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Longitudinal Wake Field Corrections in OSTI
Circular Machines *

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Abstract. In computations of longitudinal particle motions in accelerators and storage rings, the fields produced by the interactions of the beam with the cavity in which it circulates are usually calculated by multiplying Fourier components of the beam current by the appropriate impedances. This procedure neglects the slow variation with time of the Fourier coefficients and of the beam revolution frequency. When there are cavity elements with decay times that are comparable with or larger than the time during which changes in the beam parameters occur, these changes can not be neglected. Corrections for this effect have been worked out in terms of the response functions of elements in the ring. The result is expressed as a correction to the impedance which depends on the way in which the beam parameters are changing. A method is presented for correcting a numerical simulation by keeping track of the steady state and transient terms in the response of a cavity.

INTRODUCTION

In many calculations and simulations of longitudinal motion in particle accelerators we find the longitudinal electric field by first finding its Fourier transform and then inverting the transform. If we have the Fourier transform $\hat{E}(\omega)$ of the complete electric field history $E(t)$, then its inverse transform will of course give the correct field $E(t)$. However we do not know to begin with either the complete field history $E(t)$ or its transform $\hat{E}(\omega)$. That is one of the things we want to calculate, so it is available only at the end of the calculation. This problem has also been treated by J. Machlachlan[1].

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We will use the azimuthal coordinate θ to locate a point around the ring in a circular accelerator. In order to distinguish between rapidly varying quantities like θ and slowly varying quantities, we introduce a reference angle

$$\Theta = \int_0^t \Omega(t') dt', \quad (1)$$

where $\Omega(t)$ is a slowly varying reference angular velocity which is close to the angular velocities of particles in the beam. It is intended that Θ shall maintain its position relative to the beam so that the relative phase

$$\phi = \theta - \Theta \quad (2)$$

is slowly varying in comparison with the revolution time or the decay time of any circuit element around the ring.

In a circular machine, we usually know, at any time in the calculation, the linear particle density $\rho(\phi)$, which is slowly varying. We then proceed as follows. The linear charge density is $e\rho(\phi)$, where e is the charge on a particle. We write $\rho(\phi)$ in terms of its Fourier transform:

$$\rho(\phi) = \sum_{k=-\infty}^{\infty} \hat{\rho}_k e^{ik\phi}, \quad (3)$$

and use the equation of continuity for the current to get

$$\hat{J}_k = e\Omega R \hat{\rho}_k, \quad (4)$$

where $\hat{J}_k$ is the Fourier transform of the current $J(\phi)$ and $2\pi R$ is the circumference. We will express the azimuthal electric field in terms of the voltage per turn $V(\phi)$ delivered to a particle at ϕ . The Fourier transform of $V(\phi)$ may then be written:

$$\hat{V}_k = -Z(k\Omega) \hat{J}_k, \quad (5)$$

where $Z(k\Omega)$ is the impedance at the frequency $k\Omega$ associated with the Fourier component k .

The above calculation of the voltage per turn neglects changes in $\rho(\phi)$ with time. If the ring contains any element which produces an extended wake field, it may be necessary to take into account such changes. The purpose of this paper is to find the correction to formula (5) which properly treats long lasting wake fields.

THE VOLTAGE ACROSS A RING ELEMENT

The voltage response function $K(t)$ of an element of the accelerator ring to a unit current impulse at $t = 0$ may be written in the form:

$$K(t) = -K_0 e^{-\gamma t} \sin(\omega t + \zeta), \quad (6)$$

where the phase ζ is given by the condition that $\partial K/\partial t$ vanish at $t = 0$:

$$\tan \zeta = \frac{\omega}{\gamma}. \quad (7)$$

Equation(6) is written for the underdamped case, but the other cases may be handled in a similar way. The voltage across the element at any time t is

$$V(t) = \int_{-\infty}^t J(t')K(t-t')dt' = \int_0^{\infty} J(t-\tau)K(\tau)d\tau. \quad (8)$$

If we represent this element by a series RLC circuit with the voltage appearing across the capacitance, then we have the relations:

$$\begin{aligned} \gamma &= \frac{R}{2L} = \frac{\omega_0}{2Q}, \quad Q = \frac{\omega_0 L}{R}, \quad \omega_0 = \frac{1}{\sqrt{LC}}, \\ \omega &= \left(\frac{1}{LC} - \left(\frac{R}{2L} \right)^2 \right)^{1/2} = \omega_0 \left(1 - \frac{1}{4Q^2} \right)^{1/2}, \\ \tan \zeta &= (4Q^2 - 1)^{1/2}, \quad K_0 \sin \zeta = \frac{1}{C}, \quad K_0 = \frac{1}{C} \left(1 - \frac{1}{4Q^2} \right)^{-1/2}. \end{aligned} \quad (9)$$

An accelerating cavity can be represented as a set of modes, each represented by an RLC circuit as above. The complete ring can be represented as a sequence of elements, so that the total voltage per turn is the sum of the voltages of all the elements. Since the total charge density and current are sums over Fourier components as given by Eq.(3), we will consider the response of a single element to a single Fourier component of the current. Let the element j be located at the azimuth θ_j . Its response to the current

$$J_k = \hat{J}_k e^{ik\phi} = \hat{J}_k e^{ik\theta_j} e^{-ik\Theta(t)} \quad (10)$$

is given by Eq.(8):

$$V(t) = - \int_0^{\infty} \hat{J}_k(t-\tau) e^{ik\theta_j} e^{-ik \int_0^{t-\tau} \Omega(t')dt'} K_0 e^{-\gamma\tau} \sin(\omega\tau + \zeta) d\tau, \quad (11)$$

where we have explicitly indicated the possible time-dependence of $J(\phi)$ and Ω .

If we neglect these time-dependences, we get

$$V(t) = \hat{V}_k e^{ik\theta_j} e^{-ik\Omega t}, \quad (12)$$

where

$$\hat{V}_k = -\hat{J}_k Z(k\Omega), \quad (13)$$

and

$$Z(k\Omega) = \int_0^{\infty} e^{(-\gamma + ik\Omega)\tau} K_0 \sin(\omega\tau + \zeta) d\tau. \quad (14)$$

Equation (12) can be rewritten using Eq.(2) in the form:

$$V(\phi) = \hat{V}_k e^{ik\phi}, \quad (15)$$

where we have substituted $t = (\theta_j - \phi)/\Omega$, since the element j is located at θ_j . Equation (15) justifies the use of the notation $\hat{V}_k$ in Eq.(12).

The integral in Eq.(14) can be evaluated. It is not hard to show, using Eqs.(9), that

$$Z(k\Omega) = \frac{R - ik\Omega L}{1 - k^2\Omega^2 LC - iRk\Omega C}. \quad (16)$$

This is the correct formula for the impedance across a capacitance which is part of an RLC series circuit.

Now let us take into account the slow time-dependence of $J(\phi)$ and Ω . In order to write the result as a correction to Eq.(14), we substitute in Eq.(11):

$$\hat{J}_k(t - \tau) e^{-ik\Theta(t-\tau)} = \hat{J}_k(t) e^{-ik\Theta(t)} e^{ik\Omega(t)\tau} F(t, \tau), \quad (17)$$

where the function $F(t, \tau)$ approaches 1 as $\tau \rightarrow 0$ and is slowly varying in t . Equation (11) then again gives the generalization of Eq.(12):

$$V(t) = \hat{V}_k(t) e^{ik\theta_j} e^{-ik \int_0^t \Omega(t') dt'}, \quad (18)$$

where

$$\hat{V}_k(t) = -\hat{J}_k(t) Z_{\text{eff}}(k\Omega(t)) \quad (19)$$

is a slowly varying function of t and the effective impedance is

$$Z_{\text{eff}}(k\Omega) = \int_0^\infty F(t, \tau) e^{(-\gamma + ik\Omega(t))\tau} K_0 \sin(\omega\tau + \zeta) d\tau. \quad (20)$$

This formula gives the correction to the impedance taking into account the slow time-dependence of $J(\phi)$ and Ω . The impedance is now also a slowly varying function of t . The correction factor given by Eq.(17) is:

$$F(t, \tau) = \frac{\hat{J}_k(t - \tau)}{\hat{J}_k(t)} e^{ik \int_0^\tau [\Omega(t-\tau') - \Omega(t)] d\tau'}. \quad (21)$$

Note that the exponent is of order τ^2 .

IMPEDANCE CORRECTION – ANALYTIC CASE

The factors in Eq.(21) can be expanded in power series in τ :

$$\begin{aligned} F(t, \tau) &= \frac{1}{\hat{J}_k(t)} \sum_0^\infty \frac{(-\tau)^n}{n!} \frac{\partial^n \hat{J}_k(t)}{\partial t^n} e^{-ik \sum_{n=2}^\infty \frac{(-\tau)^n}{n!} \frac{\partial^{n-1} \Omega(t)}{\partial t^{n-1}}} \\ &= \sum_{n=0}^\infty r_n (-\tau)^n. \end{aligned} \quad (22)$$

The first few coefficients are

$$\begin{aligned} r_0 &= 1, \quad r_1 = \frac{\dot{J}_k}{\hat{J}_k}, \quad r_2 = \frac{1}{2} \frac{\ddot{J}_k}{\hat{J}_k} - \frac{1}{2} i k \dot{\Omega}, \\ r_3 &= \frac{1}{6} \frac{\ddot{\ddot{J}}_k}{\hat{J}_k} - \frac{1}{6} i k \ddot{\Omega} - \frac{1}{2} \frac{i k \dot{\Omega} \dot{J}_k}{\hat{J}_k}. \end{aligned} \quad (23)$$

We substitute these results into Eq.(20) to obtain

$$Z_{\text{eff}}(k\Omega) = Z(k\Omega) + \sum_{n=1}^{\infty} r_n \frac{\partial^n Z(k\Omega)}{\partial \gamma^n}, \quad (24)$$

where $Z(k\Omega)$ is the uncorrected impedance given by Eq.(14). The sum in the second term is the correction due to the time dependence of $\rho(\phi)$ and $\Omega(t)$.

IMPEDANCE CORRECTION – NUMERICAL SIMULATION

In this section we will evaluate the impedance correction for a numerical simulation using the leap-frog algorithm to move the particles. A formula derived from Eqs.(20) and (21) is given in an ANL report [2]. We present here a simpler way to handle this problem by writing the voltage across a circuit element as the sum of a steady-state term plus a transient.

In a simulation using the leap-frog algorithm, the particle positions and the resulting density $\rho_n(\phi)$ are computed at the beginning of each time step at $t_n = n\Delta t$. We will use the subscripts n and $n + 1/2$ to denote quantities evaluated at the corresponding times t_n and $t_{n+1/2} = (n+1/2)\Delta t$. The density $\rho_n(\phi)$ is taken to be constant during the time interval $t_{n-1/2} \leq t < t_{n+1/2}$. We will calculate from this density in a standard way, for example by using Fourier transforms, the steady state response $V_{ssn}(\phi) = V_{ssn}(t)$ of each circuit j during this n^{th} time interval, where ϕ and t are related by Eq.(2). We then write the total voltage as the steady state plus a transient:

$$V_n(t) = V_{ssn}(t) + V_{trn} e^{-\gamma(t-t_{n-1/2})} \sin[\omega(t-t_{n-1/2}) + \xi_{trn}], \quad (25)$$

where ω, γ, ξ (for element j) are given by Eqs.(7) and (9).

We now match the solution at the beginning of time step n to that at the end of time step $n-1$:

$$\begin{aligned} V_{n-1}(t_{n-1/2}) &= V_{ssn}(t_{n-1/2}) + V_{trn} \sin \xi_{trn}, \\ \left. \frac{\partial V_{n-1}}{\partial t} \right)_{t=t_{n-1/2}} &= \left. \frac{\partial V_{ssn}}{\partial t} \right)_{t=t_{n-1/2}} - \gamma V_{trn} \sin \xi_{trn} + \omega V_{trn} \cos \xi_{trn}. \end{aligned} \quad (26)$$

The solution of these equations for the transient during the n^{th} time step is

$$V_{trn} = [A^2 + B^2]^{1/2}, \quad \tan \xi_{trn} = \frac{A}{B}, \quad (27)$$

where

$$A = V_{n-1}(t_{n-1/2}) - V_{ssn}(t_{n-1/2}) \quad , \quad B = \omega^{-1} \left[\frac{\partial V_{n-1}}{\partial t} \right]_{t=t_{n-1/2}} + \gamma V_{n-1}(t_{n-1/2}) - \gamma V_{ssn}(t_{n-1/2}) - \frac{\partial V_{ssn}}{\partial t} \bigg|_{t=t_{n-1/2}} \quad (28)$$

The reference angular velocity $\Omega(t) = \Omega_n$ is taken as constant during the interval $t_{n-1/2} \leq t < t_{n+1/2}$. Equation (1) then gives

$$\Theta(t) = \Omega_n(t - t_{n-1/2}) + \Theta_{n-1/2}, \quad (29)$$

where

$$\Theta_{n-1/2} = \sum_{n'=1}^{n-1} \Omega_{n'} \Delta t + \Omega_0 \frac{\Delta t}{2}. \quad (30)$$

According to Eqs.(2), (30) and (29) a particle with coordinate ϕ will arrive at the element j at times given by

$$\phi = \theta_j - \Theta_{n-1/2} + \Omega_n(t - t_{n-1/2}) + 2\pi\ell, \quad (31)$$

where ℓ is any integer which puts t in the interval $t_{n-1/2} \leq t < t_{n+1/2}$. The first such time is given by

$$t - t_{n-1/2} = \delta t = \frac{\phi - \theta_j - \Theta_{n-1/2} + 2\pi\ell}{\Omega_n}, \quad (32)$$

where ℓ is the smallest integer which makes δt non-negative. The particle at ϕ thus crosses the element j at the times $t = t_{n-1/2} + \delta t + 2\pi r / \Omega_n$. The integer r runs over the range $0 \leq r \leq r_1$, where

$$r_1 = [\text{nearest integer} \leq (\Omega_n \delta t / 2\pi) - 1]. \quad (33)$$

The total voltage increment given to the particle by the transient during the n^{th} time interval is

$$\delta V = \sum_{r=0}^{r_1} V_{trn} e^{-\gamma[\delta t + (2\pi r / \Omega_n)]} \sin[\omega(\delta t + 2\pi r / \Omega_n) + \xi_{trn}]. \quad (34)$$

The energy of each particle at $t_{n+1/2}$ is obtained by adding to the energy at $t_{n-1/2}$ the increment $eV_{ssn}(\phi) + e\delta V$.

In the leap-frog algorithm, the new energies at time $t_{n+1/2}$ are then used to advance the particle phases from their values at time t_n to time t_{n+1} .

References

- [1] J. A. MacLachlan, Fundamentals of Particle Tracking for the Longitudinal Projection of Beam Phase Space in Synchrotrons, FNAL Report FN-481, April 15, 1988.
- [2] Keith Symon, Longitudinal Wake Field Corrections in Circular Machines, ANL Report NSA-96-1, August 14, 1996. [Available by request from the author.]