Quantum Groups: Geometry and Applications

C.-S. Chu
Physics Division

May 1996
Ph. D. Thesis
DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor The Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or The Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof, or The Regents of the University of California.

Ernest Orlando Lawrence Berkeley National Laboratory is an equal opportunity employer.
Quantum Groups: Geometry and Applications\textsuperscript{1}

(Ph.D. Thesis)

Chong-Sun Chu\textsuperscript{2}

\textit{Theoretical Physics Group}
\textit{Lawrence Berkeley Laboratory}
\textit{University of California}
\textit{Berkeley, California 94720}

\textsuperscript{1}This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY-9514797.

\textsuperscript{2}email address: cachu@physica.berkeley.edu
Abstract

The main subject of this thesis is the geometry of quantum groups and quantum spaces. The main tool used is the Faddeev-Reshetikhin-Takhtajan description of quantum groups. A few content-rich examples of quantum complex spaces with quantum group symmetry are treated in details. In chapter 1, we review some of the basic concepts and notions for Hopf algebras and other background materials. In chapter 2, we study the vector fields of quantum groups. A compact realization of these vector fields as pseudodifferential operators acting on the linear quantum spaces is given. In chapter 3, we describe the quantum sphere as a complex quantum manifold by means of a quantum stereographic projection. A covariant calculus is introduced. An interesting property of this calculus is the existence of a one-form realization of the exterior differential operator. The concept of a braided comodule is introduced and a braided algebra of quantum spheres is constructed. In chapter 4, we consider the more general higher dimensional quantum complex projective spaces and the quantum Grassmanian manifolds. Differential calculus, integration and braiding can be introduced as in the one dimensional case. A sufficient condition for the existence of a one-form realization of the exterior differential operators $\delta, \bar{\delta}$ is given and is applied to the case of complex projective spaces and quantum Grassmannians. Finally, in chapter 5, we study the framework of quantum principal bundle and construct the $q$-deformed Dirac monopole as a quantum principal bundle with a quantum sphere as the base and a $U(1)$ with non-commutative calculus as the fiber. The first Chern class can be introduced and integrated to give the monopole charge.

Disclaimer

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor The Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial products process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or The Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof, or The Regents of the University of California.

Lawrence Berkeley Laboratory is an equal opportunity employer.
DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.
# Contents

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Acknowledgements</td>
<td>iv</td>
</tr>
<tr>
<td></td>
<td>Introduction</td>
<td>ix</td>
</tr>
<tr>
<td>1</td>
<td>Hopf Algebras</td>
<td>1</td>
</tr>
<tr>
<td>1.1</td>
<td>Basic Definitions</td>
<td>1</td>
</tr>
<tr>
<td>1.2</td>
<td>Quasi-triangular $\mathcal{U}$ and the Dual Pair $(\mathcal{U}, A)$</td>
<td>3</td>
</tr>
<tr>
<td>1.2.1</td>
<td>Example: $U_q(su(2))$ and $SU_q(2)$</td>
<td>7</td>
</tr>
<tr>
<td>1.3</td>
<td>The Smash Product</td>
<td>10</td>
</tr>
<tr>
<td>1.3.1</td>
<td>Action and Coaction</td>
<td>10</td>
</tr>
<tr>
<td>1.3.2</td>
<td>Smash Product</td>
<td>11</td>
</tr>
<tr>
<td>1.3.3</td>
<td>Example: Quantum Group</td>
<td>11</td>
</tr>
<tr>
<td>1.4</td>
<td>Differential Calculus on an Algebra</td>
<td>13</td>
</tr>
<tr>
<td>1.4.1</td>
<td>First Order Differential Calculus on an Algebra</td>
<td>13</td>
</tr>
<tr>
<td>1.4.2</td>
<td>Universal Calculus and General Calculus</td>
<td>13</td>
</tr>
<tr>
<td>1.4.3</td>
<td>3-D Calculus on $SU_q(2)$</td>
<td>16</td>
</tr>
<tr>
<td>2</td>
<td>Pseudodifferential Operators Realization of Vector Fields</td>
<td>18</td>
</tr>
<tr>
<td>2.1</td>
<td>Vector Fields for Quantum Groups</td>
<td>18</td>
</tr>
<tr>
<td>2.2</td>
<td>$q$-Determinant for FRT Type Algebra</td>
<td>19</td>
</tr>
<tr>
<td>2.2.1</td>
<td>$q$-Determinant for Bicovariant Vector Fields $V$</td>
<td>22</td>
</tr>
<tr>
<td>2.3</td>
<td>$GL_q(N), SL_q(N)$ and $SU_q(N)$</td>
<td>24</td>
</tr>
<tr>
<td>2.3.1</td>
<td>$GL_q(N)$</td>
<td>24</td>
</tr>
<tr>
<td>2.3.2</td>
<td>$SL_q(N)$</td>
<td>28</td>
</tr>
<tr>
<td>2.3.3</td>
<td>Real Forms $U_q(N)$ and $SU_q(N)$</td>
<td>28</td>
</tr>
<tr>
<td>2.4</td>
<td>$SO_q(N)$ and $SO_q(N, R)$</td>
<td>33</td>
</tr>
<tr>
<td>2.5</td>
<td>A Few Remarks</td>
<td>38</td>
</tr>
<tr>
<td>3</td>
<td>Quantum Complex Sphere $S_q^2$</td>
<td>41</td>
</tr>
<tr>
<td>3.1</td>
<td>The Algebra $S_q^2$ and the Patch $C^+$</td>
<td>41</td>
</tr>
<tr>
<td>3.2</td>
<td>Differential Calculus</td>
<td>43</td>
</tr>
<tr>
<td>3.2.1</td>
<td>One-form Realization of the Exterior Differential Operator $d$</td>
<td>46</td>
</tr>
<tr>
<td>3.3</td>
<td>Patching Two Quantum Planes</td>
<td>47</td>
</tr>
<tr>
<td>3.4</td>
<td>Right Invariant Vector Fields on $S_q^2$</td>
<td>49</td>
</tr>
<tr>
<td>3.5</td>
<td>Braided Quantum Sphere</td>
<td>52</td>
</tr>
<tr>
<td>3.5.1</td>
<td>Braiding for Quantum Group Comodules</td>
<td>53</td>
</tr>
<tr>
<td>3.5.2</td>
<td>The Braided Sphere</td>
<td>55</td>
</tr>
<tr>
<td>3.6</td>
<td>Integration</td>
<td>57</td>
</tr>
<tr>
<td>4</td>
<td>Quantum Complex Projective Space</td>
<td>60</td>
</tr>
<tr>
<td>4.1</td>
<td>$CP_q(N)$ as a Complex Manifold</td>
<td>60</td>
</tr>
<tr>
<td>4.1.1</td>
<td>$SU_q(N + 1)$ Covariant Quantum Space $C^{N+1}_q$</td>
<td>60</td>
</tr>
<tr>
<td>4.1.2</td>
<td>Complex Projective Space $CP_q(N)$</td>
<td>63</td>
</tr>
<tr>
<td>4.1.3</td>
<td>Other Choices for the Defining Relations for $C^{N+1}_q$</td>
<td>66</td>
</tr>
<tr>
<td>4.2</td>
<td>One-Form Realization of Exterior Differential Operators</td>
<td>68</td>
</tr>
<tr>
<td>4.2.1</td>
<td>A Special One-Form</td>
<td>68</td>
</tr>
<tr>
<td>4.2.2</td>
<td>One-form Realization of the Exterior Differential for a $*$-Algebra</td>
<td>71</td>
</tr>
<tr>
<td>4.3</td>
<td>Integration</td>
<td>74</td>
</tr>
<tr>
<td>4.4</td>
<td>Braided $CP_q(N)$</td>
<td>77</td>
</tr>
<tr>
<td>4.5</td>
<td>Quantum Grassmannians $G_q^{M,N}$</td>
<td>78</td>
</tr>
<tr>
<td>4.5.1</td>
<td>The Algebra</td>
<td>78</td>
</tr>
<tr>
<td>4.5.2</td>
<td>Calculus</td>
<td>80</td>
</tr>
<tr>
<td>4.5.3</td>
<td>One-Form Realization</td>
<td>81</td>
</tr>
<tr>
<td>4.5.4</td>
<td>Braided $G_q^{M,N}$</td>
<td>85</td>
</tr>
<tr>
<td>5</td>
<td>$q$-Deformed Dirac Monopole</td>
<td>87</td>
</tr>
<tr>
<td>5.1</td>
<td>Quantum Principal Bundle</td>
<td>87</td>
</tr>
<tr>
<td>5.1.1</td>
<td>Universal Calculus</td>
<td>88</td>
</tr>
<tr>
<td>5.1.2</td>
<td>General Calculus</td>
<td>91</td>
</tr>
</tbody>
</table>
Acknowledgements

It is a great pleasure to thank my thesis adviser, Professor Bruno Zumino, for guiding me into the fascinating area of quantum groups and theoretical physics, for his very inspiring advice and informative remarks, for his kindness and support, on both academic and bureaucratic matters, for his great patience with my mistakes and ignorance, for creating an enthusiastic environment for pleasant collaboration and for his invaluable contributions to our collaboration. It is an invaluable experience to be his student.

My good friend and collaborator Pei-Ming Ho deserves special recognition, for sharing his talents and ability with me, for his constructive criticism, constant enthusiastic discussions and most importantly for sharing his friendship with me. The other members in my advisor’s research group, Bogdan Morariu and Harold Steinacker, contributed much to the stimulating and enjoyable research atmosphere at Berkeley. I have learned a lot from many interesting discussions and from collaboration with them. I thank them all for sharing their insights and their friendship with me.

I must also thank my fellow graduate students in Berkeley, who make me feel very much at home, among those I mention especially Luis Bernardo, Hein-Chia Cheng, An-Dien Nguyen, Kamran Saririan, Zheng Yin and Yi-Yen Wu as well as the postdocs Joanne Cohn, Jonathen Feng, Michael Schlieker and Nir Sochen.

Thanks go to the staff at LBL and on campus, especially Barbara Gordon, Luanne Neumann, Donna Sakima, Laura Scott and Anne Takizawa. Without their assistance, bureaucratic difficulties would be much more difficult to endure. Special thanks go to Anne, adjustment to life in the US as a foreign student was made much easier because of her ready help.

I am grateful to Professors Orlando Alvarez, Korkut Bardakci, Mary K. Gaillard, Hirosi Ooguri, Mahiko Suzuki, Bruno Zumino in the physics department and Professors Nicolai Reshetikhin, Marc Rieffel in the mathematics department for the inspiring courses they gave on quantum field theory, particle physics, general relativity, string and conformal theory, quantum groups, differential geometry, classical and quantum integrable systems, operator algebras respectively. I also want to thank Professors Korkut Bardakci, Hirosi Ooguri and Nicolai Reshetikhin for nu-
numerous helpful and stimulating discussions and particularly to Professors Bardakci
and Reshetikhin for being also on my committee and for agreeing to read my thesis.

Finally, no thanks is enough to my parents for their constant support and encour-
agement and for their trust in my choice of path. “Thanks”.

This work was supported in part by the Director, Office of Energy Research,
Office of High Energy and Nuclear Physics, Division of High Energy Physics of the
U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part
by the National Science Foundation under grant PHY-9514797.

Introduction

Quantum groups were discovered in Leningrad by studying the quantum integrable
1+1 dimensional systems, using the method of quantum inverse scattering [1, 2]. In
the Hamiltonian approach to the inverse scattering method [2, 3], one introduces
the pair of first order differential operators,

$$L = \frac{d}{dx} - U, \quad M = \frac{d}{dt} - V, \quad (0.1)$$

where $U(x, \lambda), V(x, \lambda)$ are matrices with matrix elements being functions of
the basic field variables (e.g. $\psi, \psi^*$ for the nonlinear Schrödinger system) and
their spatial derivatives, and $\lambda$ is a complex parameter. Given the Hamiltonian $H$
and a Poisson bracket $\{,\}$, one requires that the equation of motion (EOM) admits the
Lax representation

$$\text{EOM} \Leftrightarrow [L, M] = 0. \quad (0.2)$$

The Poisson bracket, written in terms of the matrix $U(x, \lambda)$ is of the form

$$\{U(x, \lambda) \circ U(y, \mu)\} = [r(\lambda - \mu), U(x, \lambda)] \circ I + I \circ [U(y, \mu)] \delta(x - y), \quad (0.3)$$

where

$$\{A \circ B\}_{\text{in}} = \{A_{\ell} \circ B_{\ell}\} \quad (0.4)$$

and $r(\lambda)$ is the classical r-matrix with spectral parameter $\lambda$ satisfying the classical
Yang-Baxter equation (CYBE),

$$[r_{12}(\lambda - \mu), r_{13}(\lambda)] + [r_{12}(\lambda), r_{23}(\mu)] = 0. \quad (0.5)$$

In a quantum theory, one often uses a lattice to regularize the theory, therefore it
is preferable to consider the above classical system on a one dimensional spatial
lattice with lattice sites labelled by integer $n$. Define $L_n(\lambda)$ by

$$\exp \int_{-\infty}^{+\infty} dx U(x, \lambda) \simeq L_n(\lambda) + [\delta x]^2, \quad (0.6)$$

then (0.3) becomes

$$\{L_n(\lambda) \circ L_m(\mu)\} = [r(\lambda - \mu), L_n(\lambda) \circ L_m(\mu)] \delta_{nm}. \quad (0.7)$$
This formula has the remarkable property that (0.7) still holds after replacing \( L_n(\lambda) \) by \( L_{n+1}(\lambda)L_n(\lambda) \). This suggest that we are dealing with a group comultiplication (loop group) that is compatible with the Poisson structure. It was this property that led Drinfeld to the notion of a Poisson-Lie group and thus revealed the mathematical structure of Poisson-Lie groups behind the theory of solvable lattice models. Upon quantization, Sklyanin, Takhtajan and Faddeev found that the operators \( \hat{L}_n(\lambda) \) satisfy the commutation relations

\[
R_{12}(\lambda - \mu)(\hat{L}_n(\lambda) \otimes I)(I \otimes \hat{L}_n(\mu)) = (I \otimes \hat{L}_n(\mu))(\hat{L}_n(\lambda) \otimes I)R_{12}(\lambda - \mu),
\]

(0.8)

where \( R(\lambda) \) is the quantum \( R \)-matrix satisfying the quantum Yang-Baxter equation (QYBE),

\[
R_{12}(\lambda - \mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda - \mu)
\]

(0.9)

and the matrices \( \hat{L}_n(\lambda) \) have commuting matrix elements for different lattice sites. (0.8) reduces to (0.7) in the classical limit of \( \hbar \rightarrow 0 \):

\[
\frac{1}{\hbar}[\hat{L}_n(\lambda), \hat{L}_m(\mu)] \rightarrow [L_n(\lambda), L_m(\mu)],
\]

(0.10)

\[
\frac{1}{\hbar}(R(\lambda) - I) \rightarrow r(\lambda).
\]

(0.11)

Remarkably, (0.8) also has the property that the matrix product \( \hat{L}_{n+1}(\lambda)\hat{L}_n(\lambda) \) satisfies the same Poisson bracket (0.8) also. This "comultiplication structure" suggests an underlying structure of bialgebra compatible with the quantum algebra (0.8) specified by the matrix \( R(\lambda) \) and eventually led Drinfeld and Jimbo [4, 5, 6, 7] to identify the relevant algebraic structure as a quasi-triangular Hopf algebra eventually. As such, quantum groups are deformations of their classical counterparts, the algebra of functions \( \text{Fun}(G) \) over a Lie group \( G \) respectively its dual, the universal enveloping algebra \( U(\mathfrak{g}) \) of a Lie algebra \( \mathfrak{g} \). The standard 1-parameter deformations of the classical groups and the universal enveloping algebras were given in [8]. The one-parameter deformation of the universal enveloping algebra was also constructed by Drinfeld and Jimbo from the point of view of quantization of Co-Poisson universal enveloping algebra (dual to the algebra of functions on the Poisson-Lie groups).

From a different point of view, Woronowicz initiated the study of quantum groups as nontrivial examples of non-commutative geometry. If \( M \) is a topological space, then the algebra \( C_0(\mathbb{R}) \) of all complex-valued continuous functions over \( M \) which vanish at \( \infty \) is commutative and is a \( C^* \)-algebra. The converse statement is also true: if \( A \) is a commutative \( C^* \)-algebra, then \( A \) is isomorphic to \( C_0(\mathbb{R}) \) for some locally compact topological space \( M \). This is the Gelfand-Naimark theorem. When \( A \) is non-commutative, such a \( M \) does not exist. However, from the point of view of category theory, it is convenient to introduce non-commutative spaces as objects of the category which is dual to the category of \( C^* \)-algebras. The theory of group structures on non-commutative spaces is quite old [9, 10, 11]. It was Woronowicz who revived the interest in this subject. He proposed a useful set of axioms [12] for compact matrix quantum (pseudo) groups and which turn out to have a very rich structure and representation theory. In [13, 14], Woronowicz introduced covariant differential calculi on quantum groups. Differential calculi on linear quantum spaces were later constructed by Wess and Zumino [15]. See for example [16, 17, 18, 19, 20, 21, 22, 23] for many interesting constructions of the differential geometry on quantum groups and quantum spaces.

Quantum groups as "symmetries" of quantum spaces were first proposed by Manin [24, 25]. The suggestion of quantizing spacetime is not new, see for example [26, 27] for early attempts. On the other hand, the idea of a quantum symmetry with a quantized spacetime as carrier is new. It is very attractive and naturally attracted a lot of activities from the physics community: extensive work has been carried out on the \( \eta \)-deformation of the Lorentz [28, 29] and the Poincare [30, 31] group. The corresponding "Lie algebras" have also been constructed [32, 33]. In addition to the \( \eta \)-deformation, there exists also the \( \kappa \)-deformation of Poincare algebra [34]. Woronowicz and Zakrzewski made some natural and reasonable requirements on how a deformation of the Lorentz group \( SL(2, \mathbb{C}) \) should look like and gave a complete classification from a Hopf \( \ast \)-algebraic point of view [35]. The classification of deformed Poincare groups has been worked out recently [36]. Wave equations on Euclidean spaces [37] and on the quantum Minkowski spaces are studied [38]. The introduction of gauge symmetry was a very important step in physics. The QFT of strong, weak and electromagnetic interactions are modelled as gauge theories nowadays. However, it is likely that a new understanding of gauge symmetry is necessary for the ultimate goal of unification of all forces and a lot of attempts were made in constructing quantum group gauge theories [39, 40, 41].

\( x \)

\( x i \)
It is commonly believed that our usual description of spacetime using the language of a smooth manifold is only a low energy classical approximation and will very likely be modified at short distances by quantum gravity effects. The use of a spacetime continuum is a basic and fundamental assumption in Einstein's geometric theory of gravity. This assumption, however, is tested experimentally only for distance much larger than the Planck length. It is natural to expect that one needs some new spacetime structures at small scales for a quantum theory of gravity. This belief is reinforced by the fact that quantum field theory of gravity is plagued by ultraviolet divergences which is not removable by the usual procedure of renormalization. Some alternative descriptions have been suggested, notably string theory [42] and non-commutative geometry [43, 44]. As an example of the framework of non-commutative geometry, the effects of electroweak unification on the nature of spacetime is studied in [45] by Lott and Connes who proposed a minimal two-point internal space and thus provided a geometric origin of the Higgs mechanism: the Higgs field appears as the component of the gauge field in the direction of the internal space. See [46] for a review of the prescription for particle physics model building in the non-commutative geometric framework, outstanding problems in this framework and many references. In particular, it is not clear how to incorporate the physics of quantum gravity using the language of non-commutative geometry.

Quantum groups and Hopf algebras make their appearance in many different branches of physics and mathematics. In physics, they occur as symmetries of lower dimensional theories such as spin chains and solvable lattice models (see for example [47]), and as hidden symmetry of WZNW models and conformal field theory [48, 49]. In some two dimensional CFT [50, 51], Mack and Schomerus found the more general quasi-Hopf algebra symmetry at certain values of the parameter space, generalizing the usual Hopf algebra symmetry. On the mathematical side, quantum group have stimulated a lot of interesting topics such as representation theory, knot theory and non-commutative geometry.

It will be very interesting to understand the role of quantum group symmetry in higher dimensional physics. The main theme of this thesis is a study of the geometry of quantum groups and quantum spaces, with the hope that they will be useful for the construction of quantum field theory with quantum group symmetry.

In this context, we study the realization of quantum group symmetries, we construct the $q$-deformed Dirac monopole as a quantum principal bundle and we study a few other content-rich examples of non-commutative geometry. It is hoped that these simple models of quantum spaces may capture some of the interesting features of the spacetime at the Planck scale.
Chapter 1

Hopf Algebras

We review in this chapter some of the most basic concepts and constructions of Hopf algebras and quantum groups.

1.1 Basic Definitions

Definition 1.1.1 (Bialgebra)
A bialgebra $A$ over the field $k$ is a unital algebra and a coalgebra, i.e. with also the two maps, comultiplication $\Delta: A \to A \otimes A$ and counit $\varepsilon: A \to k$ such that

\begin{align*}
(\Delta \otimes \text{id}) \circ \Delta(a) &= (\text{id} \otimes \Delta) \circ \Delta(a), \\
(\varepsilon \otimes \text{id}) \circ \Delta(a) &= (\text{id} \otimes \varepsilon) \circ \Delta(a) = a, \\
\Delta(ab) &= \Delta(a)\Delta(b), \quad \Delta(1) = 1 \otimes 1, \\
\varepsilon(ab) &= \varepsilon(a)\varepsilon(b), \quad \varepsilon(1) = 1.
\end{align*}

for any $a, b \in A$.

Definition 1.1.2 (Hopf Algebra)
A Hopf algebra $A$ is a bialgebra together with a map $S: A \to A$ such that

\begin{align*}
\Delta(a_1)S(a_2) = S(a_1)a_2 = 1\varepsilon(a), \quad (1.5)
\end{align*}

where $1$ is the unit of $A$.

\footnote{Here we use the Sweedler's notation $\Delta(a) = \sum a_{(1)} \otimes a_{(2)} = a_{(1)} \otimes a_{(2)}$ for simplicity.}

Definition 1.1.3 (Quasi-triangular Hopf Algebra)
A quasi-triangular Hopf algebra $A$ is a Hopf algebra together with an invertible element $R \in A \otimes A$ satisfying the relations

\begin{align*}
(\Delta \otimes \text{id})R &= R_{13}R_{23}, \\
(id \otimes \Delta)R &= R_{12}R_{23}, \\
(\tau \circ \Delta)(a) &= R\Delta(a)R^{-1},
\end{align*}

where

\begin{align*}
\tau : A \otimes A &\to A \otimes A, \\
\otimes b &\mapsto b \otimes a
\end{align*}

is the permutation map and

\begin{align*}
R_{12} = r_1 \otimes r_1 \otimes 1
\end{align*}

etc. if we denote $R = r_1 \otimes r_1$. $R$ is called the universal $R$-matrix of $A$ and satisfies the quantum Yang-Baxter equation

\begin{align*}
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}
\end{align*}

as a consequence of the definition.

Definition 1.1.4 (*$\text{-}$Hopf Algebra)
A *-Hopf algebra is a Hopf algebra together with an *-involution satisfying

\begin{align*}
(b\alpha)^* &= \beta^*a^*, \quad \forall \beta \in k, \\
(ab)^* &= b^*a^*, \\
\Delta(a^*) &= (\ast \otimes \ast)\Delta(a), \\
\varepsilon(a^*) &= \varepsilon(a)^*, \\
(S(a^*))^* &= S^{-1}(a)
\end{align*}

where $\beta^*$ is the *-conjugation of $\beta \in k$.

Definition 1.1.5 (Dual Pairing of Hopf Algebras)
Two Hopf algebras $\mathcal{H}$ and $A$ are called dually paired if there exists a non-degenerate
inner product : $\langle \cdot , \cdot \rangle : \mathcal{U} \otimes \mathcal{A} \rightarrow k$ such that
\begin{align}
\langle x, ab \rangle &= \langle x_{(1)}, a \rangle \langle x_{(2)}, b \rangle, \\
\langle xy, a \rangle &= \langle x, a_{(1)} \rangle \langle y, a_{(2)} \rangle, \\
\langle 1_{\mathcal{U}}, a \rangle &= \epsilon(a), \\
\langle x, 1_{\mathcal{A}} \rangle &= \epsilon(x), \\
\langle S(x), a \rangle &= \langle x, S(a) \rangle.
\end{align}

where $x, y \in \mathcal{U}$ and $a, b \in \mathcal{A}$.

If $\mathcal{U}$ and $\mathcal{A}$ are *-Hopf algebras, then we require also
\begin{equation}
\langle x^*, a \rangle = \langle x, (S(a))^* \rangle^*,
\end{equation}

Let $\mathcal{U}$ be a Hopf algebra and
\begin{equation}
\rho : \mathcal{U} \rightarrow M_N(k)
\end{equation}

be an $N \times N$ faithful representation of $\mathcal{U}$. We can define a new Hopf algebra $\mathcal{A}$, dual to $\mathcal{U}$ in the following way: $\mathcal{A}$ is generated by the $N^2$ matrix elements $A^i_j$ satisfying
\begin{equation}
\rho(x) = \langle x, A^i_j \rangle
\end{equation}

for $x \in \mathcal{U}$. Since $\rho$ is faithful, $A^i_j$ is well defined. The multiplication in $\mathcal{A}$ is determined by the coproduct in $\mathcal{U}$ through (1.24). As for the other Hopf structures, it is
\begin{align}
\Delta(A^i_j) &= A^i_k \otimes A^k_j, \\
\epsilon(A^i_j) &= \delta^i_j, \\
S(A^i_j) &= (A^{-1})^i_j.
\end{align}

### 1.2 Quasi-triangular $\mathcal{U}$ and the Dual Pair $(\mathcal{U}, \mathcal{A})$

When $\mathcal{U}$ is quasi-triangular, we have
\begin{equation}
\Delta'(x) = R \Delta(x) R^{-1}, \quad \forall x \in \mathcal{U},
\end{equation}

where $R$ is the universal $R$-matrix and one can determine the multiplication in $\mathcal{A}$. It is convenient to introduce the $N^2 \times N^2$ numerical $R$-matrix for the representation $\rho,$
\begin{equation}
R_{mn}^{ik} = (\rho_m^i \otimes \rho_n^k)(R) = \langle R, A^i_m \otimes A^k_n \rangle,
\end{equation}

then
\begin{align}
0 &= \langle R \Delta(x) - \Delta'(x) R, A^i_j \otimes A^k_l \rangle \\
&= \langle R, A^i_m \otimes A^k_n \rangle \langle \Delta(x), A^l_p \otimes A^p_q \rangle - \langle \Delta'(x), A^i_m \otimes A^k_n \rangle \langle R, A^l_p \otimes A^p_q \rangle \\
&= \langle x, R_{mn}^{ik} A^i_m A^k_n - A^k_n A^i_m R_{mn}^{ik} \rangle.
\end{align}

Since $x$ is arbitrary, we have[8]
\begin{equation}
R_{mn}^{ik} A^i_m A^k_n = A^k_n A^i_m R_{mn}^{ik},
\end{equation}

which can be written compactly in tensor product notation as
\begin{equation}
R_{12} A_1 A_2 = A_2 A_1 R_{12},
\end{equation}

where
\begin{equation}
R_{12} = (\rho_1 \otimes \rho_2)(R), \quad A_1 = A \otimes 1, \quad A_2 = 1 \otimes A,
\end{equation}

i.e.
\begin{equation}
(R_{12})_{ij}^{kl} = R_{ij}^{kl}, \quad (A_1)_{ij}^{kl} = A^l_k A^i_j
\end{equation}

etc.. The QYBE becomes
\begin{equation}
R_{12} R_{13} R_{23} = R_{23} R_{12} R_{13}.
\end{equation}

It is convenient to introduce the numerical $\hat{R}$-matrix,
\begin{equation}
\hat{R} = P \circ R,
\end{equation}

where $P_{ij}^{kl} = \delta^i_k \delta^j_l$ is the permutation matrix. Explicitly
\begin{equation}
\hat{R}_{ij}^{kl} = R_{ij}^{kl}.
\end{equation}
And the QYBE becomes
\[ \hat{R}_{13} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{13}, \]  
(1.36)

The numerical R-matrix for q-deformation of classical groups is given in [8], for example, it is
\[ \hat{R}_{mn}^{kl} = \delta_{m}^{l} \delta_{n}^{k} (1 + (q-1) \delta_{l}^{k}) + \lambda \delta_{n}^{l} \delta_{m}^{l} \theta(l > k), \]  
for \( GL_{q}(N) \),  
(1.37)
where \( \theta(l > k) \) is equal to 1 if \( l > k \) and is zero for \( l \leq k \).

**GL_{q}(N) R-matrix properties**

For ease of reference, we list here also some other useful properties of the \( GL_{q}(N) \) R-matrix. The numerical R-matrix for \( GL_{q}(N) \) is
\[ \hat{R}_{mn}^{kl} = \delta_{m}^{l} \delta_{n}^{k} (1 + (q-1) \delta_{l}^{k}) + \lambda \delta_{n}^{l} \delta_{m}^{l} \theta(l > k), \]  
(1.38)
where the indices run from 1 to \( N \). The step function \( \theta(l > k) \) is equal to 1 if \( l > k \) and is zero for \( l \leq k \). \( q \) is any nonzero complex number. \( \hat{R}_{12} \) satisfies
\[ \hat{R}_{12}(\hat{R}_{23} \hat{R}_{12} \hat{R}_{23} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}, \]  
(1.39)
\[ \hat{R}_{12}(q^{-1}) = \hat{R}_{12}^{-1}(q), \]  
(1.40)
\[ \hat{R}_{12}^{ij} = \hat{R}_{23}^{ij}, \]  
(1.41)
where
\[ i' = N - i + 1. \]  
(1.42)

The quantum \( \epsilon^{i_{1}...i_{N}} \) is given by
\[ \epsilon^{i_{1}...i_{N}} = (-q)^{l(\sigma)}, \]  
(1.43)
when \( (i_{1},i_{2}...,i_{N}) \) is a permutation of \( (12...N) \): \( (i_{1},i_{2}...,i_{N}) = \sigma(12...N) \) and is zero otherwise. Here \( l(\sigma) \) is the length of the permutation \( \sigma \). It is
\[ R_{0N}...R_{20} R_{01}^{12...N} = q \epsilon^{12...N} I_{0} = R_{01} R_{02}...R_{N0}^{12...N}. \]  
(1.44)
The characteristic equation is
\[ \hat{R} - \hat{R}^{-1} = \lambda I \]  
(1.45)
and the spectral representation is
\[ \hat{R} = q P_{+} - q^{-1} P_{-}, \]  
(1.46)
where
\[ P_{+} = \frac{1}{q + q^{-1}} (\hat{R} + q^{-1} I), \quad P_{-} = \frac{1}{q + q^{-1}} (\hat{R} - q I). \]  
(1.47)

One can introduce the inverse of \( (\hat{R}^{-1})^{jk} \) with respect to the indices \( i,j \),
\[ \Phi^{ij}_{kl} = \hat{R}^{ij}_{kl} q^{l(k-j)}, \]  
(1.48)
which satisfies
\[ \Phi^{ij}_{kl} (\hat{R}^{-1})^{jk} = (\hat{R}^{-1})^{ij}_{kl} \Phi^{jk}_{il} = \delta^{l}_{j} \delta^{k}_{i}. \]  
(1.49)

It satisfies the interesting "trace" properties (sum over the index \( k \)),
\[ \Phi^{ik}_{jk} = \delta^{j}_{i} q^{N-1}, \]  
(1.50)
\[ \Phi^{li}_{kj} = \delta^{j}_{i} q^{2(N-1)} + 1. \]  
(1.51)
Similarly, one can introduce the inverse of \( \hat{R}^{ik}_{lj} \) with respect to the indices \( i,j \),
\[ \Psi^{ij}_{kl} = (\hat{R}^{-1})^{ij}_{kl} q^{l(k-j)} (\hat{R}^{-1})^{ik}_{lj} q^{2(k-j)}, \]  
(1.52)
which satisfies
\[ \Psi^{ij}_{kl} \hat{R}^{ij}_{kl} = \delta^{l}_{j} \delta^{k}_{i}, \]  
(1.53)
and the "trace" properties (sum over the index \( k \)),
\[ \Psi^{ik}_{jk} = \delta^{j}_{i} q^{N-1}, \]  
(1.54)
\[ \Psi^{ik}_{kj} = \delta^{j}_{i} q^{2(N-1)} + 1. \]  
(1.55)

**Generators for \( U \)**

It is convenient to introduce the \( N \times N \) matrices \( L^{\pm} \) with entries in \( U \) by
\[ L^{+} = < R \cdot \text{id} \otimes A >, \]
\[ L^{-} = < R^{-1} \cdot A \otimes \text{id} >. \]  
(1.56)
The commutation relations[8] follow immediately from the QYBE,

\[ R_{12}L^+_2 \overset{L}{} L^+_1 = L^+_1 \overset{L}{} L^+_2 R_{12}, \]
\[ R_{12}L^-_2 \overset{L}{} L^-_1 = L^-_1 \overset{L}{} L^-_2 R_{12}, \]
\[ R_{12}L^+_2 \overset{L}{} L^-_1 = L^-_1 \overset{L}{} L^+_2 R_{12}. \]  

(1.57)

For example,
\[ 0 = \langle R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, id \otimes A_1 \otimes A_2 \rangle = R_{12}L^+_2 \overset{L}{} L^+_1 - L^+_1 \overset{L}{} L^+_2 R_{12}. \]

The Hopf structures follow from the properties of \( R \) also,

\[ \Delta(L^\pm) = L^\pm \otimes L^\pm, \]
\[ \epsilon(L^\pm) = I, \]
\[ S(L^\pm) = (L^\pm)^{-1}, \]  

(1.58)

where the notation \( (M \otimes N)^j_i = M^i_j \otimes N^j_i \) is used.

1.2.1 Example: \( U_q(su(2)) \) and \( SU_q(2) \)

A typical example of a nontrivial *-dual pair is \( \mathcal{U} = U_q(su(2)) \) and \( A = SU_q(2) \).

The Hopf Algebra \( U_q(su(2)) \)

The quantum enveloping algebra \( U_q(su(2)) \) of \( su(2) \) is the associative unital algebra generated by the Drinfeld-Jimbo generators \( H, X_+, \) and \( X_- \) modulo the commutation relations

\[ [H, X_\pm] = \pm 2X_\pm, \quad [X_+, X_-] = \frac{qH - q^{-H}}{q - q^{-1}}, \]  

(1.59)

where \( q \) is a nonzero real number. The Hopf structures are given by

\[ \Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(X_\pm) = X_\pm \otimes q^H + q^{-H} \otimes X_\pm, \]
\[ \epsilon(H) = 0, \quad \epsilon(X_\pm) = 0, \]
\[ S(H) = -H, \quad S(X_\pm) = -q^{H^2}X_\pm. \]  

(1.60)

and the *-structure is

\[ H^* = H, \quad X^*_\pm = X_\pm. \]  

(1.61)

The universal R-matrix for \( \mathcal{U} \) is given by

\[ R = q^{H \otimes H} \sum_{\nu = 0}^{\infty} \frac{(1 - q^{-2})^\nu}{[\nu]_q} (q^{H/2}X_+ \otimes q^{-H/2}X_-)^\nu \]  

(1.62)

with

\[ [n]_q! := \prod_{m=1}^{n} [m]_q \]  

(1.63)

and

\[ \lambda := q - q^{-1}. \]  

(1.64)

for integer \( n \).

The Dual Hopf algebra \( SU_q(2) \)

A faithful 2 \times 2 representation for \( U_q(su(2)) \) is given by

\[ H = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \]  

(1.65)

The numerical R-matrix in this representation is a 4 \times 4 matrix,

\[ R = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}; \]  

(1.66)

where \( \lambda := q - q^{-1}. \) \( GL_q(2) \) is the algebra generated by the matrix elements \( \alpha, \beta, \gamma, \delta \) of the quantum matrix

\[ A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \]  

(1.67)

Eqn.(1.30) then give

\[ \alpha \beta = q \beta \alpha, \quad \alpha \gamma = q \gamma \alpha, \quad \alpha \delta = \delta \alpha + (q - q^{-1})\beta \gamma, \]
\[ \beta \gamma = \gamma \beta, \quad \beta \delta = \delta \beta, \quad \gamma \delta = q \delta \gamma. \]  

(1.68)
The comultiplication is given by
\[ \Delta \left( \begin{array}{c} \alpha \\ \gamma \\ \delta \end{array} \right) = \left( \begin{array}{c} \alpha \\ \gamma \\ \delta \end{array} \right) \otimes \left( \begin{array}{c} \beta \\ \gamma \\ \delta \end{array} \right) \]
and antipode by
\[ S \left( \begin{array}{c} \alpha \\ \gamma \\ \delta \end{array} \right) = (\text{det}_q T)^{-1} \left( \begin{array}{ccc} \delta & -q^{-1} \beta \\ -q \gamma & \alpha \end{array} \right), \]
where \( \text{det}_q T := \alpha \delta - q \beta \gamma \) is central in the algebra and is group-like
\[ \Delta(\text{det}_q T) = \text{det}_q T \otimes \text{det}_q T. \]

1.3 The Smash Product

1.3.1 Action and Coaction

Definition 1.3.1 (Left \( \mathcal{A} \)-Module and Module Algebra)
Let \( \mathcal{A} \) be an algebra, we say that a vector space \( V \) is a left \( \mathcal{A} \)-module if there is a map \( \triangleright : \mathcal{A} \otimes V \to V : \mathcal{A} \otimes v \mapsto x \triangleright v, \) (1.76) such that
\[ (xy) \triangleright v = x \triangleright (y \triangleright v) \quad 1 \triangleright v = v. \]
The map \( \triangleright \) is called the left action of \( \mathcal{A} \) on \( V \).

If \( V \) is an algebra, then we can require a Hopf algebra \( \mathcal{A} \) to act on it so as to respect its algebraic structure. Thus, if in addition to (1.77), also
\[ z \triangleright (uv) = (z \triangleright u)(z \triangleright v), \quad z \triangleright 1 = 1 \triangleright (x), \]
then we say that \( V \) is a left \( \mathcal{A} \)-module algebra.

Right \( \mathcal{A} \)-module and right \( \mathcal{A} \)-module algebra can be defined similarly.

Definition 1.3.2 (Left \( \mathcal{A} \)-Comodule and Comodule Algebra)
Let \( \mathcal{A} \) be a coalgebra, we say that a vector space \( V \) is a left \( \mathcal{A} \)-comodule if there is a map \( \Delta_L : V \to \mathcal{A} \otimes V \) such that
\[ (\Delta \otimes \text{id})\Delta_L = (\text{id} \otimes \Delta_L)\Delta_L, \quad (\epsilon \otimes \text{id})\Delta_L = \text{id}. \]
The map \( \Delta_L \) is called the left coaction of \( \mathcal{A} \) on \( V \).

If \( V \) is an algebra, then we can require a Hopf algebra \( \mathcal{A} \) to coact on it so as to respect its algebraic structure. Thus, if in addition to (1.79), also
\[ \Delta_L(\triangleright uv) = \Delta_L(\triangleright u)\Delta_L(v), \quad \Delta_L(1v) = 1_A \otimes 1_V, \]
i.e. \( \Delta_L \) is an algebra map also, then we say that \( V \) is a left \( \mathcal{A} \)-comodule algebra.

Right \( \mathcal{A} \)-comodule and right \( \mathcal{A} \)-comodule algebra can be defined similarly.
1.3.2 Smash Product

Let \( \mathcal{U} \) and \( \mathcal{A} \) be two dually paired Hopf algebras. We want to construct a new associative algebra, their smash product \( \mathcal{A} \ltimes \mathcal{U} \). The smash product \( \mathcal{A} \ltimes \mathcal{U} \) is isomorphic to the tensor product \( \mathcal{A} \otimes \mathcal{U} \) as a vector space, and is equipped with the "twisted" multiplication. First, instead of the usual multiplication in tensor product space

\[(b \otimes x)(a \otimes y) = ba \otimes xy, \quad (1.81)\]

where \( b, a \in \mathcal{A}, x, y \in \mathcal{U} \), one can also introduced a twisted multiplication on \( \mathcal{A} \otimes \mathcal{U} \) as

\[(1 \otimes x)(a \otimes 1) = <a(1), a(2) > a(1) \otimes x(2) \quad (1.82)\]

for \( x \in \mathcal{U}, a \in \mathcal{A} \), where

\[\Delta_\mathcal{U}x = x(1) \otimes x(2), \quad \Delta_\mathcal{A}a = a(1) \otimes a(2) \quad (1.83)\]

are the coproducts on \( \mathcal{A} \) and \( \mathcal{U} \) respectively. This multiplication is associative and is induced from the action of \( \mathcal{U} \) on \( \mathcal{A} \)

\[x \triangleright a = a(1) < x, a(2) > \quad (1.84)\]

For simplicity, we will drop the tensor product sign \( \otimes \) and write

\[a \otimes x = ax, \quad (1.85)\]

the above relation (1.82) becomes

\[xa = a(1)x(2) < x(1), a(2) > \quad (1.86)\]

1.3.3 Example: Quantum Group

Now consider the case that \( \mathcal{U} \) is quasi-triangular and \( (\mathcal{A}, \mathcal{U}) \) be a dual pair defined by a faithful representation

\[\rho : \mathcal{U} \rightarrow \text{Mat}_N(k). \quad (1.87)\]

The relations

\[R_{12}A_1A_2 = A_2A_1R_{12} \quad (1.88)\]

and

\[R_{13}L_2^+L_1^+ = L_1^+L_2^+R_{13}, \]
\[R_{13}L_2^-L_1^- = L_1^-L_2^-R_{13}, \]
\[R_{13}L_2^+L_1^- = L_1^-L_2^+R_{13} \quad (1.89)\]

have been obtained.

Using (1.86), one obtains, for example

\[L_1^+A_1^+ = A_1^+ < L_1^+, A_1^+ > L_1^+ \]

In tensor notation, the full set of mixed relations are

\[L_1^+A_2 = A_2R_{12}L_1^+, \quad L_1^-A_2 = A_2R_{12}^-L_1^- \quad (1.90)\]

Bicovariant Generators

It will be convenient to introduce the matrix \( Y_j^i \)

\[Y = L^+ (L^-)^{-1} \quad (1.91)\]

and instead of (1.89),(1.90), we have[53, 54, 55, 56]

\[\hat{R}_{12}Y_2 \hat{R}_{12}Y_1 = Y_2 \hat{R}_{12}Y_2 \hat{R}_{12}, \quad (1.92)\]
\[Y_1A_2 = A_2R_{12}Y_2 \hat{R}_{12}. \quad (1.93)\]

One reason for introducing these generators is that the relations (1.92),(1.93) have the remarkable property that they are covariant under the transformation[53]

\[A \rightarrow A'A, \quad Y \rightarrow Y, \quad (1.94)\]

or

\[A \rightarrow AA', \quad Y \rightarrow A'^{-1}YA', \quad (1.95)\]

where \( A' \) is a second copy of the quantum group satisfying (1.30) but is inert to \( A \) and \( Y \), i.e. the matrix elements of \( A' \) commute with those of \( A \) as well as those of \( Y \). The commutation relations (1.92),(1.93) are said to be "bicovariant" and the matrix elements of \( Y \) are bicovariant vector fields on the quantum group \( \mathcal{A} \).
1.4 Differential Calculus on an Algebra

In this section, we want to introduce the notion of a differential calculus on an algebra.

1.4.1 First Order Differential Calculus on an Algebra

Classically, to introduce a differential calculus, one first has to know what is one-form. Then one can proceed to construct completely antisymmetrized tensor product of one-form and get the exterior algebra, with the wedge product. So the first thing we need to know is the properties of first order differential form in the classical case. Remember that we have the Leibniz rule

\[ d(a \otimes b) = ad(b) + (da) \otimes b, \quad a, b \in \text{Fun}(M) \]

and any one-form \( \rho \) can be written in the form \( \rho = \sum a_k \otimes b_k, a_k, b_k \in \text{Fun}(M) \). Classically, one also knows that forms and functions commute i.e. \( b(da) = (da)b \). In the deformed case, even the "functions" themselves don't commute, so one shouldn't expect this latter condition to be true. (Still, to be a useful calculus, it is required to give some sort of commutations. We will see how to do that later.) Therefore we will keep the first two properties in our definition of first order differential calculus on an algebra.

Definition 1.4.1 (First Order Differential Calculus)

Let \( A \) be an algebra with unity, \( \Gamma \) be a bimodule \(^2\) over \( A \) and \( d : A \to \Gamma \) be a linear map. We say that \((\Gamma, d)\) is a first order differential calculus over \( A \) if

1. \( \forall a, b \in A, d(ab) = adb + (da)b, \quad a, b \in A \in A \)

2. Any element \( \rho \in \Gamma \) can be written in the form \( \rho = \sum a_k \otimes b_k, a_k, b_k \in A \).

1.4.2 Universal Calculus and General Calculus

We have the definition. But it is not good for anything until we can actually construct "useful" calculus. This can be done in two steps. First, we construct a universal calculus over \( A \), and then we get a non-universal one called general calculus (the useful one).

Universal Calculus

The universal calculus \((A^2, d)\) on \( A \) can be constructed as follows. Introduce

\[ A^2 = \text{ker} \subset A \otimes A, \]

where \( \cdot : A \otimes A \to A \) is the multiplication map in \( A \) and \( d : A \to A^2 \) defined by

\[ da = 1 \otimes a - a \otimes 1. \]

\( A^2 \) can be given an \( A \) bimodule structure given by

\[ c(\sum a_k \otimes b_k) = \sum (c a_k) \otimes b_k \]

\[ (\sum a_k \otimes b_k)c = \sum a_k \otimes (b_k c) \]

for any \( \sum a_k \otimes b_k \in A^2, c \in A \). Then \( d \) clearly obeys the Leibniz rule since,

\[ d(ab) = 1 \otimes ab - ab \otimes 1, \]

\[ adb = a \otimes b - ab \otimes 1, \quad (da)b = 1 \otimes ab - a \otimes b. \]

Furthermore, it is easy to see that every element of \( A^2 \) can be represented in the form \( \sum a_k \otimes b_k \). In this way \((A^2, d)\) is indeed a first order differential calculus over \( A \) as stated. Note that the construction of universal calculus is canonical and it is constructed precisely to observe the Leibniz rule.

General Calculus

We still don't know the relations between \( adb \) and \((db)a\). Why then do we construct such an object? It is because every first order differential calculus on \( A \) can be obtained as a quotient of it. The way is to mode out a submodule in \( A^2 \) to restrict the universal calculus to a general calculus. Taking the quotient is equivalent to imposing the commutation relations. More precisely,
Proposition 1.4.1 Let $N$ be a submodule of $A^2$, define $\Gamma = A^2/N$, $\pi$ be the canonical epimorphism $A^2 \rightarrow \Gamma$ and $d' = \pi \circ d$. Then $(\Gamma, d')$ is a first order differential calculus on $A$.

For example, if we take $N = \langle adb - (db)a, a, b \in \text{Fun}(M) \rangle$, then we obtain the classical calculus.

Covariant Properties of Calculus

When $A$ is a Hopf algebra, the notion of calculus is more fruitful. Introduce the coaction on $A'$ by

$$
\Delta_L : A^2 \rightarrow A \otimes A^2 : adb \mapsto a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)},
$$

$$
\Delta_R : A^2 \rightarrow A^2 \otimes A : adb \mapsto a_{(1)}db_{(1)} \otimes a_{(2)}b_{(2)}.
$$

The first order differential calculus is said to be

1. left covariant if $\sum_k a_k db_k = 0 \Rightarrow \Delta_L(\sum_k a_k db_k) = 0$,
2. right covariant if $\sum_k a_k db_k = 0 \Rightarrow \Delta_R(\sum_k a_k db_k) = 0$,
3. bi-covariant if both.

Classically, the calculus on a Lie group is bicovariant in the sense that a zero differential form remains zero under a left or right group action.

For Hopf algebras, Woronowicz [14] gave a way to choose $N$ so that the resulting general calculus enjoy some desired properties that must be there in the classical limit:

1. The prescription is to pick a right ideal $M$ of $A$ contained in $k\text{erc}$, then $N = \kappa(A \otimes M)$ give a left covariant calculus.
2. If $M$ is $Ad_R$ invariant also, i.e. $Ad_R M \subseteq M \otimes A$, then the calculus is bicovariant.

Here the maps $\kappa$ and $Ad_R$ are,

$$
\kappa : A \otimes A \rightarrow A \otimes A, \quad \kappa(a \otimes a') = \sum aSa'_{(1)} \otimes a'_{(2)},
$$

$$
Ad_R : A \rightarrow A \otimes A, \quad Ad_R(a) = \sum a_{(2)} \otimes Sa_{(1)}a_{(3)}.
$$

Note that $Ad_R$ is just $T \rightarrow T^{-1}TT'$ in the classical.

1.4.3 3-D Calculus on $SU_q(2)$

For $SU_q(2)$, the 3-D calculus are defined [13] by picking the right ideal $M \in SU_q(2)$ which is generated by six elements,

$$
\delta + q^3 \alpha - (1 + q^2), \quad \gamma^2, \quad \beta \gamma,
$$

$$
\beta^2, \quad (\alpha - 1) \gamma, \quad (\alpha - 1) \beta.
$$

We choose the basis of the space of the left-invariant 1-forms on $P$ as

$$
\omega^0 = \delta d\beta - q^{-1} \beta d\delta,
$$

$$
\omega^1 = \delta d\alpha - q^{-1} \beta d\gamma,
$$

$$
\omega^2 = \gamma d\alpha - q^{-1} \alpha d\gamma.
$$

We have the following commutation relations between $\omega^i, i = 0, 1, 2$ and the generators of $SU_q(2)$

$$
\omega^0 \alpha = q^{-1} \omega^0, \quad \omega^0 \beta = q \omega^0,
$$

$$
\omega^1 \alpha = q^{-1} \omega^1, \quad \omega^1 \beta = q \omega^1,
$$

$$
\omega^2 \alpha = q^{-1} \omega^2, \quad \omega^2 \beta = q \omega^2.
$$

The remaining relations can be obtained by the replacement $\omega^i \rightarrow \gamma$, $\beta \rightarrow \delta$. The relation between exterior differential and basic one-forms $\omega^i$ is given by

$$
d\alpha = \alpha \omega^1 - q \beta \omega^2, \quad d\beta = \omega^0 - q^2 \beta \omega^1.
$$

(1.109)

and similarly with $\alpha$ replaced by $\gamma$ and $\beta$ replaced by $\delta$.

The higher order calculus is also determined

$$
\omega^0 \omega^0 = 0, \quad \omega^2 \omega^0 = -q^2 \omega^0 \omega^2,
$$

$$
\omega^1 \omega^1 = 0, \quad \omega^0 \omega^0 = -q \omega^0 \omega^1,
$$

$$
\omega^2 \omega^2 = 0, \quad \omega^3 \omega^1 = -q \omega^1 \omega^2.
$$

(1.110)

3 It is called 3-D calculus because the dimension of the space of one-forms is 3 in this calculus.

One can construct another calculus called the 4-D calculus on $SU(2)$ which is bicovariant. That one has a dimension of four for the space of one-forms; one too many!
We also have the Maurer-Cartan equations

\[\begin{align*}
d\omega^0 &= q^2(1 + q^2)\omega^0\omega^1, \\
d\omega^1 &= q\omega^0\omega^2, \\
d\omega^2 &= q^2(1 + q^2)\omega^1\omega^2. 
\end{align*}\] (1.111)

Chapter 2

Pseudodifferential Operators
Realization of Vector Fields

In this chapter, we consider the realization of the vector fields of the quantum Lie algebra of the quantum groups $GL_q(N), SL_q(N)$ and $SO_q(N)$ as pseudodifferential operators on the linear quantum spaces covariant under the corresponding quantum groups. Their expressions are simple and compact. These vector fields satisfy certain characteristic polynomial identities. The real forms $SU_q(N)$ and $SO_q(N, R)$ are also discussed.

2.1 Vector Fields for Quantum Groups

As seen in Chapter 1, a quantum group can be described in terms of $N \times N$ matrices $A$ with noncommuting elements satisfying the equation

\[\hat{R}_{12}A_1A_2 = A_1A_2\hat{R}_{12},\] (2.1)

with the $\hat{R}$ matrix appropriate to the particular quantum group. The matrix elements generate the algebra of functions on the group. We also derived the vector fields on the quantum group. It can be described by the matrix elements of a matrix $Y$ satisfying the commutation relation

\[\hat{R}_{12}Y_2\hat{R}_{12}Y_1 = Y_2\hat{R}_{12}Y_1\hat{R}_{12},\] (2.2)
which corresponds to the Lie algebra relations in the classical case. The action of the vector fields on the group is then given by the commutation relation
\[ Y_1 A_2 = A_2 \dot{R}_{13} Y_3 \dot{R}_{13}. \tag{2.3} \]

The quantum group matrices can coact on a quantum space, for instance by right multiplication. A point of coordinates \( x_0 \) is transformed into \( x = x_0 A \), or, more compactly,
\[ x = x_0 A. \tag{2.4} \]
Keeping the original point \( x_0 \) fixed, the action of a vector field on the quantum group induces an action on the quantum space
\[ Y_i x_2 = x_2 \dot{R}_{13} Y_3 \dot{R}_{13}, \tag{2.5} \]
i.e.
\[ Y^i_j x_k = x_m \dot{R}_{m}^{i} Y^m \dot{R}_{jk}. \tag{2.6} \]

We shall consider the case when a differential calculus covariant with respect to the coaction of the quantum group exists on the quantum space. In this case it is natural to ask whether it is possible to realize the vector fields \( Y \) as pseudodifferential operators satisfying (2.2) and (2.5). We shall show that this can be done for the quantum groups \( GL_q(N), SL_q(N) \) and \( SO_q(N) \). Their real forms are also considered.

### 2.2 q-Determinant for FRT Type Algebra

For the sake of completeness, we construct and discuss the properties of the \( q \)-determinant for FRT type algebras. The discussion is self-contained. By assuming the top form of the associated exterior quantum plane to be unique, we construct the completely antisymmetric tensor \( \epsilon \) and the \( q \)-determinant for FRT type algebras. Contrary to \cite{57}, we don’t need to assume the existence of a metric.

First we recall the definition of the FRT algebra \( \mathcal{A} \) and the associated quantum exterior algebra. FRT algebra is an associative algebra generated by 1 and \( t_j^i \); \( i, j = 1, \ldots, N \) which obey the RTT equation (2.1). Projectors can be introduced by looking at the spectral decomposition of the \( R \) matrix. In particular, the \( q \)-exterior algebra is defined by the \( q \)-symmetrizer \( P_b^g \):
\[ dx_1 dx_2 (P_b^g)^2 = 0. \tag{2.7} \]

Consider the top forms \( dx_1 dx_2 \ldots dx_{iN} \). Assuming that it is unique, then one can define an \( N \)-dimensional tensor \( \epsilon \) by,
\[ dx_1 dx_2 \ldots dx_{iN} = \epsilon_{i_1 i_2 \ldots i_N} dv \tag{2.8} \]
Like in the classical, our \( \epsilon \) is completely antisymmetric:
\[ \epsilon_{i_1 i_2 \ldots i_{N+1}} P_{i_{N+1}}^{i_1 i_2 \ldots i_N} = 0 \]
from the definition of the exterior algebra.

Now, we can prove an important property for \( \epsilon \),

**Proposition 2.2.1**
\[ \epsilon_{i_1 i_2 \ldots i_{N+1}} t_{i_1}^{h_{i_2} \ldots} t_{i_N}^{h_{i_{N+1}}} = \epsilon_{h_{i_1} \ldots h_{i_N}} D \tag{2.9} \]
for some \( D \in \mathcal{A} \).

**Proof** Consider \( dx_1 dx_2 \ldots dx_{iN} = \epsilon_{i_1 i_2 \ldots i_N} dv \), multiply both side by \( t_{i_1}^{h_{i_2} \ldots} t_{i_N}^{h_{i_{N+1}}} \), we get
\[ dx_1 dx_2 \ldots dx_{iN} \otimes t_{i_1}^{h_{i_2} \ldots} t_{i_N}^{h_{i_{N+1}}} = dv \otimes \epsilon_{i_1 i_2 \ldots i_N} t_{i_1}^{h_{i_2} \ldots} t_{i_N}^{h_{i_{N+1}}} \tag{2.10} \]
But the left side is equal to
\[ \Delta_{\mathcal{A}}(dx_1 dx_2 \ldots dx_{iN}), \tag{2.11} \]
where \( \Delta_{\mathcal{A}} \) is the left coaction of \( \mathcal{A} \) on the quantum plane
\[ \Delta_{\mathcal{A}} x_j = x_j \otimes t^i_j, \quad \Delta_{\mathcal{A}} dx_j = dx_j \otimes t^i_j. \tag{2.12} \]
Since
\[ \Delta_{\mathcal{A}}(dx_1 dx_2 \ldots dx_{iN}) = \epsilon_{i_1 i_2 \ldots i_N} \Delta_{\mathcal{A}}(dv) = \epsilon_{h_{i_1} \ldots h_{i_N}} dv \otimes D, \tag{2.13} \]
\( \epsilon \) tensor introduced here has lower indices while the one in Chapter I and Chapter 4 has upper indices. The reason is because we have used coordinates \( x_i \) with lower indices here. The two are however identical.
where we have used the fact that the top form is unique and so $\Delta_A(dv) = dv \otimes D$ for some $D \in A$. Comparing the coefficient of $dv$ in (2.10) and (2.13), we obtain $\epsilon_{ij\ldots nk} t^i_j t^j_k \ldots t^l_n = \epsilon_{ij\ldots nk} D$.

We call this $D$ the $q$-determinant. Notice that equation (2.9) can be written compactly as

$$\epsilon_{ij\ldots kn} T_1 T_2 \ldots T_N = D \epsilon_{ij\ldots kn}. \quad (2.14)$$

$D$ has the following properties.

**Proposition 2.2.2** $\epsilon(D) = 1$ and $\Delta(D) = D \otimes D$.

**Proof** The first equation is trivial. For the second one, apply $\Delta$ to (2.9)

$$\epsilon_{ij\ldots kn} \Delta(D) = \epsilon_{ij\ldots kn} t^i_j t^j_k \ldots t^l_n = D \epsilon_{ij\ldots kn} \otimes \epsilon_{ij\ldots kn} = D \otimes D \epsilon_{ij\ldots kn}. \quad Q \square$$

**Proposition 2.2.3** $D$ is central in the algebra $A$.

**Proof** First, notice that the FRT algebra admits the representations

$$\rho_+(T_1)_0 = R_{01}, \quad \rho_-(T_1)_0 = R_{01}^{-1} \quad (2.15)$$

Thus we have

$$\epsilon_{ij\ldots kn} R_{01}^{-1} R_{k1}^{-1} \ldots R_{n1}^{-1} = \rho_+(D) \epsilon_{ij\ldots kn} I_0. \quad (2.18)$$

where $I_0$ denotes the identity matrix in the zeroth vector space. We also have the inverse relation

$$\epsilon_{ij\ldots kn} R_{01}^{-1} R_{k1}^{-1} \ldots R_{n1}^{-1} I_0 = (D)^{-1} \epsilon_{ij\ldots kn} I_0. \quad (2.19)$$

Now, consider

$$\epsilon_{ij\ldots kn} R_{01}^{-1} R_{k1}^{-1} \ldots R_{n1}^{-1} I_0 = (D)^{-1} \epsilon_{ij\ldots kn} I_0. \quad (2.20)$$

Hence $t^i_j D = D t^i_j. \quad Q \square$

### 2.2.1 $q$-Determinant for Bicovariant Vector Fields $Y$

Recall that the bicovariant vector field matrix $Y^i_j$ is defined by

$$Y^i_j = \langle R_{21} R_{12}, A^i_j \otimes id \rangle \quad (2.21)$$

and satisfy

$$R_{21} Y_1 R_{12} Y_2 = Y_2 R_{21} Y_1 R_{12}, \quad (2.22)$$

or equivalently,

$$\hat{R}_{12} Y_2 \hat{R}_{12} Y_2 = Y_2 \hat{R}_{12} Y_2 \hat{R}_{12}. \quad (2.23)$$

Following [56], it will be convenient to introduce an associative $*$-product by

$$Y_1 \ast Y_2 \ast \cdots \ast Y_k := \langle R_{21} R_{12}, A_1 A_2 \cdots A_k \otimes id \rangle \quad (2.24)$$

then (2.22) can be rewritten in the FRT form

$$R_{12} Y_1 \ast Y_2 = Y_2 \ast Y_1 R_{12}, \quad (2.25)$$

and (2.23) can be rewritten in the FRT form

$$R_{12} Y_2 \ast Y_2 = Y_2 \ast Y_2 R_{12}. \quad (2.26)$$
And so one can define the $q$-determinant
\[ \varepsilon_{12 \ldots N} Y_1 \cdot Y_2 \cdots Y_N = \text{Det}_q Y_{12 \ldots N}, \] (2.26)
which is central and group-like. Remember that in chapter 1, we introduced the numerical R-matrix that gives the commutation relations between two quantum matrices in the fundamental representation
\[ R_{kN} = \left< \mathcal{R}_k A_k^{\pm} \otimes A^k_{\pm} \right>. \] (2.27)
For higher tensor product representations: $A_1 A_2, A_1 A_2 A_3, \ldots, A_1 A_2 A_3 \cdots A_m$ one can introduce the corresponding numerical R-matrix
\[ R_{I, II} = \left< \mathcal{R}_I A_I \otimes A_{II} \right> \]
where
\[ A_I \equiv A_{(1' \ldots n')} \equiv A_{1' \ldots A'_{n'}} \quad A_{II} \equiv A_{(12 \ldots m)} \equiv A_{12 \ldots A_m} \] (2.28)
satisfy
\[ R_{I, II} A_I A_{II} = A_{II} A_I R_{I, II} \] (2.29)
and $R_{I, II}$ satisfies the QYBE
\[ R_{I, II} R_{I, III} R_{II, III} = R_{II, III} R_{I, III} R_{I, II}. \] (2.30)
Eq. (2.26) can be rewritten compactly as
\[ \varepsilon_{12 \ldots N} Y_{12 \ldots N} = \text{Det}_q Y_{12 \ldots N}, \] (2.31)
where we have introduced the notation
\[ Y_{12 \ldots N} = Y_1 \cdot Y_2 \cdots \cdot Y_N \] (2.32)
in the same way as in (2.28). For simplicity, we will sometimes write $\text{Det}_q Y$ as $\text{Det} Y$. For arbitrary $I = (1'2' \ldots n')$, $II = (12 \cdots m)$, we have
\[ Y_I \cdot Y_{II} = \text{Det}_q^1 Y_I R_{I, II} Y_{II}. \] (2.33)

### 2.3 $GL_q(N), SL_q(N)$ and $SU_q(N)$

#### 2.3.1 $GL_q(N)$

The calculus for the quantum plane covariant under $GL_q(N)$ is well known[58]. The coordinates $x_i$ in the plane satisfy the commutation relations
\[ x_1 x_2 = q^{-1} x_2 x_1 \hat{R}_{12} \] (2.34)
and the derivatives $\partial^i$ satisfy
\[ \partial^i x_j = \delta^i_j + q \hat{R}_{ij}^k x_k \partial^j \] (2.35)
and
\[ \partial_2 \partial_1 = q^{-1} \hat{R}_{12} \partial_1 \partial_2. \] (2.36)
All indices run from 1 to $N$ and $\hat{R}$ is the $GL_q(N)$ matrix, which satisfies the characteristic equation
\[ \hat{R}^2 = 1 + \lambda \hat{R}, \quad \lambda = q - q^{-1}. \] (2.37)
Using (2.37) and the above commutation relations, it is easy to verify that

Construction 2.3.1 (Realisation for $GL_q(N)$) 

The differential operator
\[ d_j = q^{-2} \delta + q^{-1} \lambda \delta^i x_i \] (2.38)
satisfies (2.5),
\[ \delta_2 Y_1 = \hat{R}_{12} Y_2 \hat{R}_{12} \delta_2 \] (2.39)
and (2.2).

Characteristic Equation

Let $\mu$ be the rescaling operator in the plane
\[ \mu = 1 + q \lambda x_i \delta^i, \] (2.40)
which satisfies
\[ \mu x_i = q^2 x_i \mu, \quad \delta^i \mu = q^2 \mu \delta^i. \] (2.41)
It is obvious that \( \mu \) commutes with the elements of \( Y \).

It is very easy to verify that the matrix given by (2.38) satisfies the identity

\[
(Y - \mu)(Y - q^{-2}) = 0,
\]

(2.42)

where matrix multiplication is implied. This is a special example of polynomial characteristic equations satisfied by quantum vector fields[59]. In general these equations are of higher order but for the realization (2.38) we see that the polynomial is quadratic in \( Y \).

The invariant quantum trace of the \( k \)-th power of the matrix \( Y \) is defined as

\[
t_k = Tr D^{-1} Y^k,
\]

(2.43)

where \( D \) is the diagonal matrix \((1, q^2, ..., q^{2(N-1)})\). The \( t_k \)'s commute with the matrix elements of \( Y \). In general only the first \( r \) of them \((k = 1, 2, ..., r)\) are independent and they generate the center of the \( Y \) algebra (2.2), where \( r \) is the rank of the group[8]. For \( Y \) given by (2.38) all \( t_k \) are simply functions of \( \mu \). For instance,

\[
t_1 = [N] - 1 + \mu = q^{-2} t_0 - q^{-3} + \mu,
\]

\[
t_2 = q^{-2} t_1 - \mu q^{-2} t_0 + \mu^2,
\]

\[
t_3 = q^{-2} t_2 - \mu^2 q^{-2} t_0 + \mu^3,
\]

(2.44)

etc., where

\[
[N] = 1 + q^{-2} + q^{-4} + ... + q^{-2(N-1)}.
\]

(2.45)

\( GL_q(2) \) characteristic equation

For \( GL_q(2) \), denote

\[
Y = \begin{pmatrix}
y_1 & y_+ \\
y_- & y_2
\end{pmatrix},
\]

(2.46)

then (2.2) gives explicitly,

\[
y_1 y_+ - y_+ y_1 + \lambda q^{-1} y_+ y_2 = 0,
\]

\[
y_1 y_- - y_- y_1 - \lambda q^{-1} y_2 y_2 = 0,
\]

\[
y_2 y_+ = q^2 y_+ y_2,
\]

\[
y_2 y_- = q^{-2} y_- y_2.
\]

(2.47)

Using these, it was discovered by Professor B. Zumino [54] that a characteristic identity exists for this algebra,

\[
Y^2 - p_1 Y - p_2 = 0,
\]

(2.48)

where

\[
p_1 = Tr D^{-1} Y = y_1 + q^{-2} y_2,
\]

\[
p_2 = \frac{1}{2} (t_2 - t_1) = y_+ y_- - q^{-2} y_1 y_2 = -q^{-2} Det Y.
\]

(2.49)

Using (2.44), (2.48) reduces to (2.42) if

\[
Det Y = \mu
\]

(2.50)

holds for our realization (2.38). This is in fact true, first we have the lemma

Lemma 2.3.1

\[
Det Y z = q^2 z Det Y, \quad Det Y \partial = q^{-2} \partial Det Y.
\]

(2.51)

Proof First we need

\[
Y_I x_0 = x_0 R_0 Y_I R_0
\]

(2.52)

for any \( I = (12 \cdots m) \). This is true for \( I = (1) \). Assume it is true for \( I \) and \( II \), then

\[
Y_{I II} x_0 = Y_I \circ Y_{I II} x_0 = R_{I II} Y_I R_{I II} x_0 = R_{I II} Y_I x_0 R_{I II} R_{I II} x_0
\]

\[
= x_0 R_{I II} Y_I R_{I II} R_{I II} x_0 = x_0 R_{I II} Y_I R_{I II} x_0 R_{I II} R_{I II} x_0
\]

\[
= x_0 R_{I II} R_{I II} R_{I II} x_0 = x_0 R_{I II} R_{I II} R_{I II} x_0
\]

\[
= x_0 R_{I II} R_{I II} x_0.
\]

(2.53)
Hence
\[ \text{Det}Y_{\epsilon \epsilon_{12} - N} = \epsilon_{12} - N Y_{(12) - N} x_0 = \epsilon_{12} - N \epsilon_0 R_{(12) - N} Y_{(12) - N} R_{(12) - N} = q^2 x_0 \text{Det}Y_{\epsilon \epsilon_{12} - N}. \] (2.54)

We have used the fact (2.16)
\[ \epsilon_{12} - N R_{(12) - N} = \epsilon_0 \epsilon_{12} - N = \epsilon_{12} - N R_{(12) - N}. \] (2.55)
The proof for \( \text{Det}Y \partial = q^2 \partial \text{Det}Y \) is similar. \( \square \)

**Proposition 2.3.1** For the particular representation (2.38) of the \( Y \) matrix, it is \( \text{Det}Y = \mu. \) (2.56)

**Proof** Remember that \( \text{Det}Y \) is central in the \( Y \) algebra. Now, since the quantum traces \( t_\epsilon, k = 1, 2, \ldots, r \) generate the center of the \( Y \) algebra and \( t_\epsilon \) is a polynomial of \( \mu \) of degree \( k \), we have
\[ \text{Det}Y = \sum_{n=0}^{N} a_n \mu^n, \] (2.57)
where the numerical coefficients \( a_n \) are functions of \( q \) and \( N \). Due to lemma 2.3.1, \( \text{Det}Y/\mu \) commutes with \( x \) and \( \partial \), we have
\[ \text{Det}Y = a_\mu = a + \lambda q a \cdot \partial, \] (2.58)
where \( a \) is a constant. We claim that \( a = 1 \). First recall the definition of \( \text{Det}Y \)
\[ c_{\epsilon i \epsilon_j - j N} \text{Det}Y = c_{\epsilon i \epsilon_j - j N} (Y_{(12) - N})^{\epsilon j - i N}, \] (2.59)
where
\[ Y_{(12) - N} = (R^{-1}_{12} R^{-1}_{13} \cdots R^{-1}_{1N} Y_{11N} \cdots R_{12}). \] (2.60)

Using
\[ Y_j^i = q^{-2} \delta_j^i + q^{-1} \lambda \partial^i x_j = \delta_j^i + \lambda R_{ji}^k x_k \partial^j, \] (2.61)

one can easily extract the constant term in (2.60)
\[ (Y_{(12) - N})^{\epsilon j - i N} = \delta_j^i \delta_j^2 \cdots \delta_j^N + \cdots, \] (2.62)
where \( \cdots \) is a polynomial of \( x \) and \( \partial \) with at least one \( x \) and one \( \partial \) in each term. Substituting into (2.59),
\[ c_{\epsilon i \epsilon_j - j N} \text{Det}Y = 1 c_{\epsilon i \epsilon_j - j N} + \cdots \] (2.63)
Comparing with (2.58) shows that \( a = 1 \). \( \square \)

2.3.2 \( SL_q(N) \)
The quantum subgroup \( SL_q(N) \) can be obtained from \( GL_q(N) \) as follows[55]. For the quantum matrices one uses the standard quantum determinant \( \text{det}_q A \) and defines a new matrix
\[ T = (\text{det}_q A)^{-1/q N} A \] (2.64)
having quantum determinant equal to one. For the vector fields, one uses the determinant \( \text{Det}Y \) and defines a new matrix of vector fields[55, 56]
\[ Z = (\text{Det}Y)^{-1/q N} Y \] (2.65)
having determinant one. The number of independent elements of the matrix \( Z \) is \( N^2 - 1 \), as in the classical case.

**Construction 2.3.2 (Realization for \( SL_q(N) \))**
\[ Z = \mu^{-1/q N} Y \] (2.66)
realizes the \( SL_q(N) \) vector fields as pseudodifferential operators in the quantum plane.

2.3.3 Real Forms \( U_q(N) \) and \( SU_q(N) \)
If \( |q| = 1 \) the calculus given by (2.34-2.36) for the quantum plane can be given a reality structure[58, 60] by requiring \( x_i \) to be real
\[ x_i^* = x_i \] (2.67)
and by defining conjugate derivatives as

$$\partial^* = -q^{2i} \partial^i,$$

(2.68)

where we have introduced the notation

$$i' = N + 1 - i, \quad i = 1, 2, \ldots, N.$$  

(2.69)

Here we consider instead the case when $q$ is real and the complex conjugates of $x_i$ and of $\partial^i$ are new independent variables. The complex conjugation $*$ is an involution which inverts the order of factors in a product. It will be convenient to give them new names, i.e., we set

$$x_i^* = \dot{x}^i$$

(2.70)

and

$$\partial^* = -\partial_i.$$  

(2.71)

The commutation relations of these new variables can be obtained immediately from (2.34-2.36) by complex conjugation. Using the symmetry property

$$\delta_{ij} = \delta_{ji},$$

(2.72)

we see that

$$\dot{x}_2 \dot{x}_1 = q^{-1} \dot{R}_{12} \dot{x}_2 \dot{x}_1,$$

(2.73)

$$\dot{\partial}_1 \dot{\partial}_2 = -\partial_i + q \hat{R}_{ik} \partial_i \dot{x}^k$$

(2.74)

and

$$\dot{\partial}_1 \dot{\partial}_2 = q^{-1} \dot{R}_{ik} \partial_i \dot{x}^k.$$  

(2.75)

Eq. (2.74) can be written in a form more analogous to (2.35) if one introduces the matrix

$$\Psi_{ij}^* = (\dot{R}^{-1})_{ij} q^{2(i-j)} = (\dot{R}^{-1})_{ij} q^{2(i-j)},$$

(2.76)

which satisfies

$$\hat{R}_{ij}^* \Psi_{jk}^* = \Psi_{ik}^* \hat{R}_{kj} = \delta_i^k \delta_j^l,$$

(2.77)

$$\Psi_{ij}^* = \delta_i^k q^{-2(N-r)-1}$$

(2.78)

and

$$\Psi_{ij}^* = \delta_i^k q^{-2(N-r)-1}.$$  

(2.79)

It takes the form

$$\dot{\partial}_i \dot{x}^j = q^{-2i} + q^{-1} \Psi_{ik}^* \dot{\partial}_k \dot{x}^l,$$

(2.80)

where $i'$ is given by (2.69).

To complete the algebra of the complex calculus, we must now give commutation relations between the variables $x_i, \partial^i$ and their conjugates $\dot{x}^i, \dot{\partial}_i$. A consistent set is given by

$$\dot{\partial} \dot{x}_j = q(\dot{R}^{-1})_{ij} \dot{x}_j \dot{x}^i,$$

(2.81)

$$\partial^i \partial_j = q(\dot{R}^{-1})_{ij} \dot{\partial}_i \dot{\partial}_j,$$

(2.82)

and

$$\dot{\partial}_i \dot{\partial}_j = q^{-1} \dot{R}_{ij} \dot{\partial}_i \dot{\partial}_j.$$  

(2.83)

Consistency can be checked by verifying that all these relations braid correctly with each other.

Having the complex calculus we can now ask how the vector field realization of (2.38) acts on the conjugate variables. It is not hard to verify that

$$\dot{z}_2 Y_1 = \dot{R}_{12} Y_2 \dot{R}_{12} \dot{z}_2,$$

(2.85)

and

$$Y_1 \dot{\partial}_2 = \dot{\partial}_2 \dot{R}_{12} Y_2 \dot{R}_{12}. $$

(2.86)

On the other hand, by complex conjugation, (2.5),(2.39), (2.85) and (2.86) give

$$\dot{z}_2 Y_1^* = \dot{R}_{12}^* Y_2^* \dot{R}_{12} \dot{z}_2,$$

(2.87)

$$Y_1^* \dot{\partial}_2 = \dot{\partial}_2 \dot{R}_{12}^* Y_2^* \dot{R}_{12}, $$

(2.88)

and

$$Y_1^* \dot{z}_2 = \dot{z}_2 \dot{R}_{12}^* Y_2^* \dot{R}_{12}.$$

(2.89)

and

$$Y_1^* \dot{\partial}_2 = \dot{\partial}_2 \dot{R}_{12}^* Y_2^* \dot{R}_{12}. $$

(2.90)

29
where $Y^\dagger$ is the hermitian conjugate of the matrix $Y$

$$
(Y^\dagger)^i_j = Y^\dagger_{j^*} = q^{-1}x^i_j - x^{-1}\lambda\delta^i_j, \quad (2.91)
$$

which satisfies the equation conjugate to (2.2)

$$
\hat{R}_{12}Y_2^\dagger\hat{R}_{12}Y_1^\dagger = Y_1^\dagger\hat{R}_{12}Y_2^\dagger\hat{R}_{12}, \quad (2.92)
$$

as well as the commutation relation with $Y$

$$
\hat{R}_{12}Y_2\hat{R}_{12}^{-1}Y_1^\dagger = Y_1^\dagger\hat{R}_{12}Y_2\hat{R}_{12}^{-1}. \quad (2.93)
$$

Until now, we have considered two $GL_n$ groups complex conjugate of each other, i.e. a truly complex $GL_n$. The quantum group can be restricted to $U_q(N)$ by imposing on its matrices the unitarity condition

$$
A^\dagger = A^{-1} \quad (2.94)
$$

and to $SU_q(N)$ by further normalizing the matrices as in (2.64) so that they have quantum determinant equal to one.

The vector fields of the $U_q(N)$ subgroup can be defined as the elements of the Hermitian matrix

$$
U = YY^\dagger. \quad (2.95)
$$

Indeed, it is very easy to check that $U$ commutes with the Hermitian length

$$
\mathcal{L} = z_i\hat{z}_i = z_i\hat{z}_i^* \quad (2.96)
$$

($Y$ and $Y^\dagger$ separately do not), i.e. the $U$ vector fields leave $\mathcal{L}$ invariant. $U$ is a perfectly good matrix of vector fields and satisfies equations similar to (2.2) and (2.5)

$$
\hat{R}_{12}U_2\hat{R}_{12}U_1 = U_2\hat{R}_{12}U_1\hat{R}_{12}, \quad (2.97)
$$

$$
U_1z_2 = z_2\hat{R}_{12}U_1\hat{R}_{12}, \quad (2.98)
$$

and

$$
\hat{z}_2U_1 = \hat{R}_{12}U_2\hat{R}_{12}\hat{z}_2, \quad (2.99)
$$

as a consequence of the equations for $Y$ and $Y^\dagger$ given above. Notice that

$$
q^iU^i_j = q^{-2}\delta^i_j + q^{-1}\lambda\delta^i_j - q^{-1}\lambda\hat{\delta}^i\hat{\delta}_j - \lambda^2\delta^i\lambda\hat{\delta}_j, \quad (2.100)
$$

which will be useful to us later.

Finally we observe that, if we want to reduce the vector fields to the number appropriate to $SU_q(N)$, we must normalize $U$, i.e., take the matrix

$$
ZZ^\dagger = U/(\text{Det}U)^{1/N}. \quad (2.101)
$$

Lemma 2.3.2 It is

$$
\text{Det}U = \mu\mu^*. \quad (2.102)
$$

Proof By use of the same techniques as in lemma 2.3.1, one can verify that with

$$
U \equiv YY^\dagger \quad (2.103)
$$

then

$$
U_I = Y_IY_I^\dagger \quad (2.104)
$$

for any $I = (12 \cdots m)$. Hence

$$
\epsilon_{12 \cdots N}\text{Det}U = \epsilon_{12 \cdots N}U_{12 \cdots N} = \epsilon_{12 \cdots N}Y_{12 \cdots N}Y_{12 \cdots N} = \epsilon_{12 \cdots N}\text{Det}Y\text{Det}(Y^\dagger) = \mu\mu^*\epsilon_{12 \cdots N}. \quad (2.105)
$$

□

In addition to commuting with $Y_j^i$, the rescaling operator $\mu$ in (2.40) commutes with $\hat{z}_i, \hat{\delta}_i$ and therefore with $(Y^\dagger)^i_j$ and

$$
\mu^* = 1 - q\lambda\hat{\delta}^i\hat{\delta}_i. \quad (2.106)
$$

On the other hand $\mu^*$ commutes with $(Y^\dagger)^i_j, z_i, \theta^i, Y_j^i$ and satisfies

$$
\mu^*\hat{z}_i = q^{-2}\hat{z}_i\mu^*, \quad \hat{\delta}_i\mu^* = q^{-2}\mu^*\hat{\delta}_i. \quad (2.107)
$$

Clearly $\mu\mu^*$ commutes with $\mathcal{L}$, therefore so does $ZZ^\dagger$. $Z$ and $Z^\dagger$ satisfy equations analogous to (2.2),(2.92),(2.93). Using this fact one can show that

$$
\text{Det}ZZ^\dagger = (\text{Det}Z)(\text{Det}Z^\dagger) = 1. \quad (2.108)
$$

Notice that the vector field matrix $ZZ^\dagger$ is Hermitian, which is the natural reality condition for $SU_q(N)$. 32
2.3.3 (Realization for SUq(N))

\[ \mathcal{Z} \mu = U/(\mu \mu^*)^{1/N} \]  

(2.109)

provides the pseudodifferential operator realization for the SUq(N) vector fields.

2.4 SOq(N) and SOq(N, R)

We shall call \( T \) the quantum matrix of \( SOq(N) \), instead of \( A \). In addition to

\[ T'gT = g, \quad Tg^{-1}T' = g^{-1}, \]

(2.111)

where the numerical quantum metric matrices \( g = g_{ij} \) and \( g^{-1} = g^{ij} \) can be chosen to be equal \( g_{ij} = g^{ij} \). The \( SOq(N) \) \( \hat{R} \) matrix satisfies also orthogonality conditions

\[ (\hat{R}^{-1})^{ij}_{kl} = g^{km} \hat{R}^m_{nk} g^{ni}, \]

as well as the usual symmetry relations

\[ \hat{R}^k_{li} = \hat{R}^i_{kj}. \]

(2.113)

The \( SOq(N) \) vector field matrix, which we shall call \( Z \), satisfies

\[ \hat{R}_{12} \hat{T}_1 \hat{T}_2 = \hat{T}_1 \hat{T}_2 \hat{R}_{12}, \]

(2.110)

\[ \hat{R}_{12} \hat{T}_1 \hat{T}_2 = \hat{T}_1 \hat{T}_2 \hat{R}_{12}, \]

(2.115)

as well as an orthogonality constraint in one of the two equivalent forms[54, 56]

\[ g_{ij}(Z \hat{R}_{12} \hat{T}_2)^{lj}_{mi} = q^{1-N} g_{ml}, \]

(2.116)

\[ (Z \hat{R}_{12} \hat{T}_2)^{lj}_{mi} g^{im} = q^{1-N} g^{lj}. \]

(2.117)

Eq. (2.116) or (2.117) reduces the number of independent vector fields from \( N^2 \) to \( N(N-1)/2 \) as in the classical case.

The projector decomposition of the \( \hat{R} \) matrix for \( SOq(N) \) is

\[ \hat{R} = qP^+ - q^{-1}P^- + q^{1-N}P^0. \]

(2.118)

Here \( P^+ \) is the traceless part of the symmetrizer, \( P^- \) is the antisymmetrizer and \( P^0 \) is the trace operator. It is related to the metric by

\[ (P^0)^{ij}_{kl} = \nu g^{ij} g_{kl}, \quad \nu = \frac{\lambda}{(q^N - 1)(q^{1-N} + q^{-1})}. \]

(2.119)

The coordinates \( x_i \) of the quantum Euclidean space satisfy the commutation relations

\[ x_k x_i (P^-)^{ij}_{kl} = 0, \]

or in one of the two equivalent forms\(^3\)

\[ x_k x_i (\hat{R}^{-1})^{ij}_{kl} = q^{-1} x_k x_j - \lambda L g_{ij}, \]

(2.121)

\[ x_k x_i (\hat{R}^{-1})^{ij}_{kl} = q^{-1} x_k x_j + \lambda q^{N-2} L g_{ij}, \]

(2.122)

where the length \( L \) is defined as

\[ L = \alpha z \cdot z, \]

(2.123)

\[ \alpha = \frac{1}{1 + q^N - 2}. \]

(2.124)

As a consequence of (2.121), the length \( L \) commutes with all the coordinates,

\[ L x_i = x_i L. \]

(2.125)

Since \( g_{ij} \neq g^{ji} \), we need a convention to raise and lower the indices, we adopt

\[ m_i = g_{ij} m^j, \quad m^i = g^{ij} m_j. \]

(2.126)

\(^3\)Due to the \( SOq(N) \) characteristic equation for the numerical \( R \) matrix,

\[ \hat{R}^m_{ni} - (\hat{R}^{-1})^{lj}_{mi} \lambda(\delta^{ij} g^{km} - g^{kj} g_{ml}) \]

and

\[ \hat{R}^m_{ni} g^{m} = q^{1-N} g^{ij}. \]
A calculus on quantum Euclidean space can be obtained by introducing derivatives \( \partial^i \) which satisfy
\[
\partial^i x_j = \delta^i_j + q \hat{R}^i_{jk} x_k \partial^j \tag{2.127}
\]
and
\[
(P^-)^{ij}_k \partial^j \partial^k = 0. \tag{2.128}
\]
Explicitly, the latter is
\[
\delta_i \delta_j (\hat{R}^{-1})^i_{jk} = q^{-1} \delta_i \delta_j - \lambda \Delta g_{ij}, \tag{2.129}
\]
or equivalently
\[
\delta_i \delta_j (\hat{R}^{-1})^i_{jk} = q^{-1} \delta_i \delta_j - \lambda q^{-2} \Delta g_{ij}, \tag{2.130}
\]
where the Laplacian
\[
\Delta \equiv \alpha \partial \cdot \partial = \alpha \delta_i \delta^i. \tag{2.131}
\]
commutes with all derivatives,
\[
\Delta \partial^i = \partial^i \Delta. \tag{2.132}
\]
One can define a rescaling operator
\[
\Lambda = 1 + q \lambda x_i \partial^i + q N \lambda^2 L \Delta, \tag{2.133}
\]
which satisfies
\[
\Lambda x_i = q^2 x_i \Lambda, \quad \partial^i \Lambda = q^2 \Lambda \partial^i. \tag{2.134}
\]
A useful relation is
\[
\partial^i L = q^2 \delta^i + q^{-2} N \partial^i. \tag{2.135}
\]
The action of the vector fields \( Z \) on \( SO_q(N, R) \) induces in the standard way an action on Euclidean space analogous to (2.5)
\[
Z^i_j x_k = x_m \hat{R}^m_{kn} Z^k_j \hat{R}^r_{rk}. \tag{2.136}
\]
For \( q \) real, the quantum Euclidean space can be endowed with a reality structure as follows. For the coordinates one imposes the reality condition
\[
x_i^* = g^{ij} x_j = z^i. \tag{2.137}
\]
Let us now define derivatives \( \hat{\partial}^i \) in terms of the conjugate derivatives by
\[
\hat{\partial}^i = g^{ij} \partial^j = -q N \partial^i. \tag{2.138}
\]
The complex conjugate of (2.127) can be transformed to the form
\[
\hat{\partial}^i x_j = \delta^i_j + q^{-1} (\hat{R}^{-1})^i_{jk} x_k \partial^j \tag{2.139}
\]
and that of (2.129) to
\[
\hat{\delta}_i \hat{\delta}_j (\hat{R}^{-1})^i_{jk} = q^{-1} \delta_i \delta_j + \lambda \hat{\Delta} g_{ij}, \tag{2.140}
\]
where
\[
\hat{\Delta} \equiv \alpha q^{-2} \hat{\partial} \cdot \hat{\partial}. \tag{2.141}
\]
The relation between the derivatives \( \partial^i \) and their complex conjugates or the \( \hat{\partial}^i \) can be written[64] in the nonlinear form
\[
\hat{\partial}^i = \Lambda^{-1} (\delta^i_j + q^{-1} \lambda a z^j \delta_j) \partial^j \tag{2.142}
\]
which can be shown to satisfy (2.139). Using (2.142), one can show that
\[
\partial^i \partial^j = q \hat{R}^i_{kn} \partial^k \partial^j. \tag{2.143}
\]
We wish to find a realization for the vector fields \( Z \) of \( SO_q(N, R) \) as pseudodifferential operators on Euclidean space. It must satisfy (2.136), (2.114), the orthogonality relations (2.116, 2.117) and the reality condition for \( SO_q(N, R) \)
\[
Z^i = Z. \tag{2.144}
\]
One way to find the appropriate expression is to proceed in analogy with (2.100) by writing similar terms but adjusting the coefficients so that all relations required of \( Z \) are satisfied. It turns out that the correct formula is
\[
Z^i_j = q^{-2} \delta^i_j + q^{-1} \lambda \partial^i x_j - q^{-1} N \lambda z^j \delta_j - \lambda^2 \partial^i \partial^j. \tag{2.145}
\]
In fact, using the relations given above for the calculus on Euclidean space, one can verify that \( Z^i_j \) satisfies (2.136) as well as
\[
\partial^i Z^i_k = \hat{R}^i_{lm} Z^l_m \hat{R}^r_{kn} \partial^r \tag{2.146}
\]
and
\[
\hat{\partial}^i Z^i_k = \hat{R}^i_{lm} Z^l_m \hat{R}^r_{kn} \partial^r. \tag{2.147}
\]
Combining (2.136), (2.146) and (2.147), one finds that $Z$ satisfies also (2.114).

It is very easy to see that $Z$, as given by (2.145) satisfies the reality condition (2.144) if one observes that (2.145) can be written in the more symmetric form

$$ q^2 Z_j^i = \delta_j^i + q \lambda \lambda' x_j + q \lambda x_j \theta^\nu + \alpha q^N \lambda^2 \theta^\nu x_k \theta^\nu $$

(2.148)

by using (2.135).

Finally, the orthogonality condition is

$$ g_{ij} Z_c^c Z_c^m = q^{-N} g_{kl}. \tag{2.149} $$

Using (2.145), it is easy to get

$$ g_{ij} Z_c^c Z_c^m = q^{-N} g_{kl} \hat{Z}_c^m, \tag{2.150} $$

where the quantity $\hat{Z}_c^m$ is obtained from $Z_c^m$ by exchanging

$$ x \leftrightarrow \hat{x} \equiv x, \quad \theta \leftrightarrow \hat{\theta}, \quad q \leftrightarrow q^{-1}, \quad \hat{R} \leftrightarrow \hat{R}^{-1}, \quad g \leftrightarrow g. \tag{2.153} $$

Note that this operation exchanges (2.127) with (2.139), (2.129) with (2.140) and leaves (2.143) invariant and is therefore a symmetry of the quantum Euclidean space. We claim

**Lemma 2.4.1**

$$ Z_c^m \hat{Z}_c^m = \delta_c^m. \tag{2.154} $$

**Proof** It can be easily checked by direct computations and making use of the following useful equalities

$$ \lambda \hat{\theta}_m x^m = (q^N - 1)(q^{-1} + q^{-1}) + \lambda q^{-N} x_m \theta^m, \tag{2.155} $$

$$ \hat{\theta}_m = q^{-2} \Lambda^{-1} \theta_m + \lambda q^{-N} \Lambda^{-1} \Delta x_m, \tag{2.156} $$

$$ \hat{L} = q^{N-2} L, \tag{2.157} $$

$$ \hat{\Delta} = q^{N-4} \Lambda^{-1} \Delta, \tag{2.158} $$

$$ \hat{\Lambda} = \Lambda^{-1}. \tag{2.159} $$

Combining all these, we have

**Construction 2.4.1** (Realization for $SO_q(N, R)$)

$$ Z_j^i = q^{-2} \delta_j^i + q^{-1} \lambda \lambda' x_j - q^{-1} \lambda' x_j \theta_j - \lambda^2 \theta^j \theta_j $$

(2.160)

realizes the $SO_q(N, R)$ vector fields as pseudodifferential operators in the quantum Euclidean space.

It is remarkable that $Z$ as given by (2.160) satisfies even the orthogonality relations (2.116) and (2.117), without need for any further normalization as was necessary in (2.66) and (2.101). This is due, apparently, to the fact that the $SO_q(N)$ matrix already satisfies orthogonality relations.

On the other hand, if one does not impose (2.137) and doesn’t identify $\theta_i$, as given in (2.142), with the complex conjugate derivative $\theta_i$ by (2.138), then (2.144) will not be true. However, (2.160) would still give a realization of vector fields for the complex quantum group $SO_q(N)$ on Euclidean space.

### 2.5 A Few Remarks

1. In the differential calculus on a quantum space, one naturally introduces the differentials of the coordinates

$$ \xi_i = dx_i. \tag{2.161} $$

For quantum Euclidean space, they satisfy the commutation relations

$$ \xi_k \xi_i (P^+)^j_i = 0, \quad \xi_k \xi_i (P^-)^j_i = 0, \tag{2.162} $$

$$ x_i \xi_j = q x_i \xi_j, \tag{2.163} $$

$$ \theta^i \xi_j = q \xi_j \theta^i. \tag{2.164} $$

According to (2.137) it is natural to introduce variables $\xi_i$ related to $\xi_i$ by

$$ \xi_i^* = g^i_j \xi_j = \xi_i. \tag{2.165} $$
The complex conjugate of (2.163) can be written as
\[ \xi_i^{\dagger} = q x_i^\dagger A_{ij}. \]  
(2.166)

It was shown[64] that the \( \xi_i \) can be related to \( \xi_i^{\dagger} \) by a (nonlinear) transformation which was given explicitly there. It turns out that that transformation can be written very compactly as
\[ \xi_i = \sigma q^N A_{ij} Z_j^i, \]  
(2.167)

where A is given by (2.133). In this form one can easily verify that \( \xi \) satisfies all desired relations. For instance (2.166) follows immediately from (2.134), (2.136) and (2.163). The requirement that complex conjugation be an involution restricts \( \sigma \) to be a phase, \( |\sigma| = 1 \). Vice versa, if one knows the correct expression for \( \xi_i \), one can infer from it the formula for \( Z_j^i \).

2. All of the above equations are "covariant". This means that they go into themselves by coaction transformations. For instance, for all equations for \( GL_q(N) \) from Eq.(2.1) to (2.39), it is easy to see that the transformation
\[ A \rightarrow AB, \quad z \rightarrow zB, \quad \theta \rightarrow B^{-1} \theta, \]  
(2.168)
\[ Y \rightarrow B^{-1} YB, \quad x_i \partial^i \rightarrow x_i \partial^i \]  
(2.169)
leaves them invariant. Here the matrix elements of \( B \) are taken to commute with everything (which is the reason for using the word coaction) but \( B \) itself a quantum matrix, satisfying the analogue of Eq.(2.1). It holds similarly for the complex conjugate sector of \( GL_q(N) \),
\[ A^i \rightarrow B^i A^i, \quad \tilde{z} \rightarrow B^i \tilde{z}, \quad \tilde{\theta} \rightarrow \tilde{\theta} (B^i)^{-1}, \]  
(2.170)
\[ Y^i \rightarrow B^i Y^i (B^{-i})^{-1}, \quad \tilde{\partial}_i \tilde{z}^i \rightarrow \tilde{\partial}_i \tilde{z}^i \]  
(2.171)
(the relation \( (B^i)^{-1} = (B^{-i})^i \) is used). Analogous transformation laws leave invariant the \( SL_q(N), SO_q(N) \) equations as well as their respective real forms.

3. The realization of vector fields for \( GL_q(N) \) and \( SL_q(N) \) given above is equivalent to that given earlier[65]. The formulas given here are simpler because of a more convenient choice of notations and definitions. For instances, we use a right coaction and a corresponding more convenient lower index for the coordinates \( x_i \) and upper index for the derivatives \( \partial^i \). The same applies to a comparison between the formulas written above for \( SO_q(N) \) and earlier ones[64]. The reader should have no difficulty in establishing the correspondence between the conventions of these different references.

4. A realization of vector fields for the orthogonal group in terms of pseudodifferential operators on quantum Euclidean space has been given by Gaetano Fiore[66]. He uses the explicit description of the quantum Lie algebra by Drinfeld and Jimbo, instead of (2.114), (2.116) and (2.117) and gives explicit realizations for the vector fields in that basis. Ours is an alternative solution of the same problem which has perhaps the advantage of being more symmetric and also covariant, as explained above.
Chapter 3

Quantum Complex Sphere $S_q^2$

In this chapter, we introduce complex coordinate to the two dimensional quantum sphere of Podlés [67, 68, 69, 70] by means a stereographic projection. The action of $SU_q(2)$ on the sphere is given by fractional transformations on the complex variable in the plane, analogous to the classical ones. Left covariant differential calculus is introduced. To cover the whole sphere, we need at least one more coordinate patch. The quantum sphere appears then as the quantum deformation of the classical two-sphere described as a complex manifold. We also discuss a very interesting property of the calculus: the admission of a one-form realization of the exterior differential operator.

3.1 The Algebra $S_q^2$ and the Patch $C^+$

A family of quantum 2-spheres was introduced in [67]. There, the algebra of functions over the sphere is generated by 3 coordinates, subjected to a condition that reduces the number of independent generators to 2. The case of $c = 0$ is of special interest[71]. In this case, the algebra over the sphere $S_q^2$ is the $C^*$-algebra generated by the elements $b_+ = \alpha \beta, b_- = \alpha \delta, b_3 = \alpha \delta$, (where $\alpha, \beta, \gamma, \delta \in SU_q(2)$) with commutations

\[
(1 - b_3)b_+ = q^{-2}b_+(1 - b_3),
\]

\[
(1 - b_3)b_- = q^2b_-(1 - b_3),
\]

\[
q^{-2}b_+ b_- = q^3b_+ b_- - \lambda (b_3 - 1),
\]  

where $\lambda = q - q^{-1}$ and constraint

\[
b_3^2 = b_3 + q^{-1}b_- b_+.
\]  

The *-algebra structure is $b_+^* = -q^2b_+, b_-^* = b_3$. $q$ is a real number.

One can construct a stereographic projection to rewrite this algebra in terms of coordinates of the complex plane $z, \bar{z}$. Define

\[
z = -q b_3 (1 - b_3)^{-1} = \alpha \gamma^{-1},
\]

\[
\bar{z} = b_3 (1 - b_3)^{-1} = -\delta \beta^{-1},
\]  

which is the projection from the north pole of the sphere to the plane with coordinates $z, \bar{z}$. It is easy to derive the commutation relation

\[
z \bar{z} = q^{-2} \bar{z} z + q^{-2} - 1
\]  

and the *-structure

\[
z^* = \bar{z}.
\]  

We will denote the *-algebra generated by $z$ and $\bar{z}$ subjected to the commutation relation (3.4) and *-structure (3.5) by $C^+$. Note that the relation (3.4) for the patch $C^+$ differs from the usual quantum plane (see for example [58]) by an additional inhomogeneous constant term.

The inverse relations to (3.3) can also be obtained easily. It is

\[
b_+ = \rho^{-1} \bar{z},
\]

\[
1 - b_3 = q^2 \rho^{-1},
\]  

where $\rho$ is defined as

\[
\rho = 1 + \bar{z} z
\]  

and satisfies

\[
\rho z = q^2 z \rho,
\]

\[
\rho \bar{z} = q^{-2} \bar{z} \rho.
\]
One can check directly that Eq. (3.4) is covariant under the fractional transformation, with
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU_q(2),
\]
\[
z \to (az + b)(cz + d)^{-1}, \quad \bar{z} \to -(c - d\bar{z})(a - b\bar{z})^{-1},
\]
(3.9)
which is induced from the \(SU_q(2)\) coproduct, interpreted as a left transformation. Here \(a, b, c\) and \(d\) commute with \(z\) and \(\bar{z}\).

3.2 Differential Calculus

In Refs. [68, 69, 70], differential structures on \(S^2_q\) are studied and classified. In this section, we give a differential calculus on the sphere in terms of the complex coordinates \(z\) and \(\bar{z}\) of the patch. Just as the algebras of functions on \(C^+\) can be inferred from those of \(SU_q(2)\), so can be the differential calculus.

For \(SU_q(2)\) there are several well-known calculi [13, 14]: the 3D left- and right-covariant differential calculi, and the 4D+, 4D- bi-covariant calculi. The 4D bi-covariant calculi have one extra dimension in their space of one-forms compared with the classical case. The right-covariant calculus will not give a projection on \(C^+\) in a closed form in terms of \(z, \bar{z}\) which are defined to transform from the left. Therefore we shall choose the left-covariant differential calculus.

It is straightforward to obtain the following relations from those for \(SU_q(2)\):
\[
zd\bar{z} = q^{-2}dzc, \quad \bar{zd} = q^2d\bar{z},
\]
(3.10)
\[
zd\bar{z} = q^{-2}d\bar{z}z, \quad \bar{zd} = q^2d\bar{z},
\]
(3.11)
\[
(dx)^2 = (d\bar{z})^2 = 0,
\]
(3.12)
and
\[
dz\bar{z} = -q^{-2}d\bar{z}z.
\]
(3.13)
We can also define derivatives \(\partial, \bar{\partial}\) such that on functions,
\[
d = dz\partial + d\bar{z}\bar{\partial}.
\]
(3.14)

From the requirement \(d^2 = 0\) and the undeformed Leibniz rule for \(d\) together with Eqs. (3.10) to (3.12) it follows that:
\[
\partial z = 1 + q^{-2}z\partial, \quad \bar{\partial} = q^2z\bar{\partial},
\]
(3.15)
\[
\partial z = q^{-2}z\partial, \quad \bar{\partial} = 1 + q^2z\bar{\partial},
\]
(3.16)
and
\[
\partial \bar{\partial} = q^{-2}z\bar{\partial}.
\]
(3.17)

It can be checked explicitly that these commutation relations are covariant under the transformation (3.9) and
\[
dz \to dz(q^{-1}cz + d)^{-1}(cz + d)^{-1},
\]
(3.18)
\[
\partial \to (cz + d)(q^{-1}cz + d)\partial,
\]
(3.19)
which follow from (3.9) and the fact that \(d\) is invariant.

The \(\ast\)-structure also follows from that of \(SU_q(2)\):
\[
(dx)^\ast = dz,
\]
(3.20)
\[
\partial^\ast = -q^{-2}\partial + (1 + q^{-2})z\partial^{-1},
\]
(3.21)
\[
\bar{\partial}^\ast = -q^2\partial + (1 + q^2)\partial^{-1}z,
\]
(3.22)
where the \(\ast\)-involution inverts the order of factors in a product.

The inhomogeneous pieces on the RHS of the Eqs. (3.21) and (3.22) reflect the fact that the sphere has curvature. Incidentally all the commutation relations in this section admit another possible involution:
\[
(dx)^\ast = dz,
\]
(3.23)
\[
\partial^\ast = -q^{-2}\partial,
\]
(3.24)
\[
\bar{\partial}^\ast = -q^2\partial.
\]
(3.25)
This involution is not covariant under the fractional transformations and cannot be used for the sphere. However, it can be used when we have a quantum plane defined by the same algebra of functions and calculus. We shall take Eqs. (3.10) to (3.22) as the definition of the differential calculus on the patch \(C^+\).
Symmetries

It is interesting to note that there exist two different types of symmetries in the calculus. The first symmetry is that if we put a bar on all unbarred variables \((z, \bar{z}, dz, \bar{d}z)\), take away the bar from any barred ones and at the same time replace \(q\) by \(1/q\) in any statement about the calculus, the statement is still true. The second symmetry is the consecutive operation of the two \(*\)-involutions above, so that

\[
\theta \rightarrow -q^2\theta^* = q^2\theta - q^2(1 + q^2)\rho^{-1}\bar{z},
\]

\[
\delta \rightarrow -q^{-2}\delta^* = q^{-2}\delta - q^{-2}(1 + q^{-2})\rho^{-1}z,
\]

with \(z, \bar{z}, dz, \bar{d}z\) unchanged. This replacement can be iterated \(n\) times and gives a symmetry which resembles that of a gauge transformation on a line bundle:

\[
\theta \rightarrow \theta^{(n)} \equiv q^{4n}\theta - q^{2[2n]}\rho^{-1}\bar{z},
\]

\[
\delta \rightarrow \delta^{(n)} \equiv q^{-4n}\delta - q^{-2[2n]}\rho^{-1}z,
\]

where \([n]_q = \frac{q^n - 1}{q - 1}\). For example, we have

\[
\theta^{(0)}z = 1 + q^{-2}\bar{z}\theta^{(0)}.
\]

Making a particular choice of \(\theta, \delta\) is like fixing a gauge.

Many of the features of a calculus on a classical complex manifold are preserved. Define \(\theta = dz\delta\) and \(\delta = d\bar{z}\bar{\delta}\) as the exterior derivatives on the holomorphic and antiholomorphic functions on \(\mathbb{C}^+\) respectively. We have:

\[
[\delta, z] = dz, \quad [\delta, \bar{z}] = 0,
\]

\[
[\delta, \bar{z}] = 0, \quad \bar{[\delta, z]} = d\bar{z},
\]

\[
d = \delta + \bar{\delta}.
\]

The action of \(\delta\) and \(\bar{\delta}\) can be extended consistently on forms as follows

\[
\delta dz = d\delta z = 0, \quad \bar{\delta} \bar{d}z = d\bar{\delta} \bar{z} = 0,
\]

\[
\{\delta, dz\} = 0, \quad \{\delta, \bar{d}z\} = 0,
\]

\[
\delta = \delta^* = 0,
\]

\[
\{\delta, \bar{\delta}\} = 0,
\]

where \(\{\cdot, \cdot\}, [\cdot, \cdot\}\) are the anticommutator and commutator respectively.

### 3.2.1 One-form Realization of the Exterior Differential Operator \(d\)

The calculus described in the previous section has a very interesting property. There exists a one-form \(\Xi\) having the property that

\[
\Xi f \equiv f\Xi = \lambda df,
\]

where, as usual, the minus sign applies for functions or even forms and the plus sign for odd forms. Indeed, it is very easy to check that

\[
\Xi = \xi - \xi^* = \xi
\]

\[
\xi = qdz\rho^{-1}z
\]

satisfies Eq.(3.40) and

\[
\Xi^* = -\Xi.
\]

It is also easy to check that

\[
\Xi^2 = 2qdz\rho^{-2}dz
\]

and

\[
\Xi^2 = q\lambda d\bar{z}\rho^{-2}d\bar{z}.
\]

Suitably normalized, \(d\Xi\) is the natural area element on the quantum sphere. Notice that \(d\Xi\) commutes with all functions and forms, as required for consistency with the relation

\[
d\Xi = 0.
\]

The existence of the form \(\Xi\) within the algebra of \(z, \bar{z}, dz, d\bar{z}\) is especially interesting because no such form exists for the 3-D calculus on \(SU_q(2)\), from which we have derived the calculus on the quantum sphere (a one-form analogous to \(\Xi\) does exist for the two bicovariant calculi on \(SU_q(2)\), but we have explained before why we didn’t choose either of them). It is also interesting that \(dz\Xi\) and \(\Xi^2\) do not vanish (as the corresponding expressions do in the bicovariant calculus on the quantum groups or in the calculus on quantum Euclidean space).
The one-form \( \Xi \) is regular everywhere on the sphere, except at the point \( z = \bar{z} = \infty \), which classically corresponds to the north pole. This is discussed in [72] where we argue that the pole singularity at that point can be included by allowing forms with distribution valued coefficients. The area element \( d\Sigma \) is regular everywhere on the sphere.

### 3.3 Patching Two Quantum Planes

The variables \( z \) and \( \bar{z} \) cover the sphere with the exception of the north pole. In analogy with the classical case, we can introduce new variables \( w = z^{-1} \) and \( \bar{w} = \bar{z}^{-1} \) which describe the sphere without the south pole. These variables satisfy the commutation relation

\[
ww = q^{-2}\bar{w}w + (q^{-2} - 1)\bar{w}w^2w
\]

and the *-structure

\[
w^* = \bar{w}.
\]

It is clear that (3.47) is covariant under the transformation

\[
w \to (dw + c)(bw + a)^{-1}, \quad \bar{w} \to -(\bar{w} - b)(cw - d)^{-1}.
\]

Notice that the commutation relation (3.47) is different from that satisfied by \( z \) and \( \bar{z} \); our way of quantizing the sphere is inherently asymmetric between the north and the south pole.

The calculus in \( z \) and \( \bar{z} \) induces a calculus in \( w \) and \( \bar{w} \). It is not hard to derive the commutation relations for this \( w, \bar{w} \) calculus as well as the mixed commutation relations. For example, we have

\[
wdw = q^2\bar{w}w, \quad (3.50)
\]

\[
\partial_w w = 1 + q^2w\partial_w, \quad (3.51)
\]

and

\[
dzw = q^{-3}wdz. \quad (3.52)
\]

Denote by \( C^- \) the *-algebra generated by the variables \( w \) and \( \bar{w} \), subjecting to (3.47) and (3.48). The coordinates \( w, \bar{w} \) are related to the coordination \( b_3, b_3 \) by

\[
b_+ = \rho_w \bar{w}, \quad b_- = -q\rho_w^{-1}w, \quad b_3 = \rho_w^{-1},
\]

where \( \rho_w \) is defined as

\[
\rho_w = 1 + \bar{w}w. \quad (3.54)
\]

The relations (3.6) (respectively (3.53)) defines a *-algebra homomorphism between the algebra \( S^2_q \) and \( C^+ \) (respectively \( C^- \)) and the sphere \( S^2_q \) is covered by the two patches \( C^* \) with the transition relation

\[
wz = zw = 1.
\]

#### Singularity

Since \( w \) and \( \bar{w} \) are functions of \( z \) and \( \bar{z} \), Eq.(3.40) is valid for functions and forms in \( w \) and \( \bar{w} \), with the same \( \Xi \). In terms of \( w \) and \( \bar{w} \) the one-forms \( \xi \) and \( \xi^* \) are given by

\[
\xi = -w^{-1}dww(1 + \bar{w}w)^{-1}, \quad \xi^* = -(1 + \bar{w}w)^{-1}d\bar{w}w^{-1}.
\]

Clearly they are singular at the north pole \( w = \bar{w} = 0 \). This polar singularity is an intrinsic feature of our asymmetric quantization and of our calculus. This asymmetry is also apparent when we go to the classical limit of the Poisson sphere [73] and it seems to be unavoidable in our approach [72]. A different description of Podlěš spheres was given in an interesting paper by Štovíček[74]. He shows that the sphere can be understood as the patching of two complex quantum planes. His choice of variable is symmetric between the two planes, but the coaction of \( SU_q(2) \) is very complicated in terms of his variable. Also, Štovíček does not consider the non-commutative calculus on the sphere.

We believe that the singularity can be controlled by allowing distributions, rather than just functions as the elements of our algebra and as coefficients of differential forms. This point of view is explained in [72] for the limit of the Poisson sphere so as to avoid the need to develop the concept of distribution in the framework of noncommutative algebra.
3.4 Right Invariant Vector Fields on $S^2_q$

In this section we want to define vector fields on $S^2_q$ which generate the fractional transformation mentioned above. We will see that these vector fields can be inferred from those on $SU_q(2)$.

First let us recall some well-known facts about the vector fields on $SU_q(2)$ (see for example Ref.[53]). The enveloping algebra $U$ of $SU_q(2)$ is usually said to be generated by the left-invariant vector fields $H_L, X_L, L^+$ and $L^-$. The action of these vector fields corresponds to infinitesimal right transformation: $T \to T T'$. What we want now is the infinitesimal version of the left transformation given by Eq.(3.9), hence we shall use the right-invariant vector fields $R, X_R, L^+$. Since only the right-invariant ones will be used, we will drop the subscript $R$ hereafter.

The properties of the right-invariant vector fields are similar to those of the left-invariant ones. Note that if an $SU_q(2)$ matrix $T$ is transformed from the right by another $SU_q(2)$ matrix $T'$, then it is equivalent to say that the $SU_{1/q}(2)$ matrix $T^{-1}$ is transformed from the left by another $SU_q(2)$ matrix $T'^{-1}$. Therefore one can simply write down all properties of the left-invariant vector fields and then make the replacements: $q \to 1/q$, $T \to T^{-1}$ and left-invariant fields $\to$ right-invariant fields.

FRT Basis

Using the matrices:

$$M^+ = \begin{pmatrix} q^{-1/2} & q^{-1/2} \lambda X_+ \\ 0 & q^{1/2} \end{pmatrix}, \quad M^- = \begin{pmatrix} q^{1/2} & 0 \\ -q^{1/2} \lambda X_- & q^{-1/2} \end{pmatrix},$$

(3.57)

the commutation relations between the vector fields are given by,

$$R_{12} M^+_1 M^+_2 = M^+_1 M^+_2 R_{12},$$

(3.58)

$$R_{12} M^-_1 M^-_2 = M^-_1 M^-_2 R_{12},$$

(3.59)

$$R_{12} M^+_1 M^-_2 = M^-_1 M^+_2 R_{12},$$

(3.60)

while the commutation relations between the vector fields and the elements of the quantum matrix in the smash product of $U$ and $SU_q(2)$ are,

$$T_1 M^+_1 = M^+_1 R_{12} T_1,$$

(3.61)

$$T_1 M^-_1 = M^-_1 R_{13} T_1,$$

(3.62)

where $T$ is a $SU_q(2)$ matrix, $R = q^{-1/2} R$ and $R$ is the $GL_q(2)$ R-matrix. Clearly $M^+$ and $M^-$ are the right-invariant counterparts of $L^+$ and $L^-$. The commutation relations between the $M$'s and the $T$'s tell us how the functions on $SU_q(2)$ are transformed by the vector fields $H, X_+, X_-$. 

Basis suitable for the Patch

It is convenient to define a different basis for the vector fields,

$$Z_+ = X_+ q^{1/2},$$

(3.63)

$$Z_- = q^{1/2} X_-,$$

(3.64)

and

$$H = [H]_q = \frac{q \theta^2}{q-1}.$$  

(3.65)

They satisfy the commutation relations

$$H Z_+ - q^4 Z_+ H = (1 + q^2) Z_+,$$

(3.66)

$$Z_- H - q^4 H Z_- = (1 + q^2) Z_-,$$

(3.67)

and

$$q Z_+ Z_- - q^{-1} Z_- Z_+ = H.$$  

(3.68)

Using the expressions of $z, \bar{z}$ in terms of $\alpha, \beta, \gamma, \delta$, one can easily find the action of these vector fields on the variables $z, \bar{z}$,

$$Z_+ z = q^2 z Z_+ + q^{1/2} z^2,$$

(3.69)

$$Z_+ \bar{z} = \bar{z} Z_+ + q^{-3/2} \bar{z},$$

(3.70)

$$\bar{H} z = q^2 \bar{z} + (1 + q^2) z,$$

(3.71)

$$\bar{H} \bar{z} = q^{-3/2} \bar{H} - q^{-4} (1 + q^2) \bar{z},$$

(3.72)

and

$$Z_- \bar{z} = q^2 \bar{z} Z_- - q^{1/2} \bar{z},$$

(3.73)

$$Z_- z = \bar{z} Z_- - q^{-3/2} \bar{z}^2.$$  

(3.74)
It is clear that a $*$-involution can be given:

$$Z_+^* = Z_-, \quad \mathcal{H}^* = \mathcal{H}. \quad (3.75)$$

Since all the relations listed above are closed in the vector fields and $z$, $\xi$ (this would not be the case if we had used the left-invariant fields), we can now take these equations as the definition of the vector fields that generate the fractional transformation on $S_2^+$. We shall take our vector fields to commute with the exterior differentiation $d$. This is consistent for right-invariant vector fields in a left-covariant calculus and allows us to obtain the action of our vector fields on the differentials $dz$ and $d\xi$, as well as on the derivatives $\partial$ and $\partial$. For instance (3.69) gives

$$Z_+ dz = q^2 dz Z_+ + q^{1/2}(dz + x dz) \quad (3.76)$$

and

$$\partial Z_+ = q^2 Z_+ \partial + q^{-3/2}(1 + q^2) z \partial. \quad (3.77)$$

It is interesting to see how $\Xi$ and $d\Xi$ transform under the action of the right invariant vector fields or under the coaction of the fractional transformations (3.9). Using (3.69) to (3.74) one finds

$$Z_+ \Xi = \Xi Z_+ + q^{-1/3}dz \quad (3.78)$$

and

$$\mathcal{H} \Xi = \Xi \mathcal{H}. \quad (3.79)$$

These equations are consistent with (3.40). For instance,

$$Z_+(\lambda dz - \Xi z + z \Xi) = q^2(\lambda dz - \Xi z + z \Xi) Z_+ \quad (3.80)$$

$$+ q^{1/2}(\lambda dz^2 - \Xi z^2 + x^2 \Xi) - q^{-1/3}(dz^2 - q^2 dz x). \quad (3.81)$$

Eqs. (3.78) and (3.79) imply that $d\Xi$ commutes with $Z_+$ and $\mathcal{H}$, as expected for the invariant area element.

For the fractional transformation (3.9) one finds $\xi \rightarrow \xi'$ where

$$\xi' - \xi = -q(dz) cd^{-1}(1 + cd^{-1} x)^{-1} \quad (3.82)$$

and a similar formula for $\xi^*$. The right hand side of (3.82) is a closed one-form, since $(dz)^2 = 0$, so one could write

$$\xi' - \xi = -q d[\log_q(1 + cd^{-1} x)] \quad (3.83)$$

with a suitably defined quantum function $\log_q$. At any rate

$$d\xi' = d\xi \quad (3.84)$$

so that the area element two-form is invariant under finite transformations as well.

**Pseudo-differential Operators Realization**

It is natural to ask whether one can realize the vector fields $Z_+, Z_-, \mathcal{H}$ as pseudo-differential operators acting on $C^+$. The answer is yes. Introduce the differential operators,

$$C = 1 - \lambda q^{-1} z \partial, \quad D = 1 + \lambda q^2 \partial. \quad (3.85)$$

and

$$B = 1 - \lambda q^{-1} z \partial + \lambda q^2 \partial - \lambda^2 q^{-2} \rho \partial. \quad (3.86)$$

One finds the following realizations of $Z_+, Z_-, \mathcal{H}$ as pseudo-differential operators, which satisfy Eqs. (3.66) to (3.75):

$$q^{1/2} Z_+ = (z^2 \partial + q^2 \partial B^{-1}) C^{-1}, \quad (3.88)$$

$$-q^{1/2} Z_- = (q^2 \partial \partial + \partial B^{-1}) D^{-1} \quad (3.89)$$

and

$$\mathcal{H} = \frac{1 - B^{-2}}{1 - q^2}. \quad (3.90)$$

Equivalently,

$$q^{-1} \rho \partial = (Z_+ z \mathcal{Z}_+ - q^4 Z_+ \mathcal{Z}_+ + q^{1/2}(1 + q^2) \mathcal{Z}_+) B \quad (3.91)$$

and

$$q^{-1} \rho \partial = (q^4 Z_+ \mathcal{Z}_- - Z_+ z \mathcal{Z}_- - q^{1/2}(1 + q^2) \mathcal{Z}_-) B. \quad (3.92)$$

**3.5 Braided Quantum Sphere**

We first review the general formulation [75] for obtaining the braiding of quantum spaces in terms of the universal R-matrix of the quantum group which coacts on the quantum space.
Then the multiplication on the other hand, one start with two left-$\mathcal{V}$-comodule $\varphi$, one have:

\[(\ast) \cdot (\phi) \cdot (\psi) = \psi \otimes (\ast) \quad \text{for all } \ast, \phi, \psi \in \mathcal{V} \]

Since

\[
(\ast) \cdot (\phi) = \phi \otimes (\ast) \quad \text{for all } \ast, \phi \in \mathcal{V}
\]

Then, similarly for elements $\ast, \phi, \psi \in \mathcal{M}$,

\[
(\ast) \cdot (\phi) \cdot (\psi) = \psi \otimes (\ast) \cdot (\phi)
\]

Thus

\[
(\ast) \cdot (\phi) = (\phi) \otimes (\ast)
\]

Similarly for elements $\ast, \phi, \psi \in \mathcal{M}$.

\[
(\ast \cdot \phi) \cdot (\psi) = \psi \otimes (\ast \cdot \phi)
\]

Hence

\[
(\ast \cdot \phi) = (\phi) \otimes (\ast)
\]

Let $M$ be another left-$\mathcal{V}$-comodule algebra.

\[
(\ast) \cdot \phi = \phi \otimes (\ast)
\]

where we have used the Sweedler notation for $\mathcal{V}$.

\[
(\ast \cdot \phi) \cdot (\psi) = \psi \otimes (\ast \cdot \phi)
\]

Let $M$ be another left-$\mathcal{V}$-comodule algebra.

\[
(\ast \cdot \phi) = (\phi) \otimes (\ast)
\]

where $\ast$ is the algebra of functions on a quantum group and an algebra on which $\mathcal{V}$ acts on the left.
Anharmonic Ratio

Let \( t_i, i = 1, 2, 3, 4 \) be four braided spheres with commutation relations
\[
(z_i z_j = q^{2} z_j z_i - A q z_j, \quad i \neq j,)
\]
(3.106)

For \( A = SU_q(2) \), it is
\[
R(T^1_j, T^k_i) = q^{-1/2} R^j_i,
\]
where \( R \) is the \( GL_q(2) \) R-matrix. For example,
\[
R(a, T) = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}, \quad R(b, T) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
\]
(3.103)

3.5.2 The Braided Sphere

Since we know how \( z, z' \) and \( z' \) transform, we can use (3.93) to derive the braided commutation relations[76]. We will not repeat the derivation here but only give the results
\[
z z = q^{-2} z - \lambda q^{-1},
\]
(3.107)
\[
z z' = q^{2} z' - \lambda q z',
\]
(3.108)
\[
z z'' = q^{-2} z' - \lambda q^{-1}.
\]
(3.109)

For consistency with the \( \ast \)-involution of the braided algebra the braiding order of \( z, z', z'' \) and \( z''' \) has to be \( z < z' < z'' < z''' \) after we have fixed \( z < z' \) and \( z < z'' \) as assumed in [72]. It is crucial that we braid separately \( A = \{(1, z)\} \) with \( A' \) and \( \tilde{A}, \) and \( \tilde{A} = \{(1, \tilde{z})\} \) with \( A' \) and \( \tilde{A}' \) instead of simply braiding the whole algebra \( \{(1, z, \tilde{z})\} \) with \( \{(1, z', \tilde{z}')\} \). Otherwise we will not be able to have the usual properties of the \( \ast \)-involution (e.g. \((f(z)g(z'))^\ast = g(z')^\ast f(z)^\ast\)) for the braiding relations.

Anharmonic Ratio

Let \( z_i, i = 1, 2, 3, 4 \) be four braided spheres with commutation relations
\[
z_i z_j = q^{2} z_j z_i - \lambda q z_i^2, \quad i \neq j,
\]
(3.106)

one can verify that the anharmonic ratio \([12][24]^{-1}[34][13]^{-1}, [12][23]^{-1}[34][14]^{-1}\), as well as a number of others are invariant under the projective transformation
\[
z_i \to (a z_i + b)(c z_i + d)^{-1}.
\]
(3.107)

It is natural to ask whether these invariants are independent of each other. A detailed analysis has been carried out in [72] and the answer is that all the invariants are related and can be written as a function of any one of them. There is only one independent anharmonic ratios, exactly the same as in the undeformed case.

This interesting quantum projective invariant in the algebra of four braided spheres was first discovered by Pei-Ming Ho. Its existence was later explained by Professor Zumino. They have also worked out the similar projective invariants for the higher projective space \( CP_q(N) \) [77, 78].

Extending to the Differential Algebra

The differential calculus can also be defined on the braided spheres by imposing the Leibniz rule on the exterior derivatives \( d \) and \( d' \) so that \( d' \) acts on \( z' \) and \( f' \) in the same way \( d \) acts on \( z \) and \( f \), and
\[
d'z = zd', \quad dz = zd,
\]
(3.104)
\[
d'z' = zd', \quad dz = zd.
\]
(3.105)

For consistency with the \( \ast \)-involution of the braided algebra the braiding order of \( z, z', z'' \) and \( z''' \) has to be \( z < z' < z'' < z''' \) after we have fixed \( z < z' \) and \( z < z'' \) as assumed in [72]. It is crucial that we braid separately \( A = \{(1, z)\} \) with \( A' \) and \( \tilde{A}, \) and \( \tilde{A} = \{(1, \tilde{z})\} \) with \( A' \) and \( \tilde{A}' \) instead of simply braiding the whole algebra \( \{(1, z, \tilde{z})\} \) with \( \{(1, z', \tilde{z}')\} \). Otherwise we will not be able to have the usual properties of the \( \ast \)-involution (e.g. \((f(z)g(z'))^\ast = g(z')^\ast f(z)^\ast\)) for the braiding relations.
3.6 Integration

We want to determine the invariant integral \( \langle f \rangle \) of a function \( f(z, \bar{z}) \) over the sphere.

Using the Definition

A left-invariant integral can be defined, up to a normalization constant, by requiring invariance under the action of the right-invariant vector fields

\[ \langle xf(z, \bar{z}) \rangle = 0, \quad \text{for } x = Z_+, Z_-, \mathcal{H}. \]

Using \( \mathcal{H} \) and Eqs. (3.71) and (3.72) one finds that

\[ \langle z^k \bar{z}^l g(z, \bar{z}) \rangle = 0, \quad \text{unless } k = l. \]

(Here \( g \) is a convergence function such that \( z^k \bar{z}^l g(z, \bar{z}) \) belongs to the sphere.) Therefore we can restrict ourselves to integrals of the form \( \langle f(z, \bar{z}) \rangle \).

Eqs. (3.69) and (3.70) imply

\[ z + p = \rho z + q^{1/2} \rho \]

and

\[ z^{-1} \rho = \rho^{-1} z + q^{-3/2} [l, q^{1/2}] \rho^{-1}. \]

From \( \langle z^k \bar{z}^l \rangle > 0, \ l \geq 1 \), one finds easily the recursion formula

\[ [l + 1]_q < \rho^{-1} > [l]_q < \rho^{-1} >, \ l \geq 1, \]

which gives

\[ < \rho^{-1} > = \frac{1}{[l + 1]_q} < 1 >, \ l \geq 0. \]

Similarly

\[ < \frac{\bar{z}}{1 + \bar{z} \bar{z}^l} > = \frac{1}{[l]_q} < 1 >, \ l \geq 1. \]

We leave it to the reader to find the expression for

\[ < \left( \frac{\bar{z}}{1 + \bar{z} \bar{z}^l} \right)^k >, \ l \geq k. \]

As an application of the stereographic projection, we can define an integration on the complex quantum plane \( C_q \) by inserting an appropriate measure factor \( \rho^2 \). \( C_q \) has the same algebra (3.4) and differential calculus (3.10) to (3.17), but a different \( \ast \)-structure (3.23) to (3.25). Classically, it holds \( \int_{\mathcal{O}} dz d\bar{z} \rho^2 \rho^2 f(z, \bar{z}) = \int_{\mathcal{O}} \rho^2 f(z, \bar{z}) \).

Motivated by this, we define an integration over the quantum plane as,

\[ \int_{\mathcal{O}} f(z, \bar{z}) \equiv \langle \rho^2 f(z, \bar{z}) \rangle. \]  

We need to check that this integration is translationally invariant, namely, \( f \partial f = \bar{f} \partial \bar{f} = 0 \). It follows immediately from (3.91), (3.92) and the definition (3.116). Formally, we have

\[ \int_{\mathcal{O}} \partial f = < \rho^2 \partial f > = < Z_+ \cdots > - < Z_- \cdots > \]

and

\[ \int_{\mathcal{O}} \bar{\partial} f = < \rho^2 \bar{\partial} f > = < Z_- \cdots > - < Z_+ \cdots >, \]

which are both zero since the integral on the sphere is defined by \( < \mathcal{O} f > = 0 \) for \( \mathcal{O} = Z_+, \mathcal{H} \). So the integral defined by (3.116) is translationally invariant.

Using the Braiding

We can also compute the left-invariant integral by requiring its consistency with the braiding relations.

Since \( z' \) and \( \bar{z}' \) are always on the same side of the variables of their braided copy, \( z \) and \( \bar{z} \), in the braiding order \( (z < z' < \bar{z}' < \bar{z}) \), the integration on \( z', \bar{z}' \), has the following property:

\[ f(z', \bar{z}') g(z, \bar{z}) = \sum_i g_i(z, \bar{z}) f_i(z', \bar{z}'), \]

then

\[ < f(z', \bar{z}') > g(z, \bar{z}) = \sum_i g_i(z, \bar{z}) < f_i(z', \bar{z}') >, \]

where \( < \cdot > \) is the invariant integral on \( \mathcal{S}_q^2 \). However,

\[ f(z', \bar{z}') < g(z, \bar{z}) > \neq \sum_i g_i(z, \bar{z}) < f_i(z', \bar{z}'). \]
The above property (3.119) can be used to derive explicit integral rules. For example, consider the case of \( f(t', \bar{z}') = t' \rho'^{-n} \), where \( \rho' = 1 + t' z' \) and \( g(z, \bar{z}) = z \). Since 
\[
\bar{z}' \rho'^{-n} z = q^2 z \bar{z}' \rho'^{-n} + q^{1-n} \lambda \langle [n+1]_q - [n]_q \rangle \rho'^{-n}, \quad n \geq 0,
\]
where \([n]_q = \frac{q^n - 1}{q - 1}\), using (3.119) and \( \langle \bar{z}' \rho'^{-n} \rangle = 0 \) we get the recursion relation:
\[
[n + 1]_q < \rho'^{-n} >= [n]_q < \rho'^{-n-1} >, \quad n \geq 1. \tag{3.120}
\]
This agree with the first method.

Notice that one can also compute the same integral by using the "cyclic property" of the quantum integral \(^2\)
\[
\langle f(z, \bar{z})g(z, \bar{z}) \rangle = \langle g(z, \bar{z})f(q^2 z, q^2 \bar{z}) \rangle. \tag{3.121}
\]

---

### Chapter 4

**Quantum Complex Projective Space**

In this chapter, we define the quantum projective space \( C_{q}(N) \) in terms of both homogeneous and inhomogeneous complex coordinates and we study the differential calculus on it. \( C_{q}(N) \) is shown to be the quantum deformation of a Kähler manifold with the Fubini-Study metric.

#### 4.1 \( C_{q}(N) \) as a Complex Manifold

**4.1.1 \( SU_{q}(N-1) \) Covariant Quantum Space \( C_{q}^{N+1} \)**

First, let us define the complex quantum space \( C_{q}^{N+1} \) from which the projective space can be obtained. \( C_{q}^{N+1} \) is the algebra spanned by the coordinates \( x_i \) and its complex conjugate \( \bar{x}_j \), \( i = 0, 1, ..., N \) which satisfy the relations
\[
x_i x_j = q^{-1} \bar{R}_{ij} x_k \bar{x}_l, \tag{4.1}
\]
\[
\bar{x}_i \bar{x}_j = q(\bar{R}^{-1})_{ij} \bar{x}_k \bar{x}_l, \tag{4.2}
\]
\[
x_i \bar{x}_j = q^{-1} \bar{R}_{ij} x_k \bar{x}_l. \tag{4.3}
\]
The indices run from 0 to \( N \), instead of from 1 to \( N+1 \) because in the next section, we will introduce the inhomogeneous coordinates \( z_a \) for \( C_{q}(N) \) and we want the

---

\(^2\)Similar cyclic properties have been found by H. Steinacker\(^7\) for integrals over higher-dimensional quantum spheres in quantum Euclidean space.
indices $a$ of $z_a$ to run from 1 to $N$, instead of from 2 to $N+1$. The later is not suitable for a compact R-matrix description.

$q$ is a real number and $\tilde{R}^{ij}_{kl}$ is the $GL_q(N+1)$ $\tilde{R}$-matrix [8] with indices running from 0 to $N$. The complex conjugate coordinate $\bar{z}^i$ is related to the coordinate $z_i$ by a $*$-involution

$$z_i^* = \bar{z}^i.$$ (4.4)

It is trivial to check that the Hermitian length

$$L \equiv z_i\bar{z}^i$$ (4.5)

is real and central

$$Lz_i = z_iL.$$ (4.6)

Differential Calculus

As usual, the differential calculus can be introduced by imposing commutation relations between the functions and forms. We propose the differentials $\xi_i = dq_i, \bar{\xi}^i = (\xi^i)^*$ to satisfy

$$z_i\xi_j = q\tilde{R}^{ij}_{kl}z_k,$$ (4.7)

$$\bar{z}^i\xi_j = q(\tilde{R}^{-1})^{ij}_{kl}\bar{z}^k\xi^l.$$ (4.8)

and

$$\xi_i\xi_j = -q\tilde{R}^{ij}_{kl}\xi_k\xi_l,$$ (4.9)

$$\bar{\xi}^i\xi_j = -q(\tilde{R}^{-1})^{ij}_{kl}\bar{\xi}^k\xi^l.$$ (4.10)

We will discuss the possibilities of other choices later.

To introduce derivatives $D^i, \bar{D}_i$ acting on the "functions", 1 we require the exterior derivatives $\delta \equiv \xi^iD_i, \bar{\delta} \equiv \bar{\xi}^i\bar{D}_i$ on the holomorphic and antiholomorphic functions satisfy the undeformed Leibniz rule, $\delta^2 = \delta\delta = 0$ and $\delta z_j = x_j\delta$ etc. These imply

$$D^i z_j = \delta^i_j + q\tilde{R}^{ij}_{kl}z_kD^l,$$ (4.11)

$$D^i \bar{z}^j = q^{-1}(\tilde{R}^{-1})^{ij}_{kl}\bar{z}^k\bar{D}_l,$$ (4.12)

$$D_i z_j = q^{-1}\tilde{R}^{ij}_{kl}z_k\bar{D}_l,$$ (4.13)

$$D_i \bar{z}^j = q^{-1}(\tilde{R}^{-1})^{ij}_{kl}\bar{z}^kD_l.$$ (4.14)

The matrix $\tilde{R}^{ij}_{kl}$ is defined as

$$\tilde{R}^{ij}_{kl} = \tilde{R}^{ik}_{jl}q^{-1} = \tilde{R}^{il}_{jk}q^{-1},$$ (4.15)

which satisfies

$$\tilde{R}^{ij}_{kl} = (\tilde{R}^{-1})^{ij}_{kl},$$ (4.16)

and (sum over the index $k$)

$$\tilde{R}^{ij}_{jk} = \delta^i_jq^{2i+1}, \quad \tilde{R}^{ij}_{ki} = \delta^ijq^{2(N-2i)+1}.$$ (4.17)

Because we have chosen to run the indices from 0 to $N+1$, (4.18) is slightly different from the formulas in chapter 1.

Symmetry and $*$-involution

Using

$$\tilde{R}^{ij}_{kl}(q^{-1}) = (\tilde{R}^{-1})^{ij}_{kl}(q)$$ (4.19)

and

$$\tilde{R}^{ii}_{kl} = \tilde{R}^{ii}_{ji},$$ (4.20)

one can show that if we do the following replacement

$$q \to q^{-1},$$ (4.21)

$$z_i \to kq^{-2i}z_i, \quad \bar{z}^i \to z_i,$$ (4.22)

$$\xi_i \to kq^{-2\xi_i}, \quad \bar{\xi}^i \to k\bar{\xi}^i,$$ (4.23)

and

$$D^i \to k^{-1}q^{-i}D_i, \quad D_i \to l^{-1}D^i,$$ (4.24)

where $k$ and $l$ are arbitrary constants, then all the commutation relations just go back to themselves and the replacement (4.21) - (4.24) is hence a symmetry of the

---

1The usual symbols $\delta^a, \delta^i$ are reserved below for the derivatives on $CP_q(N)$. 
algebra. Exchanging the barred and unbarred quantities in (4.21) - (4.24), we get another symmetry which is related to the inverse of this one.

Since \( L \) commutes with \( x_i, \bar{x}^i \), a \(*\)-involution can be defined for \( D^i \)

\[
(D^i)^* = -q^{-2n} L^a \bar{D}_i L^{-a},
\]

where

\[
i' = N - i + 1
\]

for any real number \( n \). The \(*\)-involutions corresponding to different \( n \)'s are related to one another by the symmetry of conjugation by \( L \)

\[
a \rightarrow L^n a L^{-n},
\]

where \( a \) can be any function or derivative and \( m \) is the difference in the \( n \)'s. Finally, all the above relations are covariant under the transformation

\[
x_i \rightarrow x_i T_i^j, \quad \bar{x}^i \rightarrow (T^{-1})^i_j \bar{x}^j, \quad D_i \rightarrow (T^{-1})^i_j D^j, \quad \bar{D}_i \rightarrow \bar{D}_j q^{-2j} T_i^j,
\]

(4.28)  
(4.29)  
where \( T_i^j \in SU(N+1) \).

One can check that

\[
L \xi_i = q^2 \xi_i L,
\]

(4.31)  
which will be useful to us later.

### 4.1.2 Complex Projective Space \( CP_q(N) \)

Define for \( a = 1, \ldots, N \), the inhomogeneous coordinates \(^\text{2}\)

\[
x_a = x_0^{-1} x_a, \quad \bar{x}^a = \bar{x}^0 (x^0)^{-1}.
\]

(4.32)  
As a consequence of (4.1, 4.2), we have

\[
x_a \bar{x}^b = q^{-1} \hat{R}^{bc}_{a0} x_a \bar{x}^b,
\]

(4.33)  
where \( \hat{R}^{bc}_{a0} \) is the \( GL_q(N) \) \( \hat{R} \)-matrix with indices running from 1 to \( N \) and \( \lambda = q^{-1}/q \).

The formulas

\[
x_0 x_a = q x_a x_0, \quad x_0 \bar{x}^0 = \bar{x}^0 x_0
\]

(4.35)  
and

\[
x_0 \bar{x}^0 = q^{-1} \bar{x} x_0
\]

(4.36)  
are very useful to obtaining (4.33) and (4.34).

### Differential Calculus

The commutation relations for the calculus can be induced from that of \( C_q^{N+1} \). It is obvious that

\[
d z_a = x_0^{-1} (\xi_a - \xi_a x_0), \quad d \bar{z}^a = (\bar{z}^0 - \bar{z}^0 x^0) (x^0)^{-1}
\]

(4.37)  
and

\[
x_0 \xi_0 = q^2 \xi_0 x_0, \quad x_0 \bar{\xi}^0 = \bar{\xi}^0 x_0.
\]

(4.38)  
Using these and (4.7) and (4.8), we have

\[
z_a dz_3 = q \hat{R}^{a0}_{cb} dz_3, \quad \bar{d} z^a = q^{-1} (\bar{z}^a) \hat{R}^{bc}_{a0} d \bar{z}^c
\]

(4.39)  
\[
d \bar{z}^a = q^{-1} (\bar{z}^a) \hat{R}^{bc}_{a0} d \bar{z}^c
\]

(4.42)  
One can introduce derivatives \( \partial^a, \bar{\partial}_a \) by requiring \( \partial \equiv dz_a \partial^a \) and \( \bar{\partial} \equiv d\bar{z}^a \bar{\partial}_a \) to be exterior differentiations, i.e. \( \partial^2 = \bar{\partial}^2 = 0 \) and satisfy the Leibniz rule. It follows from (4.39) and (4.40) that

\[
\partial^a z_b = q \hat{R}^{a0}_{cb} dz_c, \quad \bar{\partial}^a \bar{z}^b = q^{-1} (\bar{z}^b) \hat{R}^{bc}_{a0} d \bar{z}^c
\]

(4.43)  
\[
\partial^a \bar{z}^b = q^{-1} (\bar{z}^b) \hat{R}^{bc}_{a0} d \bar{z}^c
\]

(4.44)  
\(^\text{2}\)The letters \( a, b, c, e \) etc. run from 1 to \( N \), while \( i, j, k, l \) run from 0 to \( N \).
Symmetry and $\ast$-involution

The algebra of the differential calculus on $CP(N)$ has the symmetry:

$$
\phi_{ab} = \phi_{ba}^* + q^{-1}(\hat{R}^{-1})_{a}^{b} q_{c}^{d} \phi_{cd},
$$

(4.46)

$$
\phi_{ab} = q^{-1} \hat{R}^{b}_{a} \phi_{cd},
$$

(4.47)

and

$$
\phi_{ab} = q \phi_{ba}^* \phi_{cd},
$$

(4.48)

where the $\phi$ matrix is defined by

$$
\phi_{ab} = \hat{R}^{a}_{b} q^{(d-s)} = \hat{R}^{a}_{b} q^{(d-s)}.
$$

(4.49)

Symmetry and $\ast$-involution

The algebra of the differential calculus on $CP(N)$ has the symmetry:

$$
q \to q^{-1},
$$

(4.50)

$$
z_a \to r q^{-2a} z_a, \quad z^a \to s z_a,
$$

(4.51)

dz_a \to r q^{-2a} dz_a, \quad dz^a \to s dz_a
$$

(4.52)

and

$$
\phi_a^* \to \phi_a^* r q^{-2a}, \quad \phi_a \to s^{-1} \phi_a
$$

(4.53)

where $rs = q^2$. And also another symmetry by exchanging the barred and unbarred quantities in the above.

One can define a $\ast$-involution

$$
z_a^* = z^a,
$$

(4.54)

$$
dz_a^* = dz^a
$$

(4.55)

and

$$
\phi_a^* = - q^{2n_{a}} \phi_a \rho^{-n},
$$

(4.56)

where

$$
a' = N - a + 1
$$

(4.57)

and

$$
\rho = 1 + \sum_{a=1}^{N} z_a \bar{z}_a
$$

(4.58)

for any $n$. The different choice of involution for different $n$'s are related to one another by the symmetry of conjugation by $\rho$ to some powers followed by a rescaling by appropriate powers of $q$. In particular, the correct classical limit of Hermitian conjugation with the standard measure $\rho^{-1(n+1)}$ of $CP(N)$ is reproduced by $\ast$-involution with the choice of $n = N + 1$.

Covariance

The transformation for $z_a$ is induced from (4.28) on $C_{q}^{N+1}$, it is

$$
z_a \to (T^a + T^b)(T^a + T^b).
$$

(4.59)

The differentials transform as

$$
dz_a \to dz_a M^a_b, \quad dz^a \to (M^a)_{b} dz^b,
$$

(4.60)

where $M^a_b$ is a matrix of functions in $z_a$ with coefficients in $SU(N+1)$. It is computable from (4.59) but we don't need to know their explicit from. It is $(M^a)_{b} \equiv (M^a)^{b}_{c}$. The transformation on the derivatives are

$$
\phi_{a} \to (M^{-1})^{a}_{b} \phi_{b}, \quad (\phi_{a})^{*} \to (M^{a})^{*}((M^{-1})^{a}_{b}).
$$

(4.61)

It follows from the fact that $\phi_\bar{a} \phi$ are invariant. The covariance of the $CP_q(N)$ relations under the transformation (4.59), (4.60) and (4.61) follows directly from the covariance in $C_{q}^{N+1}$.

4.1.3 Other Choices for the Defining Relations for $C_{q}^{N+1}$

One may ask why do we choose to define the $C_{q}^{N+1}$ algebra and calculus as in section 4.1.1. Keeping the other relations the same, one can use the alternative

$$
z^a z_j = q^{-1} \hat{R}^a_{ij} z_i \bar{z}_j,
$$

(4.62)

and

$$
z_a \xi_j = q^{-1} (\hat{R}^{-1})^{a}_{b} \xi_b z_i
$$

(4.63)

instead of (4.2) and (4.7). We have 4 different choices:
We have also listed the corresponding order that will yield the commutation relations. Since Type II is essentially the same as Type I (with a replacement $z \rightarrow \bar{z}, dz \rightarrow d\bar{z}$, one goes from IA to IIA, IB to IIB), we will only discuss the other choice IB here. It is easy to check that in IB, we still have $x_i L = L x_i$. But instead of (4.31), we have

$$\xi_i L = L \xi_i + \lambda q x_i \delta L$$

(4.64)

or equivalently

$$x_i \delta L = q^{-2} \delta L x_i.$$  

(4.65)

This doesn't fit with the construction for the one-form realization that we are going to introduce in the next section. Interestingly enough, with the same definitions (4.32) and (4.37) for $x_i, \delta L$ in Type IIA, we get exactly the same relations for the $CP_2(N)$ as in section 4.1.2. In particular we have the same one-form realization (see next section)

$$\rho x_a = q^{-2} x_a \rho,$$

$$\rho dz_a = dz_a \rho$$

(4.66)

and

$$\eta = -q^{-1} \delta \rho \rho^{-1},$$

$$\bar{\eta} = q \delta \rho \rho^{-1}.$$  

(4.67)

We choose to work with Type I because the corresponding relations for $C_{\mu}^{(4)}$ have nicer property. But there is really no difference at the level of projective space $CP_2(N)$.

### 4.2 One-Form Realization of Exterior Differential Operators

Let us first recall that in Connes’ non-commutative geometry[44], the calculus is quantized using the following operator representation for the differentials,

$$d\omega = F \omega - (-1)^{k} \omega F$$

(4.68)

where $\omega$ is a $k$-form and $F$ is an operator such that $F^* = F$ and $F^3 = 1$. In the bicovariant calculus of Quantum Groups[14], there exists a one-form $\eta$ with the properties $\eta^* = -\eta$, $\eta^2 = 0$ and

$$df = [\eta, f]_\pm$$

(4.69)

where $[a, b]_\pm = ab \pm ba$ is the graded commutator with plus sign only when both $a$ and $b$ are odd. It is interesting to ask when such a realization of differentials exist? And what will be the properties of this special one-form? Instead of studying the operator aspect, we will first consider these questions in the simpler algebraic sense.

#### 4.2.1 A Special One-Form

Let us first look at an example.

**Example 4.2.1 (SO(3) quantum space)** [8, 64]

The quantum matrices $T$ of $SO_3(N)$ satisfy in addition to

$$\hat{R}_{12} T_1 T_2 = T_1 T_2 \hat{R}_{12},$$

(4.70)

also the orthogonality relations[8]

$$T^* y T = g, \quad T^* \rho^{-1} T = g^{-1},$$

(4.71)

---

3 The appropriate setting is a Fredholm module $(\mathcal{H}, F)$ where all these relations take place in the Hilbert space $\mathcal{H}$.

4 For the 3D left or right covariant calculus of $SU_3(2)$[13], such a one-form doesn’t exist and has to be introduced as an additional one-form.
where the numerical quantum metric matrices $g = g_{ij}$ and $g^{-1} = g^{ij}$ can be chosen to be equal $g_{ij} = g^{ij}$. The coordinates $x_i$ of the quantum Euclidean space satisfy the commutation relations

$$x_i x_j \hat{R}^{ij}_{kl} = q x_j x_i - \lambda \alpha (x \cdot x) g_{ij},$$

(4.72)

where $L \equiv x \cdot x = x_k x_i g^{kl} = x_k x^k$ and $\alpha = \frac{1}{1 + q^{-1}}$. The differentials of the coordinates $\xi_i = dx_i$ satisfy the commutation relations

$$x_i \xi_j = q(x_i \xi_j \hat{R}^{ij}_{kl}),$$

(4.73)

It can be verified that

$$Lx_i = x_i L, \quad Ldx_i = q^2 dx_i L.$$  

(4.74)

Hence $\eta = -q^{-1} dLL^{-1}$ satisfies

$$\lambda df = [\eta, f]_k.$$  

(4.75)

Generalizing the idea, we have the following construction:

Construction 4.2.1 (Sufficient condition)

Let $A$ be an algebra generated by coordinates $x_i$. $(\Omega(A), d)$ be a differential calculus over $A$. If there exists an element $a \in A$, constants $r, s$ such that

$$ax_i = rx_ia, \quad adx_i = sdx_ia, \quad \forall i,$$

(4.76)

then

$$\lambda df = [\eta, f]_k$$  

(4.77)

with

$$\eta = \frac{\lambda}{1 - s/r} da a^{-1}.$$  

(4.78)

The normalization constant $\lambda$ is introduced such that $\lambda/(1 - s/r)$ is well defined as $r, s \to 1$.

Proof

- $adx_i = sdx_i a \Rightarrow dax_i = -sdx_i da$$
  \Rightarrow [da a^{-1}, dx_i]_k = 0.$$
- $ax_i = qx_i a \Rightarrow dax_i = ddx_i = ddx_i a + r x_i da$$
  \Rightarrow r da a^{-1} x_i + sdx_i = r dx_i + rx_i da a^{-1}$$
  \Rightarrow [(1 - s/r)^{-1} dda a^{-1}, x_i] = dx_i.$$

$\square$

It is not hard to prove that $\eta^2 = d\eta = 0$. We give a few examples.

Example 4.2.2 ($SU_q(N + 1)$ covariant quantum spaces $C_q^{N+1}$) [8]

The $SU_q(N + 1)$ symmetry is represented in the complex quantum space $C_q^{N+1}$ with coordinates $x_i, \bar{x}^i$ and differentials $\xi_i, \bar{\xi}^i$, $i = 0, 1, \ldots, N$. They satisfy the relations:

$$x_i x_j = q^{-1} x_k x_i \hat{R}^{ij}_{kl},$$

(4.79)

$$\bar{x}^i x_j = q(\hat{R}^{-1})^{ij}_{kl} \bar{x}^k \bar{x}^l,$$

(4.80)

$$x_i \xi_j = q \hat{R}^{ij}_{kl} x_k \xi_l,$$

(4.81)

$$\bar{x}^i \xi_j = q(\hat{R}^{-1})^{ij}_{kl} \bar{x}^k \xi^l,$$

(4.82)

where $\hat{R}$ is the $GL_q(N + 1)$ R matrix. The Hermitian length

$$L = x_i \bar{x}^i$$

(4.83)

satisfies

$$Lx_i = x_i L, \quad Ldx_i = q^2 dx_i L.$$  

(4.84)

Hence

$$\eta = -q^{-1} dLL^{-1}.$$  

(4.85)

Example 4.2.3 ($GL_q(N)$ quantum group) [8, 55]

The algebra is generated by the elements of the quantum matrix $T = (T^i_j), i, j = 1, N$ and the differentials $dT^i_j$. The quantum determinant $L = \det_q T$ satisfies

$$LT^i_j = T^i_j L, \quad LdT^i_j = q^2 dT^i_j L$$

(4.86)

and

$$\eta = -q^{-1} dLL^{-1}.$$  

(4.87)
### 4.2.2 One-form Realization of the Exterior Differential for a \(*\)-Algebra

In the same manner as in the construction in section 4.2.1, we have the following:

**Construction 4.2.2 (\(*\)-Algebra)**

Let \( A \) be a \(*\)-involutive algebra with coordinates \( z_i, \bar{z}_i \) and differentials \( dz_i = \delta z_i, d\bar{z}_i = \delta \bar{z}_i \) such that \( \delta z_i = z_i^*, d\bar{z}_i = (dz_i)^* \). If there exists a real element \( a \in A \) and real unequal nonvanishing constants \( r, s \) such that

\[
\begin{align*}
ax_i &= rx_i a, \quad adz_i = sdx_i a, \quad \forall i,
\end{align*}
\]

then, as easily seen,

\[
\lambda \delta f = [\eta, f]_\lambda, \quad \eta = \frac{\lambda}{1 - s/r} \delta aa^{-1},
\]

(4.89)

\[
\lambda \delta f = [\bar{\eta}, f]_\lambda, \quad \bar{\eta} = \frac{\lambda}{1 - r/s} \delta aa^{-1}
\]

(4.90)

and

\[
\lambda df = [\Xi, f]_\lambda, \quad \Xi = \eta + \bar{\eta},
\]

(4.91)

where \( \pm \) applies for odd/even forms \( f \).

Notice that (4.89) and (4.90), and therefore (4.88), imply that

\[
r\delta a = s\delta a, \quad r\delta a = s\delta a.
\]

(4.92)

**Kähler Form**

It can be proved that \( \eta^* = -\bar{\eta} \) and so \( \Xi^* = -\Xi \). It holds that \( \eta^2 = \bar{\eta}^2 = 0 \). However \( \Xi^2 = \eta\bar{\eta} + \bar{\eta}\eta = \lambda \delta \eta = \lambda \delta \bar{\eta} \) will generally be nonzero. Note that

\[
\lambda d\Xi = [\Xi, \Xi]_\lambda = 2\Xi^2.
\]

(4.93)

Define

\[
K = \delta \eta = \delta \bar{\eta}
\]

(4.94)

then

\[
K = \frac{1}{2} d\Xi.
\]

(4.95)

It follows that \( dK = 0 \) and \( K^* = K \). Thus in the case \( K \neq 0 \), we will call it a Kähler form and \( K^n \) will be non-zero and define a real volume element for an integral (invariant integral if \( K^n \) is invariant). \( K \) also has the very nice property of commuting with everything

\[
K z_a = z_a K, \quad K dz_a = dz_a K.
\]

(4.96)

We see here an example of Connes' calculus [44] of the type \( F^2 \neq 0 \) rather than \( F^3 = 0 \).

**Quantum Sphere \( S^2_q \)**

In the case of \( S^2_q \), the element \( \rho = 1 + \varepsilon z \) is introduced which satisfies

\[
\rho z = q^2 z \rho, \quad \rho dz = dz \rho.
\]

(4.97)

Therefore, we get

\[
\eta = qdz \rho^{-1} z, \quad \bar{\eta} = -qd\varepsilon \rho^{-1} \varepsilon
\]

(4.98)

and \( K \) is just the area element

\[
K = \delta \eta = -q^2 dzd\varepsilon \rho^{-3}.
\]

(4.99)

One can introduce the Kähler potential \( V \) defined by

\[
K = \delta \delta V.
\]

(4.100)

It is

\[
V = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{q^{2k-1} \delta^2 z^k}{[k]^q}
\]

(4.101)

and where the quantum number \([z]_q\) is defined as

\[
[z]_q = \frac{q^2 z^{q-1} - 1}{q^2 - 1}.
\]

(4.102)
Complex Projective Space

Such a one-form representation for the calculus exists on both \( C^{N+1}_q \) and \( CP_q(N) \). For \( C^{N+1}_q \), we saw in the above that

\[
Lx_i = x_i L, \quad L\xi_i = q^4\xi_i L \quad (4.103)
\]

and

\[
\eta_0 = -q^{-1}\delta L\eta L^{-1}, \quad \bar{\eta}_0 = q\bar{\delta}LL^{-1}. \quad (4.104)
\]

In this case, \( K \) is not the Kähler form one usually assigns to \( C^{N+1}_q \). Rather, it gives \( C^{N+1}_q \) the geometry of \( CP_q(N) \) written in homogeneous coordinates.

Similar relations hold for \( CP_q(N) \) in inhomogeneous coordinates. It is

\[
\rho x_a = q^{-3}x_a \rho, \quad \rho d z_a = d x_a \rho \quad (4.105)
\]

and therefore

\[
\eta = -q^{-1}\delta \rho \rho L^{-1}, \quad \bar{\eta} = q\bar{\delta} \rho \rho L^{-1}. \quad (4.106)
\]

One can then compute

\[
K = \delta \eta \quad (4.107)
\]

\[
= d x_a g^{ab} d \bar{z}_b \quad (4.108)
\]

where the metric \( g^{ab} \) is

\[
g^{ab} = q^{-1}\rho^{-2}(\rho \delta_{ab} - q^2 z_a z_b) \quad (4.109)
\]

with inverse \( g_{bc} \)

\[
g_{bc} g^{cd} = g^{cd} g_{db} = \delta_{ab} \quad (4.110)
\]

given by

\[
g_{bc} = q \rho (\delta_{bc} + \bar{z}_b z_c) \quad (4.111)
\]

This metric is the quantum deformation of the standard Fubini-Study metric for \( CP(N) \). It is \( K = \delta \delta V \), where the Kähler potential \( V \) is

\[
V = \sum_{k=1}^{\infty} (-1)^{k-1} q^{2k-1} \frac{2k-1}{[k]_q} \sum_{1 \leq a_1, a_2, \ldots, a_k \leq N} z_{a_1} z_{a_2} \cdots z_{a_k} \bar{z}^{a_1} \cdots \bar{z}^{a_k-1} \bar{z}^{a_k}. \quad (4.112)
\]

Notice that under the transformation (4.59)

\[
\eta \rightarrow \eta + qf^{-1}\delta f, \quad f = T^0_0 + z_0 T^0_0 \quad (4.113)
\]

and so \( K \) is invariant. From (4.60) and (4.108), it follows that

\[
g^{ab} \rightarrow (M^{-1})_a^d g^{cd} \left((M^t)^{-1}\right)_d^b \quad (4.114)
\]

\[
g_{ba} \rightarrow (M^{-1})_a^d g_{dc} M^c_b \quad (4.115)
\]

One can show that the following form \( dv_x \) in \( C^{N+1}_q \)

\[
dv_x \equiv \prod_{i=0}^N (\bar{z}^i L^{-1/2}) \prod_{i=0}^N (L^{-1/2} z_i)
\]

\[
= \rho^{-(N+1)} dz^N \cdots d\bar{z} \cdots dz_1 \cdots d\bar{z}_1 \cdots d\bar{z}_N \quad (4.116)
\]

is invariant. Using this, one can prove that

\[
dv_x \equiv \rho^{-(N+1)} dz^N \cdots d\bar{z} \cdots dz_1 \cdots d\bar{z}_N \quad (4.117)
\]

is invariant also and is in fact equal to \( K^N \) (up to a numerical factor). The factor \( \rho^{-N+1} \) justifies the choice \( n = N + 1 \) for the involution (4.56).

Having a quantum Kähler metric one can define connections, curvature, a Ricci tensor and a Hodge star operation. We shall not do it here because there seems to be no unique way to define these constructs. Still, once certain choices are made, the full differential geometry can be developed. See [80] for a very nice discussion of the quantum Riemannian case.

### 4.3 Integration

We now turn to the discussion of integration on \( CP_q(N) \). We shall use the notation \( \langle f(z, \bar{z}) \rangle \) for the right-invariant integral of a function \( f(z, \bar{z}) \) over \( CP_q(N) \). It is defined, up to a normalization factor, by requiring

\[
\langle O(f(z, \bar{z})) \rangle = 0 \quad (4.119)
\]

for any left-invariant vector field \( O \) of \( SU_q(N + 1) \). We can work out the integral by looking at the explicit action of the vector fields on functions. This approach
has been worked out for the case of the sphere but it get quite complicated for
the higher dimensional projective spaces. We shall follow a different and simpler
approach here. First we notice that the identification
\begin{equation}
  z_i/L^1/2 = T_i, \quad \bar{z}^i/L^1/2 = (T^{-1})_i, \quad i = 0, 1, \ldots, N
\end{equation}
reproduces (4.1)-(4.5). Thus if we define
\begin{equation}
  < f(x, \bar{x}) > SU_q(N+1) = \frac{< f(x, x) |_{SU_q(N)} > |_{SU_q(N+1)} > SU_q(N+1)}{< f(x, x) > SU_q(N+1)},
\end{equation}
where \(< \cdot > SU_q(N+1)\) is the Haar measure \cite{12} on \(SU_q(N + 1)\), then it follows
immediately that (4.119) is satisfied. \(\dagger\) Next we claim that
\begin{equation}
  < (z_1)^i (\bar{z}^1)^j \cdots (z_N)^i (\bar{z}^N)^j > = 0 \text{ unless } i_1 = j_1, \ldots, i_N = j_N.
\end{equation}
This is because the integral is invariant under the finite transformation (4.59). For
the particular choice \(T_j = \delta_j a_i\), with \(|a_i| = 1, \Pi_{i=q}^{N} a_i = 1\), this gives
\begin{equation}
  z_a \to (a_i/a_0) z_a
\end{equation}
and so (4.122) follows.

In \cite{12}, Woronowicz proved the following interesting property for the Haar
measure
\begin{equation}
  < f(T) g(T) > SU_q(N+1) = < g(T) f(DT) D > SU_q(N+1),
\end{equation}
where
\begin{equation}
  (DT)_{ij} = D_{ik} T_k^m D_{mj}^n
\end{equation}
and
\begin{equation}
  D_{ij} = q^{-i+j} \delta_{ij}
\end{equation}
is the \(D\) matrix for \(SU_q(N + 1)\). It follows from (4.124) that
\begin{equation}
  < f(x, \bar{x}) g(x, \bar{x}) > = < g(x, \bar{x}) f(\bar{x}, \bar{x}) >,
\end{equation}
where
\begin{equation}
  D_a^b = \delta_q^{ab}, \quad a, b = 1, 2, \ldots, N.
\end{equation}

\(\dagger\) A similar strategy of using the "angular" measure to define an integration has been employed
by H. Steinacker \cite{81} in constructing integration over quantum Euclidean space.

Introducing
\begin{equation}
  \rho_r = 1 + \sum_{a=1}^{N} z_a \bar{z}^a,
\end{equation}
one finds from (4.33) and (4.34) that
\begin{equation}
  \rho_r z_a = \begin{cases} z_a \rho_r, & r < a \\ q^{-1} z_a \rho_r, & r \geq a \end{cases},
\end{equation}
\begin{equation}
  \rho_r \rho_s = \rho_s \rho_r
\end{equation}
and
\begin{equation}
  \bar{z}^a z_a = q^{-2} \rho_a - \rho_{a-1} \quad (\text{no sum}).
\end{equation}
Because of (4.122), it is sufficient to determine integrals of the form
\begin{equation}
  < \rho_1^{-i_1} \cdots \rho_N^{-i_N} >.
\end{equation}
The values of the integers \(i_a\) for (4.133) to make sense will be determined later.

Consider
\begin{equation}
  < \bar{z}_a \rho_1^{-i_1} \cdots \rho_N^{-i_N} z_a > = < \rho_1^{-i_1} \cdots \rho_N^{-i_N} z_a (q^{-2} \bar{z}^a z_a) >
  = q^{2i_a} < \rho_1^{-i_1} \cdots \rho_N^{-i_N} (\rho_a - \rho_{a-1}) >,
\end{equation}
where (4.127) is used. Applying (4.130)
\begin{equation}
  \text{L.S.} = q^{2(i_a + a)} \rho_a^{-i_a} \cdots \rho_N^{-i_N} \bar{z}^a z_a > = q^{2i_a} < \rho_1^{-i_1} \cdots \rho_N^{-i_N} \bar{z}^a z_a >,
\end{equation}
where we have denoted
\begin{equation}
  I_a = i_a + \cdots + i_N.
\end{equation}
Using (4.132) we get the recursion formula
\begin{equation}
  < \rho_1^{-i_1} \cdots \rho_a^{-i_a} - \rho_{a-1}^{-i_{a-1}} \rho_a^{-i_a} \cdots \rho_N^{-i_N} > [I_a + a]_q
  = < \rho_1^{-i_1} \cdots \rho_{a-1}^{-i_{a-1}} \rho_a^{-i_a} \cdots \rho_N^{-i_N} > [I_a + a - 1]_q.
\end{equation}
It is obvious then that
\begin{equation}
  < \rho_1^{-i_1} \cdots \rho_a^{-i_a} > = < \rho_1^{-i_1} \cdots \rho_{a-1}^{-i_{a-1}} > \frac{[a]_q}{[I_a + a]_q}.
\end{equation}
By repeated use of the recursion formula, \( < p_1^{-i} \cdots p_n^{-i} > \) reduces finally to
\[
< p_1^{-i} > = \frac{1}{|I_1 + I_2|} < 1 > .
\]
Therefore
\[
< p_1^{-i} \cdots p_n^{-i} >= < 1 > \prod_{a=1}^{\infty} \frac{|a|}{|I_a + a|} ,
\]
for this to be positive definite, \( i_a \) should be restricted such that \( I_a + a > 0 \) for \( a = 1, \cdots, N \).

### 4.4 Braided \( CP_q(N) \)

As described in [76] and also in section 3.5.1, it is sufficient to know the transformation property of the algebra to derive the braiding. But as demonstrated in [76], it is already quite complicated in the case of one dimensional algebra. Therefore although we can derive the braiding for the \( CP_q(N) \) using the general framework of 3.5.1, we will follow a different easier path: first introduce the braiding for \( C_q^{N+1} \) quantum planes and then use it to induce a braiding on the \( CP_q(N) \)’s.

**Braided \( C_q^{N+1} \)**

Let the first copy of quantum plane be denoted by \( x_i, z_i \) and the second by \( x'_i, z'_i \) and let their commutation relations be:
\[
x_i x'_j = \tau \hat{R}_{ij} x'_j x_i ,
\]
\[
z_i z'_j = \nu (\hat{R}^{-1})_{ij} z'_j z_i ,
\]
and their \(*\)-involutions for arbitrary numbers \( \tau, \nu \). These are consistent and covariant, as one can easily check. One can choose \( \tau = \nu^{-1} \) and the Hermitian length \( L \) will be central, \( L f' = f' L \), for any function \( f' \) of \( x', z' \). However, \( L' \) does not commute with \( x, z \). In the following, we don’t need to assume that \( \tau = \nu^{-1} \).

By assuming that the exterior derivatives of the two copies satisfy the Leibniz rule
\[
\delta f = \pm f' \delta , \quad \delta f' = \pm f \delta ,
\]
where the plus (minus) signs apply for even (odd) \( f \) and \( f' \), and
\[
\delta \delta' = - \delta' \delta , \quad \delta \delta' = - \delta' \delta ,
\]
(4.145)
(4.146)
One can derive the commutation relations between functions and forms. Identify \( \delta = dx, D, \delta = dz, D \) for both copies, one can derive also the commutation relations between derivatives and functions of different copies. We will not bother to write them down here.

**Braided \( CP_q(N) \)**

Using (4.141, 4.142), one can derive the braiding relations of two braided copies of \( CP_q(N) \) in terms of the inhomogeneous coordinates
\[
x_{ai} z_{bj} = q \hat{R}_{ai} (z_{bj} - q^{-1} \lambda z_{bj}) x_{ai} ,
\]
\[
z_{ai} x_{bj} = q^{-1} (\hat{R}^{-1})_{aj} z_{bj} x_{ai} - q^{-1} \lambda \delta_{aj}
\]
(4.147)
(4.148)
and their \(*\)-involutions. Notice that these are independent of the particular choice of \( \tau \) and \( \nu \). Similarly, one can work out the commutation relations between functions and forms of different copies following the assumption that their exterior derivatives anticommute. We will not list them here.

See [77, 78] for a detailed discussion on the anharmonic ratios for braided copies of \( CP_q(N) \).

### 4.5 Quantum Grassmannians \( G_{q}^{M,N} \)

#### 4.5.1 The Algebra

Let \( C_q^i, i = 1, 2, \cdots, M, a = 1, 2, \cdots, M+N \) be an \( M \times (M+N) \) rectangular matrix satisfying the commutation relations
\[
\hat{R}^{ik} C_q^i C_q^j = C_q^i C_q^j \hat{R}^{ik} ,
\]
(4.149)
where \( \hat{R}^{ik} \) is a \( GL_q(M) \) \( \hat{R} \)-matrix, with indices \( i, j, k, l \) etc. going from 1 to \( M \) and \( \hat{R}^{ab} \) is a \( GL_q(M+N) \) \( \hat{R} \)-matrix, with indices \( a, b, c, d \) etc. going from 1 to \( M+N \).
In compact notation, it is
\[ \hat{R}_{12} C_1 C_2 = C_1 C_2 \hat{R}_{12} \]
and (4.150) is covariant under the transformation
\[ C \to CT, \]  
(4.151)
where \( T \) is a \( GL_q(M + N) \) quantum matrix and also under the transformation
\[ C \to SC, \]  
(4.152)
where \( S \) is a \( GL_q(M) \) quantum matrix. Writing
\[ C_s = (A_s, B_s) \]
(4.153)
with \( \alpha = 1, 2, \cdots, N \), we have
\[ \hat{R}_{12} A_1 A_2 = A_1 A_2 \hat{R}_{12}, \]
\[ \hat{R}_{12} B_1 B_2 = B_1 B_2 \hat{R}_{12}, \]
\[ A_1 B_2 = R_{12} B_1 A_2, \]
(4.154)
where \( R_{12} \) is a \( GL_q(N) \) \( \hat{R} \)-matrix, with indices \( \alpha, \beta, \gamma, \delta \) etc. going from 1 to \( N \).
Define the coordinates \( Z' \) for the quantum Grassmannians
\[ Z = A^{-1} B. \]  
(4.155)
\( Z \) is invariant under the transformation (4.152), while under (4.151), it transforms as
\[ Z \to (\alpha + Z \gamma)^{-1}(\beta + Z \delta), \]  
(4.156)
where \( \alpha, \beta, \gamma, \delta \) are the sub-matrices of \( T \)
\[ T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \]  
(4.157)
It follows from (4.154) that \( Z \) satisfies
\[ \hat{R}_{12} Z_1 Z_2 = Z_1 Z_2 \hat{R}_{12}. \]  
(4.158)
\[ *-structure \]
We consider \( q \) to be a real number. One can introduce the \( * \)-conjugate variables \( (C_s)'^\star \) and impose the commutation relation
\[ C_1'^\star \hat{R}_{12} C_1 = C_2 \hat{R}_{12}'^\star C_2, \]  
(4.159)
i.e.
\[ (A^{-1})_1'^\star \hat{R}_{12}(A^{-1})_2 = (A^{-1})_1 \hat{R}_{12}(A^{-1})_2'^\star, \]
\[ B_1'^\star A_2 = A_2'^\star B_1 \hat{R}_{12}' \]
\[ B_1'^\star \hat{R}_{12} B_1 = B_1 B_1'^\star \hat{R}_{12}' - \lambda I_1(AA^\dagger), \]  
(4.160)
where \( (I_1(AA^\dagger))_0 = \delta_0^0(AA^\dagger)_0 \). These implies
\[ Z_1'^\star \hat{R}_{12} Z_1 = Z_1 \hat{R}_{12}^{-1} Z_1 - \lambda I_1. \]  
(4.161)
Explicitly,
\[ (Z_1'^\star)'^\star \hat{R}_{12} Z_1'^\star = Z_1'^\star \hat{R}_{12}^{-1} Z_1'^\star - \lambda \delta_0^0. \]  
(4.162)

4.5.2 Calculus
One can introduce the following commutation relation for functions and one-forms
\[ \hat{R}_{12}^{-1} C_1 dC_2 = dC_1 C_2 \hat{R}_{12}, \]  
(4.163)
i.e.
\[ \hat{R}_{12}^{-1} A_1 dA_2 = dA_1 A_2 \hat{R}_{12}, \]
\[ \hat{R}_{12}^{-1} B_1 dB_2 = dB_1 B_2 \hat{R}_{12}, \]
\[ dA_2 B_1 = R_{12}^{-1} B_1 dA_1, \]
\[ A_1 dB_2 = R_{12}(dA_2 B_1 + \lambda dB_2 A_1 P_{12}), \]  
(4.164)
where \( (P_{12})_0 = \delta_0^0 \). Since \( Z_0 = (A^{-1})_0 B_0 \), it is easy to derive
\[ dZ = A^{-1}(dB - dA_Z) \]  
(4.165)
and

\[ Z_1 dA_2 = dA_2 R^{-1}_{21} Z_1, \]
\[ dZ_1 A_2 = A_2 R^{-1}_{21} dZ_1, \]
\[ Z_1 dB_2 = (dB_2 Z_1 - \lambda Z_1 (dA Z_2) B_2) R^{-1}_{12}. \]  

(4.166)

It follows

\[ \hat{R}^{-1}_{21} Z_1 dZ_2 = dZ_2 Z_2 \hat{R}^{-1}_{12}. \]  

(4.167)

To introduce an \(*\)-structure for the calculus, it is consistent to take \((dZ^*_a)_* = d(Z^*_a)\), this implies

\[ Z^*_1 \hat{R}^*_2 dZ_2 = dZ_2 \hat{R}^{*1}_{12}. \]  

(4.168)

4.5.3 One-Form Realization

Introduce the matrix

\[ E_j^i = C_j^i (C^d)^i_d. \]  

(4.169)

It is

\[ E^i = E \]  

(4.170)

and

\[ \hat{R}_{12}^{-1} E_1 \hat{R}_{12}^{-1} E_1 = E_1 \hat{R}_{12}^{-1} E_1 \hat{R}_{12}^{-1}. \]  

(4.171)

Since

\[ \hat{R}_{12}^{-1} (q) = \hat{R}_{21} (q^{-1}), \]  

(4.172)

one can rewrite (4.171) as

\[ \hat{R}_{12} (q^{-1}) E_2 \hat{R}_{12} (q^{-1}) E_2 = E_2 \hat{R}_{12} (q^{-1}) E_1 \hat{R}_{12} (q^{-1}). \]  

(4.173)

This shows that the commutation for the \( E \) matrix is like the bi-covariant vector fields \( Y \) (c.f. (2.23)), but with \( q^{-1} \) as its parameter. Similarly one also has

\[ C_2 E_1 = \hat{R}_{21}^{-1} (q^{-1}) E_1 \hat{R}_{21} (q^{-1}) C_2, \]
\[ dC_2 E_1 = \hat{R}_{21} (q^{-1}) E_2 \hat{R}_{21} (q^{-1}) dC_2. \]  

(4.174)

One can show that

\[ R_{12}^1 E_1 \hat{R}_{21} (q^{-1}) E_2 = E_1 \hat{R}_{21} (q^{-1}) E_2 \hat{R}_{12} (q^{-1}). \]  

(4.175)

or equivalently,

\[ R_{12}^1 E_1 \hat{R}_{21} (q^{-1}) E_2 = E_1 \hat{R}_{21} (q^{-1}) E_2 \hat{R}_{12} (q^{-1}). \]  

(4.176)

where the bullet product is defined inductively by

\[ E_1 \bullet E_2 \equiv E_1 R^{-1}_{12} E_2 R_{12}, \]  

(4.177)

for any \( I = (12 \cdots m), J = (12 \cdots n) \). Hence one can introduce the quantum determinant [56] for the generators \( E \),

\[ Det E c^{12 \cdots M} = E_{(12 \cdots M)} c^{12 \cdots M}, \]  

(4.178)

where

\[ E_{(12 \cdots M)} \equiv E_1 \bullet E_2 \bullet \cdots \bullet E_M \]  

(4.179)

and \( c^{12 \cdots M} \) is the same \( \epsilon \) tensor (1.43) for \( GL(M) \).

Using

\[ R_{(12 \cdots M)} c^{12 \cdots M} = q c^{12 \cdots M} I_0, \]  

(4.180)

one can show that \( Det E \) commutes with \( E_i \):

\[ c^{12 \cdots M} Det E E_0 = E_{(12 \cdots M)} E_0 c^{12 \cdots M} = E_{(12 \cdots M)} R^{-1}_{(12 \cdots M)} E_0 c^{12 \cdots M} = q R^{-1}_{(12 \cdots M)} E_0 R_{(12 \cdots M)} E_{(12 \cdots M)} E_0 c^{12 \cdots M} = E_0 Det E c^{12 \cdots M}. \]

Denote

\[ L = Det E. \]  

(4.181)

It follows from (4.174) that

\[ dC \hat{L} = q^{-2} L dC. \]  

(4.182)

Using the general procedure stated in section 4.2, we obtain the one-form realization on the algebra generated by \( C_j^i, dC_j^i \) and their \(*\)-conjugates,

\[ \eta = -q L^{-1} \delta L. \]  

(4.183)
To find the one-form realization for the exterior differential operating on the complex Grassmannians \( Z, dZ \), we introduce

\[
X^k = (A^{-1})^k E_j (A^{-1})^j
\]

\[
= \delta^k_0 + 2 \sigma^k_0 (Z^i)^i.
\]  

(4.184)

It is not hard to check that

\[
A_1^{-1} R_{11} R_{11}^{-1} = E_2 A_2^{-1},
\]

\[
X_1 R_{11} A_2^{-1} = R_{12} A_3^{-1} X_1
\]  

(4.185)

and hence

\[
\hat{R}_{12} X_1 \hat{R}_{12} X_2 = X_2 \hat{R}_{12} X_2 \hat{R}_{12}.
\]  

(4.186)

Since \( X \) commutes like the vector field \( Y \), the quantum determinant

\[
\rho \equiv \text{Det} X = X_{1[13\ldots M]} e^{12\ldots M}
\]  

(4.187)

is central in the algebra of \( X \). Here, the \( \circ \)-product for \( X \) is

\[
X_I \circ X_J = \hat{R}_{12}^{-1} X_I \hat{R}_{12} X_J,
\]  

(4.188)

as for the vector field \( Y \). Using the techniques as in ([56]), one can show that for any \( I = (12\ldots m) \),

\[
X_I = A_{I}^{-1} E_I (A^{-1})_I,
\]  

(4.189)

where

\[
A_I = A_1 A_2 \ldots A_m, \quad A_I^{-1} = A_m^{-1} \ldots A_2^{-1} A_1^{-1}.
\]  

(4.190)

Introducing the quantum determinant \( \text{det}(A^{-1}), \text{det}(A^{1-1}) \)

\[
\text{det}(A^{-1}) e^{12\ldots M} = A_1^{-1} \ldots A_m^{-1} A_2^{-1} e^{12\ldots M},
\]

\[
\text{det}(A^{1-1}) e^{12\ldots M} = (A^{1-1})_1 (A^{1-1})_2 \ldots (A^{1-1})_M e^{12\ldots M}
\]  

(4.191)

for the "RTT"-like \( A^{-1} \) matrix and \( A^{1-1} \) matrix

\[
\hat{R}_{12}(q^{-1}) A_1^{-1} A_2^{-1} = A_1^{-1} A_2^{-1} \hat{R}_{12}(q^{-1}),
\]

\[
\hat{R}_{12}(A^{1-1})_1 (A^{1-1})_2 = (A^{1-1})_1 (A^{1-1})_2 \hat{R}_{12},
\]  

(4.192)

it is straightforward to obtain

\[
\text{Det} X = \text{det}(A^{-1}) \text{Det} E \text{det}(A^{1-1}).
\]  

(4.193)

It is trivial to check that

\[
B \text{det}(A^{-1}) = q \text{det}(A^{-1}) B,
\]

\[
B \text{det}(A^{1-1}) = q \text{det}(A^{1-1}) B
\]  

(4.194)

and noticing that

\[
\rho = \text{det}(A^{-1}) L \text{det}(A^{1-1}),
\]  

(4.195)

we have

\[
Z \rho = q^2 \rho Z.
\]  

(4.196)

Using (4.166), one can check that

\[
d Z \text{det}(A^{-1}) = q \text{det}(A^{-1}) d Z,
\]

\[
d Z \text{det}(A^{1-1}) = q \text{det}(A^{1-1}) d Z.
\]  

(4.197)

Together with

\[
d Z L = q^{-2} L d Z,
\]  

(4.198)

we have

\[
d Z \rho = \rho d Z.
\]  

(4.199)

As a result, we have the one-form realization

\[
\eta = -q^{-1} \rho^{-1} \delta \rho
\]  

(4.200)

for the exterior operator acting on the algebra generated by \( Z_i, d Z_i \) and their \( * \)-conjugate. The Kähler form

\[
K = \delta \eta
\]  

(4.201)

is central as usual.
4.5.4 Braided \( G_q^{M,N} \)

Let \( Z, Z' \) be two copies of the quantum Grassmannians \( G_q^{M,N} \) defined by

\[
Z = A^{-1}B, \quad Z' = A'^{-1}B',
\]
(4.202)

where \( C_1' = (A_1', B_1'), C_2' = (A_2', B_2') \) both satisfy the relations (4.150). Let the mixed commutation relations be

\[
Q_{12} C_1 C_2' = C_1' C_2 R_{12},
\]
(4.203)

where \( Q \) is a numerical matrix. For (4.203) to be consistent with (4.150), we can take \( Q \) to be \( \hat{R}^{k_1} \). For either of these two choices, (4.203) is covariant under

\[
C \rightarrow CT, \quad C' \rightarrow C'T,
\]
(4.204)

where \( T \) is a GL\(_m\) quantum matrix and also under the transformation

\[
C \rightarrow SC, \quad C' \rightarrow SC',
\]
(4.205)

where \( S \) is a GL\(_N\) quantum matrix. We will pick \( Q = \hat{R} \) in the following

\[
\hat{R}_{12} C_1 C_2' = C_1' C_2 R_{12}.
\]
(4.206)

Explicitly, it is

\[
R_{13} A_1 A_2' = A_2' A_1 R_{13},
\]
\[
R_{13} B_1 B_2' = B_2' B_1 R_{13},
\]
\[
B_1 A_2' = R_{13}^{-1} A_2' B_1,
\]
\[
A_1 B_2' = R_{13}^{-1} B_2' A_1 + \lambda B_1 A_2' P_{13}.
\]
(4.207)

Since

\[
Z = A^{-1}B, \quad Z' = A'^{-1}B',
\]
(4.208)

it follows that \(^9\)

\[
Z_{1} Z'_{2} = R_{12} Z_{2} Z_{1} R_{13} - \lambda Z_{1} Z_{3} R_{13}.
\]
(4.209)

\(^9\)If we had chosen the other choice \( Q = \hat{R}^{-1} \) in the above, the relations (4.207) would be different, but (4.209) would remain the same.

One can introduce a \(*\)-structure to this braided algebra, the relation

\[
C_1' \hat{R}_{12}^{-1} C_1' = C_1' \hat{R}_{12}^{-1} C_1'
\]
(4.210)

is consistent and is covariant under

\[
C \rightarrow CT, \quad C' \rightarrow C'T,
\]
(4.211)

and also under the transformation

\[
C \rightarrow SC, \quad C' \rightarrow SC'
\]
(4.212)

with the same \( T, S \) quantum matrix as explained above. It follows immediately

\[
Z_{1} \hat{R}_{12} Z_{1} = Z_{1} \hat{R}_{12}^{k} Z_{1} - \lambda Z_{1} Z_{3}.
\]
(4.213)

One can also show that the Kähler form \( K \) of the original copy (4.201) commutes also with the \( Z', dZ', dZ' \).

This concludes our discussion for the quantum Grassmannians, with the case of complex projective spaces \( CP_2(N) = G_q^{2,N} \) as a special case.\(^{10}\)

Strictly speaking, we have given in this chapter only a local description of the complex projective spaces and the quantum Grassmannians, i.e. in a certain coordinate patch. Presumably, there is no difficulty to introduce other patches in the picture and this has been illustrated in details in Chapter 3 for the simplest case of the sphere.

\(^{10}\)Notice that for \( M = 1 \), the numerical R-matrix becomes a number: \( \hat{R}_{12} = \varphi. \)
Chapter 5

q-Deformed Dirac Monopole

A major step towards a q-deformed gauge theory is to find a suitable concept of “quantum” fiber bundles. Recently some versions of quantum bundles have been proposed [71, 82, 83, 84], where both the base space and the fiber are quantum spaces. In [71], a detailed formulation for quantum principal fiber bundle is proposed and as examples, the q-deformed Dirac monopole for charge 1 and 2 are constructed. Essentially, their construction is based on the isomorphisms

\[ S^2_q = SU_q(2)/U(1) \] (5.1)

and

\[ S^2_q = SO_q(3)/U(1). \] (5.2)

However, in the case of charge 1, their trivializations involve square root of algebra elements and are formal. In this chapter, we will construct the deformed Dirac monopole on the quantum sphere \( S^2_q \) for arbitrary charge \( n \) and show that it is a quantum principal bundle in the sense of [71]. We also get the monopole charge by integrating the curvature over the base \( S^2_q \).

5.1 Quantum Principal Bundle

The definition of a quantum principal bundle follows the motto of non-commutative geometry, dualize everything and then introduce deformation. We first review the definitions of [71].

5.1.1 Universal Calculus

Definition 5.1.1 [71, Def 4.1] \( P = P(B, A) \) is a quantum principal bundle (short: QPB) with universal differential calculus, structure quantum group \( A \) and base \( B \) if

1. \( A \) is a Hopf algebra
2. \( (P, \Delta_R) \) is a right \( A \)-comodule algebra; write \( \Delta_R(p) = p^1 \otimes p^2 \in P \otimes A \)
3. \( B = P^A = \{ u \in P : \Delta_R u = u \otimes 1 \} \)
4. \( (\cdot \otimes \text{id})(\text{id} \otimes \Delta_R) : P \otimes P \to P \otimes A \) is a surjection (freeness condition)
5. \( \ker \sim = \Gamma_{\text{hor}} \) (exactness condition for the differential envelope)

where horizontal forms \( \Gamma_{\text{hor}} \) are defined by

\[ \Gamma_{\text{hor}} = \mu(\Gamma_B) P \subseteq \Gamma_P \] (5.3)

and satisfy \( \sim(\Gamma_{\text{hor}}) = 0 \) identically. The left \( P \)-module map \( \sim \) is defined as

\[ \sim = (\cdot \otimes \text{id})(\text{id} \otimes \Delta_R)|_{P^A} : P \to P \otimes \ker \epsilon. \] (5.4)

In the dual picture, it generates the fundamental (vertical) vector fields on the bundle. We will use the same symbol \( \sim \) for the extended map in condition 4.

Trivialization

Classically, the trivialization

\[ \phi : P \to G, \] (5.5)

is given by

\[ \phi(u)a = \phi(u \cdot a). \] (5.6)

In the deformed case, let \( A \) be a Hopf algebra and \( P \) an \( A \)-comodule algebra with invariant subalgebra \( B \). Suppose that there exists a convolution invertible map \( \Phi : A \to P \) such that

\[ \Delta_R \circ \Phi = (\Phi \otimes \text{id}) \circ \Delta, \quad \Phi(1_A) = 1_P \] (5.7)

(so \( \Phi \) is an intertwiner for the right coaction). It is proved in [71] that \( P \) is a quantum principal bundle. We call \( P(B, A, \Phi) \) a trivial bundle with trivialization \( \Phi \).
Connection

The topological aspects of bundles is embodied in the set of transition functions. The notion of a connection plays the same essential role in the differential geometry of bundles. A connection defines a covariant derivatives which contains a gauge field and specifies how to parallel transport a vector in \( P \) along a curve lying in the base \( B \). Therefore we have to first be able to define what is a horizontal vector in the principal bundle \( P \). A connection \( \Pi \) on \( P \) is just a prescription to separate the tangent space \( T_u P \) into the vertical subspace \( V_u P \) and the horizontal subspace \( H_u P \) such that

1. \( T_u P = V_u P \oplus H_u P \)
2. \( H_u g = R_u H_u P, \forall u \in P, g \in G \)

This definition is geometric but not practical in computation. To achieve this, it is common to introduce a Lie-algebra valued 1-form \( w \in g \otimes T^* P \) called the connection 1-form which satisfies

1. \( w(\xi) = \xi \) for any \( \xi \in g \)
2. \( (R_u)^* \omega = ad(a^{-1}) \omega \), i.e. \( \omega((R_u)_* X) = ad(a^{-1}) \omega(X) \)

for any \( a \in g \) and any vector field \( X \).

The relation between the 2 definitions is given by

\[
H_u P = \{ X \in T_u P : \omega(X) = 0 \} \tag{5.8}
\]

In the dual formulation, one define a connection \( \Pi \) on a quantum principal bundle \( P \) as an assignment of a left \( P \)-submodule \( \Gamma_{\text{ver}} \subseteq \Gamma_P \) such that:

1. \( \Gamma_P = \Gamma_{\text{hor}} \oplus \Gamma_{\text{ver}} \),
2. projection \( \Pi : \Gamma_P \rightarrow \Gamma_{\text{ver}} \) is right invariant i.e.

\[
\Delta_R \Pi = (\Pi \otimes \text{id}) \Delta_R. \tag{5.9}
\]

A connection in \( P \) is characterized by a right-invariant left \( P \)-module map \( \sigma : P \otimes \ker \epsilon \rightarrow \Gamma_P \) splitting the exact sequence

\[
0 \rightarrow \Gamma_{\text{hor}} \rightarrow \Gamma_P \xrightarrow{\sigma} P \otimes \ker \epsilon \rightarrow 0, \tag{5.10}
\]

i.e.

\[
\tilde{\sigma} \circ \sigma = \text{id}. \tag{5.11}
\]

The connection form \( \omega : A \rightarrow \Gamma_P \) is given by

\[
\omega(a) = \sigma(1 \otimes (a - \epsilon(a))). \tag{5.12}
\]

Conversely, \( \sigma(p \otimes a) = pw(a) \) for \( p \otimes a \in P \otimes \ker \epsilon \).

Gauge Field

Classically, a gauge field is a Lie algebra valued 1-form living on the base. It is the pull back of the connection one form by some section \( \psi \). Conversely, given a gauge field \( A \) over an open set \( U \) of the base \( B \), let \( \phi : \pi^{-1}(U) \rightarrow G \) be a trivialization, then

\[
\omega = \phi^{-1} \pi^* A + \phi^{-1} d \phi \tag{5.13}
\]

give a connection 1-form.

In the dual we have Let \( \beta : A \rightarrow \Gamma_B \) be a linear map such that \( \beta(1) = 0 \). Then the map \(^1\)

\[
\omega = \phi^{-1} \ast j \circ \beta \ast \Phi + \phi^{-1} \ast d \Phi \tag{5.18}
\]

\(^1\)Here \( f \ast g \) is the convolution product of two maps. Given

\[
f_i : A \rightarrow B, \quad i = 1, 2,
\]

where \( A \) is a coalgebra and \( B \) is an algebra, we define \( f_1 \ast f_2 \) as a map from \( A \) to \( B \) by

\[
(f_1 \ast f_2)(a) = f_1(a_{(1)}) f_2(a_{(2)}). \tag{5.15}
\]

Similarly, if \( V \) is a left \( A \)-comodule and \( f_1 : A \rightarrow B, f_2 : V \rightarrow B \), then \( f_1 \ast f_2 \) is defined as the map from \( V \) to \( B \) by

\[
(f_1 \ast f_2)(v) = f_1(v^{-1}) f_2(v^{(0)}), \tag{5.16}
\]

where

\[
\Delta_V(v) = v^{-1} \otimes v^{(0)} \in A \otimes V \tag{5.17}
\]
is the left coaction of \( A \) on \( V \).
is a connection 1-form in the trivial principal bundle \( P(B, A, \Phi) \) with trivialization \( \Phi. \) \( \beta \) is called a gauge field.

### 5.1.2 General Calculus

Let \( M_A \) be a right ideal of \( A^2 \) and \( N_p \) be a sub-bimodule of \( P^2. \) Introduce a first-order differential calculus on \( A \) and \( P \) by

\[
\Gamma_A = A^2/N_A, \quad (5.19)
\]

where

\[
N_A = \kappa(A \otimes M_A) \quad (5.20)
\]

and

\[
\Gamma_p = P^2/N_p, \quad (5.21)
\]

where the map \( \kappa : A \otimes A \to A \otimes A \) is given by

\[
\kappa(a \otimes a') = \sum aS a'(1) \otimes a'(2). \quad (5.22)
\]

**Definition 5.1.2** [71, Def 4.1] We say that \( P = P(B, A, N_p, M_A) \) is a quantum principal bundle with structure quantum group \( A \) and base \( B \) and quantum differential calculi defined by \( N_p, M_A \) if:

1. \( A \) is a Hopf algebra.
2. \((P, \Delta_R)\) is a right \( A \)-comodule algebra.
3. \( B = P^A = \{ u \in P : \Delta_R u = u \otimes 1 \}. \)
4. \((\cdot \otimes id)(id \otimes \Delta_R) : P \otimes P \to P \otimes A\) is a surjection (freeness condition).
5. \( \Delta_R N_p \subset N_p \otimes A \) (right covariance of differential structure).
6. \( \gamma(N_p) \subset P \otimes M_A \) (fundamental vector fields compatibility condition)
7. \( \ker \gamma(N_p) = \Gamma_{\ker} \) (exactness condition).

The map \( \gamma(N_p) \) is induced by

\[
\gamma(N_p) = (id \otimes \pi_A) \circ (\rho u) \quad (5.23)
\]

where

\[
\pi_{N_p} : P^2 \to \Gamma_p
\]

and

\[
\pi_A : \ker \epsilon \to \ker \epsilon/M_A
\]

are the canonical epimorphisms and \( \rho u \in \pi_{N_p}^2(\rho) \) is any representative of \( \rho. \) (5.23) is well defined because of condition 6. of the definition.

**Connection**

A connection on a QPB with general calculus is again determined by a splitting \( \sigma \) of the sequence

\[
0 \to \Gamma_{\ker} \to \Gamma_p \to P \otimes \ker \epsilon/M_A \to 0. \quad (5.24)
\]

The connection form is given by

\[
\omega(\alpha) = \sigma(1 \otimes \pi_a(a - \epsilon(a))). \quad (5.25)
\]

An element \( \alpha \in \Gamma_{\ker} \) is called a **vertical form**. If there exists a connection in \( P \) then any one-form \( \alpha \in \Gamma_p \) can be uniquely written as \( \alpha \) is a sum of a horizontal and a vertical forms.

Notice that one can replace the Point (6) of the definition of QPB by the slightly stronger condition [71]

6. \( \gamma(N_p) = P \otimes M_A, \) which we will adopt in our construction.

### 5.2 Monopole Bundle: Global

The monopole bundle that we are going to construct is characterized by an integer \( n \) and we will refer to it as the charge of the bundle. Let the algebras \( P, A, B \) be

\[
P = SU(2), \quad (5.26)
\]

\[
A = k < z^ {1/2}, z^{−1/2} >= U(1), \quad (5.27)
\]

\[
B = < 1, b, b+, b_3 >, \quad (5.28)
\]
where $b_+ = \alpha \beta, b_\alpha = \gamma \delta, b_3 = \alpha \delta$. For charge $n$, define the following subalgebras,

$$p^{(n)} = \left\{ p \in SU_4(2), \text{deg}(p) = nk, k \in \mathbb{Z} \right\},$$

$$A^{(n)} = k < Z^{n/2}, Z^{-n/2} >,$$  \hspace{1cm} (5.29)

where the degree for a monomial in $SU_4(2)$ is defined as

$$\text{deg}(\alpha^a \beta^b \gamma^c \delta^d) = a + c - b - d$$  \hspace{1cm} (5.31)

irrespective of ordering. Introduce the tight coaction $\Delta_R : P \rightarrow P \otimes A$ defined by

$$\Delta_R \left( \begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \end{array} \right) = \left( \begin{array}{c} \alpha \otimes Z^{1/2} \\ \beta \otimes Z^{-1/2} \\ \gamma \otimes Z^{1/2} \\ \delta \otimes Z^{-1/2} \end{array} \right).$$  \hspace{1cm} (5.32)

It is easy to see that $B = (p^{(n)}) A^{(n)}$.  \hspace{1cm} (5.33)

As for the calculus on $P^{(n)}$ and $A^{(n)}$, take the right ideal $M_{A^{(n)}}$ generated by the six elements

$$\delta + q^2 \alpha - (1 + q^2), \gamma^2, \beta \gamma, \beta^2, (\alpha - 1) \gamma, (\alpha - 1) \beta$$  \hspace{1cm} (5.34)

and

$$M_{A^{(n)}} = \pi^{-1}(P^{(n)}) \cap \{ Z^{1/2} + q^2 Z^{1/2} - (1 + q^2) \} >.$$  \hspace{1cm} (5.35)

The projection $\pi : P \rightarrow A$ (dual of $U(1) \subset SU(2)$) is an algebra map

$$\pi \left( \begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \end{array} \right) = \left( \begin{array}{c} Z^{1/2} \\ 0 \\ 0 \\ Z^{-1/2} \end{array} \right).$$  \hspace{1cm} (5.36)

For charge $n$, take $N_{P^{(n)}} = N_{P^{(n)}} \cap (P^{(n)})^2$, $M_{A^{(n)}}$ generated by $Z^{-n/2} + q^{2n} Z^{n/2} - (1 + q^n)$, i.e. $Z^{n/2} d Z^{n/2} = q^{2n} d Z^{n/2} Z^{n/2}$ or equivalently $N_{A^{(n)}} = N_{A^{(n)}} \cap (A^{(n)})^2$.

We have

Proposition 5.2.1 $P^{(n)}(B, A^{(n)}, N_{P^{(n)}}, M_{A^{(n)}})$ is a QPB.

Proof. Condition 1-3 of the definition 5.1.2 are obviously satisfied. To show condition 4, consider the monomial $p \otimes Z^{n/2} \in P^{(n)} \otimes A^{(n)} \subset P^{(1)} \otimes A^{(1)}$ for $k \in \mathbb{Z}$. Since $P^{(1)}$ is a QPB, there exists $\sum p_i \otimes p_i \in P^{(1)} \otimes P^{(1)}$ such that $\Gamma p_i \otimes p_i = p \otimes Z^{n/2}$. Now, $\Gamma = \sum p_i \otimes p_i \subset Z^{n/2}$, $Z^{n/2}$. Therefore, $\text{deg}(p_i) = nk$, and $p_i \in P^{(n)}$ for all $i$. Also, $\text{deg}(p_i \otimes p_i) = \text{deg}(p)$ and $p \in P^{(n)}$, so $\text{deg}(p_i) \in n \mathbb{Z}$. Hence, $p_i \in P^{(n)}$ for all $i$. Surjectivity is proved. Condition 5 is obvious because of our simple definition of $\Delta_R$. For condition 6, notice that $P^{(n)}(B, A^{(n)}, N_{P^{(n)}}, M_{A^{(n)}})$ is a QPB, so

$$\text{ker} \Gamma^{(n)} = (\text{ker} \Gamma^{(n)}) \cap \Gamma P^{(n)} \subset \Gamma \cap \Gamma P^{(n)} = \Gamma.$$  \hspace{1cm} (5.37)

since $\text{deg}(B) = 0$. For the same reason, we know that $N_{P^{(n)}} \subset P^{(1)} \otimes M_{A^{(n)}}$. Therefore, $N_{P^{(n)}} \subset P^{(1)} \otimes M_{A^{(n)}}$. But $\Gamma (p_i \otimes p_i) = p_i P^{(1)} \otimes P^{(1)} - p_i \otimes p_i \otimes p_i \otimes p_i \in P^{(n)} \otimes A^{(n)}$, for all $p_i \otimes p_i \in P^{(n)}$, so

$$N_{P^{(n)}} \subset P^{(n)} \otimes M_{A^{(n)}}.$$  \hspace{1cm} (5.38)

Hence we see that $P^{(n)}(B, A^{(n)}, N_{P^{(n)}}, M_{A^{(n)}})$ is a QPB. Also, note that the 3D - calculus respects the $*-$structure [14].

A possible connection one-form on $P^{(n)}$ is given by

$$\omega(Z^{n/2}) = S((\alpha^n)(1)) d(\alpha^n)(1) = \kappa(1 \otimes (\alpha^n - 1)),$$  \hspace{1cm} (5.39)

$$\omega(Z^{-n/2}) = S((\delta^n)(1)) d(\delta^n)(1) = \kappa(1 \otimes (\delta^n - 1)).$$  \hspace{1cm} (5.40)

for $k > 0$, where $\kappa$ is defined in [71]. This $\omega$ is well defined, since $S((\alpha^n)(1)), S((\delta^n)(1)), (\alpha^n)(1), (\delta^n)(1) \in P^{(n)}$. This connection was found observing that this is the trivial connection [71] obtained from the trivialization $\Phi(Z^{n/2}) = \alpha^n$, which is a gauge transformation of the trivialization (5.54) that we will introduce later. Note that the above trivialization does not respect the $*-$structure even for $q = 1$, nevertheless it is useful to e.g. find a connection; in the 3D - calculus, it simplifies to (5.43), and for even $n$, we would have obtained the same $\omega$ using (5.54). Quite generally, gauge transformations tend to spoil the $*-$structure (and algebra structure, as pointed out in [71]) of a trivialization.

To prove that $\omega$ defines a connection, we use Proposition 4.10 in [71]. We have to show

1. $\omega(1) = 0$ and $\omega(M_{A^{(n)}}) = 0$. 

93

94
2. $\gamma_n,\alpha(a) = 1 \otimes \pi_{\alpha}(a - \epsilon(a))$ for all $a \in A$

3. $\Delta_R \circ \omega = (\omega \otimes \mathrm{id}) \circ \text{Ad}_R$

2. holds since for $k > 0$,

$$\omega(Z^k/2) = S((a^k/2) \Theta (a^{k}/2)) \otimes (a^k/2) - 1 \otimes 1$$

$$\omega(Z^{k/2}) = S((a^k/2) \Theta (a^{k}/2)) \otimes Z^{k/2} - 1 \otimes 1$$

and similarly for $k < 0$ as is easily seen from our coaction. For $3.$,

$$\Delta_R \omega(Z^{k/2}) = S((a^{k/2}) \Theta (a^{k/2})) \otimes Z^{-k/2} Z^{k/2}$$

$$= \omega(Z^{k/2}) \otimes 1 = (\omega \otimes \mathrm{id}) \text{Ad}_R(Z^{k/2})$$

(5.41)

and similarly for $k < 0$. As for $1.$, this is clear since $\omega(Z^{-n/2} + q^2Z^{n/2} - (1 + q^2)) = \kappa(1 \otimes \delta^a + q^2\omega^a - (1 + q^2)) \in \kappa(1 \otimes M_{\pi(a)})$.

Thus, $\omega$ is a connection form on the bundle $P_0(a)$ and is given by

$$\omega(Z^{k/2}) = [kn]_a \omega^1,$$

$$\omega(Z^{-k/2}) = -[-kn]_a \omega^1 = -q^{2k} \omega(Z^{k/2})$$

(5.43)

if viewed in $SU_q(2)$, where $[n]_a = \frac{a^n - 1}{a - 1}$. This generalizes the result of (71) for $n = 1$ and 2. Since $(\omega^1)^* = \omega^1$, $\omega$ is a *-map for $q = 1$ only. We have used $\omega^1 = q^{-2} \omega^1$, $\omega^1 = q^{-2} \omega^1$, where $\omega^1 = \delta \alpha - q^{-1} \beta \gamma$, is a left invariant form in $SU_q(2)$.

To our knowledge, (5.29) is also a new description of the classical Dirac monopole.

## 5.3 General Statements on Patching of Trivial QPBs

Let us first show how in general nontrivial QPB's can be obtained by "glueing" together "local" bundles. To avoid repeating ourselves too much, we will give the following statements for the case of a general calculus only; the universal calculus is recovered by putting $M_A = N_A = N_P = 0$. We first observe that the conditions 4. and 7. in definition 5.1.2 are equivalent to the exactness of the sequence (5.24).

Lemma 5.3.1 $P(B, A, N_P, M_A)$ satisfying conditions 1. to 3., 5. and 6. of the definition 5.1.2 is a QPB with general calculus if and only if the sequence (5.24) is exact.

Proof Exactness of (5.24) at $\Gamma_P$ is just the condition 7. above.

Assume first $P$ is a QPB. Then by condition 4. , for any $p \otimes a \in P \otimes \ker \epsilon/M_A$ there exists $p_1 \otimes p_2 \in P \otimes P$ with $\gamma(p_1 \otimes p_2) = p_1 p_2 = p \otimes a$. Applying $\mathrm{id} \otimes \epsilon$ to this equation we get $0 = p_1 p_2 E(p_2) = p_1 p_2$, i.e. $p_1 p_2 \in \Gamma_P$, which shows that $\gamma$ in (5.24) is surjective, so it is exact.

Conversely, suppose (5.24) is exact. Take any $p \otimes a = p \otimes (a - \epsilon(a)) + p \otimes \epsilon(a) \in P \otimes A$. Since $\gamma_n$ is surjective, there exists $p_1 p_2 \in \Gamma_P$ with $\gamma_n(p_1 p_2) = p \otimes (a - \epsilon(a)) + P \otimes M_A$. Now $\gamma(p \otimes \epsilon(a)) = p \otimes \epsilon(a)$ and from 5. $\gamma(N_P) = P \otimes M_A$, so condition 4. is satisfied.

Assume now we have 3 quantum principal bundles

$$P_0(B_0, A, N_0, M_A), P_1(B_1, A, N_1, M_A) \subseteq P_0(B_0, A, N_0, M_A)$$

(5.44)

($P_0$ corresponds to the bundle on the "overlap" $B_0$ of $B_0$ and $B_1$) and we would like to know if $P_0$ and $P_1$ can be understood as two patches of a "global" quantum bundle $P(B, A, N, M_A)$ of $P_0, P_1 \subseteq P_0$. A natural guess is that $P = P_0 \cap P_1$. In this case the coactions $\Delta_R : P_1 \to P_1 \otimes A$ certainly must agree in $P$. If we want a connection on $P$, then we should also have connection forms $\omega_i : A \to \Gamma_{P_i}$ which agree on the overlap, i.e. $\omega_i(a) = \omega_i(a) \in \Gamma_{P_0}$.

However, some care must be taken if we want to compare differential forms on different patches. First of all, the differential structures on $P_i$ must be compatible, i.e. we should have $N_0 = N_0 \cap P_2$, $N_1 = N_0 \cap P_1$ and $N_P \equiv N = N_0 \cap P_2 = N_0 \cap N_1$. But this is not enough: Suppose we have any 2 differential forms -- not necessarily connections -- $\omega_0 \in \Gamma_{P_0}$ and $\omega_1 \in \Gamma_{P_1}$ and find by doing calculations in $\Gamma_{P_0}$ that they are equal. One would certainly like to conclude, as in the classical case, that they determine a "global" form $\omega$ in $\Gamma_P$. This is not evident, it is a condition on the calculus. It motivates the following definition: The above calculi on $P_0, P_1, P_0^1$ are called admissible if

$$\omega_0 = \omega_1 + n_{01} \quad \text{for} \quad \omega_i \in \Gamma_{P_i}$$

(5.45)
implies that there exists a \( \omega \in \Gamma_F \) such that
\[
\omega = \omega_0 + n_0 = \omega_1 + n_1, \quad n_1 \in N_1.
\] (5.46)
In other words, \( \omega_0 = \omega_1 \) determines a \( \omega \in \Gamma_B \cap \Gamma_A = \Gamma_F \), where the intersection is defined as intersection of the cosets. A calculus which does not satisfy this condition would be highly unpracticable for global statements. The universal calculus is certainly admissible since \( (P_0 \otimes P_0) \cap (P_1 \otimes P_1) = P \otimes P \) implies \( \Gamma_B \cap \Gamma_A = \Gamma_F \). The calculus we will consider on the monopole - bundle will be shown to be admissible too, using a fairly general line of reasoning.

**Theorem 5.3.1** In the above situation, \( P = P_0 \cap P_1 = P(B, A, N, M_A) \) is a quantum principal bundle with base \( B = B_0 \cap B_1 \) and connection if we have admissible differential structures which satisfy \( \gamma(N) = P \otimes M_A \), connection forms \( \omega_0 = \omega_1 \) on \( P_0 \) resp. \( P_1 \), and \( \Gamma_{\text{hor}} \cap \Gamma_{\text{hor}} = \Gamma_{\text{hor}} \). Conversely, if \( P = P_0 \cap P_1 \) is a quantum principal bundle, then \( \Gamma_{\text{hor}} \cap \Gamma_{\text{hor}} = \Gamma_{\text{hor}} \).

**Proof** First suppose \( P(E, A) \) is a QPB with universal calculus, \( \gamma(N) = P \otimes M_A \), connection forms \( \omega_0 = \omega_1 \) on \( P_0 \) resp. \( P_1 \), and \( \Gamma_{\text{hor}} \cap \Gamma_{\text{hor}} = \Gamma_{\text{hor}} \). From the above definition of the differential structures condition 5. is satisfied, since \( \Delta_{B_0} : P_1 \rightarrow P_1 \) does not "leave" the bundles.

Assume \( \Gamma_{\text{hor}} \cap \Gamma_{\text{hor}} = \Gamma_{\text{hor}} \). Since \( \omega_0(a) = \omega_1(a) \) and the calculus is admissible, this defines \( \omega(a) \in \Gamma_F \) and \( \sigma(a) = \omega(a) \in \Gamma_F \) for \( (p \otimes a) \in P \otimes \ker c \). From proposition 4.10 in [71] it follows that \( \omega \) is a connection 1-form. Now \( \gamma(p \otimes a) = p \otimes a \) shows that the map \( \gamma \) in (5.24) is surjective. It remains to show ker \( \gamma = \Gamma_{\text{hor}} \). Let \( p_1 dp_2 \in \Gamma_F \). Since \( P_0 \) and \( P_1 \) are quantum bundles, \( \gamma(p_1 dp_2) = 0 \) implies \( p_1 dp_2 \in \Gamma_{\text{hor}} \cap \Gamma_{\text{hor}} = \Gamma_{\text{hor}} \) by assumption. Now Lemma 5.3.1 tells us that \( P(B, A, N, M_A) \) is a quantum principal bundle.

Conversely, assume \( P = P_0 \cap P_1 \) is a quantum principal bundle. Let \( p_1 dp_2 \in \Gamma_{\text{hor}} \cap \Gamma_{\text{hor}} \). Then \( \gamma(p_1 dp_2) = 0 \). Since \( \Gamma_{\text{hor}} \cap \Gamma_{\text{hor}} \subset \Gamma_B \cap \Gamma_A = \Gamma_F \) and \( P \) is a QPB, this implies \( p_1 dp_2 \in \Gamma_{\text{hor}} \). The other inclusion \( \Gamma_{\text{hor}} \cap \Gamma_{\text{hor}} \subset \Gamma_{\text{hor}} \) is trivial. \( \square \)

If there are several "patches" \( P_i \), then the above theorem generalizes inductively in an obvious way. One can show that if \( \gamma(N_P) = P \otimes M_A \), then \( \Gamma_{\text{hor}} \cap \Gamma_{\text{hor}} = \Gamma_{\text{hor}} \) follows from \( \Gamma_{\text{hor}} \cap \Gamma_{\text{hor}} = \Gamma_{\text{hor}} \) (universal calculus). More generally, we have

**Lemma 5.3.2** If \( P(B, A) \) is a QPB with universal calculus and we have \( N_F \) and \( M_A \) satisfying conditions 5. and 6. of definition 5.1.2, then \( P(B, A, N_F, M_A) \) is a QPB with general calculus. Conversely, if \( P(B, A, N_F, M_A) \) is a QPB and \( \gamma(n) = 0 \) for \( n \in \Gamma_F \) implies \( n \in \Gamma_{\text{hor}} \), then \( P(B, A) \) is a QPB with universal calculus.

**Proof** First suppose \( P(B, A) \) is a QPB; we have to show that \( \ker \gamma \subset \Gamma_{\text{hor}} \). Let \( \gamma(n) = 0 \). This means \( \gamma(n) = P \otimes M_A = \gamma(n_F) \) by 6. So there is a \( n \in \Gamma_F \) with \( \gamma(n - n) = 0 \). But \( P \) is a QPB with universal calculus, so it follows \( \gamma \in \Gamma_{\text{hor}} \).

The converse statement can be proved similarly. \( \square \)

It is possible to give a trivialization of our monopole bundle. This will be done for even "charge" only; for odd charge, the trivializations etc. would only be formal. We define two trivial QPBs \( P_0^{(2n)} \) and \( P_1^{(2n)} \), and then show that \( P^{(2n)} = P_0^{(2n)} \cap P_1^{(2n)} \) is the monopole of charge 2n.

### 5.4 Monopole Bundle: Patching

We now present the second construction of the Dirac monopoles for general calculus as an illustration of the general method above. This will be done for even "charge" only; for odd charge, the trivializations etc. would only be formal.

#### 5.4.1 Universal Calculus

We define two trivial QPBs \( P_0^{(2n)} \) and \( P_1^{(2n)} \), and then show that \( P^{(2n)} = P_0^{(2n)} \cap P_1^{(2n)} \) is the monopole of charge 2n.

**Patches**

For \( P_0^{(2n)} \), as motivated by the charge 2 case in [71], we now try to define the base \( B_0 \), fiber \( A^{(2n)} \) and trivial bundle \( P_0^{(2n)} \) be specified by their generators as:
\[
B_0 = \{ 1, b_{-2}, b_2, (b_0 + q^{2m} - 1)^{-1}; \ m \in \mathbb{Z} \},
\] (5.47)
\[
A^{(2n)} = \{ Z^n, Z^{-n} \},
\] (5.48)
\[
P_0^{(2n)} = B_0 \cup \{ (a^{-1} \alpha^n, (\alpha^{-1} \delta)^n) \}.
\] (5.49)


The commutation relations between the generators of $P_0$ are induced by $SU_q(2)$ through the following expressions [67]:

$$b_+ = \alpha \beta, \quad b_+ = \gamma \delta, \quad b_0 = \alpha \delta,$$

(5.50)

where $\alpha, \beta, \gamma, \delta$ are generators of $SU_q(2)$ with the well-known relations stated before. The commutation relations involving inverses are obtained by multiplying them from both sides by inverses of generators. In the classical limit $q = 1$, $B_0$ becomes the algebra of the functions on $S^2 \backslash \{ \text{south pole} \}$, and $b_0 = \pm (x^2 + y^2 + z^2) = (1/2)^2$. The somewhat complicated definition here (see [85]) will become clear below. $P_0^{(2n)}$ as a trivial bundle is generated by the base $B_0$ and the fibers, cp. (5.54).

Define a coaction $\Delta_R$ on $P_0^{(2n)}$ such that $B_0 = (P_0^{(2n)})^* \Delta_R$:

$$\Delta_R(1) = 1 \otimes 1,$$

(5.51)

$$\Delta_R(b_i) = b_i \otimes 1, \quad i = -, +, 3,$$

(5.52)

$$\Delta_R((\delta^{-1} \alpha)^{\pm n}) = (\delta^{-1} \alpha)^{\pm n} \otimes Z^{\pm n}.$$  

(5.53)

The trivialization $\Phi_0$ is defined as

$$\Phi_0(1) = 1, \quad \Phi_0(Z^{\pm n}) = (\delta^{-1} \alpha)^{\pm n},$$

(5.54)

which generalizes the trivialization in [71].

Proposition 5.4.1 $P_0^{(2n)}(B, \Delta_R, \Phi_0)$ is a trivial QPB.

Proof To see that we have a trivial QPB, we first have to show that $B_0$ is the invariant subalgebra of $P_0^{(2n)}$ under the above coaction. This is clear if any $p_0 \in P_0^{(2n)}$ can be written as a sum of terms $B_0(\delta^{-1} \alpha)^m$. Thus we must be able to commute $B_0$ through $\delta^{-1} \alpha$. Writing down the commutation relations explicitly, one can always obtain relations like $aB_0 \alpha^{-1} \in B_0$. Note that for $m \in \mathbb{Z}$, $\delta^{-2m} \alpha^{-m} b_0^{-1} \alpha^{-m} = \delta^{-2m} \delta^{-m} b_0^{-1} \delta^{-m} = (b_0 + q^{2m} - 1)^{-1} \alpha^{-m} = \delta^{-2m} \delta^{-m} b_0^{-1} \delta^{-m} = (b_0 + q^{2m} - 1)^{-1} \alpha^{-m}$ (cp. [85]) and so in general,

$$q^{-4mk}(\alpha^{-1} \delta)^{nk}(b_0 + q^{2m} - 1)^{-1}(\delta^{-1} \alpha)^{nk} = (b_0 + q^{4nk+2m} - 1)^{-1}, \quad k \in \mathbb{Z}.$$  

(5.55)

This shows that $B_0$, as defined in (5.47) is the invariant subalgebra, and one can also see the necessity to include all the generators of $B_0$.

$\Phi_0$ is convolution-invertible with $\Phi_0^{-1}(Z^{\pm n}) = (\delta^{-1} \alpha)^{\pm n}$, and is also an intertwiner: $\Delta_R \circ \Phi_0 = (\Phi_0 \otimes id) \circ \Delta_A$, where $\Delta_A(Z^n) = Z^n \otimes Z^n$ is the coproduct on $A^{(2n)}$. So $P_0^{(2n)}$ is a trivial QPB. □

The discussion on $P_1^{(2n)}$ is parallel to that on $P_0^{(2n)}$, but much easier. Therefore we just give the relevant equations:

$$B_1 = <1, b_-, b_+, b_0, (b_3 - 1)^{-1}>,$$

(5.56)

$$P_1^{(2n)} = <B_1 \cup \{(\gamma \beta^{-1})^n, (\beta \gamma^{-1})^n\}>,$$

(5.57)

$$A^{(2n)} = <\{Z^n, Z^{-n}\}>,$$

(5.58)

$$\Delta_R(b_i) = b_i \otimes 1, \quad i = -, +, 3,$$

(5.59)

$$\Delta_R((\gamma \beta^{-1})^{\pm n}) = (\gamma \beta^{-1})^{\pm n} \otimes Z^{\pm n},$$

(5.60)

$$\Phi_1(Z^{\pm n}) = (-\gamma \beta^{-1})^{\pm n},$$

(5.61)

and $P_1^{(2n)}$ is also a trivial QPB. Note again that $\deg(B_1) = 0$ and $\deg(\Phi_1(Z^n)) = 2n$.

Overlap

The “overlap” $P_0^{(2n)}$ of $P_0^{(2n)}$ and $P_1^{(2n)}$ is similarly defined by

$$B_{01} = <B_0 \cup \{b_3 - 1\}>,$$

(5.62)

$$P_{01}^{(2n)} = <B_0 \cup \{(\gamma \beta^{-1})^{\pm n}, (\beta \gamma^{-1})^{\pm n}\}>$$

(5.63)

and so on as above. On $P_0^{(2n)}$, both trivializations can be used, with the transition function

$$\gamma_{01}(Z^n) = \Phi_0(Z^n) \Phi_1^{-1}(Z^n) = (-q^{2} b_3^{-1} b_0^{-1} (b_3 - 1)^{-1} )^{n} \in B_{01}.$$  

(5.64)

It should be noted that while these trivial bundles are closed under the $*$-operation, the maps $\Phi_i$ respect this $*$-structure only for $q = 1$. This appears to be very hard to avoid in this framework, and we accept it here.

Global Bundle by Patching

Now define the Dirac - monopole bundle with charge $2n$ by

$$P^{(2n)} = P_0^{(2n)} \cap P_1^{(2n)}.$$  

(5.65)
We will now show that for even charges this construction agrees with the one in section 2. First, we need the following useful equivalent representation for $P_0^{(2n)}$.

Lemma 5.4.1

\[ P_0^{(2n)} = \{ p \in SU_q(2) \cup \{(δa)^{-1}, (δd)^{-1}\} : \deg(p) = 2kn, k \in \mathbb{Z} \} \equiv \tilde{P}_0^{(2n)}, \tag{5.66} \]
i.e. the algebra generated by $SU_q(2)$ and $(δa)^{-1}, (δd)^{-1}$, with degrees being multiples of 2n.

Proof To see this, note that $P_0^{(2n)} \subset \tilde{P}_0^{(2n)}$ because $δ^{-1}α = (δd)^{-1}α^2$ etc. and $b_0^1 = (δd)^{-1}$, so $α^{-n}b_0^{-1}α^p \in \tilde{F}_0^{(2n)}$. To see the other inclusion, we first show that $B_0$ is also the invariant subalgebra (under the coaction of $A^{(2n)}$) of $F_0^{(2n)}$: we have just seen $B_0 \subset \tilde{F}_0^{(2n)}$, and the same commutation relations as above show that indeed $B_0 = (F_0^{(2n)})^A$. But this means that $\tilde{F}_0^{(2n)}$ is a QPB with the same trivialization $\Phi_0$ as above. Thus we know (from [71, Example 4.2]) that $\tilde{F}_0^{(2n)} = B_0Φ_0(A^{(2n)}) = P_0^{(2n)}$. \(\square\)

With this, it is easy to see that the $P^{(2n)}$ defined in (5.65) agrees with the one constructed in section 2.

Proposition 5.4.2

\[ P^{(2n)} = \{ p \in SU_q(2) : \deg(p) = 2kn, k \in \mathbb{Z} \}. \tag{5.67} \]

Proof Let $p_0, p_1 \in \tilde{P}_0^{(2n)}$ resp. $F_1^{(2n)} = E$ and $p_0 = p_1$. Note that $β, β^{-1}, γ, γ^{-1}$ can be commuted through any terms by just picking up powers of $q$. Multiplying $α^{-1}δ^{-1}$ to the relation $αδ = δα + (q - 1)(βδ)γ$ appropriately from both sides, one gets relations like $δa^{-1} = α^{-1}δ + (q - 1)(...)$ and $α^{-1}e^{-1} = δ^{-1}α^{-1} + (q - 1)(...)$, i.e. one can order things in any way up to terms proportional to $(q - 1)$.

Let us define a normal form for $p_1$ as follows: bring all $β, γ$ to the right of all $α, δ$ and order $α$ to the left of $δ$, picking up terms proportional to $(q - 1)$. Then replace all terms $αδ$ by $(1 + qβγ)$. Putting $γ$ to the right of $β$, $p_1$ finally has the form either $αδβγ^2 + (q - 1)(...) + δαδγ^2 + (q - 1)(...) = βδγ^2 + (q - 1)(...)$ with $x, y \in \mathbb{Z}, n \in \mathbb{N}$.

Similarly, define a normal form for $p_0$ as follows: bring all $β, γ$ to the right of all $α, δ$, order $β$ to the left of $γ$ and replace all terms $βγ$ by $(αδ - 1)/q$. Now order $α$ to the left of $δ$ picking up terms proportional to $(q - 1)$. $p_0$ finally has the form either $α^2δβγ + (q - 1)(...) + α^2δγ^2 + (q - 1)(...) = α^2δ^2 + (q - 1)(...)$ with $x, y \in \mathbb{Z}, n \in \mathbb{N}$.

Now consider the equation

\[ p_0 = p_1. \tag{5.68} \]

and put terms in $p_1$ which do not contain inverses to the left side, in normal form for $p_0$ (only for monomials which are not proportional to $(q - 1)$, say). Then let $q = 1$ and consider both sides as classical functions on $SU(2)$. All terms proportional to $(q - 1)$ vanish, and all remaining monomials are in normal form on both sides and are easily seen to be independent as functions on $SU(2)$. This implies that all coefficients are actually zero, i.e. all terms on both sides are proportional to $(q - 1)$. (or simply: classical functions defined on both patches are defined globally on $SU(2)$). We can now cancel the greatest common power of $(q - 1)$, put regular terms to the left and apply the same argument. This cannot go on forever since the right side can be ordered completely, so both sides must be zero eventually, proving that $p_0 = p_1 \in SU_q(2)$. Using (5.68), this immediately shows that

\[ P^{(2n)} = \{ p \in SU_q(2) : \deg(p) = 2kn, k \in \mathbb{Z} \}, \tag{5.69} \]
as claimed. \(\square\)

The essence of the proof is to write things in the form ("class") + $(q - 1)$ ("quantum") and to apply classical reasoning to ("class"), which should be a fairly general strategy. Proposition 5.4.2 and (5.66) generalize the result of [71] for $n = 1$.

5.4.2 General Calculus

We can now introduce the same induced 3-D calculus on the bundles as in section 2, i.e. the calculus on the patches $P^{(2n)}_1$ is defined by

\[ N_{P^{(2n)}_1} = P^{(2n)}_1N_{P^{(2n)}_1}P^{(2n)}_1, \tag{5.70} \]

with the same ideals as in section 2. Using $\sim = (id \otimes κ^{-1})$ in a Hopf algebra one can easily see $N_1^{(2)} = P^{(1)} \otimes M_0^{(1)}$, and $N_1^{(2n)} = P^{(2n)} \otimes M_0^{(2n)}$ with a similar argument as in section 2. So $P^{(2n)}_1$ are trivial QPB with this calculus by example (4.11) in [71].
It was already shown in section 2 that
\[
\omega(Z^{kn}) = S((\alpha^{2kn})_{(1)})d(\alpha^{2kn})_{(2)} = \kappa(1 \otimes (\alpha^{2kn} - 1)),
\]
\[
\omega(Z^{-2n}) = S((\delta^{2kn})_{(1)})d(\delta^{2kn})_{(2)} = \kappa(1 \otimes (\delta^{2kn} - 1))
\]
for \(k \in \mathbb{N}\) defines a connection one-form. Any monomials of degree \(2nk\) in \(P^{(2n)}_1\) can be written in the form \(\Phi(Z^{2nk})B\) or \(B_\Phi(Z^{2nk})\), and so one can put \(\omega\) in the standard form of a connection one-form in \(P^{(2n)}_0\) and \(P^{(2n)}_1\):
\[
\omega(a) = \Phi^{-1}(a)\beta_i(a)\Phi_i(a) + \Phi^{-1}(a)d\Phi_i(a); i = 0, 1; a \in A^{(2n)},
\]
where \(\beta_i \in \Gamma_{k\text{hor}}^{(2n)}\) and \(\beta_i(1) = 0\).

Now let us show the following

**Proposition 5.4.3** The calculus on \(P^{(2n)}_0, P^{(2n)}_1, P^{(2n)}_2\) is admissible.

**Proof** The reasoning is as in the previous proposition. Assume we have \(\omega_0, \omega_1 \in \Gamma_{k\text{hor}}^{(2n)}\) resp. \(\Gamma_{k\text{hor}}^{(2n)}\) with \(\omega_0 = \omega_1 \in \Gamma_{k\text{hor}}^{(2n)}\). Since in the 3D - calculus all one - forms on \(SU(2)\) and thus on \(P^{(2n)}_1\) can be written in terms of three left - invariant Maurer - Cartan forms \(\omega^0, \omega^1, \omega^2\) which have simple commutation relations
\[
\omega^0\alpha = q^{-1}\omega^0, \quad \omega^0\beta = q\beta\omega^0,
\]
\[
\omega^1\alpha = q^{-1}\omega^2, \quad \omega^1\beta = q^2\beta\omega^1,
\]
\[
\omega^2\alpha = q^{-1}\omega^2, \quad \omega^2\beta = q\beta\omega^2,
\]
and similarly with the inverses \(\alpha^{-1}\) etc., we can commute the forms to the right and have \(\omega_0 = f_0\omega^k, \quad \omega_1 = g_0\omega^k\) (summation implied), so
\[
f_0\omega^k = g_0\omega^k.
\]
As in proposition 5.4.2 put both \(f_k\) and \(g_k\) in their respective normal form ("class") \(+(q - 1)("\text{quant})\) and bring all regular terms of \(g_k\) to the left side. Then putting \(q = 1\), the "classical" parts are all independent as one - forms since the \(\omega_i\) are and therefore vanish. Cancelling \((q - 1)\) and repeating the argument, it follows that \(\omega_0\) and \(\omega_1\) are elements of \(\Gamma_{k\text{hor}}^{(2n)}\) and in fact in \(\Gamma_{k\text{hor}}^{(2n)}\), since the degree is conserved.

Now we can use theorem 5.3.1 Suppose \(\rho \in \Gamma_{\text{hor}}^{(2n)} \cap \Gamma_{k\text{hor}}^{(2n)}\), so \(\rho \in \Gamma_{k\text{hor}}^{(2n)}\). We can expand it as above
\[
\rho = f_0\omega^0 + f_1\omega^1 + f_2\omega^2,
\]
with \(f_i \in P^{(2n)}\). But \(\omega^0\) and \(\omega^2\) are horizontal (explicitly: \(\omega^0 = \delta^2db_1 + q^2\beta^2db_2 - q^{-1}(1 + q^{-2})\beta db_3\)) and \(\omega^1 = -\gamma db_1 - q^{-2}\alpha^2db_2 + q^{-1}(1 + q^{-2})\alpha^2db_3\), while \(\omega^1\) is not. Therefore \(f_1 = 0\), and \(\rho \in \Gamma_{k\text{hor}}^{(2n)}\), since all coefficients of \(dB\) must have degree 2n. So \(P^{(2n)}_i\) is a QPB with a general differential calculus, with the same connection form \(\omega\) restricted to elements \(a \in A^{(2n)}\).

Finally we would like to mention that since the trivializations are not "real" for \(q \neq 1\), one might just go ahead and use trivializations such as \(\Phi(Z^{1/2}) = \alpha\) which do not respect the \(\ast\)-structure even for \(q = 1\), at least as computational tools. Since we know that the "global" bundle with the \(\ast\)-structure does have the correct classical limit, this may be an acceptable and useful strategy, and deserves further consideration.

### 5.5 Two Remarks

#### 5.5.1 A Note on the Gauge Transformations

A gauge transformation is a convolution invertible map \(\gamma : A \rightarrow B\):
\[
\gamma \ast \gamma^{-1} = \gamma^{-1} \ast \gamma = 1.
\]

Let us define the "primitive charge" of a monomial in \(B\) as \((n_+ - n_-)\), where \(n_\pm\) are the total powers of \(b_\pm\) appearing in the monomial or equivalently (power of \(\alpha\) - power of \(\delta\)). This is preserved by the commutation relations, as our previous degree. Suppose that \(\gamma = \sum_{k=0}^n \gamma^{(k)}\), where each \(\gamma^{(k)}\) contains only monomials that have primitive charge \(k\). Hence \(i\) and \(j\) are the minimum and maximum of the primitive charges of all monomials in \(\gamma\). Let the convolution inverse of \(\gamma\) be denoted in the same way: \(\gamma^{-1} = \sum_{k=0}^n \gamma^{(k)}i\). So
\[
1 = \epsilon(Z^{1/2}) \cdot 1 = \gamma \ast \gamma^{-1} = \sum_{k=0}^n \gamma^{(k)}i
\]
which implies that \(i + j = j + i = 0\). The only possibility that this can be true is that \(i = j = -i\) = \(-j\), which means that all monomials in \(\gamma\) have the same
primitive charge \( n \). However, this means in the classical limit that \( \gamma \) is proportional to \( e^{in\phi} \). That is, by admitting only finite sums in a convolution invertible \( \gamma \) one is restricting oneself to a very special, rigid class of gauge transformations. Thus infinite series cannot be avoided in general.

5.5.2 A Note on the Chern Class

Classically, the monopole charge \( n \) is given by an integration over the base of the first Chern class

\[
\frac{1}{2\pi i} \int F = n,
\]

(5.79)

where \( F = dA_+ = dA_- \). Here \( A_+, A_- \) are the connection form on the northern and southern hemisphere respectively and the global connection form is given in terms of trivializations as

\[
\omega = \begin{cases} A_+ + id\varphi_+, & \text{on } H_+ \\ A_- + id\varphi_-, & \text{on } H_- \end{cases}
\]

(5.80)

with \( e^{i\varphi} \) being the local trivialization.

In the deformed case, we have the global connection form \( \omega \). Suppose it is written in terms of trivialization as \([71]\)

\[
\omega = \phi^{-1}_1 \beta \phi_1 + \phi^{-1}_1 d\phi_i,
\]

(5.81)

then it is not hard to check that

\[
d\omega = \phi^{-1}_1 (d\beta_i + \beta \beta_i) \phi_1 = d\omega + \omega \omega,
\]

(5.82)

which is in fact the curvature 2-form on \( P^{(n)} \) \((86), \text{cp. } [71])\). Carrying the \( \phi_i \) through the \( d\omega \), we get

\[
d\beta_i + \beta \beta_i = q^{2n} d\omega = q^{2n}[n]_0^{-\omega} d\omega^1
\]

(5.83)

which is again equal for the two patches and explicitly horizontal. It leads us to define the deformed Chern class as

\[
\frac{1}{2\pi i} F = \frac{1}{2\pi i} (d\beta_i + \beta \beta_i) = \frac{q^{2n}}{2\pi i}[n]_0^{-\omega} d\omega^1.
\]

(5.84)

Consider the base \( B = S^2 = \langle b_+, b_-, b_3 \rangle \subset SU_q(2) \), with the calculus inherited from the 3-D calculus on \( SU_q(2) \). Denote \( \Gamma_B = BdB \) and introduce the set \( \Gamma''_B \) of 2 forms on \( B \). Notice that \( \Gamma''_B \) contains elements of the form \( Bdb_1db_j, i, j = -, +, 3 \).

Since

\[
\begin{align*}
 db_+ &= \gamma^2 \omega^0 - q^2 \beta^2 \omega^3, \\
 db_- &= \alpha^2 \omega^0 - q^2 \beta^2 \omega^3, \\
 db_3 &= \alpha \gamma \omega^0 - q^2 \beta \omega^3.
\end{align*}
\]

(5.85)

So, \( \Gamma''_B = Bq^2 \omega^3 = Bdw^1 \). Because \( dw^1 \) is a central element in \( B_01 \), under a gauge transformation \( U \in B_01 \) we have \( F \to U^{-1}FU = F \).

Notice that \( \omega \omega \) is manifestly left invariant under the coaction of \( SU_q(2) \), and is the unique top 2-form on \( B \). This allows us to introduce a linear functional

\[
\int_B: \Gamma''_B \to C,
\]

(5.86)

where \( <\cdot, SU_q(2)\rangle \) is the invariant "Haar" measure on \( SU_q(2) \).[12]. This integral is obviously left- and right- invariant under the coaction of \( SO_q(3) \) and unique as such. The normalization is chosen to give the correct classical limit. Classically, \( dw^1 = i/2d\Omega \).

Therefore the deformed monopole charge is obtained as in the classical case

\[
\frac{1}{2\pi i} \int q^{2n} \int [n]_0^{-\omega} d\omega^1 = q^{2n}[n]_0^{-\omega}.
\]

(5.87)

This is actually gauge - invariant in the sense that it does not depend on the trivialization chosen, but this appears to be the case only for our particular connection.
Bibliography


[23] P. Aschieri, P. Schupp, Vector Fields on Quantum Groups, IFUP-TH 15/95, LMU-TPW 94-14, q-alg/9505023.


