Abstract

We show that requiring diffeomorphic equivalence for one-dimensional stationary states implies that the reduced action $S_0$ satisfies the quantum Hamilton-Jacobi equation with the Planck constant playing the role of a covariantizing parameter. The construction shows the existence of a fundamental initial condition which is strictly related to the Möbius symmetry of the Legendre transform and to its involutive character. The universal nature of the initial condition implies the Schrödinger equation in any dimension.
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While general relativity is based on a simple fundamental principle, similar geometrical structures do not seem to underlie quantum mechanics. In this letter we show that requiring that for any one-dimensional stationary quantum state there is always a coordinate choice \( \tilde{q} \) in which \( \mathcal{W}(q) \equiv V(q) - E \) corresponds to \( \dot{W}(\tilde{q}) = 0 \), implies that the reduced action satisfies the quantum Hamilton-Jacobi equation with the Planck constant playing the role of covariantizing parameter. Diffeomorphic equivalence implies that \( S_0 \) is never a constant, rather there is the crucial initial condition \( S_0 = \pm \frac{1}{2} \hbar \ln q \) for \( \mathcal{W} = 0 \), implying that the Legendre transform \( T_0 = q \partial_q S_0 - S_0 \) is defined for any state. Our construction is deeply related to the Möbius symmetry of the Legendre transform and to its dual (involutive) character. The universal nature of the initial condition implies the Schrödinger equation in any dimension.

In ref. [1] we introduced the prepotential \( \mathcal{F} \) in quantum mechanics defined by \( \psi_D = \mathcal{F}'(\psi) \), where \( \psi \) and \( \psi_D \) are two linearly independent solutions of the Schrödinger equation. We showed that the space coordinate can be regarded as the Legendre transform of \( \mathcal{F} \) with respect to the probability density. Thus, the space coordinate and \( \mathcal{F} \) obtain equal status from the point of view of describing the system by the Legendre transform. This means that the wave function itself carries information on the geometry of a physical system.

Let us begin by noting the Möbius symmetry of the Legendre transform. Let us define the function \( T_0(p) \) by

\[
q = \partial_p T_0,
\]

(1)

where \( q \) is the coordinate. The Legendre transform of \( T_0 \)

\[
S_0 = p \partial_p T_0 - T_0,
\]

(2)

satisfies

\[
p = \partial_q S_0.
\]

(3)

The Legendre transform has a crucial symmetry under

\[
q = \frac{Aq + B}{Cq + D}, \quad p = (Cq + D)^2 p,
\]

(4)

where

\[
\left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in SL(2, \mathbb{C}).
\]

Eqs.(1) and (4) imply that

\[
\tilde{T}_0(\tilde{p}) = T_0(p) + ACpq^2 + BDP + 2BCpq.
\]
Therefore
\[ \tilde{T}_0(\tilde{p}) - T_0(p) = \tilde{p}q - p_q, \]
implying that \( S_0 \) is a scalar under (4)
\[ \tilde{S}_0(\tilde{q}) = S_0(q). \] (5)

The transformations (4) are equivalent to
\[ \begin{align*}
\tilde{q} \sqrt{\tilde{p}} &= \epsilon(Aq \sqrt{p} + B \sqrt{p}), \\
\sqrt{\tilde{p}} &= \epsilon(Cq \sqrt{p} + D \sqrt{p}),
\end{align*} \] (6)
where \( \epsilon = \pm 1 \). The second derivative of (2) with respect to \( s = S_0(q) \) gives
\[ \frac{1}{q \sqrt{p}} \frac{d^2(q \sqrt{p})}{ds^2} = \frac{1}{p \sqrt{q}} \frac{d^2(p \sqrt{q})}{ds^2} = -\mathcal{U}(s), \]
that is
\[ \left[ \frac{d^2}{ds^2} + \mathcal{U}(s) \right] q \sqrt{p} = 0 = \left[ \frac{d^2}{ds^2} + \mathcal{U}(s) \right] \sqrt{p}. \] (7)
Thus, we associated a second order differential equation with the Legendre transform. This method is precisely the one employed, in the reverse order, for obtaining the inversion formula [2] in \( N = 2 \) super Yang-Mills, and for the Schrödinger equation [1], which has been further investigated in [3].

Let us introduce the Schwarzian derivative
\[ \{h(x), x\} = \frac{h'''}{h'} - \frac{3}{2} \left( \frac{h''}{h'} \right)^2, \] (8)
\[ \{h(x), x(y)\} = (\partial_x y)^2 (\{h(x), y\} - \{x, y\}). \] (9)
Let \( \phi_1, \phi_2 \) be linearly independent solutions of \( \partial_x^2 \phi(x) + \mathcal{P}(x) \phi(x) = 0 \), then \( \{\phi_1/\phi_2, x\} = 2\mathcal{P} \) (note that \( \phi_2(x) = A\phi_1(x) + B\phi_1(x) \int x^{-2}, B \neq 0 \)). Therefore, by (7)
\[ \mathcal{U}(s) = \frac{1}{2} \{q \sqrt{p}/\sqrt{p}, s\} = \frac{1}{2} \{q, s\}. \] (10)

The involutive nature of the Legendre transform implies a dual version of the above equations. These are obtained by considering the correspondence
\[ S_0 \leftrightarrow T_0, \quad q \leftrightarrow p. \] (11)
In particular, the dual versions of Eqs. (7) and (10) are
\[ \left[ \frac{d^2}{dt^2} + \mathcal{V}(t) \right] p \sqrt{q} = 0 = \left[ \frac{d^2}{dt^2} + \mathcal{V}(t) \right] \sqrt{q}, \] (12)
and

\[ V(t) = \frac{1}{2} \{ p\sqrt{q}/\sqrt{q}, t \} = \frac{1}{2} \{ p, t \}, \]

where \( t = T_0(p) \).

We note the existence of a self-dual state in which the solutions of Eqs. (7)-(12) coincide. This happens for \( p = \gamma/q \), that is for \( S_0 = \gamma \ln q \), so that

\[ S_0 + T_0 = pq = \gamma, \tag{13} \]

is constant. This case is of crucial significance for obtaining the covariant construction.

Let us now consider the stationary classical Hamilton-Jacobi equation (CHJE)

\[ \frac{1}{2m} \left( \frac{\partial S_0^{cl}}{\partial q} \right)^2 + V(q) - E = 0. \tag{14} \]

As in (14) there is only one differential operator, it should be possible to consistently require covariance under diffeomorphisms. In particular, requiring that \( \tilde{S}_0^{cl}(\tilde{q}) = S_0^{cl}(q) \), we have

\[ (\partial_q S_0^{cl}(\tilde{q}))^2 (d\tilde{q})^2 = (\partial_q S_0^{cl}(q))^2 (dq)^2. \]

Therefore, \((\partial_q S_0^{cl})^2\) belongs to \( \mathcal{Q} \), the space of quadratic differentials. Requiring covariance of (14) implies that

\[ \mathcal{W}(q) \equiv V(q) - E, \tag{15} \]

belongs to \( \mathcal{Q} \), that is

\[ \tilde{\mathcal{W}}(\tilde{q})(d\tilde{q})^2 = \mathcal{W}(q)(dq)^2. \]

Covariance suggests that there exists always a coordinate choice \( \tilde{q} \) in which a state with arbitrary \( \mathcal{W}(q) \) reduces to the system with \( \tilde{\mathcal{W}}(\tilde{q}) = 0 \). We will call this "diffeomorphic equivalence principle". We now show the crucial fact that in classical mechanics, covariance and equivalence principle cannot be simultaneously satisfied. This follows from the fact that as \( \mathcal{W} \in \mathcal{Q} \) we have

\[ \mathcal{W}(q) = 0 \longrightarrow \tilde{\mathcal{W}}(\tilde{q}) = (\partial_q \tilde{q})^{-2} \mathcal{W}(q) = 0. \tag{16} \]

Therefore, due to the homogeneity of the transformation properties of quadratic differentials, a state with \( \mathcal{W} = 0 \) is a fixed point in the space of states. That is, denoting by \( \mathcal{H} \) the space of all possible \( \mathcal{W} \), we have that requiring covariance, \( \mathcal{H} \) cannot be reduced to a point after factorizing by the diffeomorphisms.
Diffeomorphic equivalence means that for each pair $W_a, W_b \in \mathcal{H}$ there is a coordinate choice such that $W_a(q) \rightarrow \tilde{W}_a(q) = W_b(\tilde{q})$. In particular, this implies that for all $W \in \mathcal{H}$, there is a coordinate choice $q_0$ such that

$$ W(q) \rightarrow \tilde{W}(q_0) = 0. $$

In the following we will derive a differential equation for $\mathcal{S}_0$ with the following properties

1. It is covariant under diffeomorphisms,

2. All the states $W \in \mathcal{H}$ are diffeomorphic equivalent,

3. In a suitable limit it reduces to the CHJE.

Condition 1. implies that the equation we are looking for has the general structure

$$ \frac{1}{2m} \left( \frac{\partial \mathcal{S}_0}{\partial q} \right)^2 + \mathcal{V}(q) + Q(q) = 0, $$

where

$$ (\mathcal{V} + Q) \in \mathcal{Q}. $$

Condition 2. implies that $W \not\in \mathcal{Q}$, so that by (18) we have $Q \not\in \mathcal{Q}$. The classical limit $Q \rightarrow 0$, for which Eq.(17) reduces to Eq.(14), corresponds to the covariance breaking limit, so that $Q$ has the geometrical nature of a covariantizing term.

Let $\mathcal{K}$ be the space of all possible $\mathcal{S}_0$. By 2. and Eq.(17), which provides a correspondence between the $\mathcal{H}$ and $\mathcal{K}$ spaces, it follows that all the states in $\mathcal{K}$ are related by diffeomorphisms. Let us consider the $\mathcal{V} = 0$ case, so that $(\partial_q \mathcal{S}_0)^2 = -2mQ$. Observe that as $(\partial_q \mathcal{S}_0)^2 \in \mathcal{Q}$, and $Q \not\in \mathcal{Q}$, covariance would apparently imply $Q = 0$. Furthermore, in this case Eq.(17) gives $\mathcal{S}_0 = \text{cnst}$ and as $\tilde{\mathcal{S}}(\tilde{q}) = \mathcal{S}_0(q)$, we have that starting from $\mathcal{S}_0 = \text{cnst}$, for any choice of coordinates we will always have $\tilde{\mathcal{S}}(\tilde{q}) = \text{cnst}$, so that, in contradiction with 2. $\mathcal{S}_0 = \text{cnst}$ would be a fixed point in $\mathcal{K}$.

Note that also the duality between $\mathcal{S}_0$ and $\mathcal{T}_0$ considered above breaks down when $\mathcal{S}_0$ is a constant. Furthermore, as $\mathcal{T}_0 = \text{cnst}$ is meaningless, it is clear that full $\mathcal{S}_0$-$\mathcal{T}_0$ duality would imply a reconsideration of $\mathcal{S}_0$.

We now introduce the crucial identity

$$ \left( \frac{\partial \mathcal{S}_0}{\partial q} \right)^2 = \frac{\beta^2}{2} \{ e^{\frac{\beta^2}{2} \mathcal{S}_0}, q \} - \frac{\beta^2}{2} \{ \mathcal{S}_0, q \}, $$

(19)
expressing the quadratic differential \((\partial_q S_0)^2\) as the difference of two Schwarzian derivatives. This forces us to use the exponential of \(S_0\) and therefore to introduce the dimensional constant \(\beta\) and the imaginary factor "i".

By (17) and (19) we have

\[
W(q) = \frac{\beta^2}{4m} \{S_0, q\} - \frac{\beta^2}{4m} \{e^{\frac{3}{2}S_0}, q\} - Q(q). 
\]

We know that in the CHJE there is no universal constant with the dimension of an action. Furthermore, \(\beta\) is the only natural parameter we can use in order to reach the covariance breaking phase in which \(Q = 0\). On the other hand by (19) the quantity \(\frac{\beta^2}{4} \{e^{\frac{3}{2}S_0}, q\} - \frac{\beta^2}{4} \{S_0, q\}\) is \(\beta\)-independent so that it is insensitive to the \(\beta \to 0\) limit.

The scalar nature of \(S_0\) apparently breaks both \(S_0 - \mathcal{T}_0\) duality and diffeomorphic equivalence. There is a remarkable mechanism that solves both problems and fixes \(Q\) in a unique way. To see this let us first set

\[
Q(q) = \frac{\beta^2}{4m} \{S_0, q\},
\]

so that by (20)

\[
W(q) = -\frac{\beta^2}{4m} \{e^{\frac{3}{2}S_0}, q\},
\]

and the differential equation (17) becomes

\[
\frac{1}{2m} \left( \frac{\partial S_0}{\partial q} \right)^2 + V(q) - E + \frac{\beta^2}{4m} \{S_0, q\} = 0.
\]

By (21) in the \(\beta \to 0\) limit, the \(Q\) term disappears so that by breaking covariance we obtain the CHJE (14).

Eq. (21) provides a mechanism, related to the properties of the Schwarzian derivative, which solves the problems due to the scalar nature of \(S_0\). The key point is that for \(W = 0\), besides \(S_0 = \text{cnst}\) the equation

\[
\left( \frac{\partial S_0}{\partial q} \right)^2 = -\frac{\beta^2}{2} \{S_0, q\},
\]

has the dual solution \(S_0 = \pm \frac{3}{2} \beta \ln q\), (this also follows from (22) and the fact that besides \(S_0 = \text{cnst}\) the solution of \(\{e^{\frac{3}{2}S_0}, q\} = 0\) is \(S_0 = \pm \frac{3}{2} \beta \ln q\). In this context we observe the crucial property of the Schwarzian derivative: even if \(\{h, x\}\) vanishes in the coordinate \(x\), by (9) \(\{h, y\}\) is in general non-zero. The above problems are then solved by admitting also imaginary values for \(S_0\). In particular, in the case in which \(W = 0\) we have \(S_0 = \pm \frac{3}{2} \beta \ln q\), so that all the \(W \in \mathcal{H}\) are connected by a coordinate transformation.
Eq. (22) together with the relationship between the Schwartzian derivative and the second order differential equations illustrated above, imply that

$$e^{\frac{2\beta}{3}S_0} = \frac{\psi_1}{\psi_2},$$  \hspace{1cm} (25)

where $\psi_1$ and $\psi_2$ are two arbitrary linearly independent solutions of the equation

$$\left[ -\frac{\beta^2}{2m} \frac{\partial^2}{\partial q^2} + V(q) \right] \psi = E\psi,$$ \hspace{1cm} (26)

which is the stationary Schrödinger equation. Thus, the "covariantizing parameter" $\beta$ is the Planck constant

$$\beta = \hbar.$$ \hspace{1cm} (27)

By (25) a general solution of (26) has the form

$$\psi = \frac{1}{\sqrt{S_0'}} \left( A e^{-\frac{\hbar}{2}S_0} + B e^{\frac{\hbar}{2}S_0} \right).$$ \hspace{1cm} (28)

To make the connection with the standard notation we distinguish the two cases $\bar{\psi}_1 \neq \psi_1$ and $\bar{\psi}_1 \propto \psi_1$. In the first one we can take $\psi_2 = \bar{\psi}_1$, that is

$$\psi_1(q) = R(q) e^{\frac{\hbar}{2}S_0(q)}, \quad \psi_2(q) = R(q) e^{-\frac{\hbar}{2}S_0(q)},$$ \hspace{1cm} (29)

and by Wronskian arguments $R = 1/\sqrt{S_0}$, so that

$$Q(q) = \frac{\hbar^2}{4m} \{ S_0, q \} = -\frac{\hbar^2}{2m} \frac{\partial^2 R}{R},$$ \hspace{1cm} (30)

and (26) is equivalent to (23). In the other case we have

$$\psi_1(q) = R(q),$$ \hspace{1cm} (31)

which, up to a possible trivial global constant, is real. As $\psi_2(q) = R(q) \int^q R^{-2}$ is also real, by (25) $S_0$ is purely imaginary. That is $S_0 = \frac{i}{2} \hbar \ln \int^q R^{-2}$ and

$$\left( \frac{\partial S_0}{\partial q} \right)^2 + \frac{\hbar^2}{2} \{ S_0, q \} = -\hbar^2 \frac{\partial^2 R}{R},$$ \hspace{1cm} (32)

so that Eq. (23) is equivalent to

$$V(q) - E - \frac{\hbar^2}{2m} \frac{\partial^2 R}{R} = 0.$$ \hspace{1cm} (33)
The above two cases are both equivalent to (23) and can be set in the standard form

\[ \frac{1}{2m} \left( \frac{\partial S_0}{\partial q} \right)^2 + V(q) - E - \frac{\hbar^2}{2m} \frac{\partial^2 R}{R} = 0, \]  

\[ \partial_q \left( R^2 \partial_q S_0 \right) = 0. \]  

In the real case the continuity equation (35) gives \( R = 1/\sqrt{S_0} \) and by (30), Eq.(34) corresponds to Eq.(23). In the purely imaginary case Eq.(35) degenerates and (34) is equivalent to (33) (and then to (23)).

This setting solves the problems related to the scalar nature of \( S_0 \) simply by requiring that \( S_0 = \text{cnst} \notin \mathcal{K} \)! Remarkably, it also solves the problem of the \( S_0 - T_0 \) duality related to the indefinability of the Legendre transform when \( S_0 = \text{cnst} \). Therefore, the fundamental differential equation (23) with the initial condition

\[ S_0 = \pm \frac{i}{2} \hbar \ln q, \quad \text{for} \quad W = 0, \]  

rather than \( S_0 = \text{cnst} \), provides the way of describing physical structures in a covariant and diffeomorphic equivalent way. This implies the full \( S_0 - T_0 \) duality and fixes the universal self-dual state to be (13) with

\[ \gamma = \pm \frac{i}{2} \hbar. \]  

Observe that by (30) and (32) the quantum potential \(-\frac{\hbar^2}{2m} \frac{\partial^2 R}{R}\) is \( Q(q) = \frac{\hbar^2}{4m} \{ S_0, q \} \) for \( S_0 \) real and \( \frac{1}{2m} \left( \frac{\partial S_0}{\partial q} \right)^2 + \frac{\hbar^2}{4m} \{ S_0, q \} \) for \( S_0 \) purely imaginary. In this context we stress again the relevance of the identity (19) which contains both the classical and quantum parts \( W \) and \( Q \) respectively. In particular, note that it includes in the same equation both \( e^{\frac{i}{\hbar} S_0} \) and \( S_0 \). If one considers \( S_0 \) as a scalar field operator, then the vertex \( e^{\frac{i}{\hbar} S_0} \) resembles the bosonization of a fermionic operator. It is amusing that inspired by duality in SUSY Yang-Mills [1, 2], we obtained a quantum mechanical expression reminiscent of supersymmetry. This would suggest that fermion-boson correspondence is related to diffeomorphic equivalence.

Diffeomorphic equivalence means that starting from an arbitrary state \( W \in \mathcal{H} \) and choosing the coordinate

\[ \tilde{q} = e^{\frac{i}{\hbar} S_0(q)}, \]  

gives \( \tilde{W}(\tilde{q}) = 0 \), and the Schrödinger equation becomes

\[(\partial_q^2 + W)\psi(q) = (\partial_q \tilde{q})^{3/2} \partial_{\tilde{q}}^2 \tilde{\psi}(\tilde{q}) = 0, \]  

where by (28) \( \tilde{\psi}(\tilde{q})(d\tilde{q})^{-1/2} = \psi(q)(dq)^{-1/2} \).
Observe that if $S_0$ is purely imaginary, then by (38) $\tilde{q} = \int q R^{-2}$ is real, whereas if $S_0$ takes real values, we have $\tilde{q} = e^{\frac{i}{\hbar} S_0(q)}$ which is complex. On the other hand, a phase can be transformed to the real axis by a Möbius transformation (Cayley map), which does not change the Schwarzian derivative, so that $\tilde{W}(\tilde{q}) = 0$ is invariant. Therefore, in the case in which $S_0$ is real, instead of (38) we can choose

$$\tilde{q} = \frac{e^{\frac{i}{\hbar} S_0(q)} + i}{ie^{\frac{i}{\hbar} S_0(q)} + 1},$$

(40)

or any real Möbius transformation of it. In general, we have that the existence of the self-dual state makes it possible to find a trivializing coordinate $\tilde{q}$, a solution of the differential equation

$$\{\tilde{q}, q\} + \frac{4m}{\hbar^2}(V(q) - E) = 0,$$

(41)

in which any $\mathcal{W} \in \mathcal{H}$ reduces to $\tilde{W}(\tilde{q}) = 0$.

The quantum correction can also be seen as a modification of the CHJE, Eq. (14), by a "conformal factor"

$$\frac{1}{2m} \left( \frac{\partial S_0}{\partial q} \right)^2 \left[ 1 - \hbar^2 U(S_0) \right] + V(q) - E = 0.$$

(42)

The quantum correction $U(S_0)$ in the conformal factor exactly corresponds to the potential in (7) and (10) related to the Möbius symmetry of the Legendre transform where no quantum mechanical aspects were manifest. Eq.(42) illustrates the fundamental role of the self-dual state $S_0 = \pm \frac{i}{\hbar} \ln \sqrt{q}$ as in this case $(1 - \hbar^2 U(\pm \frac{i}{\hbar} \ln \sqrt{q})) = 0$ which is the solution associated to $\mathcal{W} = 0$ and fully characterizes quantum mechanics.

We note that a non vanishing additive term $F(S_0)$ in the right hand side of (21) implies a differential equation for $S_0$ which could not satisfy conditions 1.-3. above. In particular, the self-dual state would not be solution of $(\partial_q S_0)^2 = -2mQ$, the differential equation in the $\mathcal{W} = 0$ case. We observe that in literature, in considering the solutions of $\partial_q^2 \psi = 0$ one sets $\psi = Re^{\frac{i}{\hbar} S_0}$ with $R = A + Bq$ and $S_0 = cnst$. However, the equivalence principle implies that $(S_0 = cnst) \notin \mathcal{K}$ and that the solution is rather $S_0 = \pm \frac{i}{\hbar} \ln q$ which by (28) still gives $\psi = A + Bq$.

The remarkable point is that by (22) the solution associated to the $\mathcal{W} = 0$ state is not only $S_0 = cnst$ (which gives $\{cnst, q\} = 0$) but also $S_0 = \pm \frac{i}{\hbar} \ln q$ (which gives $\{q, q\} = \{(Aq + B)/(Cq + D), q\} = 0$). Whereas $S_0 = cnst$ gives $\mathcal{W}(q) = -\frac{\hbar^2}{4m} \{e^{\frac{i}{\hbar} S_0}, q\} = 0$ for any coordinate choice, choosing $S_0 = \pm \frac{i}{\hbar} \ln q$, $\tilde{W}(\tilde{q})$ would in general be non–zero even if $\mathcal{W}(q) = 0$. We can characterize quantum mechanics by requiring that for $\mathcal{W} = 0$, the
Hamilton-Jacobi equation admits the non-vanishing solution corresponding to the self-dual state $S_0 = \pm \frac{1}{2} \hbar \ln q$. We have actually seen that this solution, rather than $S_0 = \text{cnst}$, characterizes the quantum mechanical phase.

The existence of the self-dual solution is a universal property that should hold in any dimension. This allows the full $S_0 - T_0$ duality as the Legendre transform

$$S_0 = \sum_{k=1}^{D} p_k \frac{\partial T_0}{\partial p_k} - T_0,$$

is always defined. The fact that the apparently innocuous state $W = 0$, corresponds to the non-trivial universal solution $S_0 = \pm \frac{1}{2} \hbar \sum_{k=1}^{D} \ln q_k$ implies that also in higher dimensions one obtains the Schrödinger equation.

Finally, in the time dependent case the form of the equation for the action $S$ is determined by considering that in the classical limit it should correspond to the Hamilton-Jacobi equation and that in the time independent case it reproduces the above results. This implies the quantum Hamilton-Jacobi equation in the general case, so that it reproduces the time-dependent Schrödinger equation.

Acknowledgments. We would like to thank G. Bertoldi, G. Bonelli, L. Bonora, R. Carroll, G.F. Dell’Antonio, E.S. Fradkin, E. Gozzi, F. Illuminati, A. Kholodenko, A. Kitaev, P.A. Marchetti, J. Ng, P. Pasti, S. Shatashvili, M. Tonin and R. Zucchini, for discussions.

Work supported in part by DOE Grant No. DE-FG-0586ER40272 (AEF) and by the European Commission TMR programme ERBFMRX-CT96-0045 (MM).

References

