\Delta \sigma = E \Delta \epsilon$$

elseif $\epsilon_k^n < (\epsilon_c - \lambda)$ then

$$\Delta \epsilon_y = \epsilon_c - \lambda - \epsilon^{n-1} \text{ and}$$

$$\Delta \sigma = E_y [1 + \rho(\eta - 1)] \Delta \epsilon$$

elseif $\epsilon_k^n > (\epsilon_c + \lambda)$ then

$$\Delta \epsilon_y = \epsilon_c + \lambda - \epsilon^{n-1} \text{ and}$$

$$\Delta \sigma = E_y [1 + \rho(\eta - 1)] \Delta \epsilon$$

if $y_{indint} = 1 \dots$

if $\epsilon_k^n \geq (\epsilon_c + \lambda)$ then

$$\Delta \sigma = E_y \Delta \epsilon$$

elseif $\epsilon_c - \lambda \leq \epsilon_k^n < (\epsilon_c + \lambda)$ then

$$\Delta \sigma = E \Delta \epsilon$$

elseif $\epsilon_k^n < (\epsilon_c - \lambda)$ then

$$\Delta \epsilon_y = \epsilon_c - \lambda - \epsilon^{n-1} \text{ and}$$

FIGURE 16. Flowchart for implementation of elasto-plastic model in a Newton-Raphson solution framework

sists of using an approximation of the true tangent stiffness when yielding occurs. Thus instead of using $\frac{d\sigma}{d\varepsilon} = E_y$ to characterize the instantaneous stiffness, an approximation is used, i.e.

$$\frac{d\sigma}{d\varepsilon} \Rightarrow \frac{E_y + \tau E}{2} \quad (\text{EQ 119})$$

Utilization of the approximate expression in EQ. 119 significantly improved the convergence of the Newton-Raphson scheme at the global finite element level. This approximation does lead to higher compute costs as a result of an increased number of equilibrium iterations. Since the true instantaneous stiffness is not being utilized, convergence is no longer second order. Based on numerical experimentation, appropriate values of τ , which enhance the stability of the Newton-Rapson procedure, with a minimal impact on convergence rate, are in the range of 0.5 to 0.7.

6.0 Nonlinear truss element

The development of the SUSPNDRS truss element will begin with consideration of a long slender bar element as indicated in Figure 17. The bar is subjected only to axial forces directed along the longitudinal axis of the bar, and based on physical observations of bars subjected to axial load the following fundamental observations can be made;

- Sections plane and perpendicular to the longitudinal fibers of the bar before deformation remain plane and perpendicular to the longitudinal fibers of the bar after deformation
- Cross sections of the bar orthogonal to the longitudinal axis of the bar do not rotate when the bar is subjected to pure axial load
- The transverse strains in the bar cross section are negligibly small and the cross section dimensions can be assumed constant (i.e. $\epsilon_y \approx \epsilon_z \approx 0$)

6.1 Truss theory kinematics

To develop a simple theory for the bar behavior, a set of reference axes will be defined as indicated in Figure 17. The location of the axes is arbitrary, but for simplicity sake the axes will be assumed to pass through the centroid of the cross section. The displacement of the

FIGURE 17. Deformation of a fiber located on the reference axis

initial end of a fiber located on the reference axis is given by

$$\hat{\delta} = u(x)\hat{i} + v(x)\hat{j} + w(x)\hat{k} \quad (\text{EQ 120})$$

and based on a Taylor series expansion, the displacement at the far end of the fiber is given by

$$\begin{aligned}\hat{\delta}(x+dx) = & \left(u + \frac{\partial u}{\partial x}dx + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)dx^2 + \dots \right) \hat{i} + \\ & \left(v + \frac{\partial v}{\partial x}dx + \frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2}\right)dx^2 + \dots \right) \hat{j} + \\ & \left(w + \frac{\partial w}{\partial x}dx + \frac{1}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)dx^2 + \dots \right) \hat{j}\end{aligned}\quad (\text{EQ 121})$$

The deformed length of the fiber is given by

$$l_{new} = |\hat{\Psi}(x)| = \sqrt{\left(dx + u + \frac{du}{dx}dx - u\right)^2 + \left(v + \frac{dv}{dx}dx - v\right)^2 + \left(w + \frac{dw}{dx}dx - w\right)^2} \quad (\text{EQ 122})$$

or

$$l_{new} = \sqrt{\left(dx + \frac{du}{dx}dx\right)^2 + \left(\frac{dv}{dx}dx\right)^2 + \left(\frac{dw}{dx}dx\right)^2} - dx \quad (\text{EQ 123})$$

From EQ. 123, the fiber strain can be written

$$\epsilon_x = \sqrt{\left(1 + \frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} - 1 \quad (\text{EQ 124})$$

or

$$\epsilon_x = \sqrt{\left(1 + 2\frac{du}{dx} + \left(\frac{du}{dx}\right)^2\right) + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} - 1 \quad (\text{EQ 125})$$

Neglecting the higher order term $\left(\frac{du}{dx}\right)^2$ and retaining the higher order terms $\left(\frac{dv}{dx}\right)^2$ and $\left(\frac{dw}{dx}\right)^2$ (these terms will provide the element geometric stiffness component), the strain of the fiber is given by

$$\epsilon_x = \sqrt{\left(1 + 2\frac{du}{dx} + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right)} - 1 \quad (\text{EQ 126})$$

Expanding EQ. 126 using the binomial expansion¹ yields

1. $(1 \pm x)^{\frac{1}{2}} = 1 \pm \frac{1}{2}x - \frac{1}{8}x^2 \pm \dots$

$$\epsilon_x = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \quad (\text{EQ 127})$$

6.2 Truss theory stress resultants

The stress resultant for the simple truss is shown in Figure 18, and the resultant is given by

FIGURE 18. Truss stress resultants

$$F_x = \int_A \sigma_x dA \quad (\text{EQ 128})$$

For the case of pure axial extension and compression, F_x is the only nonzero stress resultant in the bar.

6.3 The truss finite element

A simple finite displacement truss element has been implemented in the SUSPNDRS program. The two node element is tracked in space with an element updated Lagrangian coordinate system which extends from element Node I to element Node J, as shown in Figure 19, to account for gross geometric changes associated with large displacements of the structural system. The element can accommodate both linear elastic and elasto-plastic (classical kinematic hardening) constitutive behavior.

Considering the element shown in Figure 19, the element contribution to the internal resisting force vector and the instantaneous stiffness matrix can be obtained by consider-

$$\epsilon_x = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \quad (\text{EQ 127})$$

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For the case of pure axial extension and compression, F_x is the only nonzero stress resultant in the bar.

6.3 The truss finite element

A simple finite displacement truss element has been implemented in the SUSPNDRS program. The two node element is tracked in space with an element updated Lagrangian coordinate system which extends from element Node I to element Node J, as shown in Figure 19, to account for gross geometric changes associated with large displacements of the structural system. The element can accommodate both linear elastic and elasto-plastic (classical kinematic hardening) constitutive behavior.

Considering the element shown in Figure 19, the element contribution to the internal resisting force vector and the instantaneous stiffness matrix can be obtained by consider-

ation of the principal of virtual displacements. For any vector of imposed virtual displacements δd , the internal work must equal the external work,

$$\int_0^l F_x(x') \delta \epsilon_x(x') dx = q_1 \delta d_1 + q_2 \delta d_2 + q_3 \delta d_3 + q_4 \delta d_4 + q_5 \delta d_5 + q_6 \delta d_6 \quad (\text{EQ 129})$$

or

$$\int_0^l F_x(x') \delta \epsilon_x(x') dx = \{q\}^T \{\delta d\} \quad (\text{EQ 130})$$

where

$$\{q\}^T = [q_1 \ q_2 \ q_3 \ q_4 \ q_5 \ q_6] \quad (\text{EQ 131})$$

$$\{\delta d\} = [\delta d_1 \ \delta d_2 \ \delta d_3 \ \delta d_4 \ \delta d_5 \ \delta d_6] \quad (\text{EQ 132})$$

To facilitate the computation of the element residual and stiffness contributions, a transformation is made from element to natural coordinates. The relationship between physical and natural coordinates (Figure 20) is given by

FIGURE 20. Truss element coordinate systems

$$\xi = \frac{2x}{l} - 1 \quad (\text{EQ 133})$$

$$d\xi = \frac{2}{l} dx \quad (\text{EQ 134})$$

The transformation to the semidiscrete system is made by introducing the displacement field approximations

$$u = \sum_{i=1}^2 N_i(\xi) u_i \quad (\text{EQ 135})$$

$$v = \sum_{i=1}^2 N_i(\xi) v_i \quad (\text{EQ 136})$$

$$w = \sum_{i=1}^2 N_i(\xi) w_i \quad (\text{EQ 137})$$

where

$$N_1(\xi) = \frac{1}{2}(1 - \xi) \quad (\text{EQ 138})$$

$$N_2(\xi) = \frac{1}{2}(1 + \xi) \quad (\text{EQ 139})$$

In terms of the natural coordinates, the element strain is given by

$$\epsilon = \left(\frac{du d\xi}{d\xi dx} \right) + \frac{1}{2} \left(\frac{dv d\xi}{d\xi dx} \right)^2 + \frac{1}{2} \left(\frac{dw d\xi}{d\xi dx} \right)^2 \quad (\text{EQ 140})$$

or

$$\epsilon = \left(\frac{du}{d\xi} \frac{2}{l} \right) + \frac{1}{2} \left(\frac{dv}{d\xi} \frac{2}{l} \right)^2 + \frac{1}{2} \left(\frac{dw}{d\xi} \frac{2}{l} \right)^2 \quad (\text{EQ 141})$$

Introducing the strain expression into the PVD statement of EQ. 129 provides

$$\begin{aligned} \int_{-1}^1 F_x(\xi) \left[\left(\sum_{i=1}^2 N_i'(\xi) \delta u_i J^{-1} \right) + \left(\sum_{i=1}^2 N_i' v_i J^{-1} \right) \left(\sum_{i=1}^2 N_i' \delta v_i J^{-1} \right) \right. \\ \left. + \left(\sum_{i=1}^2 N_i' w_i J^{-1} \right) \left(\sum_{i=1}^2 N_i' \delta w_i J^{-1} \right) \right] J d\xi = q_1 \delta d_1 + \dots q_6 \delta d_6 \end{aligned} \quad (\text{EQ 142})$$

which can be written in matrix form as

$$\int_{-1}^1 F_x \{ \{ B_1 \} + \{ B_{1_g}(\{ d \}) \} \} \{ \delta d \} d\xi = \{ q \}^T \{ \delta d \} \quad (\text{EQ 143})$$

where

$$\{ B_1 \} \equiv \begin{bmatrix} (N_1' J^{-1}) & 0 & 0 & (N_2' J^{-1}) & 0 & 0 \end{bmatrix} \quad (\text{EQ 144})$$

$$\{ B_{1_g}(\{ d \}) \} \equiv \begin{bmatrix} 0 & (S_1 N_1' J^{-1}) & (S_2 N_1' J^{-1}) & 0 & (S_1 N_2' J^{-1}) & (S_2 N_2' J^{-1}) \end{bmatrix} \quad (\text{EQ 145})$$

and

$$S_1 = \sum_{i=1}^2 N_i'(\xi) v_i J^{-1} \quad (\text{EQ 146})$$

$$S_2 = \sum_{i=1}^2 N_i'(\xi) w_i J^{-1} \quad (\text{EQ 147})$$

The principal of virtual displacements can then be written

$$\int_{-1}^1 F_x \{ \{B\} + \{B_G(\{d\})\} \} \{ \delta d \} d\xi = \{q\}^T \{ \delta d \} \quad (\text{EQ 148})$$

where

$$\{B\} = \begin{bmatrix} (N_1' J^{-1}) & 0 & 0 & (N_2' J^{-1}) & 0 & 0 \end{bmatrix} \quad (\text{EQ 149})$$

$$\{B_G(\{d\})\} = \quad (\text{EQ 150})$$

$$\begin{bmatrix} 0 & (S_1 N_1' J^{-1}) & (S_2 N_1' J^{-1}) & 0 & (S_1 N_2' J^{-1}) & (S_2 N_2' J^{-1}) \end{bmatrix}$$

EQ. 143 must hold for any virtual displacement $\{ \delta d \}$ thus

$$\{q\} = \int_{-1}^1 \{ \{B\} + \{B_G(\{d\})\} \}^T F_x J d\xi \quad (\text{EQ 151})$$

EQ. 151 provides the element end forces in the element local coordinate system. By definition, the element instantaneous stiffness matrix consists of the instantaneous rate of change of the element end forces with respect to element end displacements,

$$[k_f] = \begin{bmatrix} \left(\frac{\partial q_1}{\partial d_1} \right) \left(\frac{\partial q_1}{\partial d_2} \right) \left(\frac{\partial q_1}{\partial d_3} \right) & \dots & \dots & \left(\frac{\partial q_1}{\partial d_6} \right) \\ \left(\frac{\partial q_2}{\partial d_1} \right) \left(\frac{\partial q_2}{\partial d_2} \right) & \dots & \dots & \dots \\ \left(\frac{\partial q_3}{\partial d_1} \right) & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \left(\frac{\partial q_4}{\partial d_6} \right) \\ \dots & \dots & \dots & \left(\frac{\partial q_5}{\partial d_5} \right) \left(\frac{\partial q_5}{\partial d_6} \right) \\ \left(\frac{\partial q_6}{\partial d_1} \right) & \dots & \dots & \left(\frac{\partial q_4}{\partial d_6} \right) \left(\frac{\partial q_5}{\partial d_6} \right) \left(\frac{\partial q_6}{\partial d_6} \right) \end{bmatrix} \quad (\text{EQ 152})$$

EQ. 152 can be written

$$[k_I] = \begin{bmatrix} \left\{ \frac{\partial q}{\partial d_1} \right\} & \left\{ \frac{\partial q}{\partial d_2} \right\} & \left\{ \frac{\partial q}{\partial d_3} \right\} & \dots & \left\{ \frac{\partial q}{\partial d_6} \right\} \end{bmatrix} \quad (\text{EQ 153})$$

and from EQ. 151,

$$\left\{ \frac{\partial q}{\partial d_1} \right\} = \int_{-1}^1 \left[\{ \{B\} + \{B_G(\{d\}) \} \}^T \frac{\partial F_x}{\partial d_1} + \left\{ \frac{\partial}{\partial d_1} B_G(\{d\}) \right\}^T F_x \right] J d\xi \quad (\text{EQ 154})$$

$$\left\{ \frac{\partial q}{\partial d_2} \right\} = \int_{-1}^1 \left[\{ \{B\} + \{B_G(\{d\}) \} \}^T \frac{\partial F_x}{\partial d_2} + \left\{ \frac{\partial}{\partial d_2} B_G(\{d\}) \right\}^T F_x \right] J d\xi \quad (\text{EQ 155})$$

and so on for d_3 through d_6 . The element instantaneous stiffness matrix can then be written

$$[k_I] = \left[\frac{\partial q}{\partial d} \right] \quad (\text{EQ 156})$$

$$= \int_{-1}^1 \left[\{ \{B\} + \{B_G(\{d\}) \} \}^T \left(\frac{\partial F_x}{\partial d} \right) + \left[\left\{ \frac{\partial}{\partial d_k} B_G(\{d\}) \right\}^T F_x \right]_{k=1,6} \right] J(d\xi)$$

where,

$$\left\{ \frac{\partial F}{\partial d} \right\} = \begin{bmatrix} \left(\frac{\partial F_x}{\partial d_1} \right) & \left(\frac{\partial F_x}{\partial d_2} \right) & \dots & \dots & \left(\frac{\partial F_x}{\partial d_6} \right) \end{bmatrix} \quad (\text{EQ 157})$$

and

$$\begin{aligned} \left[\left\{ \frac{\partial}{\partial d_k} B_G(\{d\}) \right\}^T \{F\} \right]_{k=1,6} &= \\ \left\{ \left[\frac{\partial}{\partial d_1} B_G(\{d\}) \right]^T \{F\} \right\} &\left\{ \left[\frac{\partial}{\partial d_1} B_G(\{d\}) \right]^T \{F\} \right\} \dots \left\{ \left[\frac{\partial}{\partial d_6} B_G(\{d\}) \right]^T \{F\} \right\} \end{aligned} \quad (\text{EQ 158})$$

The chain rule of differentiation can be applied to EQ. 157 to yield

$$\left\{ \frac{\partial F}{\partial d} \right\} = \frac{\partial F_x}{\partial \epsilon_x} \left\{ \frac{\partial \epsilon_x}{\partial d} \right\} \quad (\text{EQ 159})$$

By comparison of EQ. 130 and EQ. 143,

$$\left\{ \frac{\partial \epsilon_x}{\partial d} \right\} = \{ \{B\} + \{B_G(\{d\})\} \} \quad (\text{EQ 160})$$

and the element instantaneous stiffness is given by

$$\begin{aligned} [k_I] &= \left[\frac{\partial q}{\partial d} \right] \quad (\text{EQ 161}) \\ &= \int_{-1}^1 \left\{ \{B\} + \{B_G(\{d\})\} \right\}^T \left(\frac{\partial F_x}{\partial \epsilon_x} \right) \{ \{B\} + \{B_G(\{d\})\} \} + \left[\left\{ \frac{\partial}{\partial d_k} B_G(\{d\}) \right\}^T F_x \right]_{k=1,6} J d\xi \end{aligned}$$

or

$$[k_I] = [k_i] + [k_{G_1}] + [k_{G_2}] + [k_{G_3}] + [k_{G_4}] \quad (\text{EQ 162})$$

where

$$[k_i] = \int_{-1}^1 \{B\}^T \left(\frac{\partial F_x}{\partial \epsilon_x} \right) \{B\} J d\xi \quad (\text{EQ 163})$$

$$[k_{G_1}] = \int_{-1}^1 \{B_G(\{d\})\}^T \left(\frac{\partial F_x}{\partial \epsilon_x} \right) \{B\} J d\xi \quad (\text{EQ 164})$$

$$[k_{G_2}] = \int_{-1}^1 \{B\}^T \left(\frac{\partial F_x}{\partial \epsilon_x} \right) \{B_G(\{d\})\} J d\xi \quad (\text{EQ 165})$$

$$[k_{G_3}] = \int_{-1}^1 \{B_G(\{d\})\}^T \left(\frac{\partial F_x}{\partial \epsilon_x} \right) \{B_G(\{d\})\} J d\xi \quad (\text{EQ 166})$$

$$[k_{G_4}] = \int_{-1}^1 \left[\left\{ \frac{\partial}{\partial d_k} B_G(\{d\}) \right\}^T F_x \right]_{k=1,6} J d\xi \quad (\text{EQ 167})$$

In the updated Lagrangian coordinate system, Figure 19, the truss element nodal displacements are identically zero, Thus, the matrix $\{B_{1_G}(\{d\})\}$ vanishes and the element residual and instantaneous stiffness terms are given by

$$\{q\} = \int_{-1}^1 \{B\}^T F_x J d\xi \quad (\text{EQ 168})$$

$$[k_I] = \int_{-1}^1 \{B\}^T \left(\frac{\partial F_x}{\partial \epsilon_x} \right) \{B\} J d\xi + \int_{-1}^1 \left[\left\{ \frac{\partial}{\partial d_k} B_G(\{d\}) \right\}^T \{F\} \right]_{k=1,6} J d\xi \quad (\text{EQ 169})$$

6.4 Elasto-plastic constitutive model

The constitutive behavior of the truss material has yet to be defined. For the SUSPNDRS program, the truss is allowed either linear elastic or elasto-plastic with classical kinematic hardening. For bilinear plasticity with kinematic hardening, the truss constitutive relationship can be stated (see Figure 21)

$$\frac{\partial F_x}{\partial \epsilon_x} = E_o A \quad (\text{EQ 170})$$

when $F_x < F_y$, and

$$\frac{\partial F_x}{\partial \epsilon_x} = E_y A \quad (\text{EQ 171})$$

when $F_x \geq F_y$.

FIGURE 21. Elasto-plastic behavior of the truss elements

The truss element force F_x is obtained from path independent integration of the elasto-plastic constitutive equations using the integration scheme developed in section 5.0.

6.5 Implementation of the elasto-plastic truss element

The Newton-Raphson based incremental, iterative global solution algorithm requires the element contributions to the global residual vector and instantaneous stiffness matrix (see EQ. 6 and EQ. 8). Based on the developments in the previous sections, the components necessary for development of the truss element residual and instantaneous stiffness matrix are now available. The element end forces in the element updated Lagrangian coordinate system are given by

$$\{q\} = \int_{-1}^1 \{B\}^T F_x J d\xi$$

where from EQ. 149,

$$\{B\} = \begin{bmatrix} (N_1' J^{-1}) & 0 & 0 & (N_2' J^{-1}) & 0 & 0 \end{bmatrix}$$

The element contribution, for element i, to the global resisting force vector is then found from the simple coordinate transformation

$$\{Q\}_i = [T]^T \int_{-1}^1 \{B\}^T F_x J d\xi \quad (\text{EQ 172})$$

where $[T]^T$ is the transformation matrix between the current Lagrangian coordinate system and the global coordinate system (see EQ. 35). The element instantaneous stiffness is provided by EQ. 169

$$[k_I] = \int_{-1}^1 \{B\}^T \left(\frac{\partial F_x}{\partial \epsilon_x} \right) \{B\} J d\xi + \int_{-1}^1 \left[\left\{ \frac{\partial}{\partial d_k} B_G(\{d\}) \right\}^T \{F\} \right]_{k=1,6} J d\xi$$

where,

$$\{B_G(\{d\})\} = \begin{bmatrix} 0 & (S_1 N_1' J^{-1}) & (S_2 N_1' J^{-1}) & 0 & (S_1 N_2' J^{-1}) & (S_2 N_2' J^{-1}) \end{bmatrix}$$

The element contribution to the global instantaneous stiffness matrix is found from the linear coordinate transformation between the element Lagrangian and global coordinates,

$$[k_I] = [T]^T \left(\int_{-1}^1 \{B\}^T \left(\frac{\partial F_x}{\partial \epsilon_x} \right) \{B\} J d\xi + \int_{-1}^1 \left[\left\{ \frac{\partial}{\partial d_k} B_G(\{d\}) \right\}^T \{F\} \right]_{k=1,6} J d\xi \right) [T] \quad (\text{EQ 173})$$

The integrations indicated in EQ. 173, are performed analytically for the SUSPNDRS program implementation.

7.0 Nonlinear beam element

7.1 Engineering theory of beam bending

Initially, a long slender beam¹ will be considered in which the two transverse dimensions are much smaller than the longitudinal dimension as shown in Figure 22. The beam is subjected to both transverse and longitudinal forces. Based on physical observations of flexure of slender beams, the following fundamental observations about slender beam bending can be made;

- Sections plane and perpendicular to longitudinal fibers of the beam before deformation remain plane and perpendicular to the longitudinal fibers of the beam after deformation

FIGURE 22. Slender beam subjected to transverse and longitudinal loads and the beam reference axes

- The transverse strains in the beam cross section are negligibly small and the cross section dimensions can be assumed constant (i.e. $\epsilon_y \approx \epsilon_z \approx 0$)

The kinematics which result from these observations allows the development of a simple engineering theory of beam bending. Exploitation of these observations allows reduction of slender beam bending from a problem of three dimensional solid mechanics to a simple one dimensional engineering theory of beam bending.

1. For the discussion here, slender refers to beams for which the length is at least ten times the largest transverse dimension.

7.2 Beam theory kinematics

To develop the simple bending theory, a set of reference axes will be defined as shown in Figure 23. To begin with, the location of the reference axes is arbitrary, the axes simply originate somewhere in the beam cross section at one end of the beam, and they are aligned with the primary beam directions as shown in Figure 23.

The displacement of the beam is defined in terms of the displacements of the reference axis, as given by the three displacement components u, v and w as shown in Figure 23. The displacement of any point located off of the reference axes can be described in terms of the reference axis displacements. Consider any cross section of the beam undergoing deformation as shown in Figure 23. Taking into account the fact that transverse strains are neg-

FIGURE 23. Displacements of points on and off the reference axes

ligibly small (the second fundamental observation of slender beam behavior) the off axis displacements in the y and z directions are taken equal to the reference axis displacements and thus only the longitudinal displacement is a function of all three spatial coordinates x , y , and z . Thus,

$$\bar{u} = f(x, y, z) \quad (\text{EQ 174})$$

$$u = f(x) \text{ only} \quad (\text{EQ 175})$$

$$\bar{v} = v = f(x) \text{ only} \quad (\text{EQ 176})$$

$$\bar{w} = w = f(x) \text{ only} \quad (\text{EQ 177})$$

The displacement of any point on the reference axis is denoted by

$$\hat{\delta} = u(x)\hat{i} + v(x)\hat{j} + w(x)\hat{k} \quad (\text{EQ 178})$$

and the displacement of any point off the reference axis is denoted by

$$\bar{\delta} = \bar{u}(x, y, z)\hat{i} + \bar{v}(x)\hat{j} + \bar{w}(x)\hat{k} \quad (\text{EQ 179})$$

Where the overbars are used to indicate a point off of the reference axis.

The longitudinal strain in fibers located either off or on the longitudinal axis can be written in terms of the displacement quantities. Consider two fiber segments, each with initial length dx , undergoing deformation as shown in Figure 24. The displacement of an axis fiber at the initial end is given by

$$\hat{\delta}(x) = u(x)\hat{i} + v(x)\hat{j} + w(x)\hat{k} \quad (\text{EQ 180})$$

and the displacement at the far end of the fiber is given by

$$\begin{aligned} \hat{\delta}(x+dx) = & \left(u + \frac{\partial u}{\partial x}dx + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)dx^2 + \dots \right)\hat{i} + \\ & \left(v + \frac{\partial v}{\partial x}dx + \frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2}\right)dx^2 + \dots \right)\hat{j} + \\ & \left(w + \frac{\partial w}{\partial x}dx + \frac{1}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)dx^2 + \dots \right)\hat{k} \end{aligned} \quad (\text{EQ 181})$$

Where the displacements in EQ. 181 have been expanded about the initial end of the fiber in a Taylor series. From vector addition, the vector defining the new fiber location can be found,

$$\hat{\delta}(x) + \bar{\Psi}(x) = dx\hat{i} + \hat{\delta}(x+dx) \quad (\text{EQ 182})$$

or

$$\bar{\Psi}(x) = dx\hat{i} + \hat{\delta}(x+dx) - \hat{\delta}(x) \quad (\text{EQ 183})$$

Neglecting higher order terms, the new length of the fiber is approximately equal to,

$$l_{new} = |\bar{\Psi}(x)| = \sqrt{\left(dx + u + \frac{du}{dx}dx - u\right)^2 + \left(v + \frac{dv}{dx}dx - v\right)^2 + \left(w + \frac{dw}{dx}dx - w\right)^2} \quad (\text{EQ 184})$$

or

$$l_{new} = \sqrt{\left(dx + \frac{du}{dx}dx\right)^2 + \left(\frac{dv}{dx}dx\right)^2 + \left(\frac{dw}{dx}dx\right)^2} - dx \quad (\text{EQ 185})$$

and the fiber strain is given by,

FIGURE 24. Deformation of fibers located on and off of the reference axis

$$\epsilon_x = \sqrt{\left(1 + \frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} - 1 \quad (\text{EQ 186})$$

or

$$\epsilon_x = \sqrt{\left(1 + 2\frac{du}{dx} + \left(\frac{du}{dx}\right)^2\right) + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} - 1 \quad (\text{EQ 187})$$

Neglecting the higher order term $\left(\frac{du}{dx}\right)^2$ but keeping the higher order terms $\left(\frac{dv}{dx}\right)^2$ and $\left(\frac{dw}{dx}\right)^2$ (this term will ultimately lead to inclusion of the geometric stiffness), the strain of the fiber is given by

$$\epsilon_x = \sqrt{\left(1 + \left(2\frac{du}{dx} + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right)\right)} - 1 \quad (\text{EQ 188})$$

Expanding EQ. 188 using the binomial expansion,¹

$$\epsilon_x = 1 + \frac{1}{2}\left(2\frac{du}{dx} + \left(\frac{dv}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right) + (\text{higher order terms}) - 1 \quad (\text{EQ 189})$$

1. $(1 \pm x)^{\frac{1}{2}} = 1 \pm \frac{1}{2}x - \frac{1}{8}x^2 \pm \dots$

or

$$\epsilon_x = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \quad (\text{EQ 190})$$

Similarly, the new length of the off-axis fiber is given by,

$$l_{new} = \sqrt{\left(dx + \bar{u} + \frac{\partial \bar{u}}{\partial x} dx - \bar{u}\right)^2 + \left(\bar{v} + \frac{d}{dx}\bar{v}dx - \bar{v}\right)^2 + \left(\bar{w} + \frac{d}{dx}\bar{w}dx - \bar{w}\right)^2} \quad (\text{EQ 191})$$

Which after similar expansion finally yields,

$$\bar{\epsilon}_x = \frac{\partial \bar{u}}{\partial x} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \quad (\text{EQ 192})$$

Where use has been made of the fact that $\bar{v} = v$ and $\bar{w} = w$.

The off axis displacement can be written in terms of the reference axis displacement,

$$\bar{u} = u - \theta_y y + \theta_z z \quad (\text{EQ 193})$$

For small deformations and small displacements, the rotations can be approximated by

$$\theta_y \approx \frac{dw}{dx} \quad (\text{EQ 194})$$

$$\theta_z \approx \frac{dv}{dx} \quad (\text{EQ 195})$$

EQ. 194 and EQ. 195, EQ. 193 can be written

$$\bar{u} = u - \frac{dv}{dx}y - \frac{dw}{dx}z \quad (\text{EQ 196})$$

Based on EQ. 196, EQ. 176 and EQ. 177, the displacements at points located off of the reference axis have been completely stated in terms of reference axis displacements. This is a direct consequence of the law of plane sections and is the key to reduction from a three dimensional continuum theory to a one dimensional engineering theory. EQ. 192 then becomes,

$$\bar{\epsilon}_x = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 - \left(\frac{d^2 v}{dx^2}\right)y - \left(\frac{d^2 w}{dx^2}\right)z \quad (\text{EQ 197})$$

which can be written,

$$\bar{\epsilon}_x = \epsilon_x - \kappa_y y - \kappa_z z \quad (\text{EQ 198})$$

7.3 Beam theory stress resultants

The beam theory stress resultants are shown in Figure 25. The stress resultants are given by

$$F_x = \int_A \bar{\sigma}_x dA \quad (\text{EQ 199})$$

$$M_y = -\int_A \bar{\sigma}_x y dA \quad (\text{EQ 200})$$

$$M_z = -\int_A \bar{\sigma}_x z dA \quad (\text{EQ 201})$$

$$T_x = \int_A (\bar{\tau}_{xz} y - \bar{\tau}_{xy} z) dA \quad (\text{EQ 202})$$

FIGURE 25. Beam stress resultants

Where A is the cross sectional area of the beam and the bars on the stress terms denote the stresses at locations off of the reference axis. For the specialized case of a linear elastic material, the stress-strain constitutive behavior is provided by a classical Hooke's law,

$$\bar{\epsilon}_x = \frac{1}{E} [\bar{\sigma}_x - \nu(\bar{\sigma}_y + \bar{\sigma}_z)] \quad (\text{EQ 203})$$

For slender beams, the longitudinal fiber stresses are significantly larger than either of the transverse stress components, and Hooke's law can be approximated by,

$$\bar{\epsilon}_x = \frac{1}{E} \bar{\sigma}_x \quad (\text{EQ 204})$$

Combining EQ. 204 and EQ. 198 gives,

$$\bar{\sigma}_x = E\epsilon_x - E\kappa_y y - E\kappa_z z \quad (\text{EQ 205})$$

Substituting this stress equation into the stress resultant expressions of EQ. 199 to EQ. 201 yields,

$$F_x = \int_A (E\epsilon_x - E\kappa_y y - E\kappa_z z) dA \quad (\text{EQ 206})$$

$$M_y = - \int_A (E\epsilon_x - E\kappa_y y - E\kappa_z z) y dA \quad (\text{EQ 207})$$

$$M_z = - \int_A (E\epsilon_x - E\kappa_y y - E\kappa_z z) z dA \quad (\text{EQ 208})$$

$$T_x = \int_A (G\gamma_{xz} y - G\gamma_{xy} z) dA \quad (\text{EQ 209})$$

For a homogeneous, elastic material, these relationships can be rewritten,

$$F_x = EA\epsilon_x - E\kappa_y \int_A y dA - E\kappa_z \int_A z dA \quad (\text{EQ 210})$$

$$M_y = -E\epsilon_x \int_A y dA + E\kappa_y \int_A y^2 dA + E\kappa_z \int_A zy dA \quad (\text{EQ 211})$$

$$M_z = -E\epsilon_x \int_A z dA + E\kappa_y \int_A yz dA + E\kappa_z \int_A z^2 dA \quad (\text{EQ 212})$$

$$T_x = G \int_A (\gamma_{xz} y - \gamma_{xy} z) dA \quad (\text{EQ 213})$$

EQ. 210 to EQ. 213 do not depend on any specific location of the reference axes. For a linear elastic material however, if the reference axes are taken to coincide with the centroidal axes of the beam, the relationships reduce to the familiar strength of materials expressions for simple Bernoulli-Euler beam bending. For example, if the reference axes correspond to the centroidal axes, by definition,

$$\int_A y dA = 0 \quad (\text{EQ 214})$$

$$\int_A z dA = 0 \quad (\text{EQ 215})$$

$$\int_A y^2 dA = I_{yy} \quad (\text{EQ 216})$$

$$\int_A z^2 dA = I_{zz} \quad (\text{EQ 217})$$

$$\int_A xy dA = 0 \quad (\text{EQ 218})$$

and EQ. 210 through EQ. 213 reduce to the standard expressions

$$F_x = EA\epsilon_x \quad (\text{EQ 219})$$

$$M_y = EI_{yy}\kappa_y \quad (\text{EQ 220})$$

$$M_z = EI_{zz}\kappa_z \quad (\text{EQ 221})$$

$$T_x = GJ\gamma \quad (\text{EQ 222})$$

From EQ. 219 it is evident that if the axial force applied to the beam is zero, the reference axis strain is zero and thus the reference axis (and centroidal axis for this case) is also the neutral axis of the beam. When the reference axes correspond to the centroidal axes the beam axial force is only a function of the reference axis strain and the moments are only a function of the reference axis curvature. In the consideration of material nonlinearity in a subsequent section, it will be found that the uncoupling of the axial stress resultant from the curvature terms and the uncoupling of the moments from the reference axis strain will not generally be achievable because the neutral axis of the beam translates through the beam section as portions of the beam cross section undergo nonlinear material response.

7.4 Beam finite element

In typical civil engineering applications, the structure consists of a frame or truss system in which a large number of beam elements constitute the structural system. For seismic analyses of typical structures, many regions of the structure may remain entirely in the elastic range while other regions of the structure experience nonlinear material behavior. Most often, the regions of nonlinearity are not known a priori and depend on the complex dynamic transient response of the entire structural system. It is thus desirable to have a structural model which can efficiently and accurately represent both linear and nonlinear behavior. With this concept in mind, a nonlinear beam element has been developed based on a cubic Hermite polynomial displacement field approximation. This element allows adequate representation of linear beam bending with a single element between nodes of the model. This is in contrast to some existing linear beam elements (see for example NIKE3D [Ref 6] beam element technology) which require multiple element discretizations of structural elements in order to accurately represent the linear bending characteristics of the beam. Discretization with linear displacement field elements can result in an excessively large number of model degrees of freedom when representing a long-span bridge.

Consider the one dimensional flexural element shown in Figure 26. The element contributions to the internal resisting force vector and instantaneous stiffness matrix can be obtained by consideration of the principle of virtual displacements. For any vector of imposed virtual displacements δd , the internal work must equal the external work,

$$\int_0^l [F_x(x')\delta\epsilon_x(x') + M_y(x')\delta\kappa_y(x') + M_z(x')\delta\kappa_z(x') + T_x(x')\delta\Gamma_x(x')]dx' \quad (\text{EQ 223})$$

$$= q_1\delta d_1 + q_2\delta d_2 + q_3\delta d_3 + q_4\delta d_4 + q_5\delta d_5 + q_6\delta d_6 + q_7\delta d_7 +$$

$$q_8\delta d_8 + q_9\delta d_9 + q_{10}\delta d_{10} + q_{11}\delta d_{11} + q_{12}\delta d_{12}$$

or

$$\int_0^l [\{F(x')\}^T \{\delta\epsilon(x')\}]dx' = \{q\}^T \{\delta d\} \quad (\text{EQ 224})$$

where

$[k_{G_1}]$, $[k_{G_2}]$, and $[k_{G_3}]$ are identically zero. For the updated corotational system, the ele-

FIGURE 28. Flexural element and convected corotational coordinate system

ment residual and instantaneous stiffness contributions are thus given by,

$$\{q\} = \int_{-1}^1 [B]^T \{F\} J d\xi \quad (\text{EQ 284})$$

$$[k_I] = \int_{-1}^1 [B]^T \left[\frac{\partial F}{\partial \epsilon} \right] [B] J d\xi + \int_{-1}^1 \left[\frac{\partial}{\partial d_k} B_G(\{d\}) \right]^T \{F\} J d\xi \quad (\text{EQ 285})$$

$k = 1, 12$

Up to this point in the development, the nonlinear constitutive behavior of the element material has been left completely arbitrary. If the material is linear elastic, the constitutive matrix defining the rate of change of stress resultants with respect to strains (EQ. 274) reduces to a very simple form for the case in which the reference axes correspond to the beam centroidal axes. For this case, the constitutive matrix reduces to

FIGURE 27. Flexural element coordinate systems

$$d\xi = \frac{2}{l}dx \quad (\text{EQ 230})$$

The transformation to a semidiscrete system is made by introduction of the displacement field approximations,

$$u = \sum_{i=1}^2 N_i(\xi) u_i \quad (\text{EQ 231})$$

$$v = \sum_{i=1}^4 M_i(\xi) v_i \quad (\text{EQ 232})$$

$$w = \sum_{i=1}^4 L_i(\xi) w_i \quad (\text{EQ 233})$$

$$\theta = \sum_{i=1}^2 N_i(\xi) \theta_i \quad (\text{EQ 234})$$

For computational expediency, the quadratic terms of EQ. 197 (i.e. $\left(\frac{dv}{dx}\right)^2$ and $\left(\frac{dw}{dx}\right)^2$) will be represented by a lower order approximation (see Przemieniecki [Ref 7]). A second order approximation for the slope terms would result if the displacement approximations of EQ. 232 and EQ. 233 were used directly, and the corresponding instantaneous stiffness and residual terms would include high order terms. The approximate expressions for the slope terms are given by,

$$\frac{dv}{d\xi} \equiv \sum_{i=1}^2 N_i'(\xi) v_i \quad (\text{EQ 235})$$

$$\frac{dw}{d\xi} \equiv \sum_{i=1}^2 N_i'(\xi) w_i \quad (\text{EQ 236})$$

In EQ. 231 through EQ. 234, the u_i , v_i , w_i and θ_i terms are displacements and rotations at the ends of the element, which are equivalent to the corresponding “ d ” terms in Figure 26 (for example $v_1 = d_2$, $v_2 = d_6$, $v_3 = d_8$, $v_4 = d_{12}$) and

$$N_1 = \frac{1}{2}(1 - \xi) \quad (\text{EQ 237})$$

$$N_2 = \frac{1}{2}(1 + \xi) \quad (\text{EQ 238})$$

$$M_1 = \frac{1}{4}(1 - \xi)^2(2 + \xi) \quad (\text{EQ 239})$$

$$M_2 = \frac{l}{8}(\xi + 1)(1 - \xi)^2 \quad (\text{EQ 240})$$

$$M_3 = \frac{1}{4}(\xi + 1)^2(2 - \xi) \quad (\text{EQ 241})$$

$$M_4 = \frac{l}{8}(\xi + 1)^2(\xi - 1) \quad (\text{EQ 242})$$

$$L_1 = \frac{1}{4}(1 - \xi)^2(2 + \xi) \quad (\text{EQ 243})$$

$$L_2 = -\frac{l}{8}(\xi + 1)(1 - \xi)^2 \quad (\text{EQ 244})$$

$$L_3 = \frac{1}{4}(\xi + 1)^2(2 - \xi) \quad (\text{EQ 245})$$

$$L_4 = -\frac{l}{8}(\xi + 1)^2(\xi - 1) \quad (\text{EQ 246})$$

In terms of natural coordinates the element strains can be written,

$$\epsilon_x = \left(\frac{du d\xi}{d\xi dx} \right) + \frac{1}{2} \left(\frac{dv d\xi}{d\xi dx} \right)^2 + \frac{1}{2} \left(\frac{dw d\xi}{d\xi dx} \right)^2 \quad (\text{EQ 247})$$

or

$$\epsilon_x = \left(\frac{du}{d\xi} \frac{2}{l} \right) + \frac{1}{2} \left(\frac{dv}{d\xi} \frac{2}{l} \right)^2 + \frac{1}{2} \left(\frac{dw}{d\xi} \frac{2}{l} \right)^2 \quad (\text{EQ 248})$$

similarly,

$$\kappa_y = \frac{d^2 v}{d\xi^2} \frac{4}{l^2} \quad (\text{EQ 249})$$

$$\kappa_z = \frac{d^2 w}{d\xi^2} \frac{4}{l^2} \quad (\text{EQ 250})$$

$$\Gamma = \frac{d\theta}{d\xi} \frac{2}{l} \quad (\text{EQ 251})$$

Introducing these strain expressions into the PVD statement of EQ. 223 and converting to natural coordinates yields,

$$\begin{aligned}
& \int_{-1}^1 \left[F_x(\xi) \left[\left(\sum_{i=1}^2 N_i'(\xi) \delta u_i J^{-1} \right) + \left(\sum_{i=1}^2 N_i' v_i J^{-1} \right) \left(\sum_{i=1}^2 N_i' \delta v_i J^{-1} \right) \right. \right. \\
& + \left. \left(\sum_{i=1}^2 N_i' w_i J^{-1} \right) \left(\sum_{i=1}^2 N_i' \delta w_i J^{-1} \right) \right] + M_y \left(\sum_{i=1}^4 M_i''(\xi) \delta v_i (J^{-1})^2 \right) \\
& + M_z \left(\sum_{i=1}^4 L_i''(\xi) \delta w_i (J^{-1})^2 \right) + T_x \left(\sum_{i=1}^2 N_i'(\xi) \delta u_i (J^{-1})^2 \right) \Big] J d\xi \\
& = q_1 \delta d_1 + \dots q_{12} \delta d_{12}
\end{aligned} \tag{EQ 252}$$

which can be written in matrix form,

$$\begin{aligned}
& \int_{-1}^1 [F_x \{ \{B_1\} + \{B_{1_G}(\{d\}) \} \} \{ \delta d \} + M_y \{B_2\} \{ \delta d \} + M_z \{B_3\} \{ \delta d \} \\
& + T_x \{B_4\} \{ \delta d \}] J d\xi = \{q\}^T \{ \delta d \}
\end{aligned} \tag{EQ 253}$$

where,

$$\{B_1\} = \begin{bmatrix} (N_1' J^{-1}) & 0 & 0 & 0 & 0 & 0 & (N_2' J^{-1}) & 0 & 0 & 0 & 0 \end{bmatrix} \tag{EQ 254}$$

$$\{B_{1_G}(\{d\})\} = \begin{bmatrix} 0 & (S_1 N_1' J^{-1}) & (S_2 N_1' J^{-1}) & 0 & 0 & 0 & 0 & (S_1 N_2' J^{-1}) & (S_2 N_2' J^{-1}) & 0 & 0 & 0 \end{bmatrix} \tag{EQ 255}$$

$$\{B_2\} = \begin{bmatrix} 0 & (M_1'' (J^{-1})^2) & 0 & 0 & 0 & (M_2'' (J^{-1})^2) & 0 & (M_3'' (J^{-1})^2) & 0 & 0 & 0 & (M_4'' (J^{-1})^2) \end{bmatrix} \tag{EQ 256}$$

$$\{B_3\} = \begin{bmatrix} 0 & 0 & (L_1'' (J^{-1})^2) & 0 & (L_2'' (J^{-1})^2) & 0 & 0 & 0 & (L_3'' (J^{-1})^2) & 0 & (L_4'' (J^{-1})^2) & 0 \end{bmatrix} \tag{EQ 257}$$

$$\{B_4\} = \begin{bmatrix} 0 & 0 & 0 & (N_1' J^{-1}) & 0 & 0 & 0 & 0 & 0 & (N_2' J^{-1}) & 0 & 0 \end{bmatrix} \tag{EQ 258}$$

and

$$S_1 = \sum_{i=1}^2 N_i'(\xi) v_i J^{-1} \tag{EQ 259}$$

$$S_2 = \sum_{i=1}^2 N_i'(\xi) w_i J^{-1} \tag{EQ 260}$$

The principal of virtual displacement can then be written,

$$\int_{-1}^1 \{F\}^T [[B] + [B_G(\{d\})]] \{ \delta d \} J d\xi = \{q\}^T \{ \delta d \} \tag{EQ 261}$$

Comparing EQ. 261 and EQ. 224, the relationship between strain and displacements is given by

$$\{ \delta \epsilon \} = [[B] + [B_G(\{d\})]] \{ \delta d \} \tag{EQ 262}$$

where,

$$[B] = \begin{bmatrix} N_1' J^{-1} & 0 & 0 & 0 & 0 & 0 & N_2' J^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & M_1'' (J^{-1})^2 & 0 & 0 & 0 & M_2'' (J^{-1})^2 & 0 & M_3'' (J^{-1})^2 & 0 & 0 & 0 & M_4'' (J^{-1})^2 \\ 0 & 0 & L_1'' (J^{-1})^2 & 0 & L_2'' (J^{-1})^2 & 0 & 0 & 0 & L_3'' (J^{-1})^2 & 0 & L_4'' (J^{-1})^2 & 0 \\ 0 & 0 & 0 & N_1' J^{-1} & 0 & 0 & 0 & 0 & 0 & N_2' J^{-1} & 0 & 0 \end{bmatrix} \quad (\text{EQ 263})$$

$$[B_G(\{d\})] = \begin{bmatrix} 0 & (S_1 N_1' J^{-1}) & (S_2 N_1' J^{-1}) & 0 & 0 & 0 & 0 & (S_1 N_2' J^{-1}) & (S_2 N_2' J^{-1}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (\text{EQ 264})$$

EQ. 261 must hold for any $\{\delta d\}$, thus,

$$\{q\} = \int_{-1}^1 [[B] + [B_G(\{d\})]]^T \{F\} J d\xi \quad (\text{EQ 265})$$

EQ. 265 provides the element end forces in the element local coordinate system. By definition, the element instantaneous stiffness matrix consists of the instantaneous rate of change of the element end forces with respect to element end displacements, i.e.

$$[k_I] = \begin{bmatrix} \left(\frac{\partial q_1}{\partial d_1}\right) & \left(\frac{\partial q_1}{\partial d_2}\right) & \left(\frac{\partial q_1}{\partial d_3}\right) & \dots & \dots & \left(\frac{\partial q_1}{\partial d_{12}}\right) \\ \left(\frac{\partial q_2}{\partial d_1}\right) & \left(\frac{\partial q_2}{\partial d_2}\right) & \dots & \dots & \dots & \dots \\ \left(\frac{\partial q_3}{\partial d_1}\right) & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \left(\frac{\partial q_{10}}{\partial d_{12}}\right) \\ \dots & \dots & \dots & \dots & \left(\frac{\partial q_{11}}{\partial d_{11}}\right) & \left(\frac{\partial q_{11}}{\partial d_{12}}\right) \\ \left(\frac{\partial q_{12}}{\partial d_1}\right) & \dots & \dots & \left(\frac{\partial q_{12}}{\partial d_{10}}\right) & \left(\frac{\partial q_{12}}{\partial d_{11}}\right) & \left(\frac{\partial q_{12}}{\partial d_{12}}\right) \end{bmatrix} \quad (\text{EQ 266})$$

EQ. 266 can be written,

$$[k_I] = \left[\left\{ \frac{\partial q}{\partial d_1} \right\} \left\{ \frac{\partial q}{\partial d_2} \right\} \left\{ \frac{\partial q}{\partial d_3} \right\} \dots \left\{ \frac{\partial q}{\partial d_{12}} \right\} \right] \quad (\text{EQ 267})$$

From EQ. 265,

$$\left\{ \frac{\partial q}{\partial d_1} \right\} = \int_{-1}^1 \left[[[B] + [B_G(\{d\})]]^T \left\{ \frac{\partial F}{\partial d_1} \right\} + \left[\frac{\partial}{\partial d_1} B_G(\{d\}) \right]^T \{F\} \right] J d\xi \quad (\text{EQ 268})$$

$$\left\{ \frac{\partial q}{\partial d_2} \right\} = \int_{-1}^1 \left[([B] + [B_G(\{d\})])^T \left\{ \frac{\partial F}{\partial d_2} \right\} + \left[\frac{\partial}{\partial d_2} B_G(\{d\}) \right]^T \{F\} \right] J d\xi \quad (\text{EQ 269})$$

similar expressions exist for d_3 through d_{12} , and the element instantaneous stiffness can thus be written,

$$[k_I] = \left[\frac{\partial q}{\partial d} \right] \quad (\text{EQ 270})$$

$$= \int_{-1}^1 \left[([B] + [B_G(\{d\})])^T \left[\frac{\partial F}{\partial d} \right] + \left[\frac{\partial}{\partial d_k} B_G(\{d\}) \right]^T \{F\} \right]_{k=1,12} J d\xi$$

where

$$\left[\frac{\partial F}{\partial d} \right] = \begin{bmatrix} \left(\frac{\partial F_x}{\partial d_1} \right) & \left(\frac{\partial F_x}{\partial d_2} \right) & \dots & \dots & \left(\frac{\partial F_x}{\partial d_{12}} \right) \\ \left(\frac{\partial M_y}{\partial d_1} \right) & \dots & \dots & \dots & \dots \\ \left(\frac{\partial M_z}{\partial d_1} \right) & \dots & \dots & \dots & \left(\frac{\partial M_x}{\partial d_{12}} \right) \\ \left(\frac{\partial T}{\partial d_1} \right) & \dots & \dots & \left(\frac{\partial T}{\partial d_{11}} \right) & \left(\frac{\partial T}{\partial d_{12}} \right) \end{bmatrix} \quad (\text{EQ 271})$$

and

$$\left[\frac{\partial}{\partial d_k} B_G(\{d\}) \right]^T \{F\} \Big|_{k=1,12} = \left\{ \left[\frac{\partial}{\partial d_1} B_G(\{d\}) \right]^T \{F\} \right\} \left\{ \left[\frac{\partial}{\partial d_1} B_G(\{d\}) \right]^T \{F\} \right\} \dots \left\{ \left[\frac{\partial}{\partial d_{12}} B_G(\{d\}) \right]^T \{F\} \right\} \quad (\text{EQ 272})$$

The chain rule of differentiation can be applied to EQ. 271 to yield

$$\left[\frac{\partial F}{\partial d} \right] = \left[\frac{\partial F}{\partial \epsilon} \right] \left[\frac{\partial \epsilon}{\partial d} \right] \quad (\text{EQ 273})$$

where

$$\left[\frac{\partial F}{\partial \epsilon} \right] = \begin{bmatrix} \left(\frac{\partial F}{\partial \epsilon_x} \right) & \left(\frac{\partial F}{\partial \kappa_y} \right) & \left(\frac{\partial F}{\partial \kappa_z} \right) & \left(\frac{\partial F}{\partial \Gamma} \right) \\ \left(\frac{\partial M_y}{\partial \epsilon_x} \right) & \left(\frac{\partial M_y}{\partial \kappa_y} \right) & \left(\frac{\partial M_y}{\partial \kappa_z} \right) & \left(\frac{\partial M_y}{\partial \Gamma} \right) \\ \left(\frac{\partial M_z}{\partial \epsilon_x} \right) & \left(\frac{\partial M_z}{\partial \kappa_y} \right) & \left(\frac{\partial M_z}{\partial \kappa_z} \right) & \left(\frac{\partial M_z}{\partial \Gamma} \right) \\ \left(\frac{\partial T}{\partial \epsilon_x} \right) & \left(\frac{\partial T}{\partial \kappa_y} \right) & \left(\frac{\partial T}{\partial \kappa_z} \right) & \left(\frac{\partial T}{\partial \Gamma} \right) \end{bmatrix} \quad (\text{EQ 274})$$

and,

$$\left[\frac{\partial \epsilon}{\partial d} \right] = \begin{bmatrix} \left(\frac{\partial \epsilon_x}{\partial d_1} \right) \left(\frac{\partial \epsilon_x}{\partial d_2} \right) \cdots \cdots \left(\frac{\partial \epsilon_x}{\partial d_{12}} \right) \\ \left(\frac{\partial \kappa_y}{\partial d_1} \right) \cdots \cdots \cdots \left(\frac{\partial \kappa_y}{\partial d_{12}} \right) \\ \left(\frac{\partial \kappa_z}{\partial d_1} \right) \cdots \cdots \cdots \left(\frac{\partial \kappa_z}{\partial d_{12}} \right) \\ \left(\frac{\partial \Gamma}{\partial d_1} \right) \cdots \cdots \left(\frac{\partial \Gamma}{\partial d_{11}} \right) \left(\frac{\partial \Gamma}{\partial d_{12}} \right) \end{bmatrix} \quad (\text{EQ 275})$$

Comparison of EQ. 275 and EQ. 262 shows that EQ. 275 can be written,

$$\left[\frac{\partial \epsilon}{\partial d} \right] = [[B] + [B_G(\{d\})]] \quad (\text{EQ 276})$$

The instantaneous stiffness matrix given by EQ. 270 can then be written,

$$[k_I] = \left[\frac{\partial q}{\partial d} \right] = \int_{-1}^1 \left[[[B] + [B_G(\{d\})]]^T \left[\frac{\partial F}{\partial \epsilon} \right] [[B] + [B_G(\{d\})]] + \left[\left[\frac{\partial}{\partial d_k} B_G(\{d\}) \right]^T \{F\} \right]_{k=1,12} \right] J d\xi \quad (\text{EQ 277})$$

or

$$[k_I] = [k_i] + [k_{G_1}] + [k_{G_2}] + [k_{G_3}] + [k_{G_4}] \quad (\text{EQ 278})$$

and

$$[k_i] = \int_{-1}^1 [B]^T \left[\frac{\partial F}{\partial \epsilon} \right] [B] J d\xi \quad (\text{EQ 279})$$

$$[k_{G_1}] = \int_{-1}^1 [B_G(\{d\})]^T \left[\frac{\partial F}{\partial \epsilon} \right] [B] J d\xi \quad (\text{EQ 280})$$

$$[k_{G_2}] = \int_{-1}^1 [B]^T \left[\frac{\partial F}{\partial \epsilon} \right] [B_G(\{d\})] J d\xi \quad (\text{EQ 281})$$

$$[k_{G_3}] = \int_{-1}^1 [B_G(\{d\})]^T \left[\frac{\partial F}{\partial \epsilon} \right] [B_G(\{d\})] J d\xi \quad (\text{EQ 282})$$

$$[k_{G_4}] = \int_{-1}^1 \left[\left[\frac{\partial}{\partial d_k} B_G(\{d\}) \right]^T \{F\} \right]_{k=1,12} J d\xi \quad (\text{EQ 283})$$

In the updated corotational coordinate system developed for the beam element (section 4.0.1), the element end displacements in the updated coordinate system are identically zero (see Figure 28 below). Thus, with the simple slope approximation given by EQ. 235 and EQ. 236, the matrix $[B_G(\{d\})]$ vanishes, and the instantaneous stiffness terms

$[k_{G_1}]$, $[k_{G_2}]$, and $[k_{G_3}]$ are identically zero. For the updated corotational system, the ele-

FIGURE 28. Flexural element and convected corotational coordinate system

ment residual and instantaneous stiffness contributions are thus given by,

$$\{q\} = \int_{-1}^1 [B]^T \{F\} J d\xi \quad (\text{EQ 284})$$

$$[k_I] = \int_{-1}^1 [B]^T \left[\frac{\partial F}{\partial \epsilon} \right] [B] J d\xi + \int_{-1}^1 \left[\left[\frac{\partial}{\partial d_k} B_G(\{d\}) \right]^T \{F\} \right]_{k=1,12} J d\xi \quad (\text{EQ 285})$$

Up to this point in the development, the nonlinear constitutive behavior of the element material has been left completely arbitrary. If the material is linear elastic, the constitutive matrix defining the rate of change of stress resultants with respect to strains (EQ. 274) reduces to a very simple form for the case in which the reference axes correspond to the beam centroidal axes. For this case, the constitutive matrix reduces to

$$\left[\frac{\partial F}{\partial \epsilon} \right] = \begin{bmatrix} \left(\frac{\partial F}{\partial \epsilon_x} \right) & \left(\frac{\partial F}{\partial \kappa_y} \right) & \left(\frac{\partial F}{\partial \kappa_z} \right) & \left(\frac{\partial F}{\partial \Gamma} \right) \\ \left(\frac{\partial M_y}{\partial \epsilon_x} \right) & \left(\frac{\partial M_y}{\partial \kappa_y} \right) & \left(\frac{\partial M_y}{\partial \kappa_z} \right) & \left(\frac{\partial M_y}{\partial \Gamma} \right) \\ \left(\frac{\partial M_z}{\partial \epsilon_x} \right) & \left(\frac{\partial M_z}{\partial \kappa_y} \right) & \left(\frac{\partial M_z}{\partial \kappa_z} \right) & \left(\frac{\partial M_z}{\partial \Gamma} \right) \\ \left(\frac{\partial T}{\partial \epsilon_x} \right) & \left(\frac{\partial T}{\partial \kappa_y} \right) & \left(\frac{\partial T}{\partial \kappa_z} \right) & \left(\frac{\partial T}{\partial \Gamma} \right) \end{bmatrix} = \begin{bmatrix} EA & 0 & 0 & 0 \\ 0 & EI_{yy} & 0 & 0 \\ 0 & 0 & EI_{zz} & 0 \\ 0 & 0 & 0 & GJ \end{bmatrix} \quad (\text{EQ 286})$$

7.4.1 Finite fiber beam cross section

To accomodate general nonlinear material behavior in the beam element, the beam cross section will be subdivided into a number of finite fibers¹ as indicated in Figure 29, and the stress-strain relationship for the materials of the beam will be defined seperately for each finite fiber.

Henceforth in this development, the reference axes (i.e. the y and z axes in Figure 29) will be assumed to correspond to the centroidal axes of the beam cross section.

The force resultants of the beam can now be defined based on the finite fibers,

$$F_x = \sum_{i=1}^{NFIBR} \sigma_{x_i} A_i \quad (\text{EQ 287})$$

$$M_y = - \sum_{i=1}^{NFIBR} \sigma_{x_i} y_i A_i \quad (\text{EQ 288})$$

$$M_z = - \sum_{i=1}^{NFIBR} \sigma_{x_i} z_i A_i \quad (\text{EQ 289})$$

$$T_x = GJ\Gamma \quad (\text{EQ 290})$$

The constitutive matrix of the beam can be determined by differentiation of the relationships given in EQ. 287 through EQ. 290. The constitutive relationship for the axial force resultant is given by,

$$\frac{\partial F_x}{\partial \epsilon_x} = \sum_{i=1}^{NFIBR} \frac{d\sigma_{x_i}}{d\bar{\epsilon}_{x_i}} \left(\frac{\partial \bar{\epsilon}_{x_i}}{\partial \epsilon_x} \right) A_i \quad (\text{EQ 291})$$

1. The term "finite" as used here refers to the finite dimensions of each defined fiber section, thus distinguishing from the typical connotation of an infinitesimal dimension fiber.

FIGURE 29. Subdivision of beam cross section into finite fibers. (a) Wide flange beam cross section; (b) finite fibers for the cross section; (c) material constitutive behavior at a finite fiber cross section.

from EQ. 198, $\frac{\partial \bar{\epsilon}_x}{\partial \epsilon_x} = 1$, and EQ. 291 becomes

$$\frac{\partial F_x}{\partial \epsilon_x} = \sum_{i=1}^{NFIBR} \frac{d\sigma_{x_i}}{d\bar{\epsilon}_{x_i}} A_i \quad (\text{EQ 292})$$

similarly,

$$\frac{\partial F_x}{\partial \kappa_y} = \sum_{i=1}^{NFIBR} \frac{d\sigma_{x_i}}{d\bar{\epsilon}_{x_i}} \left(\frac{\partial \bar{\epsilon}_{x_i}}{\partial \kappa_y} \right) A_i \quad (\text{EQ 293})$$

from EQ. 198, $\frac{\partial \bar{\epsilon}_x}{\partial \kappa_y} = -y$ and

$$\frac{\partial F_x}{\partial \kappa_y} = - \sum_{i=1}^{NFIBR} \frac{d\sigma_{x_i}}{d\bar{\epsilon}_{x_i}} y_i A_i \quad (\text{EQ 294})$$

and

$$\frac{\partial F_x}{\partial \kappa_z} = \sum_{i=1}^{NFIBR} \frac{d\sigma_{x_i}}{d\bar{\epsilon}_{x_i}} \left(\frac{\partial \bar{\epsilon}_{x_i}}{\partial \kappa_z} \right) A_i \quad (\text{EQ 295})$$

from EQ. 198, $\frac{\partial \bar{\epsilon}_x}{\partial \kappa_y} = -z$ thus

$$\frac{\partial F_x}{\partial \kappa_z} = - \sum_{i=1}^{NFIBR} \frac{d\sigma_{x_i}}{d\bar{\epsilon}_{x_i}} z_i A_i \quad (\text{EQ 296})$$

and

$$\frac{\partial F_x}{\partial \Gamma} = 0 \quad (\text{EQ 297})$$

Similarly,

$$\frac{\partial M_y}{\partial \epsilon_x} = - \sum_{i=1}^{NFIBR} \frac{d\sigma_{x_i}}{d\bar{\epsilon}_x} \left(\frac{\partial \bar{\epsilon}_x}{\partial \epsilon_x} \right) y_i A_i \quad (\text{EQ 298})$$

$$\frac{\partial M_y}{\partial \kappa_y} = - \sum_{i=1}^{NFIBR} \frac{d\sigma_{x_i}}{d\bar{\epsilon}_x} \left(\frac{\partial \bar{\epsilon}_x}{\partial \kappa_y} \right) y_i A_i \quad (\text{EQ 299})$$

$$\frac{\partial M_y}{\partial \kappa_z} = - \sum_{i=1}^{NFIBR} \frac{d\sigma_{x_i}}{d\bar{\epsilon}_x} \left(\frac{\partial \bar{\epsilon}_x}{\partial \kappa_z} \right) y_i A_i \quad (\text{EQ 300})$$

$$\frac{\partial M_y}{\partial \Gamma} = 0 \quad (\text{EQ 301})$$

EQ. 298 through EQ. 301 can be rewritten

$$\frac{\partial M_y}{\partial \epsilon_x} = - \sum_{i=1}^{NFIBR} \frac{d\sigma_{x_i}}{d\bar{\epsilon}_x} y_i A_i \quad (\text{EQ 302})$$

$$\frac{\partial M_y}{\partial \kappa_y} = \sum_{i=1}^{NFIBR} \frac{d\sigma_{x_i}}{d\bar{\epsilon}_x} y_i^2 A_i \quad (\text{EQ 303})$$

$$\frac{\partial M_y}{\partial \kappa_z} = \sum_{i=1}^{NFIBR} \frac{d\sigma_{x_i}}{d\bar{\epsilon}_x} y_i z_i A_i \quad (\text{EQ 304})$$

$$\frac{\partial M_y}{\partial \Gamma} = 0 \quad (\text{EQ 305})$$

Similarly,

$$\frac{\partial M_z}{\partial \epsilon_x} = - \sum_{i=1}^{NFIBR} \frac{d\sigma_{x_i}}{d\bar{\epsilon}_x} \left(\frac{\partial \bar{\epsilon}_x}{\partial \epsilon_x} \right) z_i A_i \quad (\text{EQ 306})$$

$$\frac{\partial M_z}{\partial \kappa_y} = - \sum_{i=1}^{NFIBR} \frac{d\sigma_x}{d\bar{\epsilon}_x} \left(\frac{\partial \bar{\epsilon}_x}{\partial \kappa_y} \right) z_i A_i \quad (\text{EQ 307})$$

$$\frac{\partial M_z}{\partial \kappa_z} = - \sum_{i=1}^{NFIBR} \frac{d\sigma_x}{d\bar{\epsilon}_x} \left(\frac{\partial \bar{\epsilon}_x}{\partial \kappa_z} \right) z_i A_i \quad (\text{EQ 308})$$

$$\frac{\partial M_y}{\partial \Gamma} = 0 \quad (\text{EQ 309})$$

and EQ. 306 through EQ. 309 can be written,

$$\frac{\partial M_z}{\partial \epsilon_x} = - \sum_{i=1}^{NFIBR} \frac{d\sigma_x}{d\bar{\epsilon}_x} z_i A_i \quad (\text{EQ 310})$$

$$\frac{\partial M_z}{\partial \kappa_y} = \sum_{i=1}^{NFIBR} \frac{d\sigma_x}{d\bar{\epsilon}_x} y_i z_i A_i \quad (\text{EQ 311})$$

$$\frac{\partial M_z}{\partial \kappa_z} = \sum_{i=1}^{NFIBR} \frac{d\sigma_x}{d\bar{\epsilon}_x} z_i^2 A_i \quad (\text{EQ 312})$$

$$\frac{\partial M_y}{\partial \Gamma} = 0 \quad (\text{EQ 313})$$

7.4.2 Elasto-plastic constitutive model

The basic material which will be represented here is a classical elasto-plastic model with kinematic hardening. For this model, the rate of change of stress with respect to strain is linear with the slope determined by whether or not the finite fiber is in a yield condition. For the elasto-plastic material, the rate of change of stress with respect to strain is given simply by,

$$\frac{d\sigma_{x_i}}{d\bar{\epsilon}_{x_i}} = E_0 \quad (\text{EQ 314})$$

when $\sigma_{x_i} < \sigma_y$, and

$$\frac{d\sigma_{x_i}}{d\bar{\epsilon}_{x_i}} = E_y \quad (\text{EQ 315})$$

when $\sigma_{x_i} \geq \sigma_y$. The relationships in EQ. 314 and EQ. 315 can simply be written

$$\frac{d\sigma_{x_i}}{d\bar{\epsilon}_{x_i}} = E(\epsilon_i) \quad (\text{EQ 316})$$

where $E(\epsilon_i) = E_0$ when $\sigma_{x_i} < \sigma_y$ and $E(\epsilon_i) = E_y$ when $\sigma_{x_i} \geq \sigma_y$.

7.4.3 Implementation of the finite fiber elastoplastic element

The Newton-Raphson based incremental, iterative global solution algorithm requires the element contributions to the global residual vector and instantaneous stiffness matrix (see EQ. 6 and EQ. 8). Based on the developments in the previous sections, the components necessary for development of the beam element residual and instantaneous stiffness matrix are now available. The element end forces in the element updated corotational coordinate system are given by EQ. 284,

$$\{q\} = \int_{-1}^1 [B]^T \{F\} J d\xi$$

where, from EQ. 263,

$$[B] = \begin{bmatrix} N_1' J^{-1} & 0 & 0 & 0 & 0 & 0 & N_2' J^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & M_1'' (J^{-1})^2 & 0 & 0 & 0 & M_2'' (J^{-1})^2 & 0 & M_3'' (J^{-1})^2 & 0 & 0 & 0 & M_4'' (J^{-1})^2 \\ 0 & 0 & L_1'' (J^{-1})^2 & 0 & L_2'' (J^{-1})^2 & 0 & 0 & 0 & L_3'' (J^{-1})^2 & 0 & L_4'' (J^{-1})^2 & 0 \\ 0 & 0 & 0 & N_1' J^{-1} & 0 & 0 & 0 & 0 & 0 & N_2' J^{-1} & 0 & 0 \end{bmatrix}$$

The element contribution, for element i , to the global resisting force vector is then found from the simple coordinate transformation

$$\{Q\}_i = [T]^T \int_{-1}^1 [B]^T \{F\} J d\xi \quad (\text{EQ 317})$$

where $[T]^T$ is the transformation matrix between the current corotational coordinate system for the element and the global coordinate system (see EQ. 35).

The element instantaneous stiffness matrix is provided by EQ. 285,

$$[k_I] = \int_{-1}^1 [B]^T \left[\frac{\partial F}{\partial \epsilon} \right] [B] J d\xi + \int_{-1}^1 \left[\left[\frac{\partial}{\partial d_k} B_G(\{d\}) \right]^T \{F\} \right]_{k=1,12} J d\xi$$

where,

$$[B_G(\{d\})] = \begin{bmatrix} 0 & (S_1 N_1' J^{-1}) & (S_2 N_1' J^{-1}) & 0 & 0 & 0 & 0 & (S_1 N_2' J^{-1}) & (S_2 N_2' J^{-1}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and from EQ. 274 and EQ. 292 through EQ. 313 the symmetric constitutive matrix is given by,

$$\left[\frac{\partial F}{\partial \epsilon} \right] = \begin{bmatrix} \left(\frac{\partial F}{\partial \epsilon_x} \right) & \left(\frac{\partial F}{\partial \kappa_y} \right) & \left(\frac{\partial F}{\partial \kappa_z} \right) & \left(\frac{\partial F}{\partial \Gamma} \right) \\ \left(\frac{\partial M_y}{\partial \epsilon_x} \right) & \left(\frac{\partial M_y}{\partial \kappa_y} \right) & \left(\frac{\partial M_y}{\partial \kappa_z} \right) & \left(\frac{\partial M_y}{\partial \Gamma} \right) \\ \left(\frac{\partial M_z}{\partial \epsilon_x} \right) & \left(\frac{\partial M_z}{\partial \kappa_y} \right) & \left(\frac{\partial M_z}{\partial \kappa_z} \right) & \left(\frac{\partial M_z}{\partial \Gamma} \right) \\ \left(\frac{\partial T}{\partial \epsilon_x} \right) & \left(\frac{\partial T}{\partial \kappa_y} \right) & \left(\frac{\partial T}{\partial \kappa_z} \right) & \left(\frac{\partial T}{\partial \Gamma} \right) \end{bmatrix} = \quad (EQ 318)$$

$$\begin{bmatrix} \left(\sum_{i=1}^{NFIBR} E(\epsilon_i) A_i \right) & \left(- \sum_{i=1}^{NFIBR} E(\epsilon_i) y_i A_i \right) & \left(- \sum_{i=1}^{NFIBR} E(\epsilon_i) z_i A_i \right) & 0 \\ \left(- \sum_{i=1}^{NFIBR} E(\epsilon_i) y_i A_i \right) & \left(\sum_{i=1}^{NFIBR} E(\epsilon_i) y_i^2 A_i \right) & \left(\sum_{i=1}^{NFIBR} E(\epsilon_i) y_i z_i A_i \right) & 0 \\ \left(- \sum_{i=1}^{NFIBR} E(\epsilon_i) z_i A_i \right) & \left(\sum_{i=1}^{NFIBR} E(\epsilon_i) y_i z_i A_i \right) & \left(\sum_{i=1}^{NFIBR} E(\epsilon_i) z_i^2 A_i \right) & 0 \\ 0 & 0 & 0 & GJ \end{bmatrix}$$

The element contribution to the global instantaneous stiffness matrix is found from the simple coordinate transformation between element corotational and global coordinates,

$$[K]_i = [T]^T \left(\int_{-1}^1 [B]^T \left[\frac{\partial F}{\partial \epsilon} \right] [B] J d\xi + \int_{-1}^1 \left[\frac{\partial}{\partial d_k} B_G(\{d\}) \right]^T \{F\} J d\xi \right) [T] \quad (EQ 319)$$

$k = 1, 12$

The integrations indicated in EQ. 319 are performed using a quadrature numerical integration. The well known Gaussian quadrature provides the most economical quadrature rule and for an n-point Gaussian integration rule, a polynomial of degree 2n-1 is exactly integrated. However, for the nonlinear beam element developed here an alternative quadrature approach is employed. The alternative approach is based on Lobatto quadrature formulas [Ref 8]. The distinction of the Lobatto quadrature is that the quadrature points include the extreme ends of the integration interval. This allows capture of the initiation of inelastic action which occurs at the ends of the beam elements, something which Gaussian quadrature will generally miss because the Gauss points are interior to the element. For Lobatto integration, an n-point rule will exactly integrate a polynomial of order 2n-3. For the element integrations indicated in EQ. 319 and EQ. 317, the highest order terms are quadratic

in ξ , (i.e. $\text{order}(\xi^2)$), thus a three point Lobatto integration will provide exact integration. The Lobatto quadrature points and corresponding weights are shown in Figure 30.

FIGURE 30. Lobatto quadrature points and weights

8.0 Nonlinear cable element

The nonlinear cable element is equivalent to the simple truss element with the exception that the element is allowed to provide tensile force only. If the member is found to enter the compression regime in the analysis, the element is essentially removed from the model and the element instantaneous stiffness and residual are not include in any of the equilibrium iterations until the members regains a tensile condition.

For suspension bridge analysis, it appears based on the work of others that the main suspender cables will remain in tension, but that the vertical suspenders may go slack during strong earthquake shaking. In the SUSPNDRS models, the vertical suspenders will be represented by a single cable element, and thus simply temporarily removing a slackened element will not leave intermediate degrees of freedom for which there is no stiffness contribution in the global model as indicated in Figure 31.

FIGURE 31. Removal of a vertical cable which slackens

To render the global system stiffness matrix positive definite in the first load step, the cable system must be initialized with the geometric stiffness matrices of the cable elements activated. This is accomplished by including a cable element force approximation in the SUSPNDRS program input file. After the first equilibrium iteration of the first load step, the program automatically computes the element forces and includes the geometric stiffness for the cable elements.

9.0 Deck system model

In lattice bridges which have a diagonal bracing system in the transverse direction (i.e. in planes normal to the roadway) the bridge deck support structure is essentially a space truss. In this type of structure, the axial forces in the individual elements are the primary, first order contributors to member stresses and any contribution from bending of the individual elements is of secondary significance. For this type of structure, a finite element discretization of the deck support system can be adequately and accurately represented using truss (2-force) elements (Figure 32). The truss system requires three degrees of freedom (x, y, and z displacements) at each joint of the structural model.

FIGURE 32. Truss model for a space truss deck system (stringers and other secondary members neglected for clarity)

In lattice structures without transverse bracing, transverse loads must be resisted by frame action of the deck support structure, and thus bending becomes of primary importance

(Figure 33). In this case, simple truss elements are clearly not adequate to model the entire

FIGURE 33. Beam model for a deck system without transverse bracing (stringers and other secondary members neglected for clarity)

deck system. One option is to represent each member of the lattice system with a flexural beam finite element. This would require six degrees of freedom per joint, as indicated in Figure 33. A beam element discretization would result in a very large number of global degrees of freedom for a typical long-span structure with a corresponding tremendous computational effort. The San Francisco-Oakland Bay Bridge is a classic example of a long-span bridge structure which does not have a transverse bracing system (see Figure 34). The upper and lower decks of this bridge precluded the use of a transverse bracing system and lateral loads must be resisted by frame action.

FIGURE 34. Deck system of the Oakland-San Francisco Bay Bridge

9.1 The sway stiffness element

In the current study, an approximate methodology is developed which maintains the relatively economical 3 degree of freedom per node truss discretization, while accounting for the primary flexural effects with a “sway stiffness” element. Consider a truss discretization

of the bridge deck support structure and the four nodes at the edges of a truss segment as shown in Figure 35. The truss elements alone do not constitute a stable structural system since there is no resistance to transverse deformation. To approximately account for the sway stiffness resulting from frame bending, a four node sway stiffness element is included in the finite element model as indicated in Figure 35. The sway stiffness element provides nodal forces to the truss system model which are a function of the in-plane sway or shear deformation of the frame cross section. The element does not generate nodal forces as a result of in-plane extensions or contractions, these deformations are resisted by the existing truss elements.

FIGURE 35. Sway stiffness element, nodal forces and nodal displacements

The element sway deformation components in the element coordinate system are given by

$$\gamma_1 = \left(\left(\frac{d_4 + d_6}{2} \right) - \left(\frac{d_8 + d_2}{2} \right) \right) / W \quad (\text{EQ 320})$$

$$\gamma_2 = \left(\left(\frac{d_5 + d_7}{2} \right) - \left(\frac{d_3 + d_1}{2} \right) \right) / D \quad (\text{EQ 321})$$

Where the average of the two nodal displacements is used since these displacement quantities are not identical (the truss elements will have axial deformation).

These expressions can be written in matrix form

$$\gamma_1 = \begin{bmatrix} 0 & -\frac{1}{2W} & 0 & \frac{1}{2W} & 0 & \frac{1}{2W} & 0 & -\frac{1}{2W} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{bmatrix} \quad (\text{EQ 322})$$

$$\gamma_2 = \begin{bmatrix} -\frac{1}{2D} & 0 & -\frac{1}{2D} & 0 & \frac{1}{2D} & 0 & \frac{1}{2D} & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{bmatrix} \quad (\text{EQ 323})$$

Thus the element sway deformation is given by,

$$\gamma_s = \gamma_1 + \gamma_2 \quad (\text{EQ 324})$$

Combining EQ. 322 and EQ. 323, the sway deformation is given by

$$\gamma_s = \begin{bmatrix} -\frac{1}{2D} & -\frac{1}{2W} & -\frac{1}{2D} & \frac{1}{2W} & \frac{1}{2D} & \frac{1}{2W} & \frac{1}{2D} & -\frac{1}{2W} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{bmatrix} \quad (\text{EQ 325})$$

Initially, the force-displacement behavior of the frame system will be taken as linear elastic. The equivalent nodal forces associated with a sway deformation of the frame can be determined from an analysis of a detailed flexural model of the frame as indicated in Figure 36. From the detailed analysis, the nodal forces are related to the shear deformation, i.e.

$$\begin{bmatrix} F_H \\ F_V \end{bmatrix} = \begin{bmatrix} k_h \\ k_v \end{bmatrix} \gamma_s \quad (\text{EQ 326})$$

FIGURE 36. Determination of equivalent nodal forces as a function of sway deformation

The equivalent nodal forces of the sway element can be written in terms of F_H and F_V ,

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} F_H \\ F_V \end{bmatrix} \quad (\text{EQ 327})$$

Combining EQ. 326 and EQ. 327,

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{bmatrix} = \begin{bmatrix} -k_h \\ -k_v \\ -k_h \\ k_v \\ k_h \\ k_v \\ k_h \\ -k_v \end{bmatrix} \gamma_s \quad (\text{EQ 328})$$

Combining EQ. 325 and EQ. 328 yields the equivalent nodal forces in terms of nodal displacements,

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{bmatrix} = \begin{bmatrix} \frac{k_h}{2D} & \frac{k_h}{2W} & \frac{k_h}{2D} & -\frac{k_h}{2W} & -\frac{k_h}{2D} & -\frac{k_h}{2W} & -\frac{k_h}{2D} & \frac{k_h}{2W} \\ \frac{k_v}{2D} & \frac{k_v}{2W} & \frac{k_v}{2D} & -\frac{k_v}{2W} & -\frac{k_v}{2D} & -\frac{k_v}{2W} & -\frac{k_v}{2D} & \frac{k_v}{2W} \\ \frac{k_h}{2D} & \frac{k_h}{2W} & \frac{k_h}{2D} & -\frac{k_h}{2W} & -\frac{k_h}{2D} & -\frac{k_h}{2W} & -\frac{k_h}{2D} & \frac{k_h}{2W} \\ -\frac{k_v}{2D} & -\frac{k_v}{2W} & -\frac{k_v}{2D} & \frac{k_v}{2W} & \frac{k_v}{2D} & \frac{k_v}{2W} & \frac{k_v}{2D} & -\frac{k_v}{2W} \\ -\frac{k_h}{2D} & -\frac{k_h}{2W} & -\frac{k_h}{2D} & \frac{k_h}{2W} & \frac{k_h}{2D} & \frac{k_h}{2W} & \frac{k_h}{2D} & -\frac{k_h}{2W} \\ -\frac{k_v}{2D} & -\frac{k_v}{2W} & -\frac{k_v}{2D} & \frac{k_v}{2W} & \frac{k_v}{2D} & \frac{k_v}{2W} & \frac{k_v}{2D} & -\frac{k_v}{2W} \\ \frac{k_h}{2D} & \frac{k_h}{2W} & \frac{k_h}{2D} & -\frac{k_h}{2W} & -\frac{k_h}{2D} & -\frac{k_h}{2W} & -\frac{k_h}{2D} & \frac{k_h}{2W} \\ \frac{k_v}{2D} & \frac{k_v}{2W} & \frac{k_v}{2D} & -\frac{k_v}{2W} & -\frac{k_v}{2D} & -\frac{k_v}{2W} & -\frac{k_v}{2D} & \frac{k_v}{2W} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{bmatrix} \quad (\text{EQ 329})$$

From equilibrium considerations (Figure 36), the horizontal and vertical nodal forces resulting from sway deformation can be related,

$$F_V = F_H \cdot \frac{D}{W} \quad (\text{EQ 330})$$

and therefore,

$$k_v = k_h \cdot \frac{D}{W} \quad (\text{EQ 331})$$

Utilizing EQ. 331, EQ. 329 can be written

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{bmatrix} = \frac{k_h}{2} \begin{bmatrix} \frac{1}{D} & \frac{1}{W} & \frac{1}{D} & -\frac{1}{W} & -\frac{1}{D} & -\frac{1}{W} & -\frac{1}{D} & \frac{1}{W} \\ \frac{1}{W} & \frac{D}{W^2} & \frac{1}{W} & -\frac{D}{W^2} & -\frac{1}{W} & -\frac{D}{W^2} & -\frac{1}{W} & \frac{D}{W^2} \\ \frac{1}{D} & \frac{1}{W} & \frac{1}{D} & -\frac{1}{W} & -\frac{1}{D} & -\frac{1}{W} & -\frac{1}{D} & \frac{1}{W} \\ -\frac{1}{W} & -\frac{D}{W^2} & -\frac{1}{W} & \frac{D}{W^2} & \frac{1}{W} & \frac{D}{W^2} & \frac{1}{W} & -\frac{D}{W^2} \\ -\frac{1}{D} & -\frac{1}{W} & -\frac{1}{D} & \frac{1}{W} & \frac{1}{D} & \frac{1}{W} & \frac{1}{D} & -\frac{1}{W} \\ -\frac{1}{W} & -\frac{D}{W^2} & -\frac{1}{W} & \frac{D}{W^2} & \frac{1}{W} & \frac{D}{W^2} & \frac{1}{W} & -\frac{D}{W^2} \\ -\frac{1}{D} & -\frac{1}{W} & -\frac{1}{D} & \frac{1}{W} & \frac{1}{D} & \frac{1}{W} & \frac{1}{D} & -\frac{1}{W} \\ \frac{1}{W} & \frac{D}{W^2} & \frac{1}{W} & -\frac{D}{W^2} & -\frac{1}{W} & -\frac{D}{W^2} & -\frac{1}{W} & \frac{D}{W^2} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{bmatrix} \quad (\text{EQ 332})$$

Finally, letting the ratio of the depth to the width of the frame be denoted by

$$\eta = \frac{D}{W} \quad (\text{EQ 333})$$

EQ. 332 can be written

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{bmatrix} = \frac{k_h}{2D} \begin{bmatrix} 1 & \eta & 1 & -\eta & -1 & -\eta & -1 & \eta \\ \eta & \eta^2 & \eta & -\eta^2 & -\eta & -\eta^2 & -\eta & \eta^2 \\ 1 & \eta & 1 & -\eta & -1 & -\eta & -1 & \eta \\ -\eta & -\eta^2 & -\eta & \eta^2 & \eta & \eta^2 & \eta & -\eta^2 \\ -1 & -\eta & -1 & \eta & 1 & \eta & 1 & -\eta \\ -\eta & -\eta^2 & -\eta & \eta^2 & \eta & \eta^2 & \eta & -\eta^2 \\ -1 & -\eta & -1 & \eta & 1 & \eta & 1 & -\eta \\ \eta & \eta^2 & \eta & -\eta^2 & -\eta & -\eta^2 & -\eta & \eta^2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{bmatrix} \quad (\text{EQ 334})$$

EQ. 334 can be written,

$$\{q\} = [k_s]\{d\} \quad (\text{EQ 335})$$

EQ. 335 provides the sway element nodal forces in terms of the nodal displacements in the local element coordinate system.

9.1.1 Sway element updated coordinate system

The sway element must properly account for large displacements in the bridge system. To accomodate large displacements, an updated local coordinate system is defined which tracks with the sway element as the element deforms in space as indicated in Figure 37. As the deck truss displaces, the four corner nodes will not all explicitly lie in the same plane since there will be some warping of the deck truss cross section. However, the approximation is made that the sway element remains planar and the updated plane of the sway element is defined by the two vectors directed from Node I to Node J and from Node I to Node L in the deformed configuration (Figure 37). The vector $\vec{V}_3$ is obtained from the cross product,

FIGURE 37. Updated coordinate system for the sway element

$$\vec{V}_3 = \vec{V}_1 \otimes \vec{V}_2 \quad (\text{EQ 336})$$

The unit vectors defining the updated coordinate system are obtained by unitizing the $\vec{V}_1$ and $\vec{V}_3$ vectors i.e.,

$$\hat{i}' = \frac{1}{|\hat{V}_1|} \cdot \hat{V}_1 \quad (\text{EQ 337})$$

$$\hat{k}' = \frac{1}{|\hat{V}_3|} \cdot \hat{V}_3 \quad (\text{EQ 338})$$

and by taking the cross product of the $\hat{i}'$ and $\hat{k}'$ vectors, i.e.

$$\hat{j}' = \hat{k}' \otimes \hat{i}' \quad (\text{EQ 339})$$

The element transformations between the local element updated coordinate system and the global coordinates are given by,

$$\begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = \begin{bmatrix} l_{x'} & l_{y'} \\ m_{x'} & m_{y'} \\ n_{x'} & n_{y'} \end{bmatrix} \begin{bmatrix} d_{x'} \\ d_{y'} \end{bmatrix} \quad (\text{EQ 340})$$

and

$$\begin{bmatrix} d_{x'} \\ d_{y'} \end{bmatrix} = \begin{bmatrix} l_{x'} & m_{x'} & n_{x'} \\ l_{y'} & m_{y'} & n_{y'} \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} \quad (\text{EQ 341})$$

EQ. 340 and EQ. 341 can be written,

$$\{d'\} = [T]\{d\} \quad (\text{EQ 342})$$

$$\{d\} = [T]^T\{d'\} \quad (\text{EQ 343})$$

and the sway element stiffness matrix in global coordinates is then given by,

$$[K_s] = [T]^T[k_s][T] \quad (\text{EQ 344})$$

For a linear elastic material, the element forces can be determined from the element stiffness matrix. In local coordinates, the sway element nodal forces are given by

$$\{q\} = [k_s]\{d\} \quad (\text{EQ 345})$$

and the element contribution to the global residual vector becomes

$$\{Q\} = [T]^T[k_s]\{d\} \quad (\text{EQ 346})$$

It is noted that for the updated coordinate system which has been defined, the only non-zero terms for the element force state determination will be d_3 , d_5 , d_6 , d_7 and d_8 . To the first order, the element deformation can be obtained from EQ. 323,

$$\gamma_2 = \frac{d_5 + d_7}{2D} \quad (\text{EQ 347})$$

The displacements in the updated system can be obtained by determining the components of the $\vec{V}_2$ and $\vec{V}_4$ vectors in the direction of the x' axis, i.e.

$$d_5 = \vec{V}_4 \cdot \hat{i} \quad (\text{EQ 348})$$

$$d_7 = \vec{V}_2 \cdot \hat{i} \quad (\text{EQ 349})$$

9.2 The deck membrane element

Bridge roadway deck slab systems will generally exhibit significantly different cross sectional properties in the longitudinal and transverse direction and consequently the in-plane behavior of the deck system can be idealized as an orthotropic structural system. The roadway deck slab can be approximately represented in a structural model with simple anisotropic membrane stiffness elements. These elements are essentially plane stress elements in which the bending stiffness of the deck slab is neglected. The membrane element in an updated local coordinate system is shown in Figure 38.

Assuming linear elastic material behavior, the element generalized orthotropic stress-strain relationships are given by Hooke's law (a procedure for determination of the elastic constants will be discussed below),

$$\epsilon_{x'} = C_{11}\sigma_{x'} - C_{12}\sigma_{y'} \quad (\text{EQ 350})$$

$$\epsilon_{y'} = C_{21}\sigma_{y'} - C_{22}\sigma_{x'} \quad (\text{EQ 351})$$

$$\gamma_{xy'} = C_{33}\tau_{xy'} \quad (\text{EQ 352})$$

or

$$\begin{bmatrix} \epsilon_{x'} \\ \epsilon_{y'} \\ \gamma_{xy'} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \begin{bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{xy'} \end{bmatrix} \quad (\text{EQ 353})$$

Inversion of EQ. 353 yields,

$$\begin{bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{xy'} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & E_{33} \end{bmatrix} \begin{bmatrix} \epsilon_{x'} \\ \epsilon_{y'} \\ \gamma_{xy'} \end{bmatrix} \quad (\text{EQ 354})$$

or

$$\{\sigma\} = [E]\{\epsilon\} \quad (\text{EQ 355})$$

FIGURE 38. Membrane element in the element updated coordinate system

The element instantaneous stiffness matrix and residual vector can be obtained by invoking the principal of virtual work. For any virtual displacement, the external work must equal the internal work, thus

$$\iint \{\sigma\}^T \{\delta\epsilon\} t dA = \{q\}^T \{\delta d\} \quad (\text{EQ 356})$$

or

$$\iint \{\sigma\}^T \{\delta\epsilon\} t dx dy = \{q\}^T \{\delta d\} \quad (\text{EQ 357})$$

Substituting EQ. 355 into EQ. 357 yields,

$$\iint \{\epsilon\}^T [E] \{\delta\epsilon\} t dx' dy' = \{q\}^T \{\delta d\} \quad (\text{EQ 358})$$

Since the evaluation of these integrals is generally not tractable in x-y coordinates, the transformation to element natural coordinates is made (see Figure 38). EQ. 358 then becomes,

$$\int_{-1}^1 \int_{-1}^1 \{\epsilon\}^T [E] \{\delta\epsilon\} t \text{Det} J d\xi d\eta = \{q\}^T \{\delta d\} \quad (\text{EQ 359})$$

and $\text{Det} J$ is the determinant of the Jacobian matrix of the transformation between physical cartesian coordinates and the element natural coordinates¹. The Jacobian will be specified in detail below.

To complete the formulation of the element matrices, the strains must be related to element nodal displacements. In the current development, a classical four node isoparametric formulation will be employed in which the interpolation of the displacement fields and geometry are given by the same approximations, thus

$$u = \sum_{i=1}^4 N_i(\xi, \eta) u_i \quad (\text{EQ 360})$$

$$v = \sum_{i=1}^4 N_i(\xi, \eta) v_i \quad (\text{EQ 361})$$

$$x = \sum_{i=1}^4 N_i(\xi, \eta) x_i \quad (\text{EQ 362})$$

$$y = \sum_{i=1}^4 N_i(\xi, \eta) y_i \quad (\text{EQ 363})$$

In EQ. 360 and EQ. 361 the u_i 's and the v_i 's are the nodal displacements (for example $u_1 = d_1$, $u_2 = d_2$, etc.) and the x_i 's and y_i 's represent the current global coordinate system coordinates of the element nodes. The element shape functions are the classic isoparametric shape functions,

$$N_1 = \frac{1}{4}(1 - \xi)(1 - \eta) \quad (\text{EQ 364})$$

$$N_2 = \frac{1}{4}(1 + \xi)(1 - \eta) \quad (\text{EQ 365})$$

$$N_3 = \frac{1}{4}(1 + \xi)(1 + \eta) \quad (\text{EQ 366})$$

$$N_4 = \frac{1}{4}(1 - \xi)(1 + \eta) \quad (\text{EQ 367})$$

The element strains are given by,

1. A nice physical interpretation of the transformation is given by Irons and Ahmad on Page 62 of their text ... *Techniques of Finite Elements*, Ellis Horwood, 1984.

$$\epsilon_x = \sum_{i=1}^4 \frac{\partial N_i}{\partial x} u_i \quad (\text{EQ 368})$$

$$\epsilon_y = \sum_{i=1}^4 \frac{\partial N_i}{\partial y} v_i \quad (\text{EQ 369})$$

$$\gamma_{xy} = \sum_{i=1}^4 \frac{\partial N_i}{\partial y} u_i + \sum_{i=1}^4 \frac{\partial N_i}{\partial x} v_i \quad (\text{EQ 370})$$

or in matrix form,

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{bmatrix} \quad (\text{EQ 371})$$

or

$$\{\epsilon\} = [B]\{d\} \quad (\text{EQ 372})$$

where $[B]$ is the strain-displacement matrix. Applying the chain rule of differentiation, the partials in EQ. 371 are given by,

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial N_i}{\partial \eta} \frac{\partial \eta}{\partial x} \quad (\text{EQ 373})$$

$$\frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial N_i}{\partial \eta} \frac{\partial \eta}{\partial y} \quad (\text{EQ 374})$$

However, since the inverse relationships are not available to give ξ and η as a function of x and y , the partial derivatives specified in EQ. 373 and EQ. 374 cannot be obtained directly. An alternate approach, which backs into obtaining $\frac{\partial N_i}{\partial x}$ and $\frac{\partial N_i}{\partial y}$, is to first differentiate with respect to ξ and η ,

$$\frac{\partial N_i}{\partial \xi} = \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \xi} \quad (\text{EQ 375})$$

$$\frac{\partial N_i}{\partial \eta} = \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial \eta} \quad (\text{EQ 376})$$

EQ. 375 and EQ. 376 can be written

$$\begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \left(\sum_{i=1}^4 \frac{\partial N_i}{\partial \xi} x_i \right) \left(\sum_{i=1}^4 \frac{\partial N_i}{\partial \xi} y_i \right) \\ \left(\sum_{i=1}^4 \frac{\partial N_i}{\partial \xi} x_i \right) \left(\sum_{i=1}^4 \frac{\partial N_i}{\partial \xi} y_i \right) \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} \quad (\text{EQ 377})$$

or,

$$\begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} \quad (\text{EQ 378})$$

Where $[J]$ is the Jacobian matrix. The desired partials of the shape functions with respect to x and y can then be obtained by inversion of EQ. 378, i.e.

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = [J]^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix} \quad (\text{EQ 379})$$

and the inverse of the Jacobian matrix is given by,

$$[J]^{-1} = \frac{1}{\text{Det } J} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \quad (\text{EQ 380})$$

In EQ. 380 $\text{Det } J$ denotes the determinant of the matrix and is given by,

$$\text{Det } J = J_{11}J_{22} - J_{21}J_{12} \quad (\text{EQ 381})$$

Utilizing EQ. 379 and EQ. 371 the element strain-displacement matrix is now readily computed.

Substituting the strain-displacement relationship of EQ. 372 into the principal of virtual displacements given by EQ. 359 yields,

$$\int_{-1}^1 \int_{-1}^1 \{d\}^T [B]^T [E] [B] \{\delta d\} t \text{Det} J d\xi d\eta = \{q\}^T \{\delta d\} \quad (\text{EQ 382})$$

or

$$\int_{-1}^1 \int_{-1}^1 \{d\}^T [B]^T [E] [B] t \text{Det} J d\xi d\eta \{\delta d\} = \{q\}^T \{\delta d\} \quad (\text{EQ 383})$$

thus,

$$\{q\} = \left[\int_{-1}^1 \int_{-1}^1 [B]^T [E] [B] t \text{Det} J d\xi d\eta \right] \{d\} \quad (\text{EQ 384})$$

and the element instantaneous stiffness matrix in element coordinates is given by

$$[k_I] = \int_{-1}^1 \int_{-1}^1 [B]^T [E] [B] t \text{Det} J d\xi d\eta \quad (\text{EQ 385})$$

EQ. 384 and EQ. 385 provide the membrane element contributions to the global residual and instantaneous stiffness matrix. In the SUSPNDRS finite element program, the double integrations indicated in EQ. 385 are replaced by Gaussian numerical integration. For the membrane element, standard four point Gauss integration was employed.

9.2.1 Determination of the elastic constants for a bridge deck system

The deck membrane element requires the elastic constants which characterize the stress-strain behavior of the deck system (see EQ. 353 and EQ. 354). These constants can be estimated numerically by considering the in-plane force-displacement behavior of a typical segment of the deck of the bridge under consideration. For determination of these constants for the Oakland-San Francisco Bay Bridge for example, computational models of segments of the upper and lower deck systems were constructed as shown in Figure 40.

For a deck system like the Bay Bridge's, the determination of the effective longitudinal stiffness is most critical because the deck is relatively weakly coupled to the stiffening trusses and the membrane stiffness of the deck slab does not contribute fully to the longitudinal stiffness of the deck system. The deck slab does not extend fully to the stiffening truss chords, the edge of the deck slab is approximately 5 feet inboard of the truss chords and longitudinal shear can only be transferred between the deck slab and the stiffening chord via the weak bending axis of the deck girders (see photo in Figure 39). In addition, the deck slab contains expansion joints approximately every four bays and thus the deck cannot transfer tensile membrane forces across the expansion joint. The expansion joints are incorporated in the deck models as indicated in Figure 40. The relatively weak coupling between the deck slab and the stiffening trusses is evident in the displaced shape plots of Figure 40.

FIGURE 39. Consideration of load path; longitudinal shear transfer between deck slab and stiffening truss

Based on the computed force-displacement behavior of the deck slab, the effective elastic constants of the deck system can be determined by analogy with an elastic membrane.

FIGURE 41. Deck segment and an elastic continuum membrane

For the deck segment in Figure 41, the force-displacement behavior of the deck is given by,

$$P_1 = \left[\frac{E_{\text{effective}} A_{\text{deckslab}} + 2(EA_{\text{chords}})}{L} \right] \Delta_1 \quad (\text{EQ 386})$$

Where $E_{\text{effective}}$ represents the effective elastic modulus of the deck slab system.

FIGURE 40. Models of upper and lower deck segments of the Oakland-San Francisco Bay Bridge

The chord stiffness will be represented in the deck model with truss elements, thus the membrane element must correctly account for the effective axial stiffness of the deck slab/

stringer/deck girder system. The finite element model of the deck segment is loaded to get the relationship between P_1 and Δ_1 . Then from EQ. 386,

$$E_{effective} = \frac{\frac{P_1 L}{\Delta_1} - 2(EA_{chords})}{A_{decksab}} \quad (EQ. 387)$$

EQ. 387 provides the effective elastic modulus for the continuum representation of the deck system when the continuum is assigned an area equal to the cross sectional area of the actual deck slab.

9.2.2 Membrane element updated coordinate system

As in the case of the sway stiffness element, the membrane element must account for large displacements of the bridge superstructure. Similarly to the sway element, an updated element coordinate system is employed which tracks with the membrane element as shown in Figure 42. The updated coordinate system for the membrane element is identical to the

FIGURE 42. Updated coordinate system for the membrane element

system derived for the sway stiffness element and the element transformation matrices are

identical to those for the sway stiffness element. For simplicity sake, it is assumed that the membrane element is initially of rectangular geometry in the undeformed configuration and thus the element deformations are readily obtained from the element nodal locations in the updated coordinate system. For the element coordinate system indicated in Figure 42, $d_1=d_2=d_3=d_4=0$.

10.0 Contact/impact element

Structural details which lead to load path discontinuities in structural systems often introduce the possibility of contact and impact between adjacent, disjoint structural elements when a structure is subjected to seismic excitation. For example, structural details employed to accommodate thermal expansion often result in gaps and load path discontinuities where impact is a possibility. For the Oakland-San Francisco Bay Bridge, the deck systems have slip joints to accommodate longitudinal expansion and contraction at the ends of the decks. Figure 43 for example shows the connectivity detail for the deck system at the central anchorage. Under seismic excitation, the suspended deck system can translate longitudinally with the potential for significant impact against the center anchorage caisson. Other locations of potential impact include the points where the deck system connects

FIGURE 43. Connectivity between deck system and center anchorage shows the potential for impact across the expansion gap slip joint

to the towers as indicated in Figure 44. The deck systems can potentially impact the towers when the decks swing longitudinally.

Longitudinal swaying of the entire 2310 foot deck systems can generate an enormous amount of energy with the potential for serious damage in deck system elements if the amplitude of motions are sufficient to cause impact at the end of the deck systems. In addition, impact can drastically effect the global dynamic response of the bridge system. For both these reasons, it is necessary to incorporate features for representation of dynamic impact in the computational bridge model.

FIGURE 44. Potential impact location for large amplitude seismic excitation

10.1 The contact/impact element

A simple penalty function based approach is used in the SUSPNDRS program to represent potential impact between adjacent structural systems. The methodology is based on consideration of the proximity of two nodes of the computational model as indicated in Figure 45. The distance δ_0 represents the initial standoff distance between the two nodes of the model, δ_1 represents the separation distance at which the interface stiffness element is introduced into the tangent stiffness matrix of the structural system (as explained below) and the distance δ_c represents the separation of the two nodes when actual physical impact occurs and thus the interface stiffness and residual vector contributions are invoked. In the SUSPNDRS bridge model representation, the structure is idealized based on wire frame geometry, thus due to the true physical dimension of the structure, for example the 1/2 width of the tower legs, the model nodes will never actually physically touch and the impact must be based on the proximity of the adjacent nodes.

FIGURE 45. Distance definitions for contact and impact

For the special case of the Bay Bridge, it is also sufficient to consider a simple node-to-node contact rather than a surface-to-surface contact because the true physical contact is isolated to a very small region as indicated in Figure 43. This greatly simplifies the consideration of contact and obviates the need to have fully three dimensional contact surfaces with the associated element-by-element and node-by-node contact searches.

The method for developing contact forces and interface stiffnesses consists of insertion of an interface spring between nodes I and J when the element displacements are such that the initial stand-off distance is reduced enough to close the physical gap between the structural components. The actual physical gap will have reduced to zero when the separation distance between the two nodes decreases from the initial value of δ_0 to the value of δ_C . When the separation distance is reduced to δ_C , an interface spring is introduced which operates in the direction defined by a vector extending from Node I to Node J as shown in Figure 46.

FIGURE 46. Contact element in the element coordinate system

To assist in the derivation of the contributions of the contact element to the global stiffness and residual, two gap distances will be defined,

$$\delta gap_1 \equiv \delta_0 - \delta_I \quad (EQ\ 388)$$

and

$$\delta gap_2 = \delta_0 - \delta_C \quad (EQ\ 389)$$

Letting the spring stiffness of the interface spring be denoted by k_{is} , the contact forces are given by,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = k_{is} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} - k_{is} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \delta gap_2 \quad (EQ\ 390)$$

or

$$[c] = [k]\{d\} - k_{is}\{\delta gap_2\} \quad (EQ\ 391)$$

The element instantaneous stiffness matrix is given by the rate of change of contact forces with respect to element nodal displacements,

$$[k_c] = \begin{bmatrix} \frac{\partial c_1}{\partial d_1} & \frac{\partial c_1}{\partial d_2} \\ \frac{\partial c_2}{\partial d_1} & \frac{\partial c_2}{\partial d_2} \end{bmatrix} \quad (EQ\ 392)$$

From EQ. 392 and EQ. 391 the contact element instantaneous stiffness in local coordinates is given by

$$[k_c] = k_{is} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (EQ\ 393)$$

and by comparing EQ. 393 and EQ. 391, the contact element forces can be written

$$[c] = [k_c]\{d\} - k_{is}\{\delta gap_2\} \quad (EQ\ 394)$$

The transformation between contact element local coordinates and global coordinates is obtained from the current orientation of the contact element, as defined by the unit vector extending from Node I to Node J (see Figure 46). The transformation matrix is given by

$$\begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} = \begin{bmatrix} l_{x'} \\ m_{x'} \\ n_{x'} \end{bmatrix} d_{x'} \quad (EQ\ 395)$$

and

$$d_x = \begin{bmatrix} l_x & m_x & n_x \end{bmatrix} \begin{bmatrix} d_x \\ d_y \\ d_z \end{bmatrix} \quad (\text{EQ 396})$$

EQ. 396 and EQ. 395 can be written at both ends of the element to give,

$$\{d'\} = [T]\{d\} \quad (\text{EQ 397})$$

$$\{d\} = [T]^T \{d'\} \quad (\text{EQ 398})$$

The contact element 6x6 instantaneous stiffness in global coordinates is then given by,

$$[K_c] = [T]^T [k_c] [T] \quad (\text{EQ 399})$$

and the contact element contribution to the global residual vector is given by,

$$\{C\} = [T]^T \{c\} \quad (\text{EQ 400})$$

Based on previous contact work by McCallen and Romstad [Ref 2] it has been found that the stability of the global solution algorithm can be significantly enhanced if the contact stiffness is partially invoked prior to full physical contact. This is achieved by introducing the contact instantaneous stiffness matrix into the global instantaneous stiffness matrix prior to the development of full physical contact, without including the contact residual (i.e. right hand side) contribution. In this manner the correct solution will always be achieved since the residual vector is accurate, but numerical difficulties associated with sudden dramatic stiffness changes can be mitigated. The numerical implementation of this feature is based on the definition of a contact "closure" distance within which the contact stiffness matrix will be invoked with the interface stiffness linearly dependent on the distance from full contact. The point at which the interface stiffness will be invoked is denoted by δ_1 in Figure 45, and the interface stiffness which will be employed varies linearly from zero at δ_1 to the full k_{is} when the nodal separation distance reaches δ_c .

11.0 Bridge model initialization - the appropriate gravity configuration

Suspension and cable-stayed bridge systems rely on the large tensile forces which are developed in the cables of the structure to provide stability for both gravity and lateral live loads. The tensile forces developed in the cable systems depend on the manner in which the bridge was constructed, and the final dead load geometric configuration of the bridge system. When constructing a linear, small displacement model of a tension bridge system, the effect of cable system forces can be approximated by constructing a computational model based on the as-built geometry of the bridge system (i.e. the bridge geometry defined in the bridge plans) and invoking geometric stiffnesses for the cable elements using analytically determined approximations of the cable system forces. The significant simplification in this approach is the fact that the cable forces are input directly and do not depend on the structural system deformations. This approach is expedient, however it does not accurately represent the precise geometric configuration of the bridge system when large displacements occur, nor does it capture the change in cable system forces resulting from structural displacements. These geometric effects can become very important when considering the response of these structures to large earthquakes, where displacements can be significant, and these effects can only be accurately captured with a finite displacement, geometrically nonlinear model.

When a fully nonlinear, finite displacement model is employed in the modeling of suspension and cable stayed bridge systems, the model construction process is significantly more complicated. The bridge system dead load configuration cannot be defined immediately because the cable system forces explicitly depend on the displaced shape of the structural system. With a finite displacement model, it is necessary to define an initial bridge geometry, "turn on" gravity, and allow the bridge system model to achieve the appropriate equilibrium configuration through solution of the static equilibrium equations. It is noted that the bridge geometry defined in the as-built plans is the final target geometry for the dead load configuration, but it cannot be used as the initial model geometry in the finite deformation model. If it were used as the initial geometry, the bridge model would distort and deform to an erroneous geometry when gravity loads were invoked. The fundamental problem of model construction for a finite displacement model is the definition of an initial model geometry which will allow the model to achieve the appropriate dead load configuration when gravity forces are applied to the model.

11.1 Definition of initial geometry

In tension structures, the stresses in the individual structural elements can be highly dependent on the manner in which the structure was actually constructed. For example, in suspension bridges the design and construction procedures are typically predicated on the desire to have most of the deck stiffening truss members essentially stress free under gravity dead load with the stiffening truss thus reserved for distributing the live load (i.e. vehicular loads) to the cable system. Ideally, the diagonals and the chords of the stiffening truss would be stress-free under gravity load, with the vertical posts of the truss only being stressed under gravity dead load.

In the particular case of the Oakland-San Francisco Bay Bridge for example, the construction procedure consisted first of construction of the anchorage and towers (see Figure 47) then the main suspension cables were spun. The deck system stiffening truss was then lifted segmentally into place and the individual truss segments were "tacked" together at the top chord joints by insertion of a few alignment bolts. In the initial stages of truss lifting, the bottom chords of adjacent truss segments were separated by as much as one foot due to the sag of the suspension cable system. The truss segments continued to be lifted out towards the towers until the complete deck stiffening truss had been lifted. When the deck truss had been completely lifted in place and attached to the vertical cables, the truss joints had yet to be rigidly riveted and thus the deck truss was simply hanging under self weight from the vertical suspenders. At this point only the vertical posts were subjected to significant stress. As the steel stringers for the deck system were attached to the stiffening truss skeleton, the added weight of the deck steel pressed the chords of the stiffening truss together and the joints were rigidly connected with rivets as the deck steel was emplaced. Once the deck truss joints had been riveted, the ends of the deck truss were connected to the pins in the rockers arms at the towers (with "nominal" jacking forces). At this point the deck stiffening truss became a fully active structural system. Finally, the concrete deck was poured on the roadways. Thus the stiffening truss and cable system only acted as a composite structural system for the dead load of the deck concrete¹.

The deck system stiffening truss experienced significant displacements during the construction process. When the deck system was started at the suspended midspans, the construction documentation notes that the main deck system dropped as much as ten to fifteen feet below the final deadload elevation and when the deck stiffening truss was progressing out near the towers (prior to placement of the deck steel and concrete deck) the deck system rose on the order of ten to fifteen feet above the final dead load elevation.

In order to emulate the bridge construction process, a model construction procedure has been developed which relies on the SUSPNDRS program to perform equilibrium iterations to determine the appropriate dead load geometry and tensile forces in the bridge system. The procedure, which is summarized in Figure 48, consists of three main steps. In the first preprocessing step, the unstretched length of the main suspension cables is calculated. The unstretched length of the main suspension cables can be accurately estimated from the original bridge construction survey notes which describe the cable geometry after spinning of the cables. For the initial length computation, the cable geometry can be adequately idealized as a parabola for typical suspension bridges. Irvine [Ref 9] states that the true catenary shape of a suspended cable can be accurately idealized as a parabola for sag to span ratios of 1 to 8 or less. The Oakland-San Francisco Bay Bridge, for example, has a sag to span ratio of approximately 1 to 9. For the initial cable length computation, the stretch of the main cable can be neglected with small error since the cable exhibits little stretching (in comparison to its total length) until the full deck load is applied.

In step 2, a computational model consisting of the main suspension cable and supporting towers is analyzed in order to estimate the final dead load geometry of the bridge. For the

1. It appears that other investigators have neglected this fact and assumed the diagonals and chords to be stress free under dead load.

Construction of caissons and towers

Spinning of cables and attachment of vertical suspenders

Lifting of deck stiffening truss segments (joints "tacked")

Completion of deck system - joints firmly riveted and concrete deck poured

FIGURE 47. Construction sequence of the Oakland-San Francisco Bay Bridge.

Bridge geometry from cable spinning survey record

Step I. Determine the initial length of the main suspension cables

a) Analytically compute the initial unstretched length of the main suspension cables based on a parabolic approximation of cable geometry. The stretch of the main cables under self-weight is neglected for simplicity sake (the extension of the cables under self weight has a very minor effect on cable sag geometry for self-weight loading, if desired, the influence of cable extension can be accounted for by iteration with the cable model).

If the sag to span ratio is 1:8 or less, Irvine [Ref 9] provides

$$y = 4d \left\{ \frac{x}{l} - \left(\frac{x}{l} \right)^2 \right\}$$

$$H = \frac{mgl^2}{8d}$$

$$L = \int_0^l \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\} dx$$

$$a) \approx l \left\{ 1 + \frac{8}{3} \left(\frac{d}{l} \right)^2 - \frac{32}{5} \left(\frac{d}{l} \right)^4 + \dots \right\}$$

where H is the horizontal component of cable tension, L is the initial cable length, and d is the sag of the cable.

Define linear cable geometry of total length L

Gravity turned on → 5-10 equilibrium iterations

Step II. Compute the location of the main cables under full dead load (i.e. cable weights plus deck load) and determine the lengths of the vertical suspender cables

a) Perform a nonlinear analysis with initially straight cable segments, cable initial lengths are set equal to the lengths determined from the parabolic approximation.

b) Write cable nodal locations to a data file

c) Computed required length of vertical suspenders based on main cable sag shape and required deck elevations (assuming uniform load distribution in vertical suspenders)

FIGURE 48. Determination of the dead load configuration of a suspension bridge system.

Figure 48 cont. Determination of the dead load configuration of a suspension bridge system.

dead load analysis, the full dead weight of the completed bridge deck and cable system is uniformly distributed to the main suspension cables. In this analysis, simple linear segments are utilized to define the initial cable geometry with the constraint that the linear segments have the same total length as the unstretched main cable. The definition of linear cable segments is employed to make the definition of the initial geometry quite simple in the mesh generation preprocessor. Tension only, finite displacement cable elements are utilized to model the suspension cables and the stiffness of the cables elements is initial-

ized with an estimation of the final cable forces in order to render the initial global stiffness positive definite.

Once the appropriate initial length, linear cable segments are defined, gravity can be turned on with full bridge dead load, and the geometry of the main suspension cables can be obtained as indicated in Figure 48. Once the dead-load geometry of the cables is determined, the required length of vertical suspenders can be calculated. The length of each vertical suspender cable can be calculated based on the deformed location of the main suspender cable and the required elevation of the bridge deck. The required final suspender length to achieve the specified deck geometry, denoted L_f in, in combination with the load carried in the vertical suspender P_v , can be used to determine the initial, unstretched vertical cable length,

$$L_0 = L_f \left(1 + \frac{P_v}{EA_c} \right) \quad (\text{EQ 401})$$

Where L_0 is the desired initial length of the cable, E is the effective modulus and A_c is the effective aggregate area of one group of vertical suspenders. In utilizing EQ. 401, it is assumed that each suspender carries the same vertical load, which is determined by tributary area dead load of the deck and stiffening truss. This assumption is valid because of the construction sequence in which the truss joints are not rigidly coupled until much of the dead load is applied to the suspension system, and thus the stiffening truss does not appreciably effect the distribution of forces in the vertical suspension cables.

With the initial, undeformed length of the main suspension cables and vertical suspenders determined, a dead load analysis of the entire bridge can be computed. Step three of the model generation sequence consists of the bridge dead load analysis (Figure 48). The initial geometry for the dead load analysis prescribes the simple linear geometry for the main suspension cables and each vertical suspender is provided with the appropriate undeformed length as calculated from EQ. 401. For the deck stiffening truss, the vertical post truss elements are inserted with a geometric stiffness to ensure positive definiteness of the global stiffness. "Virtual" truss elements are defined for the deck truss chords and diagonals for the dead load analysis. The virtual elements allow definition of the element connectivities for these elements, but the element stiffnesses and element forces are not included in the global matrices. This allows the deck truss system to deform without introducing stresses in the diagonals and chords of the truss. The virtual elements are effectively "along for the ride" in the dead load analysis and contribute nothing to the global stiffness. This insures that only the vertical posts of the truss will be stressed under dead load, and that the vertical suspenders and posts will be perfectly vertical after the application of dead load.

For the dead load analysis, the main suspender cables are master-slaved to the tops of the towers only in the vertical direction. This ensures that the cables can slip horizontally relative to the tower tops and thus the towers will be perfectly vertical, without any horizontal shear forces, at the completion of the dead load process.

The indicated procedure for the bridge dead load initialization guarantees that;

- The bridge deck will have the appropriate geometry at the end of the dead load application
- The bridge towers will be vertical and straight at the end of the dead load application and will be subjected only to concentric axial load
- The deck stiffening truss system will have stresses in the vertical posts and the diagonals and the chords will be stress free
- The cable system will have the appropriate tensile forces

The final step in the dead load initialization consists of coupling the cables to the towers in the horizontal direction through introduction of a penalty stiffness, and turning the virtual truss elements to real elements by introduction of their stiffnesses. To ensure that the virtual elements remain stress free, the SUSPNDRS program computes a precise initial length of the chords and diagonals in the dead load configuration to ensure that the elements will not be stressed in the dead load geometric configuration.

12.0 Example problems

The computational framework and element technologies developed in the previous sections have been implemented in the SUSPNDRS special purpose finite element program at the Lawrence Livermore National Laboratory. In this section, example problems are presented which illustrate the capabilities of the SUSPNDRS program and which provide validation of the SUSPNDRS capabilities by comparison with independent nonlinear analysis results and experimental data.

12.1 Nonlinear analysis of an elasto-plastic truss structure undergoing large displacements.

The first example problem considers the nonlinear response of a cantilever truss structure subjected to a tip load (Figure 49). This test problem has been examined previously by McCallen and Romstad [Ref 10], and has proven to be an excellent problem for evaluation of the global solution algorithm in a finite element model. Difficulties created by this problem include the fact that the forces in yielded truss members change sign as the loading progresses and the problem changes from a softening behavior, dominated by progressive yielding of the individual truss elements, to a stiffening behavior, dominated by the overall geometric change of the structure and shortening of the moment arm of the applied load.

The cantilever truss was subjected to extreme loads with the NIKE3D general purpose nonlinear finite element program and with the SUSPNDRS finite element program. The force-deflection behavior computed with the respective programs are shown in Figure 49 and displaced shapes at selected load steps are shown in Figure 50. Figure 49 also shows the computed results from a third independent program CONTINUA, which analyzes the truss structure as an equivalent continuum structure (see McCallen and Romstad Ref 10). The figures illustrate the good comparison between the three programs. It is noted that the NIKE3D model is somewhat more flexible than the SUSPNDRS and CONTINUA models. This is most likely due to the fact that the NIKE3D truss element accounts for finite deformation as well as finite displacements and therefore represents the reduction in the cross sectional area of the individual truss elements under this extreme loading.

TABLE 1. Displacements of nonlinear truss

Load	Computer Program		
	SUSPNDRS	NIKE3D	CONTINUA
20000 lb	62.4 in	64.3 in	64.5 in
40000	106.0	111.3	109.6
60000	129.1	135.3	132.3
80000	142.7	149.2	145.3
60000	137.3	144.8	140.1
40000	120.6	128.4	123.5
20000	87.8	93.5	90.0
0	19.1	22.3	19.4

FIGURE 49. Force - displacement behavior of a planar truss with geometric and material nonlinearities

FIGURE 50. Displaced shape of planar truss at selected load steps. a) NIKE3D solution; b) SUSPNDRS solution (displacement scale factor=1.0)

12.2 Nonlinear analysis of an elastic frame undergoing large rotations

This problem considered a simple bent undergoing large displacements and large rotations. The purpose of the analysis was to validate the beam element rotation update procedure developed and implement in the SUSPNDRS program. The simple bent was subjected to an out of plane load at one corner of the bent as shown in Figure 51 and the load was increased until extreme displacements and rotations occurred in the bent. Beam element models of the bent were constructed for both the NIKE3D and the SUSPNDRS programs. The force-displacement behavior computed with each model is shown in Figure 51 and the displaced shape of the bent computed with the respective models is shown in Figure 51.

The two models exhibit good agreement for the geometrically nonlinear frame analysis.

TABLE 1. Displacements of nonlinear beam

Load	Computer Program					
	SUSPNDRS			NIKE3D		
	x	y	z	x	y	z
50,000 lb	-1.3 in	-9.1	55.0	-2.0	-9.0	54.5
100,000	-4.5	-27.0	92.2	-5.6	-26.8	91.7
150,000	-7.27	-44.1	114.8	-8.4	-43.9	114.4
200,000	-9.14	-58.2	129.1	-10.3	-58.2	129.0
250,000	-10.3	-69.7	138.8	-11.3	-69.8	138.7
300,000	-11.0	-79.1	145.8	-12.0	-79.3	145.7
350,000	-11.4	-87.0	151.1	-12.3	-87.3	151.1

FIGURE 51. Force-displacement behavior of an elastic beam undergoing large displacements and large rotations

FIGURE 51. Displaced shape of a simple bent subjected to out of plane loading (displacement scale factor =1.0)

12.3 Nonlinear analysis of an elasto-plastic beam of single curvature

The ability of the cubic fiber beam element to model elasto-plastic behavior was investigated by analyzing a wide flange column subjected to a transverse tip load as shown in Figure 52. SUSPNDRS fiber beam element models were compared to a very detailed, three dimensional elasto-plastic shell element model of the wide flange column. The shell element model was constructed for the NIKE3D finite element program.

A number of beam element discretizations were investigated in the SUSPNDRS model. The force-displacement results for different beam element discretizations are shown in Figure 53 to Figure 55. Figure 53 indicates that a single beam element discretization results in a model which is measurably too stiff. With two beam elements the computed nonlinear behavior is very close to that of the shell model. Higher order discretizations continue to provide excellent correlation with the shell element model.

FIGURE 52. Elasto-plastic analysis of a W 14x176 steel column. a) Displacement scale factor =1.0; b) displacement scale factor=5.0)

FIGURE 53. Comparisons of SUSPNDRS fiber beam element and the NIKE3D shell element model for the plastic analysis of a W14x176 steel column. a) One beam element discretization; b) two beam element discretization

FIGURE 54. Comparisons of SUSPNDRS fiber beam element and the NIKE3D shell element model for the plastic analysis of a W14x176 steel column. a) Three beam element discretization; b) five beam element discretization

FIGURE 55. Comparisons of SUSPNDRS fiber beam element and the NIKE3D shell element model for the plastic analysis of a W14x176 steel column. a) Ten beam element discretization; b) Overlay of one, two, three, four, five and ten element results

12.4 Nonlinear analysis of an elasto-plastic beam with compound curvature

The second beam element problem considered the plastic behavior of a W14x176 column in which complete rotation and axial restraints are provided at the top of the column (Figure 56). The NIKE3D shell element model response and the SUSPNDRS fiber beam responses for four and five element discretizations are shown in Figure 57. As a result of axial restraint, there is a degree of tensile membrane behavior occurring at large displacements and this manifests itself as stiffening in the load-deflection curve as is evident in Figure 57. The elasto-plastic behavior computed with the SUSPNDRS beam element is in good agreement with the NIKE3D elasto-plastic shell model.

FIGURE 57. Comparisons of SUSPNDRS fiber beam element and the NIKE3D shell element model for the plastic analysis of a W14x176 steel column

a)

b)

FIGURE 56. Elasto-plastic analysis of a W 14x176 steel column. a) Displacement scale factor =1.0; b) displacement scale factor=5.0)

12.5 Nonlinear analysis of a cable segment

Appropriate modeling of the cable systems in suspension or cable stayed bridges requires that the geometrically nonlinear response of the tensioned cable system be accurately represented. In tensioned cable systems, the global stiffness of the system is highly dependent on the system displacements and therefore the deformed geometry of the system must be accurately tracked. The equilibrium configuration of the cable system under static, gravity loading must be obtained prior to performance of a transient dynamic analysis. In suspension bridges, the geometry of the massive main suspension cables must be established in the initial computation.

To assess the ability of the simple tension-only cable element to represent the geometry of a cable subjected to gravity loading, an analysis was performed for a length of cable subjected to gravity loading. Irvine et. al. [Ref 11] performed measurements of the deformed sagging shape of a cable under dead load (Irvine and his coworkers actually added some uniformly distributed weight to the cable to enhance the weight of the small cable since the cable's own deadweight was not enough to remove some of the initial kinks in the cable). The initial guess for the cable geometry which was used in the finite element analysis was a crude bi-linear geometry as shown in Figure 58a. After a small number of equilibrium iterations based on full Newton stiffness updates, the gravity deformed shape is obtained very precisely as indicated in Figure 58b, where the experimentally measured geometry and the computed geometry agree very well. Subsequent to obtaining the gravity load shape, Irvine and his coworkers applied a 9.8 Newton point load on the left half of the cable and then increased the point load to 19.6 Newtons. The point loads were imposed on the finite element model and the measured and computed deformed shapes are shown in Figure 58c and Figure 58d. An excellent correlation between measured and computed deformations was obtained.

FIGURE 58. Deformed shape of a sagging cable segment. a) initial simple approximation of the cable geometry; b) deformed shape under gravity loading; c) deformed shape under gravity plus a 9.8 Newton point force; d) deformed shape under gravity plus a 19.6 Newton point load

12.6 Slacking/tensioning of a cable segment

To investigate the effect of slackening and tensioning of a cable segment, the problem shown in Figure 59 was analyzed with the SUSPNDRS program. The problem consisted of a 5" x 5" steel column with a 1/4" diameter steel cable attached to the upper end. In this analysis, the structure was first subjected to gravity loading to initialize the cable geometry and, subsequent to the gravity initialization, a transverse tip load was applied at the top of the cantilever. The tip load was first applied in the direction of the cable, and then in a direction opposite the cable. During the first part of the loading the cable continues to sag and does not resist transverse loading (See "B" in Figure 59), after the load is reversed and the magnitude of the load is increased, the slack is finally removed from the cable (see "D" in Figure 59) and a sudden increase in stiffness is observed as the stiffness of the cable is then engaged. During the initial loading the system stiffness is essentially the stiffness of the steel column and after the cable sag is removed the effective system stiffness is the stiffness of the column plus the axial stiffness of the steel cable. A comparison of the finite element model computed stiffnesses with the analytical stiffnesses is shown in Table 2, .

TABLE 2. Computational and analytical stiffnesses of the steel column/cable system

Portion of loading regime	Stiffness from computational model	Stiffness from analytical expressions
Cable sagging K_1	562 lb/in	566 lb/in
Cable tensioned K_2	13,333 lb/in	13,300 lb/in

FIGURE 59. Slacking and tensioning of a hanging cable

12.7 Linear-elastic analysis of a deck segment

The objective of the sway stiffness and membrane elements is to allow representation of the transverse frame action in a bridge deck without the explicit use of beam elements to capture the bending which occurs in individual lattice elements. The use of the sway element can significantly reduce the global degrees of freedom relative to the degrees of freedom required in a full beam element based model.

To investigate the accuracy of the sway stiffness element approximations, a multibay segment of bridge deck was analyzed (Figure 60 shows the elements of detailed and reduced order models of a deck segment). One model consisted of a full beam and shell element characterization of the deck lattice and the second model consisted of a truss element model with sway elements to represent the frame action and membrane elements to represent the deck slabs. The section properties of the various members were set to the section properties of the western span of the Bay Bridge. The natural vibration characteristics computed from each of the models are shown in Figure 61. The two models exhibit good agreement with the character of the corresponding first five modeshapes agreeing quite well.

FIGURE 60. Computational models of a two deck bridge. A) Detailed model with beam and shell elements; b) reduced order model with truss, membrane and sway stiffness elements

FIGURE 61. Comparison of modeshapes from reduced order and detailed models for a twenty bay, simply supported deck segment

FIGURE 61. Comparison of modeshapes from reduced order and detailed models for a twenty bay, simply supported deck segment

FIGURE 61. Comparison of modeshapes from reduced order and detailed models for a twenty bay, simply supported deck segment

12.8 Nonlinear analysis of a deck segment

The ability of the SUSPNDRS model of a suspended deck segment to attain appropriate numerical convergence for a deck segment undergoing nonlinear, large rigid body displacements was evaluated by modeling a 20 bay deck segment suspended from a cable system as shown in Figure 62. The main suspension cables were fixed against translation at the points indicated, and the edge of the deck was prevented from translating horizontally at one end of the deck. The first load increment established the gravity deformed shape of the structure, and subsequently two transverse point loads were applied at the free end of the deck. The deck translated to large global displacements and rotation and quick convergence was attained at each load level.

FIGURE 62. Deformed shape of a suspended deck segment

12.9 Contact of disjoint model parts

The ability to represent contact between disjoint parts of a model was investigated with the simple beam model shown in Figure 63. A lateral load was applied to the left column, pushing the left column toward the right column. For the first analysis, the columns were

Contact assumed to occur when column separation decreases to 40"

FIGURE 63. Loading of adjacent cantilever columns.

not coupled, for the second analysis a contact element coupled the two tops of the columns in the horizontal direction. The contact element was invoked when the separation between the two columns decrease to 40 inches. The displaced shapes for the two analyses are shown in Figure 64. The contact element appropriately enforced compatibility when the two columns came within 40 inches, and when the load was reversed the contact element appropriately released the connectivity between the two columns.

The force-displacement behavior exhibited by the two models is shown in Figure 65. The coupled system provides an effective stiffness which is nearly double the stiffness of the decoupled system. The stiffness increase when contact occurs is slightly less than the ideal factor of two because the displacements of the columns are large and geometric stiffening is occurring with the large displacements in the single column of the non-contacting case. Evidence of the stiffening with displacement amplitude is evident in Figure 65 where the force-displacement curves have some curvature in the extreme displacement region.

No
contact
element

Including
contact
element

FIGURE 64. Coupling between disjoint parts of a model.

FIGURE 64. Coupling between disjoint parts of a model.

FIGURE 64. Coupling between disjoint parts of a model.

FIGURE 65. Force-displacement behavior of contacting and noncontacting systems.

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Acknowledgements

This research was funded by the University of California Directed Research and Development Fund through the Campus-Laboratory Collaboration Program at the Lawrence Livermore National Laboratory. This support is gratefully acknowledged. The authors wish to thank Dr. Paul Kasameyer, Principal Investigator for the Advanced Earthquake Hazards Research at LLNL, for his continued support. The authors also wish to thank Dr. Claire Max of the Campus-Laboratory-Collaboration Program at LLNL for her continued interest and support.

A portion of this work was performed under the auspices of the United States Department of Energy, at LLNL, under contract W-7405-Eng-48.

Appendix I - Beam element shape functions

The beam element shape functions

$$N_1 = \frac{1}{2}(1 - \xi)$$

$$N_1' = -\frac{1}{2}$$

$$N_2 = \frac{1}{2}(1 + \xi)$$

$$N_2' = \frac{1}{2}$$

$$M_1 = \frac{1}{4}(1 - \xi)^2(2 + \xi)$$

$$M_1' = \frac{3}{4}\xi^2 - \frac{3}{4}$$

$$M_1'' = \frac{3}{2}\xi$$

$$M_2 = \frac{l}{8}(\xi + 1)(1 - \xi)^2$$

$$M_2' = \frac{3l}{8}\xi^2 - \frac{l}{4}\xi - \frac{l}{8}$$

$$M_2'' = \frac{3l}{4}\xi - \frac{l}{4}$$

$$M_3 = \frac{1}{4}(\xi + 1)^2(2 - \xi)$$

$$M_3' = -\frac{3}{4}\xi^2 + \frac{3}{4}$$

$$M_3'' = -\frac{3}{2}\xi$$

$$M_4 = \frac{l}{8}(\xi + 1)^2(\xi - 1)$$

The beam element shape functions

$$M_4' = \frac{3l}{8}\xi^2 + \frac{l}{4}\xi - \frac{l}{8}$$

$$M_4'' = \frac{3l}{4}\xi + \frac{l}{4}$$

$$L_1 = \frac{1}{4}(1 - \xi)^2(2 + \xi)$$

$$L_1' = \frac{3}{4}\xi^2 - \frac{3}{4}$$

$$L_1'' = \frac{3}{2}\xi$$

$$L_2 = -\frac{l}{8}(\xi + 1)(1 - \xi)^2$$

$$L_2' = -\frac{3l}{8}\xi^2 + \frac{l}{4}\xi + \frac{l}{8}$$

$$L_2'' = -\frac{3l}{4}\xi + \frac{l}{4}$$

$$L_3 = \frac{1}{4}(\xi + 1)^2(2 - \xi)$$

$$L_3' = -\frac{3}{4}\xi^2 + \frac{3}{4}$$

$$L_3'' = -\frac{3}{2}\xi$$

$$L_4 = -\frac{l}{8}(\xi + 1)^2(\xi - 1)$$

$$L_4' = -\frac{3l}{8}\xi^2 - \frac{l}{4}\xi + \frac{l}{8}$$

$$L_4'' = -\frac{3l}{4}\xi - \frac{l}{4}$$

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