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THEORETICAL AND NUMERICAL STUDIES IN MAGNETIC FUSION

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Faraday Rotation

It is possible in principle to measure the poloidal magnetic field profile, and hence the profile of the plasma current, by measuring the change in the polarization of a sequence of microwave beams that pass through the plasma. Actual measurements of the plasma current profile would be very interesting in connection with lower-hybrid or electron cyclotron resonance current drive experiments since such measurements would provide direct information on modification of the current profile by the application of RF power.

The theory of polarimetry has been developed using an eikonal representation of the electromagnetic field and assuming that $\Omega_{ce}/\omega \ll 1$. The results have been compared with the prior work of De Marco and Segre. The results have been adapted to measuring the poloidal field in Tokomaks, and applications to MTX have been considered.

Ion Transport in Tokamaks

Normally Tokamak transport is treated in the limit of ion thermal speed much greater than toroidal rotation speed of the plasma. Usually, however, once a Tokamak plasma has been established, or during and immediately after neutral beam injection, large toroidal rotation may develop. The relaxation of this rotation via a radial flux of toroidal angular momentum requires careful evaluation via transport theory, if only to determine what anomalous viscous effects are actually present in a working device.

Until the theoretical work of Hinton and Wong¹, attempts to describe toroidal rotation were restricted to small rotation speeds or were of limited applicability. We have formulated the problem of collisional transport, both classical and neoclassical, in a form which is applicable to all regimes of collisionality in a Tokamak. It admits toroidal rotation speeds of the order of the ion thermal speed, arbitrary aspect ratios, arbitrary cross sections, and general poloidal magnetic fields. It extends the work of Hinton and Wong in the following ways: the derivations are considerably simplified and a somewhat more general reduced kinetic equation is obtained; classical transport is retained; more general expressions for the transport coefficients are found, particularly in the Pfirsch-Schluter regime where restrictions on the angular velocity are removed. These results have been published².

Kinetic Theory

Of central interest in the theoretical treatment of high temperature plasmas of fusion interest is the solution of the Boltzmann and Fokker-Planck kinetic equations involving spatial boundaries in the presence of electric and strong magnetic fields.

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{Ze}{m} \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \cdot \nabla_{\mathbf{v}} f = C_{\text{charged}} + C_{\text{Boltzmann}} \quad (1)$$

This requires the stipulation on the boundary of the distribution functions of those particles entering the system. Collisions then determine the distribution of those that exit. The full problem, three dimensional in position, and in a strong magnetic field two dimensional in velocity, has not yet yielded to analytic techniques. Thus, in order to develop insight, and generate numerical methods, the work done on this grant has been restricted to model problems which are one dimensional in position, and which are or can be reduced to one dimension in velocity.

The complexity of the full problem is clear from the form of the charged particle - charged particle collision term

$$C_{\text{charged}} = - \sum_{\substack{\text{primed} \\ \text{species}}} \nabla_{\mathbf{v}} \cdot \Gamma' \quad (2)$$

where if $g = |\mathbf{v} - \mathbf{v}'|$, $f = f_j(\mathbf{r}, \mathbf{v}, t)$, $f' = f_{j'}(\mathbf{r}, \mathbf{v}', t)$,

$$-\Gamma' = \frac{2\pi Z^2 Z'^2 e^4 \ln \Lambda}{m^2} \int d^3 \mathbf{v}' \left(f' \nabla_{\mathbf{v}} f - \frac{m}{m'} f \nabla_{\mathbf{v}'} f' \right) \cdot \nabla_{\mathbf{v}} \nabla_{\mathbf{v}'} g \quad (3)$$

and the Boltzmann collision term

$$C_{\text{Boltzmann}} = \int d^3 \mathbf{v}' d\Omega g \sigma(g, \theta) (\bar{f} \bar{f}' - f f') + s(\mathbf{r}, \mathbf{v}) \quad (4)$$

The Lorentz Gas

The so-called Lorentz collision term is a simple mathematical model which describes the scattering of electrons by infinitely massive ions, namely

$$\nabla_v \cdot \left(\frac{v^2 \mathbf{I} - \mathbf{v} \mathbf{v}}{\tau} \cdot \nabla_v f \right) \quad (5)$$

where

$$\tau = \frac{m^2 v^3}{2\pi \sum_i N_i(\mathbf{r}, t) Z_i^2 e^4 \ln \Lambda} \quad (6)$$

In the case of one space dimension z measured in units of the thickness of the slab of plasma, the steady state kinetic equation governing the distribution function f is

$$\mu \frac{\partial f}{\partial x} = \alpha \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial f}{\partial \mu} \right] \quad (7)$$

where $\mu = \mathbf{e}_z \cdot \mathbf{v} / v$ and

$$\alpha = \frac{2\pi \int_0^L dz \sum_i N_i(z) Z_i^2 e^4 \ln \Lambda(z)}{m^2 v^4} \quad (8)$$

This equation has been analyzed analytically and solved numerically.

Figure 1 Diagram of the domain in x, μ indicating where the boundary conditions are assigned.

Analytic approximations have been developed for $f(x, v, 0)$ in the limits of mean-free-path long and short compared with the system size. These have been shown to be good, and to provide a useful starting point for the numerical procedure employed, which iterates on $f(x, v, 0)$. Their success suggests that counterpart approximations may work well for more complicated steady state kinetic problems of interest in the physics of the edge in proposed controlled fusion reactors, the diffusion of neutrons, the Brownian motion of heavy molecules, and the structure of shocks. It has also been demonstrated that the determination of $f(x, v, 0)$ can be reduced to the solution of a one dimensional integral equation which can readily be dealt with numerically. Typical results are shown in the figure below.

$$\alpha = 0.01$$

$$\alpha = 1$$

$$\alpha = 10$$

$$f(0, \mu) = \exp\left[-\frac{(1-\mu)^2}{100}\right] \quad 0 \leq \mu \leq 1$$

$$f(1, \mu) = 0 \quad -1 \leq \mu \leq 0$$

Fig 2-Incident Beam

$$\alpha = 0.01$$

$$\alpha = 1$$

$$\alpha = 10$$

$$f(0, \mu) = 1 \quad 0 \leq \mu \leq 1$$

$$f(1, \mu) = 0 \quad -1 \leq \mu \leq 0$$

Fig. 3-Incident Flat Distribution

A Model Kinetic Equation

Consider the model kinetic equation

$$u \frac{\partial f}{\partial z} - \frac{Ze}{m} \frac{\partial \Phi(z)}{\partial z} \frac{\partial f}{\partial u} = \frac{1}{w} \frac{\partial}{\partial w} \left[\frac{w}{\tau} \left(w f + \frac{T}{m} \frac{\partial f}{\partial w} \right) \right] + \frac{\partial}{\partial u} \left[\frac{1}{\tau} \left(u f + \frac{T}{m} \frac{\partial f}{\partial u} \right) \right] \quad (9)$$

where T is constant and τ is a given function of v , for example that given by (1).

In the limit of short mean free path (formally $\tau \rightarrow 0$) where transport theory obtains one can employ (9) to find the transport coefficients. Experiments with the form of τ suggests that if the constants in τ are adjusted to make the electrical conductivity agree with the Spitzer value, the thermal conductivity is off by something of order 40%. This suggests that (9), while it and related model equations may provide useful qualitative information, they are of limited utility for quantitative information.

The evaporation of heavy molecules from a droplet into a light gas

Consider particles evaporating from a spherical droplet of radius a into a gas of much lighter molecules.

Figure 4 The coordinate system

The Fokker-Planck equation governing the steady state distribution function $f(\mathbf{r}, \mathbf{v})$ of the heavy molecules in the light gas outside the droplet is

$$\mathbf{v} \cdot \nabla f = \nabla_{\mathbf{v}} \cdot \left[\frac{1}{\tau} \left(\mathbf{v} f + \frac{T}{m} \nabla_{\mathbf{v}} f \right) \right] \quad (10)$$

where the temperature T (which has the dimensions of energy), and the mean free time τ are constants. The density $N(\mathbf{r})$ and mean velocity $\mathbf{V}(\mathbf{r})$ are defined via

$$N = \int d^3\mathbf{v} f \quad (11)$$

$$N\mathbf{V} = \int d^3\mathbf{v} \mathbf{v} f \quad (11)$$

In the diffusion limit where $v\tau \ll 1$ it has been shown that asymptotically

$$f = \frac{k}{|\mathbf{r} - \mathbf{v}\tau|} \left(\frac{m}{2\pi T} \right)^{3/2} \exp\left(-\frac{m\mathbf{v}^2}{T}\right) \quad (12)$$

Eq. (12) leads to the diffusion result

$$N\mathbf{V} = -\tau \frac{T}{m} \nabla N \quad (13)$$

Analytic results have been found in the limit of long mean free path ($v\tau \gg r$). The problem has been solved numerically in the limit of intermediate mean free path. The results will be compared with experiment. If F denotes the total flux emitted the result for F is as shown below.

Fig 5 Total flux F vs. a/λ

A Neutron Diffusion Problem

We have treated a very simple Boltzmann kinetic equation

$$\mathbf{v} \cdot \nabla f = \int d^3 v' d\Omega g \sigma(g, \theta) (\bar{f} \bar{f}' - f f') + s(\mathbf{r}, \mathbf{v})$$

in the limit where the scatterers are infinitely massive and the mean free time is constant. This has the virtues of being almost completely solvable, and of displaying explicitly most of the qualitative features common to all boundary value problems in kinetic theory. These limiting cases include the diffusion limit for the situation where the mean free path is much less than the system size; spacial boundary layers on the order of the mean free path in thickness near walls on which boundary conditions are assigned which match these boundary conditions to the diffusion solution; and the Knudsen flow limit where the mean free path is much larger than the system size and a boundary layer in velocity space occurs to match the solutions elsewhere which are almost unaffected by collisions.

If x denotes the single spacial variable measured in units of the mean free path, the kinetic equation reduces to

$$\mu \frac{\partial f(x, \mu)}{\partial x} = \alpha [g(x) - f(x, \mu)] + s(x, \mu) \quad (14)$$

where α is the ratio of the system size to the mean free path, $\mu = \mathbf{v} \cdot \mathbf{e}_x / v$ and

$$g(x) = \frac{1}{2} \int_{-1}^1 d\mu f(x, \mu) \quad (15)$$

The problem can be reduced to solving the integral equation

$$\begin{aligned} 2g(x) = & \int_0^1 d\mu e^{-\alpha x/\mu} \left\{ f(0, \mu) + \frac{1}{\mu} \int_0^x dy e^{\alpha y/\mu} [\alpha g(y) + s(y, \mu)] \right\} \\ & + \int_{-1}^0 d\mu e^{\alpha(1-x)/\mu} \left\{ f(1, \mu) - \frac{1}{\mu} \int_x^1 dy e^{-\alpha(1-y)/\mu} [\alpha g(y) + s(y, \mu)] \right\} \end{aligned} \quad (16)$$

which can be readily solved numerically. Once $g(x)$ is known one can find the distribution function from

$$f(x, \mu) = f(0, \mu) e^{-\alpha x / \mu} + \frac{1}{\mu} \int_0^x dy [\alpha g(y) + s(y, \mu)] e^{-\alpha(x-y)/\mu}$$

$$f(x, \mu) = f(1, \mu) e^{\alpha(1-x)/\mu} - \frac{1}{\mu} \int_x^1 dy [\alpha g(y) + s(y, \mu)] e^{-\alpha(x-y)/\mu}$$
(17)

Figure 6 Plot of $f(x, \mu)$ for $\alpha = 0.01$

Figure 7 Plot near $\mu = 0$ of $f(x, \mu)$ for $\alpha = 0.01$

Approximate analytic solutions have been found in the limits of long mean free path ($\alpha \ll 1$) the Knudsen regime, and of short mean free path ($\alpha \gg 1$) the diffusion limit.

Particles Trapped in a Square Potential Well

Figure 8 The $x-u$ phase plane

Consider the case of a square well potential as indicated schematically in the phase plane diagram above. In the limit $\alpha \ll 1$ there will be two boundary layers straddling the lines $u = \pm \hat{u}$, where $\hat{u}$ is the height in u of the well. The boundary layer equation governing the distribution function in a strip straddling the line $u = \hat{u} > 0$.

$$\hat{u} \frac{\partial f}{\partial x} = \alpha \frac{\partial^2 f}{\partial u^2} \quad (18)$$

This problem can be reduced to consideration of a one-dimensional integral equation of the second kind which can be easily solved numerically.

Particles Trapped in a General Potential Well

Consider the reduced version of () where τ is constant and the potential is trapping.

$$u \frac{\partial f(x,u)}{\partial x} - \frac{Ze}{m} \frac{d\Phi(x)}{dx} \frac{\partial f(x,u)}{\partial u} = \frac{\partial}{\partial u} \left\{ \frac{1}{\tau(x,u)} \left[uf(x,u) + \frac{T}{m} \frac{\partial f(x,u)}{\partial u} \right] \right\} \quad (19)$$

Figure 9 The $x - u$ phase plane.

It has been proven that the assignment of f on the spacial boundaries leads to a unique solution. Let $\alpha = \frac{L}{\tau \sqrt{T/m}}$, where L is the distance between the spacial boundaries, and let

$H = \frac{u^2}{2} + \Phi(x)$. Then () becomes

$$\pm \frac{\partial f^\pm}{\partial x} = \alpha \frac{\partial}{\partial H} \left[\frac{|u|}{\tau} \left(f^\pm + \frac{\partial f^\pm}{\partial H} \right) \right] \quad (20)$$

where the $\pm$ is taken according to the sign of $u(H, x)$.

Figure 10 The $H - x$ plane

Then in the limit $\alpha \rightarrow 0$ it has been shown that the trapped particle distribution function is

$$f = Ce^{-H/T} \quad (21)$$

where C is constant. It has also been demonstrated that in this limit

$$C = \frac{a+b}{2} \quad (22)$$

where a and b are as shown in figure (9). It has also been demonstrated that in this limit on restoring the time derivative in the kinetic equation () that

$$\frac{\partial}{\partial t} \left(f_0 g \oint \frac{dx}{u} \right) = \frac{\partial}{\partial H} \left(f_0 \frac{\partial g}{\partial H} \oint \frac{\alpha u}{\tau} \right) \quad (23)$$

which permits an estimate of the time required to fill the trap.

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