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STRENGTH DISTRIBUTION AND SIZE EFFECTS FOR THE FRACTURE OF FIBROUS COMPOSITE MATERIALS

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ABSTRACT

Random network models have recently been developed in the physics literature to explain the strength and size effect in heterogeneous materials. Applications have included the breakdown of random fuse networks, dielectric breakdown and brittle fracture. Unfortunately, conventional scaling approaches of statistical mechanics have yielded incorrect predictions, and new approaches have been proposed which build on field enhancement occurring near the tips of critical, random clusters together with the statistical theory of extremes. New distributions and size scalings for strength have been proposed and supported through Monte Carlo simulation. Here we consider an idealized, one-dimensional model for the failure of such networks where elements of constant strength may be initially present or absent at random. Our idealized rule for local stress redistribution near breaks reflects features we find in a discrete mechanics model that has limiting forms consistent with continuum theories for cracks. We obtain rigorous asymptotic results for the strength distribution and size effect with constants and exponents that are known. The validity of various analytical approximations in the literature is then discussed.

1 INTRODUCTION

The phenomenon known as the size effect in brittle material strength has been known for almost 500 years. In unpublished work, Leonardo da Vinci (ca. 1500) observed that wires weaken with increasing length, and in perhaps the first published work on the subject, Galileo noted that for structures which are geometrically similar, their strength decreased as their dimensions increased (see Timoshenko (1953)). By 1975, hundreds of works on the subject had appeared, most of which were surveyed by Harter.

For several decades, many models have been developed for studying size effects in the strength of brittle materials containing flaws. Over fifty years ago Weibull (1939, 1951) presented a statistical theory for strength built on a weakest-link concept coupled with statistical variation in the strength of a unit link or volume element. On the basis of experimental data he argued for a

particular statistical distribution in strength which includes dependency on the total number of such unit volume elements. Today this distribution bears his name, and the two-parameter Weibull distribution is given by

$$F(\sigma) = 1 - \exp(-V(\sigma/\sigma_0)^p), \quad \sigma \geq 0, \quad (1)$$

where σ is the stress level over the volume V , and the parameters p and σ_0 are the shape and scale parameters respectively, the latter measured at volume V_0 . In Weibull's model the strength declines and scales algebraically with volume as $\sigma_0 V^{-1/p}$.

As pointed out by Epstein (1948), models such as Eq. (1) provide a natural connection to the statistical theory of extremes (Castillo, 1988). In such models the results depend heavily on the assumed shape of the strength distribution at low failure stresses and low failure probabilities for the unit volume elements, which are typically viewed as containing small crack-like flaws (Freudenthal, 1968). Yet obtaining firm guidance on specific, analytical choices for shapes of the lower tails has proven elusive both from physical fundamentals and from experimental observations. Strictly speaking, even the monolithic ceramics and glassy materials in early size effect studies are heterogeneous when viewed at the microscale, containing random distributions of flaws and grain sizes. Moreover advanced materials engineered today are not only heterogeneous (at larger size scales), but are multiphase, allowing designers room to optimize their microstructural features. Such materials along with technological advances in microscopy have introduced new levels of complexity and opened new challenges in modelling size effects in material strength. Therefore, making further progress in characterizing the size effect has required delving into the microstructural details of local material heterogeneity, e.g. weak grain boundaries, flaws, voids, inclusion, and its statistical character.

One approach to accounting for microstructural detail has been to develop discrete network or lattice models of failure based

on fiber bundles theories. One branch of these has borrowed the classic work of Daniels (1945) on the failure of simple bundles of fibers under equal load-sharing among surviving fibers, an example of which is the series-parallel model of Gücer and Gurland for particulate composites (1962). Asymptotic distributions for the strength of chains of Daniels bundles in terms of changes in dimensions have been discussed by Smith and Phoenix (1981). Coleman (1957) also pursued a time dependent version to explain the breakdown character of polymer fibers. Though analytically tractable, these models and their 'global load-sharing' generalizations (Curtin, 1991; Phoenix *et al.*, 1997) are much more applicable to the strength of weakly-bonded fibrous materials than to tightly-bonded fibrous materials displaying brittle-like failure. These equal load-sharing models can be viewed as mean-field models, whose continuum versions form a branch of nonlinear continuum mechanics called continuum damage mechanics (Kachanov, 1986; Lemaitre, 1992); though in the latter, important statistical fluctuations are typically ignored.

In the following sections, we discuss issues in modelling size effects associated with probability distributions for failure. The probabilistic failure model developed in this work is motivated by fibrous composites, wherein the strength of the fibers follows Eq. (1). Therefore we begin with previous theories in this area, followed by those developed in the physics literature with similar scaling features. In the latter, materials of interest exhibiting size effects have been dielectric materials with conducting particles, random resistor networks, and elastic lattices with flaws.

1.1 Failure Models For Fibrous Composites

The technological development of fibrous composites has motivated other versions of network models for failure (Smith, 1980, 1983; Harlow and Phoenix, 1981; Smith *et al.* 1983; Harlow, 1985; Kuo and Phoenix, 1987), where the load-sharing is much more localized. In such models the basic fiber elements are assumed to follow some distribution for strength such as Eq. (1). Elements that fail are assumed to redistribute their loads locally onto nearby unfailed neighbors, increasing their probability of failure and the likelihood of crack nucleation. Subsequent load increases eventually lead to the formation of an unstable crack and finally catastrophic failure. In this respect, these models go farther than traditional fracture mechanics models, which typically bypass the crack nucleation stage by postulating the existence of a microcrack and analyzing the macroscopic consequences upon its growth. Typically the assumptions on geometry and local stress redistribution have been highly idealized in order to make the models analytically tractable.

For planar versions of these local load-sharing models (which are really extended one-dimensional models), a variety of recursive (Harlow and Phoenix, 1981; Harlow, 1985; Kuo and Phoenix, 1987) and asymptotic approaches (Smith, 1980, 1983; Smith *et al.* 1983) have been developed to uncover the fundamental structure of the failure distributions and size effect. One major result, under broad assumptions on the distribution for strength of fiber elements, is that the distribution function for failure is given by

$$F(\sigma) \equiv 1 - [1 - W(\sigma)]^V = 1 - \exp\{-VW^*(\sigma)\}, \quad (2)$$

where $\sigma \geq 0$ is the applied stress, V is the total number of fiber elements, $W^*(\sigma) = -\log\{1 - W(\sigma)\}$, and $W(\sigma)$ is one minus the largest eigenvalue of a certain transition matrix describing the

failure process. The error results from boundary and size residue effects (from the smaller eigenvalues), both of which diminish extremely rapidly in their effects as V increases. Unfortunately, for most choices of the strength distribution for fiber elements, $W(\sigma)$ has not generally had a simple analytical form in terms of classical functions, so it has been possible only to approximate the size effect. Nevertheless, Smith (1980, 1983) was able to argue that for fiber elements following Eq. (1) and stress redistribution laterally to nearest non-failed neighbors, the median strength σ^* decreases logarithmically with volume V according to

$$\sigma^* \equiv \rho 2^{1-1/\rho} \sigma_0 (\log V)^{-1}, \quad (3)$$

where the error decreases as ρ increases.

1.2 Network Failure Models In Statistical Physics

In the past decade, network or lattice models of failure of heterogeneous materials with disordered microstructures have received considerable attention in the physics literature, particularly in connection with percolation theory. In contrast to the usual percolation models for transport properties, models have been developed to treat failure, such as conductivity breakdown in random resistor or fuse networks Arcangelis *et al.*, 1985, 1986; Kahng *et al.*, 1987; Duxbury and Leath, 1987; Duxbury *et al.*, 1987; Li and Duxbury, 1987, 1989; Roux *et al.*, 1988), dielectric breakdown in materials with randomly dispersed conducting inclusions (Roux *et al.*, 1988; Beale and Duxbury, 1988), critical currents in superconducting networks (de Arcangelis *et al.*, 1986; Kim and Duxbury, 1991; Sahimi and Goddard, 1986), and catastrophic failure of elastic lattices with random element strength (Sahimi and Goddard, 1986; Beale and Srolovitz, 1988; Hansen *et al.*, 1989; Herrmann and Roux, 1990). (A monograph edited by Herrmann and Roux (1990) is almost totally devoted to the last subject.) This new feature requires proposing failure criteria and calculating field distributions until breakdown occurs or when the lattice fails under the externally applied field.

The random fuse model, introduced by de Arcangelis *et al.* (1985), has proven to be a good prototypical model of fracture because of its relative simplicity. One version takes the form of a planar square lattice of size $L \times L$ where the conducting elements are initially resistors or fuses with probability p or insulators with probability $1-p$, where interest is primarily in the regime above the percolation threshold p_c , above which an infinite network is initially conducting. Each fuse has constant resistance when the applied voltage across it is less than a critical value v_c , but burns out to become an insulator when its applied voltage exceeds v_c . (Without loss in generality we take $v_c = 1$ in later discussion.) The interest has been in scaling relations, in terms of L and p , for the breakdown process of the full network (transition to an insulator) as the applied voltage V to the network is increased.

Despite its simplicity, analytical results have been difficult to obtain mainly because calculation of the currents in all the surviving fuses for an arbitrary array of burned out fuses has to be done numerically. Thus, Monte Carlo simulations of sample networks through numerical solution of Kirchhoff's laws were carried out by de Arcangelis *et al.* (1985) by continually adjusting the external voltage to fail the 'hottest' fuse when the voltage across it exceeded v_c . One quantity studied was the normalized applied

voltage $V = V/L$ generating progressions of local resistor failure sequences, which have the features of a wandering transverse cracks. Because of the computational complexity, results were obtained only for relatively small lattices (80×80) and for $p_c < p < 1$. In this range of p , they noted a difference between the voltage for initiation, V_1 , versus the maximum required to sustain propagation to failure, V_b , though the process appeared to be self sustaining for the most homogeneous networks (p near 1) once the first bond failed. They observed apparent power-law dependence of various characteristic quantities with exponents which were not related to known exponents of percolation. In a related paper, where fuse breakdown was not modeled, de Arcangelis *et al.* (1986) pointed out that a conventional scaling approach does not provide a complete description of the random resistor network, as an infinite hierarchy of exponents is needed to characterize the moments of the voltage distribution. For random fuse networks and elastic lattices, Li and Duxbury (1989) pointed out that conductivity and elastic moduli are related to the second moment of the corresponding field distribution in a disordered material, whereas breakdown is related to the very high moments of this distribution. Therefore the moment spectrum quantifies the crossover between these two very different classes of properties. These and other results (1987), underscored a stark contrast and apparent anomalies in scaling properties of fracture versus mean field properties typically studied in the statistical mechanics literature.

Duxbury, Leath and Beale (1987) considered in detail a particular anomaly, namely size dependence in two normalized breakdown voltages of interest, namely $V_1 = V_1/L$, and $V_b = V_b/L$. The size effects were clearly demonstrated from the results of a Monte Carlo simulation of two-dimensional sample square lattices of size $L \times L$ up to 200×200 . To understand the scaling character (for $p > p_c$) they considered Lifshitz-type arguments on the effect of 'defects' in the form of a contiguous transverse row of missing or burned out fuse elements (reminiscent of observed 'cracks') focusing on the current enhancement in the fuses at the row 'tips'. By estimating the probability that such a transverse defect would occur at a given location, they developed a scaling relationship similar to Eq. (3) where $V = L^2$. They also drew a connection to the statistical theory of extremes through calculation of probabilities of the largest 'critical' defect cluster in the lattice. They again focused on the notion that in the dilute limit (p near 1), the defect of the most critical shape and orientation is the defect with the most current enhancement at its edges. This defect cluster was a transverse slit. Using a continuum approach involving the solution to Laplace's equation they determined the current enhancements (Duxbury *et al.*, 1987)

$$i_{\text{tip}} \sim i(1 + k_2 j^{1/2}) \quad (4)$$

where j is the number of missing bonds in the cluster and i is the externally applied current per fuse. Similar relations also apply to local enhancement in the voltage V . On the other hand, they appreciated the possible importance of a defect shape in which current enhancement in the most critical fuse is approximately proportional to j . They suggested that this would occur in the one intact bond sandwiched between two colinear cracks of size j . Further study by Li and Duxbury [28] revised this proportionality to

$$i_{\text{tip}} \sim ij/\log j. \quad (5)$$

The key point is that when such a bond fails and consequently, the two j cracks coalesce to form a larger crack, the current enhancement at the new edges is actually lower than on the failing bond. As a result, an even larger external voltage is needed to fail the network. This suggested that V_1 and V_b must have different scaling behavior.

Next they focused on the calculation of $P(j)$, the probability that j transversely adjacent bonds will be missing *somewhere* in the $L \times L$ network. Using spiral boundary conditions to reduce the problem to a one dimensional problem, they calculated the complementary probability $P(j+1)$ that none of the horizontal rows contain a critical cluster of $j+1$ or more contiguous missing bonds, obtaining the result

$$\begin{aligned} P(j+1) &\sim [1 - p(1-p)^{j+1}]L^2 \\ &\sim \exp[-p(1-p)L^2 \exp(-jk)] \end{aligned} \quad (6)$$

where $k = -\log[(1-p)]$. Combining (4) to (6) with scaling arguments from the statistical theory of extremes, they argued that the distribution functions for the normalized breakdown voltages V_1 and V_b must have the forms (in 2-d)

$$F(V_s) = 1 - \exp\{-c_s L^2 \exp[-k_s V_s^{-1/\alpha_s}]\}, \quad (7)$$

for $s = 1, b$, and where c_s and k_s are constants depending on p , and the α_s is an exponent independent of p . The above calculations suggest $\alpha_1 \equiv 1$, $k_1 \equiv -\log(1-p)$ and $c_1 \equiv p(1-p)$ for the dilute case of $p \approx 1$. The Lifshitz arguments leading to (4) suggested $\alpha_1 \equiv 1$ and $\alpha_b \equiv 1/2$, and (5) and (6) suggest k_b is proportional to $-\log(1-p)$. Generally, however, they recognized difficulties in obtaining analytical expressions for these various constants and exponents. They also anticipated difficulties in developing versions of (6) in higher dimensions because of the wealth of possible shapes for a defect cluster, which might dramatically affect k .

For the size effect, upon solving $F(V_s^*) = 1/2$ in (6), one obtains

$$V_s^* = 1/[A_s(p) + B_s(p) \log L]^{\alpha_s}, \quad s = 1, b \quad (8)$$

where

$$A_s(p) = [\log(c_s) - \log \log 2]/k_s, \quad (9)$$

and

$$B_s(p) = 1/k_s. \quad (10)$$

For $s = 1$, Eq. (8) is asymptotically of the same form as Eq. (3) as L increases. The above distributional form and size effects in V_1 and V_b were largely substantiated by Monte Carlo simulation of networks up to 200×200 in size (Duxbury *et al.*, 1987).

Duxbury and co-workers also considered a network model of dielectric breakdown in metal-loaded dielectrics (Duxbury and Leath, 1987; Duxbury *et al.*, 1987; Beale and Duxbury, 1988; Kim

and Duxbury, 1991), where p is the probability that a bond element is a conductor, and p_c is the percolation threshold above which a large network is automatically conducting. They considered the scaling of the applied breakdown fields for first bond breakdown E_1 and for complete (conducting path) breakdown E_b in the range $p < p_c$. The results were similar in form to those above for the random resistor network (with E replacing V) except that the critical defects were roughly close pairs of collinear conducting clusters of elements oriented *parallel* to the applied field, and different constants and exponents were thought to be involved. In this case, the field enhancement was suggested to be proportional to the defect length. Also Beale and Duxbury (1988) suggested that E_1 and E_b are essentially the same, and $\alpha = 1$ independent of the network dimensions.

Building on the earlier work of Duxbury and co-workers (Duxbury and Leath, 1987; Duxbury *et al.*, 1987), Beale and Srolovitz (1988) studied elastic failure in a two-dimensional triangular lattice with nearest neighbor bonds in the form of harmonic springs (with no bond bending). A spring is present with probability p and fails irreversibly once its strain exceeds a certain fixed threshold value the same for all springs. The analysis was carried out for $p > p_r$, the rigidity percolation threshold below which the elastic modulus vanishes, even though the lattice is macroscopically connected. Their interest was in σ_1 , the applied stress where the first bond breaks, and σ_b , the stress at final breakdown. These were suggested to be *equivalent* based on peak stresses observed in the Monte Carlo simulation of planar lattices up to size 70×70 . As in the other material networks, they also argued that the critical defect was an arrangement of two collinear cracks of total length j in close proximity and oriented perpendicular to the applied stress. Their numerical results indicated that the enhancement on a bond element *between* the collinear cracks grew in proportion to j , whereas it grew as $j^{1/2}$ on a bond at the edge of a *single* crack. They also mentioned the relatively slow decay in the strain field ahead of the single crack tip proportional to $r^{-1/2}$, where r is distance from the tip, a result also consistent with elastic continuum fracture mechanics. Again the distributions for σ_b ($\approx \sigma_1$) were proposed to be $F(\sigma)$ of Eq. (7) but with different constants and exponents as compared to the random fuse network. Despite their emphasis on the critical collinear arrangement, they argued that $1 \leq \alpha_b \leq 2$ because of the significant contribution of single cracks (rather than collinear) and indeed found that the values 1 and 2 both produced equally appealing fits to the data. Evidently the limited sizes of the lattices precluded resolving such issues.

Despite the anomalous behavior mentioned above, the general size scalings and distributional forms obtained by Duxbury and coworkers (Duxbury *et al.*, 1987; Duxbury and Leath, 1987; Li and Duxbury, 1987; Li and Duxbury, 1989; Beale and Duxbury, 1988; Kim and Duxbury, 1991; Sahimi and Goddard, 1991; Beale and Srolovitz, 1988) have not been universally observed in simulations. Sahimi and Goddard (1986) considered a model similar to that of Beale and Srolovitz (1988) except that the bond element spring constants and failure strains were modified to be distributed quantities. They did not uncover the size scaling discussed in the above works for lattices of linear dimension up to $L = 40$. Hansen (1990) gives a nice review of percolation theory for disordered systems of the type discussed above, but largely for transport properties in the case where bonds are intact with probability p and have infinite strength. At the end of the article he comments on failure in elastic networks where the distribution (probability

density function) of local failure strengths σ is of the form $P_\alpha(\sigma) = (1-\alpha)\sigma^{-\alpha}$ for $0 < \sigma \leq 1$ and α near 1. In an infinite system he argues that the strength will have properties similar to the usual percolation scalings but with modified exponents, thus precluding the possibility of size effect scalings such as Eq. (3) and Eq. (8). On the other hand Hansen *et al.* (1989) numerically studied elastic lattices over sizes 4×4 to 24×24 assuming the continuous, uniform distribution $P(\sigma) = 1$ for $0 < \sigma \leq 1$ and on the basis of the simulation results argued for power-law (algebraic) scaling in applied failure force per unit width $F_{\max}/L$ is proportional to $L^{\beta-1}$ where $\beta \approx 3/4$, thus disputing the logarithmic scaling Eq. (3) or Eq. (8). In fact $\beta \approx 3/4$ was argued to be insensitive to the choice of $P(\sigma)$. They suggested that the disagreement with the scaling of Eq. (8) results from the difference in assumptions of the form of the element failure strength distribution, which in the case of Eq. (8) is distinctly discrete with only two possible strengths, 0 or 1; that is $P(\sigma) = p\delta(\sigma-1) + (1-p)\delta(\sigma)$ where δ is a Dirac delta function. (We remark that this argument cannot be invoked to explain Eq. (3) based on Eq. (1).) In later work Hansen *et al.* (1991) argued that for $p > p_c$ rescaling through a renormalization argument leads to the disappearance of disorder as the effective value of p defined at scale L converges to 1 as L goes to infinity. Thus such models were implied to be asymptotically equivalent to a disorderless system, which would have a finite average strength in an infinite lattice limit, and so, observations to the contrary, such as those by Duxbury and coworkers as in Eq. (8), were argued to be transients.

Thus, there has been much debate about the particular form of the size scaling. A main origin of the controversy is that the size effect is sufficiently weak that many orders of magnitude in sample dimensions in simulations are really necessary to arrive at definitive conclusions. In most cases such sizes have been inaccessible by Monte Carlo simulation alone as lattices approaching 500×500 rapidly become very time consuming in large replications. We will consider a model in this paper where lattices of this size are much too small to reveal ultimate large scale behavior. Real structural components, of course, will have 10^8 to 10^{20} elements.

1.3 The Trend To Rigorous Analysis On Idealized Models

Recently, investigators on both sides of the controversy have turned to rigorous study of idealized, one-dimensional models of failure (Harlow and Phoenix, 1991; Duxbury and Leath, 1994a,b; Leath and Duxbury, 1994; Zhang and Ding, 1994a,b,1996; Kloster *et al.*, 1997) in order to put approximate analyses of the more complex models on firmer ground. These simpler models are variations on a local-load sharing (LLS) model in Harlow and Phoenix (1991) (see earlier papers referenced therein) and are analytically solvable, having been tackled by various recursive, asymptotic, and numerical analysis methods supplemented on occasion with Monte Carlo simulation. They have been constructed to reflect the essence of stress redistribution in the various network models discussed above. Understanding their method of solution and subtleties of prediction is key to providing guidance in tackling more complex models, and they appear to be providing an accurate picture of the dominant forms of scaling for a broad range of assumed element strength distributions. In most cases the results support the size scaling in Eq. (3) and Eq. (8), that the

average strength in such models goes to zero logarithmically. Interestingly, algebraic (power-law) size scaling has been found to be a transient effect for certain *small-scale* samples (Leath and Duxbury, 1994a) and not the other way around (Hansen *et al.*, 1989). Nonetheless, such results are dependent on the load-sharing scheme as well as the assumed form of the distribution for element strength. This is clear from hierarchical bundle models studied by Newman *et al.* (1994), which in certain cases show strength varying inversely with volume V as $(\log \log V)^{1/\nu}$ where ν is a known exponent, but not in other cases where the differences are subtle.

1.4 Focus Of The Present Work

This paper will continue the study of series-parallel models in two dimensions with the basic analytical structure of one-dimensional, LLS models but modified to reflect a more diffuse tapered load-sharing (TLS) rule whereby 1/3 of the load of a failed fiber is redistributed equally onto next-nearest unfailed neighbors. Again we will begin with the simplest percolation-type assumption that fibers have strength 1 with probability p and strength zero with probability $q = 1-p$. We will motivate the important aspects of the load-sharing rule in the context of aligned fibrous composites through use of results for transversely aligned patterns of broken fibers based on recent extensions the two-dimensional, classic shear-lag model of Hedgepeth (Hedgepeth, 1961; Fichter, 1969; Beyerlein *et al.*, 1996; Beyerlein and Phoenix, 1997a,b; Hikami and Chou, 1990). One focus of the investigation will be the assertion discussed above that the most critical defect is not a single cluster of n vacancies but rather a double cluster (double co-linear crack) of n vacancies separated by a single occupied site located anywhere within the $n+1$ adjacent sites, with load on it depending on n (or $n/\log n$) rather than $n^{1/2}$ at the edge of a single, tight cluster. We will investigate the extent to which first element failure is virtually equivalent to total failure through a spanning transverse crack, as was suggested in the literature reviewed above.

We will obtain analytically precise asymptotic results for the distribution functions and size scalings for the applied loads at which the first fiber element fails and the full network fails, giving error estimates for both. In particular, we will be able to evaluate analytically all the constants in the model in terms of volume $V = L^2$ and p , and we will compare the results to Eq. (8) and Eq. (9) where we will see important departures caused by the additional complexity. The analysis is based on the Stein-Chen method of Poisson approximation as described in Arratia *et al.* (1990) and Barbour *et al.* (1984) and used earlier by Harlow and Phoenix (1991) in the simple LLS case. These problems have a strong connection with probabilities for long head runs in coin tossing experiments (Gordon *et al.*, 1986).

2 HEDGEPEETH LOAD-SHARING MODEL

2.1 Analytical Forms And Connection To Continuum Fracture Mechanics

A simple but accurate model for calculating load concentrations around broken fibers in a two-dimensional, unidirectional, planar composite sheet is the classic shear-lag model of Hedgepeth (1961). In this model, the equispaced fibers are elastic, deform and carry loads only in tension, and are well-bonded to the matrix. The matrix is also elastic but deforms and

carries load only in simple shear, and thus, is the vehicle for transmitting the tensile load of broken fibers to its intact neighbors. In his original work, Hedgepeth considered stress concentrations produced by an aligned row of t contiguous fiber breaks perpendicular to the fiber and loading direction. It turns out that the magnitudes of these stress concentrations along the transverse plane of the breaks are independent of the stiffness moduli, spacings, and cross-sectional areas of the fibers and matrix (though the longitudinal length scale over which the transfer takes place does depend on these quantities).

Certain results obtained by Hedgepeth and subsequent investigators (Hedgepeth, 1961; Fichter, 1969; Beyerlein *et al.*, 1996; Beyerlein and Phoenix, 1997a,b; Hikami and Chou, 1990) are motivation for the load-sharing model we use in later analysis. For a row or cluster of t consecutive fiber breaks, let z be the distance of an intact fiber from the last break; that is, $z = 1$ for the adjacent fiber, $z = 2$ for the sub-adjacent fiber and so on. Hedgepeth's most quoted result is for the peak load concentration $K(t,1)$ on the first intact fiber adjacent to the cluster, which is $K(0,1) = 1$ and

$$K(t,1) = \frac{(4)(6) \dots (2t+2)}{(3)(5) \dots (2t+1)}, \quad t = 1, 2, 3, \dots \quad (11)$$

evaluating to $K(1,1) = 1.333$ and $K(2,1) = 1.600$, $K(3,1) = 1.829$ Another result is for the load concentration around one break,

$$K(1,z) = 1 + \frac{1}{4n^2 - 1}, \quad (12)$$

which evaluates to $K(1,1) = 4/3$, $K(1,2) = 16/15$, and $K(1,3) = 36/35$. More recently, Hikami and Chou (1990) derived a closed form expression for $K(t,z)$ which is

$$K(t,z) = (t+2z-1) \frac{(2z)(2z+2)(2z+4) \dots (2s+2t-2)}{(2z-1)(2z+1)(2z+3) \dots (2z+2t-1)}. \quad (13)$$

Recently Beyerlein *et al.* (1996) compared these discrete stress concentrations from the Hedgepeth shear-lag analysis with predictions from classical elastic fracture mechanics for a transverse crack of length t in a continuum. They obtained

$$K(t,1) = \sqrt{\pi(t+1)} / 2 \quad (14)$$

with very small error even for small t , and for $z \gg t$, they also showed

$$K(t,z) \approx K(t,1) / \sqrt{\pi(s-1)}. \quad (15)$$

Both these asymptotic results are of the same form as the continuum results for the K -field at the tip of a crack, and both Eqs. (14) and (15) reflect the continuum solution above the length scale of the fiber spacing. Thus the Hedgepeth model gives discrete lattice results for loads surrounding a single transverse crack, which are in close agreement with results from linear elasticity but without being singular. A key advantage, however, is that it allows us to efficiently calculate loads for large arbitrary arrays of broken fibers in the sheet (not necessarily aligned) (Beyerlein and Phoenix, 1997a,b), a task that is impractical for the continuum model. In later analysis, our load-sharing model will be one-

dimensional in the sense that we will draw connection to this Hedgepeth two-dimensional model by focusing only on loads around configurations of breaks in a single transverse plane.

2.2 Motivation For Idealizations In Load-Sharing Rules

We now turn to important features of load redistribution that determine the critical failure configurations studied in later sections. Throughout this paper, we assume all intact fibers have the same strength. Previous discussion mentioned the important role of fiber elements flanked by two clusters of breaks. Figure 1 shows the stress concentrations in the lone intact 'bridging' fiber between two t -fiber break clusters (open circles), calculated using the Hedgepeth shear-lag model, and those in the surviving fibers at the tips. Also shown is $K(2t+1,1)$ according to Eq. (11) for a $2t+1$ straight crack (solid circles) without any bridging fiber. These results show that while the single bridging fiber effectively reduces the stress concentration at the tip as compared to an $2t+1$ (and also $2t$) straight crack, it sustains a divergingly higher stress concentration growing roughly linearly in $(\pi)^{1/2}/2[t/\log(t+1)] - \pi/2$. For $t = 6$, for example, when this bridging fiber breaks under the stress concentration of 4.65, the stress concentration at the ends increases from 2.66 to 3.35 for a cluster of 13 breaks, which is also larger than the value 3.22 for a tight cluster of 12 breaks. It also turns out that this single bridging fiber can be placed at any of the $2t-1$ locations within the span of $2t-1$ fibers (with $2t$ broken) and a similar situation arises.

As a second example, we consider the effect of two adjacent bridging fibers flanked by t broken fibers on each side (total of $2t+2$) and compare this situation to a crack of $2t+2$ broken fibers. In Fig. 2 we show the stress concentrations in the bridging pair (open circles), and we also compare the tip stress concentrations for a $2t+2$ crack (solid circles) with that for the $2t$ breaks centrally bridged by the intact pair (triangles). Figure 2 shows that these two bridging fibers powerfully suppress the stress concentrations at the ends, and would require a higher load to fail than the end fibers up to

Figure 1. Stress concentrations predicted by the shear-lag model for two t -sized fiber break clusters separated by a single intact fiber.

Figure 2. Stress concentrations predicted by the shear-lag model for two t -sized fiber break clusters separated by a pair of intact fibers.

Figure 3. Symmetric configuration of breaks corresponding to stress calculations in Fig. 4.

Figure 4. Ratio of the stress concentration in the first intact fiber after $2t+1$ breaks over that in the first intact fiber ahead of a $2N^*+1$ straight crack calculated using the shear-lag model.

quite a long contiguous crack of 18 fiber breaks. Thus pairs of bridging fibers act as effective crack arrestors for moderate t length

cracks. Also it can be shown that triplets of bridging fibers are even more effective for much longer cracks.

As a third example, we consider a symmetric cluster which contains $2N+1$ breaks in total arranged as shown in Fig. 3. It has a central core cluster of $2t+1$ adjacent breaks flanked on each side by $2(N-t)$ alternating intact and broken fibers giving a total length $2t+1 + 4(N-t) = 2\{N+(N-t)\}+1 = 2N^*+1$, so $N^* = N + (N-t)$. For $2N+1 = 21$ and 51 , Fig. 4 plots the ratio of the stress concentration on the fiber at the edge of the $2t+1$ core in a $2N^*+1$ fiber cluster, K_{2t+1} , over the stress concentration at the edge of a full crack of $2N^*+1$ breaks, K_{2N^*+1} , (after all the bridging fibers have subsequently broken) versus t . When the ratio is greater than one, failure of the two fibers at the core edge and all subsequent bridging fibers thereafter occurs at lower loads than that required to cause catastrophic failure in a $2N^*+1$ contiguous crack (assuming all intact fibers have the same strength). Thus, for a given number of initial broken fibers, an 'extended' cluster can be configured as an initiating core cluster flanked with a diluted fringe of a few alternating bridging fibers, such that this cluster is considerably weaker than a crack of $2N^*+1$ adjacent breaks. Thus, the diluted cluster with 21 (51) breaks, has the strength of a contiguous crack of about 26 (63) breaks, representing effectively a 25% expansion. This arrangement is by no means the weakest arrangement of $2N+1$ breaks that can be constructed.

The key points of these illustrations are that first isolated surviving fibers will fail at much lower loads than the cracks they become part of. Secondly, initial clusters consisting of a core of contiguous breaks flanked by fringes of a few isolated bridging fibers will cause catastrophic failure at significantly lower loads than initial 'cracks' with the same number of contiguous breaks. Third, isolated pairs of intact fibers can act as crack arrestors for much longer failure configurations surrounding them. One of the main tasks of this work will be to enumerate the probabilities for the occurrence of various types of clusters.

3 LOAD-SHARING RULE AND CRITICAL CONFIGURATIONS

3.1 Tapered Load-Sharing Rule

In this section, we introduce an idealized rule called tapered load sharing (TLS), which we create to reflect the features of load redistribution observed for critical configurations seen in the previous section. First consider a linear bundle with fibers numbered $1, 2, 3, \dots, n$, from left to right, which contains some broken and surviving fibers. Consider a surviving fiber that is adjacent to at least one other survivor. Suppose this fiber is directly adjacent to $i \geq 0$ contiguous fiber breaks while being sub-adjacent to $j \geq 0$ such breaks (which for $i = 0$ could occur as contiguous strings on both sides where we take the sum). By the term 'sub-adjacent breaks', we mean that there is exactly one surviving fiber between the contiguous string of sub-adjacent breaks and the surviving fiber in question. For fixed θ satisfying $2/3 \leq \theta \leq 1$, its load concentration factor, $K_{i,j}$, is given by

$$K_{i,j} = 1 + i\theta/2 + j(1-\theta)/2, \quad (16)$$

where $i, j \geq 0$. In the case where a surviving fiber is *isolated* or adjacent to i contiguous breaks counting on both sides, its load concentration factor is given by

$$K_i^* = 1 + i/2. \quad (17)$$

Otherwise a fiber's load-sharing constant is $K_{0,0} = 1$. Denoting intact fibers by '1' and failed fibers by '0' we illustrate some possibilities:

$$\begin{array}{cccccccccccc} & ? & & & K_{3,0} & K_{0,3} & K_{0,0} & K_{0,2} & K_{2,0} & & ? \\ \cdots & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \cdots \\ & ? & & & K_{3,0} & K_{0,5} & K_{2,0} & & & & ? \\ \cdots & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \cdots \\ & ? & & & K_{3,2} & K_{2,3} & & & & & ? \\ \cdots & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \cdots \\ & ? & & & K_5^* & & & & & & ? \\ \cdots & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \cdots \end{array}$$

In Eq. (16), a broken fiber shifts $\theta/2$ of its load to each of the closest survivors on each side and $(1-\theta)/2$ to each of the next closest survivors (provided these are directly adjacent to those closest). If, however, a closest survivor is isolated by such breaks, it takes $1/2$ the shifted load, giving Eq. (17). Moreover, by setting $\theta = 1$, TLS collapses to the local load-sharing (LLS) rule of much earlier work (Harlow and Phoenix, 1991; Duxbury and Leath, 1994a,b; Zhang and Ding, 1994a,b, 1996) so that a failed fiber shifts half of its load onto each of its two flanking survivors. In the other extreme, case $\theta = 2/3$, the fibers adjacent, sub-adjacent, and sub-sub-adjacent to a single break have load-concentration factors $4/3$, $7/6$ and 1 , respectively, as compared to $4/3$, $15/16$, and $36/35$ in the case of Hedgepeth where load is more diffusely distributed. One can appreciate that this extended load sharing rule, TLS, while still an idealization necessary to quickly obtain load concentration factors on survivors, also drastically extends the number of possible failure configurations.

Under LLS in Harlow and Phoenix (1991) and the assumption that all surviving fibers had the same strength, the strength of a bundle was determined by the *largest* load concentration factor initially found in the bundle, since this most overloaded fiber always resulted in an even higher load on its survivors and thus catastrophic collapse. In the present setting, however, for some configurations and loads x , the most overloaded fibers can fail without the bundle failing. In this situation, after these fibers fail under x , the resulting configuration produces a maximum load concentration factor K , such that $Kx < 1$ and the bundle is stable. For example, for $\theta = 2/3$, the configuration

$$\begin{array}{cccccccccccc} & & & & & & K_{i+j}^* & & & & & & \\ 1 & 1 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 & 1 & 1 \\ & & & \leftarrow & i & \rightarrow & & \leftarrow & i & \rightarrow & & & \end{array}$$

changes upon failure of the fiber under K_{i+j}^* to

$$\begin{array}{ccccccc}
 & & K_{i+j+1,0} & & & & K_{i+j+1,0} \\
 1 & 1 & 1 & 0 & \dots & 0 & 1 & 1 & 1 \\
 & & \leftarrow & & i+j & & \rightarrow & &
 \end{array}$$

So when $1/K_{i+j}^* \leq x < 1/K_{i+j+1,0}$, the configuration above will be stable. On the other hand, for the configuration

$$\begin{array}{ccccccc}
 & & & & K_{i,j} & K_{j,i} & & & \\
 1 & 1 & 1 & 0 \dots 0 & 1 & 1 & 0 \dots 0 & 1 & 1 & 1 \\
 & & \leftarrow i \rightarrow & & & & \leftarrow j \rightarrow & & &
 \end{array}$$

where $i \geq j$, failure of the fiber under $K_{i,j}$ will produce a maximum load concentration factor, which is at least $K_{i,j}$ in magnitude. It can be easily checked that this will be true for all subsequent configurations too so that the bundle will immediately collapse. Also if $j = 0$, failures under $K_{i,0}$ will always lead to higher load concentration factors subsequently. For this reason, we think of a pair of survivors or pictorially, adjacent '1's as a crack arrestor pair; once one of the pair fails the bundle collapses catastrophically. For a given load x , to determine the bundle strength, we consider only failure configurations or those configurations where failure is initiated and sustained to catastrophic failure. Therefore it is those configurations where the largest load concentration factor is of the type $K_{i,j}^*$ that require careful study.

Three other points should be made: First, the load-sharing rule is monotone; that is, the load concentration factor on any given survivor is never decreased by the failure of other fibers. Second, all fibers loaded beyond their strength at any stage fail instantly; that is, we need not be concerned with the order of failures. Third, a fiber sub-adjacent to broken fibers can never be the most heavily overloaded fiber since $K_{i,j} \geq K_{j,i}$ for $i \geq j$.

Unless otherwise stated we will assume $\theta = 2/3$ throughout the paper, since the contrast with LLS will be most striking. In this case, Eqs. (16) and (17) indicate that the various load concentration factors have the possible values $1+k/6$ for $k = 1, 2, 3, \dots$. For $k \geq 2$, all the values of k are connected with certain values of $K_{i,j}^*$ and $K_{i,j}$ which may be the largest load concentration factors in certain configurations. In particular $K_{i,j}^* = 1+3i/6$ for $i = 2, 3, 4, 5, \dots$ which corresponds to $k = 6, 9, 12, 15, \dots$, and $K_{i,0} = 1+2i/6$ for $i = 1, 2, 3, 4, \dots$ which corresponds to $k = 2, 4, 6, 8, \dots$. Also $K_{i,1} = 1+(2i+1)/6$ for $i = 1, 2, 3, 4, \dots$ corresponding to $k = 3, 5, 7, 9, \dots$. Note also that

$$K_{j,0} = K_{j-1,2} = K_{j-2,4} = \dots \quad (18)$$

and

$$K_{j,1} = K_{j-1,3} = K_{j-2,5} = \dots \quad (19)$$

where our interest is in all such cases in which there are more adjacent than sub-adjacent breaks, or the first subscript is greater than or equal to the second in value.

For a bundle of n fibers we let the random variable X_n denote its strength, that is, the smallest load under which the bundle fails.

We let $G_n(x)$ be the distribution function for X_n . Since the strength of all intact fibers is 1, the strength of a bundle must be some load x satisfying $Kx = 1$ where K is some maximum load concentration factor for a configuration which becomes catastrophically unstable. Thus X_n has the possible outcomes $x_0 = 1$, and

$$x_k = 6/(6+k) \quad \text{for } k = 2, 3, 4, 5, \dots, \quad (20)$$

and thus, its distribution function $G_n(x)$ is discrete.

We begin our analysis by fixing a load x on the bundle, where $0 < x \leq 1$, and calculating the probability that the bundle strength $X_n \leq x$. It turns out to be convenient to partition the possible values of the load x . The first partition involves *load regions* where load region k is the set of loads x for which

$$6/(6+k) \leq x < 6/(6+k-1), \quad k = 1, 2, 3, \dots \quad (21)$$

Thus under such a load x the bundle fails if its strength X_n is $6/(6+k)$ or less but survives if its strength is at least $6/[6+(k-1)]$. The second partition involves much larger *load spans*. Load span s covers loads x satisfying

$$1/(1+s) \leq x < 1/(1+s-1), \quad s = 1, 2, 3, \dots, \quad (22)$$

and is the union of load regions $k = 6s+1, 6s+2, \dots, 6s+6$. Furthermore, for studying repeating patterns in configurations, it is useful to refer to a relative sub-span r in a load span; that is, any load range k may be written as $6s+r$ for some sub-span r in span $s \geq 0$, where r has the possible values $1, 2, \dots, 6$. Thus Eq. (21) can be written as

$$6/(6+6s+r) \leq x < 6/(6+6s+r-1). \quad (23)$$

It turns out that for a given r value, certain types of failure configurations will have certain similarities independent of s ; therefore, r will identify a useful position for load x in a span.

Given load x , the following relationships hold:

$$s = \lfloor (1-x)/x \rfloor = \lfloor 1/x \rfloor - 1,$$

$$k = \lfloor 6(1-x)/x \rfloor = \lfloor 6/x \rfloor - 6,$$

and

$$\begin{aligned}
 r &= \lfloor 6(1-x)/x \rfloor - 6 \lfloor (1-x)/x \rfloor \\
 &= \lfloor 6/x \rfloor - 6 \lfloor 1/x \rfloor = k - 6s
 \end{aligned}$$

where $\lfloor \cdot \rfloor$ is the 'integer part' of a real number.

3.2 Critical Failure Configurations

Our analysis is an extension of the method used by Harlow and Phoenix [39], except that the failure configurations are vastly more complicated. The basic method is to consider a load x and a given fiber location say i in the bundle, and to determine uniquely all the

local catastrophic failure configurations that can occur critically within a certain subset of fibers surrounding fiber i . These failure configurations must be constructed and enumerated very carefully, so as to avoid double counting over the whole bundle. The idea will be that if such a critical configuration occurs for some fiber i , the bundle fails, but if none of the configurations occur over all fibers i , the bundle survives. Once constructed, we use the Chen-Stein method for Poisson approximation to arrive at the distribution function $G_n(x)$ for failure of the bundle and subsequently the size effect. The key difficulty is that while the initial strengths of the fibers (either '0' or '1') are independently distributed, locally the configurations overlap so that calculating their probabilities of occurrence is greatly complicated.

One type of critical configuration is of the form

$$10 \cdots 011 \overset{i}{\underset{\cdot}{X}} \cdots \underset{\cdot}{X} 10 \cdots 010 \cdots 0X \cdots X \quad (24)$$

where $X \cdots X$ denotes all configurations of '0's and '1's but where no two '1's are adjacent, and $\underset{\cdot}{X} \cdots \underset{\cdot}{X}$ has the same meaning as $X \cdots X$ except that the first and last elements must be '0'. The sub-configuration $0 \cdots 010 \cdots 0$ acts like an initiator with the isolated '1' failing first, followed by sequential failure of the '1's in the configurations $\underset{\cdot}{X} \cdots \underset{\cdot}{X}$ and $X \cdots X$. Then with the 'help' of loads shifted from the sub-adjacent string $10 \cdots 0$ on the left and the loads from all adjacent broken fibers on the right, fiber i of the crack arrestor pair finally succumbs causing catastrophic failure. All the strings and the configurational patterns therein must satisfy certain constraints involving, r and s which identify the applied load x , in order to cause failure but also to avoid double counting.

Another type of critical configuration is

$$10 \cdots 011 \overset{i}{\underset{\cdot}{X}} \cdots \underset{\cdot}{X} 10 \cdots 010 \cdots 0X \cdots \underset{\cdot}{X} 110 \cdots 0. \quad (25)$$

In this case the sub-configuration $0 \cdots 010 \cdots 0$ again acts like an initiator with the isolated '1' failing first, followed by sequential failure of the '1's in the configurations $\underset{\cdot}{X} \cdots \underset{\cdot}{X}$ and $X \cdots X$, where the last element in the latter must be an '0'. At this stage, however, the sub-adjacent string $10 \cdots 0$ on the left is not long enough to provide sufficient load to supplement the load from all adjacent broken fibers on the right in order to fail fiber i , but the string $0 \cdots 0$ on the right is in fact long enough to cause fiber i' to fail in the second crack arrestor pair, so catastrophic failure ensues by that mechanism. Again all the strings must satisfy certain constraints in r and s to cause such failure and to avoid double counting.

It turns out there are many other types of configurations, but these are typically longer and require two initiating configurations, or, they contain smaller configurations which are associated with some fiber $i'' > i$ and thus can be ruled out to avoid double counting. As the bundle size n grows large, and thus, values of the load x of interest become small so that the failure configurations become long, the sum of probabilities for these other types configurations will become negligible compared the sums of the probabilities for those described above.

4 PROBABILITY DISTRIBUTION FOR BUNDLE STRENGTH

4.1 Failure Probabilities For Larger Bundles n (Larger s)

The probability analysis is very lengthy and cannot be presented here. (See Phoenix and Beyerlein (1997) for a full analysis.) Thus we summarize a few of the key ideas and results.

First we consider the probability $X_k = P\{X \cdots X\}$, which is the probability that no two adjacent '1's initially appear in a string of k fibers. Calculating this probability involves solution of the recursive system $X_0 = X_1 = 1$, $X_2 = 1 - p^2$ and $X_k = qX_{k-1} + pqX_{k-2}$ for $k \geq 2$ which yields

$$X_k = \frac{\delta^2}{q^2} \left(\frac{\delta}{\delta + 2p} \right) \delta^k + \frac{\delta^2}{q^2} \left(\frac{\delta}{\delta + 2p} \right) \left(\frac{\delta - 1}{\delta + p} \right) \delta^{*k} \quad (26)$$

where

$$\delta = \frac{q + \sqrt{q^2 + 4pq}}{2}, \quad (27a)$$

and

$$\delta^* = \frac{q - \sqrt{q^2 + 4pq}}{2}. \quad (27b)$$

Note that $\delta > \delta^*$ so that the first term dominates as k increases. The results for $P\{\underset{\cdot}{X} \cdots \underset{\cdot}{X}\}$ and $P\{X \cdots X\}$ are similar with the same δ and δ^* .

Next let $P(s, r)$ be the probability that at load x one of the critical failure configurations described by Eqs. (24) and (25) occurs at fiber position i , where r and s are given by Eqs. (22) and (23). Then upon enumerating all possibilities it turns out that $P(s, r) = \Lambda(k)(1 + O(1/s))$ where

$$\Lambda(k) = 2s^2 \Phi(s) \Theta(r), \quad (28)$$

$$\Phi(s) = q^{2s+1} \delta^s p^5 \left(\frac{\delta}{\delta + 2p} \right)^2 \left(\frac{\delta}{q^2} \right), \quad (29)$$

and

$$\Theta(r) = q^b \delta^{a-b} \left\{ \frac{\delta + q\delta^{1-c}}{\delta - q^2} + \left(\frac{q^2 p}{\delta^2} \right) \left[\frac{1}{\delta - q^2} - \frac{q^{2-c}}{\delta - q^4} \right] \right\}. \quad (30)$$

In the above expressions, $a = \lfloor (r-1)/2 \rfloor$, $b = \lfloor (r-1)/3 \rfloor$, $c = 0$ for $r = 2, 4, 6$, and $c = 1$ for $r = 1, 3, 5$. Then, neglecting edge effects (which are absent in the case of circular bundles), an asymptotic formula for the bundle strength distribution applicable to large n (large s , small x) is

$$G_n(x) \equiv 1 - \exp\{-n\Lambda(k(x))\}, \quad (31)$$

where we recall $k(x) = \lfloor 6(1-x)/x \rfloor$ and the error is $n\Lambda(k(x))O(1/s(x))$ which decreases in relative terms as n increases and x decreases.

4.2 Asymptotic Failure Distribution For Large Bundles n

We recall $s = \lfloor 1/x \rfloor - 1$, and $k = \lfloor 6/x \rfloor - 6$ and the fact that the bundle strength distribution is discrete in x (that is, it decreases stepwise in decreasing x at points $x_k = 6/(6+k)$, for $k = 2, 3, 4, 5, \dots$). We may construct tight, continuous, upper and lower bounds on the distribution for bundle failure $G_n(x)$. These are based on the 'lowest' and 'highest' values $\Lambda(k(x))$ takes as r goes from 1 to 6 in load span s relative to a continuous approximation based on x alone. The lower bounding function in x is

$$\Lambda_l(x) = 2\left(\frac{1}{x}\right)^2 q^{2/x} \delta^{1/x} \left(\frac{\delta}{\delta + 2p}\right)^2 \left(\frac{p^5}{q^5}\right) \Theta(1). \quad (32)$$

Except when $x = 1/(1+s)$, $s = 1, 2, 3, \dots$, which occurs at the beginning of load span s , this approximation asymptotically (large s) underestimates the probability of failure. The maximum underestimate is by the factor $q\Theta(1)/\Theta(2)$. Thus an upper bounding function in x is

$$\begin{aligned} \Lambda_u(x) &= 2\left(\frac{1}{x}\right)^2 q^{2/x} \delta^{1/x} \left(\frac{p^5}{q^4}\right) \left(\frac{\delta}{\delta + 2p}\right)^2 \Theta(2) \\ &= \frac{\Theta(2)}{q\Theta(1)} \Lambda_l(k(x)). \end{aligned} \quad (33)$$

We can then write the approximation

$$\Lambda(k(x)) \equiv \left(\frac{1}{x}\right)^2 (q^2 \delta)^{1/x} \Pi(q) \Delta(1/x) \{1 + O(x)\}, \quad (34)$$

where

$$\Pi(q) = 2\left(\frac{p^5}{q^4}\right) \left(\frac{\delta}{\delta + 2p}\right)^2 \Theta(2), \quad (35)$$

and

$$\Delta(1/x) = (x \lfloor 1/x \rfloor - x)^2 (q^2 \delta)^{\lfloor 1/x \rfloor - 1/x} q \Theta(r(x))/\Theta(2). \quad (36)$$

We note that $\Delta(1/x)$ is asymptotically (as $x \rightarrow 0^+$) periodic in $1/x$, varying between 1 and $q\Theta(1)/\Theta(2)$ with period 1, corresponding to increases in s as the load x decreases. The final asymptotic result is that as n grows large and thus the values of x of interest become small

$$G_n(x) \equiv 1 - \exp\left\{-n \left(\frac{1}{x}\right)^2 (q^2 \delta)^{1/x} \Pi(q) \Delta(1/x)\right\}, \quad (37)$$

where the relative error (absolute error divided by $G_n(x)$) is $O(x)$.

The size effect is easily seen from this result. The median failure load of the composite is x_{med} , which decreases as

$$x_{\text{med}} \equiv -\log(q^2 \delta) / (\log n). \quad (38)$$

A similar approximation can be developed for the distribution function for the first fiber to fail. This distribution is

$$G_{n,1}(x) \equiv 1 - \exp\left\{-n \left(\frac{2}{x}\right)(q)^{2/x} (p/q)^2 \Delta_1(2/x)\right\}. \quad (39)$$

where $\Delta_1(2/x)$ is asymptotically (as $x \rightarrow 0^+$) periodic in $2/x$, varying between 1 and q with period $1/2$ as x decreases. In this case the median load at first fiber failure, $x_{1,\text{med}}$, decreases as

$$x_{\text{med}} \equiv -2(\log q) / (\log n). \quad (40)$$

5 DISCUSSION AND CONCLUSIONS

5.1 Comparison To Literature Results For First Element Breakdown

In comparing the structure of our results with those of previous investigators as discussed in the introduction, we find various differences. First, our resulting distributions Eqs. (37) and (39) do not have the Weibull form Eq. (1). Nor is the size effect the same in that the Weibull distribution predicts strength decreasing inversely as volume to a power, whereas our results have strength decreasing inversely as the log of the volume. To compare our results to those of Duxbury and coworkers, Eqs. (7) through (10), we first compare $G_{n,1}(x)$ for the first fiber to fail to $F_1(V_1)$. First we note that $q^{2/x} = \exp\{-2\log(1-p)(-1/x)\}$. So the structure is similar, with α_1 in Eq. (7) corresponding to 1 in our results, except our result has the prefactor $(1/x)$ to this exponential. The size effect on strength in our case, Eqs. (38) and (40), have the same inverse dependence on the logarithm of the volume as in theirs Eq. (8). What should be noted, however, is that we assumed the load on an isolated fiber grows linearly with the number of adjacent fiber breaks t counting on both sides Eq. (17) (with t in place of i); whereas our numerical results in Fig. 1 show that this load grows more realistically as $t/\log t$ for a fiber interrupting a string of t breaks near the middle. With this modification in our derivations, our resulting forms Eqs. (37) and (39) will change. In particular, while we do not know the exact form of the size effect, it is not given asymptotically by the forms Eqs. (38) and (40) or Eq. (8). Specifically, as the volume increases, the ratio of the predicted strengths diverges. Thus with respect to the distribution for strength and the size scaling for first element failure, the behavior is more complicated than our approximations indicate even in this simple 1-d model.

5.2 Comparison To Literature Results For Total Failure

Next, we compare $G_n(x)$ for complete failure Eq. (37) to $F_b(V_b)$ of Eq. (7). First we note that $(q^2 \delta)^{1/x} = \exp\{-\log(q^2 \delta)(-1/x)\}$, and because $\log(q^2 \delta)$ does not have the same behavior in $p = 1 - q$ as $-\log(1-p)$, we have an immediate difference in the dependence of the constants on p . This difference stems from the fact that the most critical breakdown configurations for a given number of fiber breaks occurring locally are not contiguous strings of breaks but rather extended strings with interrupting survivors, especially near

the fringes. Beyond our idealization, the more realistic Hedgepeth model in Section 2 indicates that such a feature will persist in more complex models. Second, the factor α_b , in Eqs. (7) and (8), is approximately 1/2, whereas in our analysis the corresponding exponent is 1. This difference comes from assumptions on how the load at the edges of strings of breaks scales with the number of breaks. In the spirit of Eqs. (14) and (15) in Hedgepeth's model, we could have modified our idealized rule, say K_{ij} in Eq. (16), to reflect load concentrations on survivors proportional to the square root of the adjacent or sub-adjacent strings of breaks, but leaving K^* of Eq. (17) alone, and appreciating that some shifted load from failures would become unaccounted for. We then would have obtained a similar dependence on the square of the applied load as in Eqs. (7) and (8). But more realistic load-sharing rules capturing the essence of load redistribution in Hedgepeth's model will introduce further complications. For example, given a fixed number of breaks locally, there may be a tendency for the most detrimental configurations to have almost no bridging fibers near the center, and a continually increasing density of bridging fibers as one moves out to the fringe, again spreading the cluster relative to a 'tight' configuration involving the same number of breaks. One can anticipate that a broader idealization of this situation would require several eigenvalue problems to solve rather than one, thus changing both the factors $\log(q^2\delta)$ and $(1/x)^2$ to something more complicated, perhaps involving the load x in some complex way. Thus, for complete failure, it is an open question whether the form of the size effect Eq. (8) will prevail.

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