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Constrained Minimization for Monotonic Reconstruction *

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Abstract

We present several innovations in a method for monotonic reconstructions. It is based on the application of constrained minimization techniques for the imposition of monotonicity on a reconstruction. In addition, we present extensions of several classical "TVD" limiters [1] to a genuinely multidimensional setting. In this case the linear least squares reconstruction method is expanded upon. We also clarify data dependent weighting techniques used with the minimization process.

I. Motivation

During the last twenty years, higher order monotone advection schemes have not only become accepted, but become ubiquitous in the CFD literature [2, 3, 4, 5, 6]. The use of these methods has largely supplanted first-order upwinding as the method of choice when non-physical oscillations cannot be tolerated. The design of one-dimensional methods is quite well developed and the ability to apply a particular amount of numerical viscosity has reached an advanced state of development.

The vast majority of applications of this technology rely upon the basic one-dimensional implementation of the basic ideas set forth in designing monotone methods. A number of truly multidimensional methods have appeared, but their use is by no means widespread as the one-dimensional methods.

With the advent of increasingly complex computational grids, the need for truly multidimensional methods has become more acute. We prefer to use meth-

ods that do not impose any concepts from operator splitting-based methodology. Grids that pose significant problems for the basically one-dimensional methods include triangular/tetrahedral [7], tree-based adaptive [8] and arbitrary connectivity polygonal types [9].

We take the multidimensional "k-exact" methods as described by Barth as our starting point [10, 11, 12, 13, 14]. Specifically, we base our work on the least-squares form introduced by Barth. The definition and application of monotonicity and other concepts is sometimes difficult to apply in a completely multidimensional context. Monotonicity as introduced by Dukowicz [15, 16] has become the standard. This is strongly related to Zalesak's multidimensional FCT algorithm [17]. Dukowicz's basic algorithm has formed the basis of Barth's application of monotonicity.

The basic method we start with is Barth's least squares minimization based interpolation method. Rather than implement a scalar imposition of monotonicity ala Dukowicz/Barth, our monotonicity imposition will take the form of an inequality constraint and result in a vector correction to the reconstruction. Secondly, we discuss an extension of the basic ingredients in one-dimensional slope limiters implemented in a completely multidimensional manner. Finally, we discuss the issue of using a weighted least squares approach in a manner that integrates expected numerical error into the interpolation process. Results are given to substantiate the benefits of this approach.

II. Methodology

II.1 Least Squares Reconstruction

Our method is based on a least squares formulation of the reconstruction in a control volume. This formulation allows a great deal of freedom in the solution of the reconstruction. Our goal is to solve for the expansion coefficients for a polynomial reconstruction of ϕ

$$\phi_i(x, y) = \phi_i + \nabla_x \phi (x - x_{1,i}) + \nabla_y \phi (y - y_{1,i}) + \nabla_x^2 \phi (x^2 - x_{2,i}) + \dots, \quad (1)$$

where the constants $x_{q,i}$ are defined so that

$$\int_{\Omega} \phi_i(x, y) d\Omega = \phi_i.$$

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This is the product of the following weighted least squares problem in two dimensions

$$\min \|w^T (Ax - b)\|_2 \rightarrow A^T w^T w Ax = A^T w^T w b \quad (2)$$

where

$$A = \begin{pmatrix} (x_{1,k} - x_{1,i}) & (y_{1,k} - y_{1,i}) & (x_{2,k} - x_{2,i}) & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ (x_{1,n} - x_{1,i}) & (y_{1,n} - y_{1,i}) & (x_{2,n} - x_{2,i}) & \cdots \end{pmatrix},$$

this will be abbreviated as

$$A = \begin{pmatrix} \Delta x_{1,k} & \Delta y_{1,k} & \Delta x_{2,k} & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ \Delta x_{1,n} & \Delta y_{1,n} & \Delta x_{2,n} & \cdots \end{pmatrix},$$

$$b = \begin{pmatrix} (\phi_k - \phi_i) \\ \vdots \\ (\phi_n - \phi_i) \end{pmatrix},$$

This is abbreviated as

$$b = \begin{pmatrix} \Delta \phi_k \\ \vdots \\ \Delta \phi_n \end{pmatrix},$$

$$w = \begin{pmatrix} w_k \\ \vdots \\ w_n \end{pmatrix},$$

and

$$x = \begin{pmatrix} \nabla_x \phi_i \\ \nabla_y \phi_i \\ \nabla_x^2 \phi_i \\ \vdots \end{pmatrix}.$$

Finally, we shall briefly mention the caution one should take in constructing the local linear systems of equations. The systems can with weighting become quite ill-conditioned. Care should be taken in both constructing and solving the resulting equations so that errors are not introduced in the process. In the base case, on regular grids solving the normal equations using a standard method is usually sufficient, but in the more general case this can be disastrous. To avoid this problem we employ a QR-factorization in cases where we suspect the systems to potentially be ill-conditions. Additionally, if the minimization problem becomes rank-deficient one can regularize it or solve the linear system

(-1,1,6)	(0,1,8)	(1,1,15)
(-1,0,3)	(0,0,5)	(1,0,13)
(-1,-1,1)	(0,-1,4)	(1,-1,10)

(x,y,ϕ)

Figure 1: The raw data for the example problem used in this section.

Figure 2: Gradients computed from various pairs of raw data and the regular and distance squared weighted least squares problem.

with a SVD algorithm. We choose to regularize the solution with a small parameter, $\epsilon \approx 1.0 \times 10^{-6}$ multiplied by the sign of the entry which is added to the entry (A Tikhonov-type method).

Throughout this section we will make use of an example problem. This will illustrate some of the concepts we introduce here. The basic raw data used is shown in Figure 1. The raw gradients are computed using a set of eight nearest neighbor triangles on the given data. The initial least squares solution is shown in Figure 2. Also shown is a distance squared weighted least squares solution which is discussed at the end of the section.

II.2 Monotonic Constrained Minimization

We believe that it is useful to strongly maintain the following distinction when thinking of monotonicity constraints, *monotonicity is a discrete concept while the high-order method is accuracy driven*. To put this another way, *an accurate derivative requires knowledge of the data surrounding a cell and where it is located, but monotonicity only requires one to know the data and the geometry of the cell*.

To review, the scalar monotonicity algorithm works in the following fashion: Our goal is to find a constant α to be applied to the interpolant

$$\phi_i(x, y) = \phi_i + \alpha(\nabla_x \phi(x - x_{1,i}) + \nabla_y \phi(y - y_{1,i}) + \nabla_x^2 \phi(x^2 - x_{2,i}) + \dots,$$

such that the reconstruction is monotone with respect to the local data.

1. Find the local minimum and maximum of the dependent data $\phi^{\min}$ and $\phi^{\max}$.
2. Find the reconstruction minimum and maximum in Ω , $\phi_{\Omega}^{\min}$ and $\phi_{\Omega}^{\max}$.
3. If the $\phi_{\Omega}^{\min} < \phi^{\min}$ compute α such that $\phi_{\Omega}^{\min} = \phi^{\min}$, otherwise $\alpha = 1$,

$$\alpha = \min \left(1, \frac{\phi - \phi^{\min}}{\phi - \phi_{\Omega}^{\min}} \right)$$

4. If the $\phi_{\Omega}^{\max} > \phi^{\max}$ compute α such that $\phi_{\Omega}^{\max} = \phi^{\max}$, otherwise $\alpha = 1$.

$$\alpha = \min \left(1, \frac{\phi^{\max} - \phi}{\phi_{\Omega}^{\max} - \phi} \right)$$

5. Choose α to be the smallest of the available choices.

By relying upon the extensive work done in solving least squares problems [18] we can extend the utility of this approach. The formulation of the monotonicity satisfying interpolation with an inequality constraint is quite similar. Basically, the least squares problem is modified by any of the constraining inequalities that is violated by the base solution. This recasts the minimization as

$$\min \|w^T(Ax - b)\|_2 \text{ subject to } Cx = d. \quad (3)$$

If none of the inequalities are violated than C is null. At most the rank of C is equal to x . For example, if there are two unknowns than up to two constraints can be active and their solution will determine the system. Were one constraint active than a minimization would take place.

We find the active constraint in a process similar to scalar algorithm. The difference is that more than one constraint can be used to determine the overall limiter. The process proceeds as follows:

1. Find the local minimum and maximum of the dependent data $\phi^{\min}$ and $\phi^{\max}$.
2. Compute the smallest positive difference $\Delta\phi^{\min} = \min(\phi - \phi^{\min}, \phi^{\max} - \phi)$.
3. Find the reconstruction minimum and maximum in Ω , $\phi_{\Omega}^{\min}$ and $\phi_{\Omega}^{\max}$.
4. Check to see if the reconstruction maxima and minima violate the monotonicity constraint. If $\phi - \phi_{\Omega}^{\min} > \Delta\phi^{\min}$ this is an active constraint, and if $\phi_{\Omega}^{\max} - \phi > \Delta\phi^{\min}$ this is an active constraint.

For simple rectangular mesh cells several simplifications can be made. The constraints that can be active are

$$\begin{aligned} h_x \eta_x \nabla_x \phi + h_y \eta_y \nabla_y \phi &\leq 2\Delta\phi^{\min}, \\ h_x \eta_x \nabla_x \phi - h_y \eta_y \nabla_y \phi &\leq 2\Delta\phi^{\min}, \end{aligned}$$

and

$$-h_x \eta_x \nabla_x \phi + h_y \eta_y \nabla_y \phi \leq 2\Delta\phi^{\min},$$

where $\eta_x = \text{sign}(\nabla_x \phi)$ and $\eta_y = \text{sign}(\nabla_y \phi)$.

The equation can be applied to the minimization problem in the following way: suppose we are computing the following interpolant,

$$\phi_i + \nabla_x \phi(x - x_{1,i}) + \nabla_y \phi(y - y_{1,i}),$$

and we have a constraint

$$h_x \nabla_x \phi + h_y \nabla_y \phi = \Delta\phi^{\lim}.$$

We have two options in applying this constraint to the interpolation problem: the method of weighting and algebraic elimination. The method of weighting would take some constant λ and add the constraint equation to the matrix A as a new row and an entry in b as

$$\lambda(h_x \nabla_x \phi + h_y \nabla_y \phi - \Delta\phi^{\lim}).$$

The algebraic elimination would solve for one of the unknowns in terms of the other(s) and then a reduced system would be minimized. In this example we could take

$$\nabla_x \phi = \frac{\Delta\phi^{\lim} - h_y \nabla_y \phi}{h_x}.$$

The result of this process is a vector scaling of the interpolant such that the constraint is satisfied and the error is minimized.

Within the constrained least squared minimization framework several variations can be constructed. One can take the view that we are now filtering the stencil in order to accentuate certain properties in the data. The practical effect will be to either remove or add dissipation from the solution. In this way much of the flexibility inherent in one-dimensional high-resolution methods can be applied to multidimensional methods on arbitrary grids.

II.3 Extending Classical TVD Limiters

We begin with the statement that the procedures given above produce multidimensional extensions of a monotone Fromm's (or centered) method¹. Two other important limiters are the minmod² and superbee³ limiter. By making the weights data dependent, these limiters can easily be implemented. Each of these multidimensional extensions reduces to the corresponding one-dimensional limiter if the flow becomes one-dimensional (mesh aligned on an orthogonal grid).

The minmod limiter in one dimension chooses the minimum of the available slopes, and the superbee chooses the maximum with the condition that the choice be monotonicity preserving. One way to generalize this is to make the choice based on the average slope. Others [19, 20] have done the same, but with the interpretation of superbee as the largest of the available slopes. These choices are implemented by setting the weights to zero if the data does not meet the selection criteria. Thus the minmod limiter generalizes by choosing the data that is smaller than the average and the superbee through choosing data that is larger. The for a two-dimensional linear reconstruction algorithm would be organized as follows:

1. Solve Equation (2) with centered data.
2. For each cell in the stencil compute $\delta_k \phi = (x_{1,i} - x_{1,k}) \nabla_x \phi + (y_{1,i} - y_{1,k}) \nabla_y \phi$.
3. For the minmod limiter if $|\Delta_k \phi| > |\delta_k \phi|$ then $w_k = 0$, otherwise it is unchanged.
4. For the superbee limiter if $|\Delta_k \phi| < |\delta_k \phi|$ then $w_k = 0$, otherwise it is unchanged.
5. Solve the reweighted least square problem again.
6. Impose the monotonicity constraints on the resulting solution.

For the example problem, we show the solution for the minmod and superbee limiters applied in multidimensional least squares fashion. In this case the minmod limiter results in a gradient that is smaller in magnitude than any of the raw gradients. The minmod gradient uses the following four data points, $(-1, -1), (-1, 0), (-1, 1), (0, -1)$. The superbee limiter provides a gradient that is near the size of the largest raw gradients. The superbee gradient uses the following four data points, $(0, 1), (1, -1), (1, 0), (1, 1)$.

In Figure 4 we show the result of applying the monotonicity constraints to the superbee limiter. One inequality constraint is active, $\nabla_x \phi + \nabla_y \phi = 8$, although

¹In the standard description of TVD limiter, $Q(r) = \minmod[\frac{1}{2}(1+r), 2, 2r]$

² $Q(r) = \minmod(1, r)$.

³ $Q(r) = \minmod[\max(1, r), 2, 2r]$

Figure 3: Computed minmod and superbee least squares gradients for the given data.

Figure 4: The standard scalar and constrained limiter applied to the non-monotone superbee gradients. Two of the four lines formed by the constraints are shown.

a second constraint is shown, the superbee gradient does not violate it. As can be observed the scalar limiter simply uniformly scales the gradient onto the line formed by the equality form of the constraint and differs significantly from the minimized constrained solution.

Another important concept is slope steepening (related strongly to artificial compression [21, 22]). One-dimensional piecewise linear schemes have been ex-

pressed in a rather completely by Huynh [23] who introduced an interesting slope steepener. We first discuss this in a simplified setting from that given by Huynh, but then move the concept to a purely multidimensional implementation. Huynh defines the scheme using a constant κ to determine the compression. The left and right slopes are differenced and multiplied by κ then made monotone. In the standard form

$$Q(r) = \min\left(\max\left(\frac{1}{2}(1+r), \kappa|1-r|\right), 2, 2r\right).$$

In multidimensions we replace the term, $\kappa|1-r|$, by $\kappa(\phi^{\max} - 2\phi + \phi^{\min})$. The other terms are also substituted as the implementation of the van Leer scheme in multiple dimensions suggests.

Two other important limiters are the harmonic mean limiter [24]⁴ and van Albada's limiter [25]⁵. By choosing the weights to be inversely proportional to the absolute value of size of the data, the harmonic mean limiter is generalized. This simply requires the weight vector be scaled by $w_k := w_k / |\Delta_k \phi|$. If this choice is the inverse of the size of the data squared, the van Albada limiter is generated. Again, this requires the weight vector be scaled, in this case by $w_k := w_k / (\Delta_k \phi)^2$. These limiters may be useful in fine-tuning the resolution and characteristics of the interpolant to a given application or situation.

Yet another small modification of the weighting used to extend the harmonic mean and van Albada limiters can be used to implement a L_1 minimization. This is accomplished through using an iteratively reweighted least squares calculation where the weights are inversely proportional to the residual. Thus, the van Leer limited scheme can be used to compute the first guess, then the inverse of the residual can be used to weight the data and the solution is found again.

We now will discuss weighting in a broader sense. The use of least squares methods in computing the functional dependence of data is well known. It seems sensible to apply more or less weight to data points depending on their reliability. The above generalization of limiters could be viewed in this context. At a given point is space for the purposes that we have employed the least squares machinery the appropriate weighting is inversely proportional to the distance. In [26] the form is discussed with relation to the expected error. There a linear interpolant was determined for some data using the model,

$$y = a + bx + \epsilon \Delta y,$$

where the last term, $\epsilon \Delta y$, is the expected error. Since we are essentially approximating a Taylor series expansion, the general form of the expected error is known (where

⁴ $Q(r) = (r + |r|) / (1 + |r|)$.

⁵ $Q(r) = (r + r^2) / (1 + r^2)$.

the function is smooth). For a linear expansion the error terms scale with the square of the distance, thus the geometric portion of the weight should be inversely proportional to the distance squared. This relation will change as the order of the interpolant increases. As a result the norm of the residual will be a reasonable estimate of the truncation error of the method.

III. Results

We will present the results of the above described methods on two test problems: a smooth double sine wave, and a circular region both on a periodic grid. The sine wave test will show the order of accuracy of the method and the circular region will show the behavior of the method at discontinuities and the distortion of a smooth body.

Table 1 shows the error for the sine wave test on a series of grids 16×16 through 64×64 . The fine grid error and the order of convergence are shown. Several conclusions can be drawn from these results, the multidimensional reconstructions are superior and constraint based limiters improve the solution.

We show the methods described above in Figures 5 through 7. The solutions are computed with a monotone unsplit differencing method similar to the one in [27]. Each is computed at a CFL number of one-half on a 50×50 grid. We believe the close-ups of the details of the solution provide critical details to judge the fidelity of the calculations

In general the purely multidimensional methods provide a significantly less grid-dependent solution. The general features of the one-dimensional limiters have been replicated in a consistent two-dimensional manner. The superbee limiter shows small 3-4 cell transitions and the Huynh compressive limiter is even sharper. While the constraint-based limiting is somewhat superior to scalar limiting, its superbee implementation shows some evidence of increased grid dependence. This may be due to a decrease in dissipation, and the interface thickness while consistent in thickness in both methods is sharper with the constraint-based limiting.

IV. Conclusions

Here we have presented several extensions of existing methods for reconstructing functions for the purpose of constructing a Godunov algorithm. We have demonstrated that these methods are genuinely multidimensional and naturally extend to arbitrary grids. Furthermore, the procedures improve the accuracy and quality of solutions. These methods are also more flexible than existing multidimensional methods. We feel that this is fertile ground for further investigation.

Table 1: Order of accuracy for various reconstruction methods for a double sine wave advection.

Method	Fine Grid Error L_1	Fine Grid Error L_∞	Order L_1	Order L_∞
1-D Fromm	1.03×10^{-3}	1.19×10^{-2}	1.97	1.42
2-D Fromm	4.28×10^{-4}	1.13×10^{-2}	2.68	1.53
2-D Constrained Fromm	4.02×10^{-4}	1.13×10^{-2}	2.72	1.51
2-D Minmod	6.30×10^{-3}	3.81×10^{-2}	1.68	1.14
2-D Harmonic	7.91×10^{-3}	3.64×10^{-2}	1.60	1.25
2-D L_1	7.24×10^{-4}	1.72×10^{-2}	2.36	1.38

Figure 5: Advection of a circle by a unsplit advection scheme using various Fromm-van Leer methods.

Figure 6: Advection of a circle by an unsplit advection scheme using various Superbee methods.

Figure 7: Advection of a circle by a unsplit advection scheme using various multidimensional slope reconstruction methods.

References

- [1] P. K. Sweby. High-resolution schemes using flux limiters for hyperbolic conservation laws. *SIAM Journal on Numerical Analysis*, 21:995–1011, 1984.
- [2] C. Hirsch. *Numerical Computation of Internal and External Flows: Volume 1*. Wiley-Interscience, 1988.
- [3] C. Hirsch. *Numerical Computation of Internal and External Flows: Volume 2*. Wiley-Interscience, 1988.
- [4] C. A. J. Fletcher. *Computational Techniques for Fluid Dynamics: Volume I*. Springer-Verlag, 1988.
- [5] C. A. J. Fletcher. *Computational Techniques for Fluid Dynamics: Volume II*. Springer-Verlag, 1988.
- [6] R. J. LeVeque. *Numerical Methods for Conservation Laws*. Birkhäuser, 1990.
- [7] V. Venkatakrishnan. Convergence to steady state solutions of the Euler equations on unstructured grids with limiters. *Journal of Computational Physics*, 118:120–130, 1995.
- [8] K. G. Powell. A tree-based adaptive scheme for the solution of the equations of gas dynamics and magnetohydrodynamics. *Applied Numerical Mathematics*, 14:327–352, 1994.
- [9] S. J. Mosso, B. K. Swartz, D. B. Kothe, and R. C. Ferrell. A parallel, volume tracking algorithm for unstructured meshes. In P. Schiano, editor, *Parallel Computational Fluid Dynamics '96*, Italy, 1996.
- [10] T. J. Barth and D. C. Jespersen. The design and application of upwind schemes on unstructured meshes. In *27th Aerospace Sciences Meeting and Exhibit*, Reno, Nevada, 1989. AIAA-89-0366.
- [11] T. J. Barth and P. O. Frederickson. Higher order solution of the Euler equations on unstructured grids using quadratic reconstruction. In *28th Aerospace Sciences Meeting and Exhibit*, Reno, Nevada, 1990. AIAA-90-0013.
- [12] T. J. Barth. Aspects of unstructured grids and finite-volume solvers for euler and navier-stokes equations, 1995. VKI/NASA/AGARD Special Course on Unstructured Grid Methods for Advection Dominated Flows AGARD Publication R-787.
- [13] T. J. Barth. Recent developments in high order K-exact reconstruction on unstructured meshes. In *31st Aerospace Sciences Meeting and Exhibit*, Reno, Nevada, 1993. AIAA-93-0668.
- [14] T. J. Barth. Parallel cfd algorithms on unstructured meshes, 1995. VKI/NASA/AGARD Lecture Series on Parallel Computing. AGARD Publication R-807, available via T. J. Barth's WWW Home Page.
- [15] J. K. Dukowicz. New methods for conservative rezoning (remapping) for general quadrilateral meshes in *rezoning workshop 1983*. Technical Report LA-10112-C, Los Alamos National Laboratory, 1984.
- [16] J. K. Dukowicz and J. W. Kodis. Accurate conservative remapping (rezoning) for arbitrary Lagrangian-Eulerian computations. *SIAM Journal on Scientific and Statistical Computing*, 8:305–321, 1987.
- [17] S. T. Zalesak. Fully multidimensional flux-corrected transport algorithms for fluids. *Journal of Computational Physics*, 31:335–362, 1979.
- [18] Å. Björk. *Numerical Methods for Least Squares Problems*. SIAM, 1996.
- [19] L. J. Durlofsky, B. Engquist, and S. Osher. Triangle based adaptive stencils for the solution of hyperbolic conservation laws. *Journal of Computational Physics*, 98:64–73, 1992.
- [20] P. Batten, C. Lambert, and D. M. Causon. Positively conservative high-resolution convection schemes for unstructured elements. *International Journal for Numerical Methods in Engineering*, 39:1821–1838, 1996.
- [21] A. Harten. The artificial compression method for computation of shocks and contact discontinuities. I. single conservation laws. *Communications on Pure and Applied Mathematics*, 30:611–638, 1977.
- [22] H. Yang. An artificial compression method for ENO schemes: The slope modification method. *Journal of Computational Physics*, 89:125–160, 1990.
- [23] H. T. Huynh. Accurate upwind methods for the Euler equations. *SIAM Journal on Numerical Analysis*, 32:1565–1619, 1995.
- [24] B. van Leer. Towards the ultimate conservative difference scheme. III. Upstream-centered finite-difference schemes for ideal compressible flow. *Journal of Computational Physics*, 23:263–275, 1977.
- [25] G. D. van Albada, B. van Leer, and W. W. Roberts. A comparative study of computational methods in cosmic gas dynamics. *Astronomy and Astrophysics*, 108:76–84, 1982.

- [26] I. R. Gatland. A weight-watcher's guide to least-squares fitting. *Computers in Physics*, 7(3):280–285, 1993.
- [27] J. B. Bell, C. N. Dawson, and G. R. Shubin. An unsplit, higher order Godunov method for scalar conservation laws in multiple dimensions. *Journal of Computational Physics*, 74:1–24, 1988.