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DOE/ER/54241--143

DOE/ER/542-41-143

FRCR #462

**Dimensionless parameters, scaling laws,
and the implications for ETG**

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April 20, 1995

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Contents

1	Introduction	1
2	Dimensionless parameters	3
3	Comparison of different devices	4
3.1	Description of calculations for ranges	4
3.1.1	ETG calculations	5
3.1.2	Spherical Tokamak Reactor	5
3.1.3	ITER	6
3.1.4	START	6
3.2	Results	6
4	Dimensionless parameter scans	8
4.1	ρ_* scan: will ETG determine the difference between Bohm and gyro-Bohm ?	10
4.2	q_ψ scan	13
4.3	ν_* scan	13
4.4	β scan	13
5	Experimental strategy for determining the confinement function	13
6	Energy confinement power laws and dimensionless quantities	14
6.1	General method for converting power laws	16
6.2	Example	18
6.3	Bohm and gyro-Bohm power laws	21
7	Conclusions	24
A	Spreadsheet Formulary	26
A.1	Basic formulas	26
A.2	Constraints	29
B	Confinement scaling laws	31

List of Figures

1	Range of plasma beta, β , for various experimental devices. Note that the most optimistic conditions for ETG yield similar β values to those of a reactor.	7
2	Range of scaled Debye length, λ , for various experimental devices. Note that the ETG range is closest to that of a reactor.	7
3	Collisionality ranges of different experimental devices. The range for ETG more closely approximates that of a reactor than START. This is mainly a function of the higher temperatures possible in ETG.	8
4	Range of scaled ion gyroradius values, ρ_* , attainable in ETG versus other experimental devices. The significant fact is that the wide range available for ETG should help in discriminating between Bohm or gyro-Bohm scaling.	9
5	Electron density, $\bar{n}_{e20}$, for ρ_* scaling sequence as a function of ρ_*	11
6	Plasma current, I_p , (in MA) for ρ_* scaling sequence as a function of ρ_*	11
7	Auxiliary power input, P_{aux} , (in MW) for ρ_* scaling sequence as a function of ρ_*	12
8	Toroidal field, B_ϕ , (in Tesla) for ρ_* scaling sequence as a function of ρ_*	13

List of Tables

1	Exponents of engineering parameters from various empirical power laws and the equivalent exponents of the dimensionless variables β , $\bar{p}_*$, q and $\bar{v}_*$	22
2	Exponents of a , R , and κ for various empirical power laws. . .	23

Abstract

ETG will be useful in resolving several physics issues relevant to Spherical Tokamak Reactor concepts. First, it will provide a test of whether transport is Bohm or gyro-Bohm in nature. The second point is that ETG will operate in a completely different range of ρ_* space from other high performance machines, opening up a previously inaccessible region of parameter space. ETG is also a (very) high- β machine. It would be the only device that would have all of its parameters except ρ_* similar to those of a Spherical Tokamak Reactor. If it turns out that the transport scales definitively as either Bohm or gyro-Bohm, then extrapolation to reactor conditions with significantly lower values of ρ_* would become more credible. It is also shown that in general one cannot obtain a power law relation in the dimensionless variables for the confinement time from a power law fit to the engineering variables. It is shown, however, that if T_i/T_e and n_i/n_e are constant or if a modified definition of certain dimensionless variables is adopted, then such a power law conversion is possible.

1 Introduction

The equilibria (in the time-independent sense, rather than the thermodynamic sense) of an experimental plasma physics device presumably can be described by a set of differential equations involving only a finite set of dimensionless parameters. Since only these dimensionless parameters appear in the equilibrium equations, one is tempted to conclude that any physical quantity associated with the plasma in equilibrium is a function of these parameters only. The catch here is that the function may be multi-valued. This fact arises from the nonlinearity of the governing equations. The implication is that the particular value of a given physical quantity "in equilibrium" will depend upon the initial conditions and/or the past history of source functions, etc. Given this caveat, it still might be useful to analyze experimental data for say, confinement times in terms of the set of dimensionless parameters assumed to be important for the experiment in question.

There are three main sets of dimensionless parameters that could be important in determining the equilibrium value of a given physical quantity. The first set of parameters includes the natural dimensionless quantities that appear explicitly in the homogeneous versions of the governing equations. The number of such dimensionless parameters will generally depend upon the plasma physics model adopted, but there are at least four parameters that may be important: plasma beta, β , scaled electron Debye length, $\lambda = \lambda_D/a$, scaled ion gyroradius, $\rho_* = \rho_i/a$, and the collisionality, ν_* .

The second set of parameters involve the dimensionless quantities at the plasma boundary. These may include (but are not restricted to) such things as the impurity sources, recycling coefficients, and wall conductivity. These terms are sufficient to include all radiation effects.

The third set of dimensionless parameters include the constant source rates for mass, momentum, and energy. These can be cast in terms of a dimensionless gas feed rate, and fractional power contributions from Ohmic, ECRH and beams.

In principle, the dimensionless physical parameter of interest can only be a function of these dimensionless parameters. The exciting implication is that if one believes that the equilibrium solution is a function of N dimensionless parameters, then the scaled confinement time, $\omega_i \tau_E$, can be at most a function of these N parameters. So if the confinement time is known for a particular set of dimensionless parameters, the same relation must hold for a next-

generation device with the same dimensionless parameters. That is, the fact that

$$\omega_i \tau_E = f(x_1, x_2, \dots, x_n) \quad (1)$$

for a particular set of dimensionless values $\vec{x} = (x_1, x_2, \dots, x_n)$ is machine-independent. It turns out that the *form* of the governing equations imposes additional restrictions upon the number of independent variables that a given physical quantity may depend upon. If one insists that the equations be invariant under a scale transformation, then one finds that a finite number of independent scale transformations exist that leave the equations invariant. In this work we will not try to reduce the dimensionality by such techniques.

The differential equations that govern the plasma equilibrium are the Maxwell equations, coupled with a (truncated) set of either kinetic equations or fluid equations plus some sort of equation of state for closure. It seems that one might wish to consider a set of kinetic equations rather than fluid equations simply because the allowable physical content is richer. For elucidation purposes, however, we will use the resistive MHD equations due to their relative simplicity. A typical system of such equations is given by:

$$\nabla \cdot E' = \sigma' / \epsilon_0 \quad (2)$$

$$\nabla \times B' = \mu_0 J' \quad (3)$$

$$\nabla \times E' = 0 \quad (4)$$

$$\nabla \cdot B' = 0 \quad (5)$$

The source-free Resistive MHD equations can be written in dimensionless form as

$$\frac{\partial \rho'}{\partial \tau} + \vec{\nabla} \cdot (\rho' \vec{V}') = 0$$

$$\frac{\partial(\rho' \vec{V}')}{\partial \tau} + \vec{\nabla} \left[\frac{3}{2} \rho_* \left(1 + \frac{B'^2}{2\beta} \right) \right] = \mu^{3/2} \lambda (\vec{\nabla} \cdot E') E' + J' \times B'$$

$$\frac{\partial J'}{\partial \tau} + \mu^{1/2} \rho_* \vec{\nabla} \left[\frac{3}{2} \rho_* \left(1 + \frac{B'^2}{2\beta} \right) \right] = E' + V' \times B' + \left[\frac{4\pi \mu^{1/2} \rho_* \nu_*}{A^{5/2} q_{cyl}} \right] J'.$$

The above equations assume the following scalings:

$$E' = \frac{ae\rho_*\mu^{1/2}}{T_e} E$$

$$J' = \frac{1}{en_e v_{th}} J$$

$$B' = B/B_\phi$$

$$\vec{V}' = \vec{V}/v_{th}$$

$$\mu = m_e/m_i$$

The first four equations are Maxwell's equations, and the last three equations are the equations of conservation for mass, momentum, and charge. Here ρ' is the scaled mass density. Notice that the scaled Debye length, λ , only appears in the term involving $\nabla \cdot E$. That is, if quasineutrality is invoked ($\sigma' = 0$), then the resistive MHD equations have no dependence whatsoever upon λ . Although I have not explicitly included an energy equation, it is also a function of these four parameters. As alluded to earlier, the source terms (which have been omitted) would contain a variety of different dimensionless parameters involving various atomic physics and NB or ECRH characteristics.

2 Dimensionless parameters

If the five dimensionless parameters β , ρ_* , ν_* , q and λ and the dimensionless physical quantity of interest, say $\omega_i \tau_E$, take on a particular set of values in a given experiment, then one should be able to say with confidence that if the four dimensionless quantities are the same in a reactor, then the scaled confinement time should also be the same. For these reasons it may be fruitful to compare the regions of dimensionless parameter space expected to be obtained in ETG with proposed spherical tokamak designs and/or other toroidal fusion experiments. If ETG will cover similar regions of parameter space compared to those of a compact reactor, it should be possible to predict the larger machine's performance based on values obtained from ETG. The

following section shows a comparison of the four parameters β , ρ_* , ν_* , and λ for various devices. The definitions of the dimensionless parameters are as follows (units are MKS unless otherwise specified): The plasma beta is given by

$$\beta = \frac{2\mu_o \int \frac{3}{2} (n_i T_i + n_e T_e) dV}{B_\phi^2 2\pi^2 a^2 \kappa R}. \quad (6)$$

In this document we will interchangeably use the symbols β and $\langle \beta \rangle$ to refer to the toroidal beta. The scaled ion gyroradius is

$$\rho_* = \frac{m_i v_{Ti}(0)}{Z_i e B_\phi a}. \quad (7)$$

The scaled electron Debye length is

$$\lambda = \left(\frac{T_e(0) \epsilon_o}{e^2 a^2 n_e(0)} \right)^{1/2}. \quad (8)$$

The collisionality parameter is

$$\nu_* = \frac{\nu_e(0)}{v_{Te}(0)} R^{5/2} a^{-3/2} q_{cyl}. \quad (9)$$

Here the electron collision frequency is given by

$$\nu_e = \frac{n_e e^4}{2\pi \epsilon_o^2 m_e^2 v_{Te}^3} \log \Lambda \quad (10)$$

and the so-called "cylindrical q " is

$$q_{cyl} = \frac{2.5 B_\phi a^2}{R I_p} (1 + \kappa^2 (1 + 2\delta^2 - 1.2\delta^3)). \quad (11)$$

Here I_p is in MA. We will also be somewhat cavalier about using q , q_ψ , and q_{cyl} , but when the distinction is important we will make a note.

3 Comparison of different devices

3.1 Description of calculations for ranges

The four dimensionless parameters, β , ν_* , ρ_* and λ were calculated for four existing experimental devices (JT-60, TFTR, DIII-D, and START) and three

proposed devices (ETG, a Spherical Tokamak Reactor, and ITER). The dimensionless parameters were calculated as prescribed in the formulas listed previously. For three machines (JT-60, TFTR, DIII-D) the values of the minima and maxima of the dimensionless parameters were obtained from neutral beam heated L-mode discharge data stored in the **Mdi** database [3]. A search was performed over all discharges and then the minimum and maximum values were determined. The comparisons shown are meant to indicate as best as possible the relative positions of the various machines in parameter space. Each of the machines other than ETG has a larger operating space than is shown. However, due to lack of availability of discharge data, it was not possible to determine the extrema of the dimensionless parameters for these machines. The START data represents a single discharge condition. This is simply due to the sparse amount of data available presently from START. The proposed Spherical Tokamak Reactor is also represented by one discharge condition.

3.1.1 ETG calculations

The values given for ETG were determined by fixing these parameters: triangularity, $\delta = 0.3$, elongation, $\kappa = 2.16$, H-factor, $H = 1.6$, profile shape factors, $\alpha_n = 0.7$, $\alpha_T = 1.0$, effective mass, $M_i = 2.0$, minor radius, $a = 0.49\text{ m}$, major radius, $R = 0.7\text{ m}$, toroidal field, $B_\phi = 0.1 - 0.5\text{ T}$, and atomic mass of working gas, $A_i = 2$. Z_{eff} was determined from the impurity ion fractions given in Appendix A. So effectively the dimensionless plasma parameters are completely determined by fixing the plasma current, auxiliary heating power, toroidal field, and electron density. For the results presented below, the plasma current in ETG was assumed to vary between $I_p = 0.3 - 1.0\text{ MA}$. The auxiliary heating power was allowed to vary between $P_{aux} = 0 - 3.5\text{ MW}$. All of the computations necessary for determining the dimensionless parameters were carried out according to the algorithm described in Appendix A.

3.1.2 Spherical Tokamak Reactor

The values given for a Spherical Tokamak Reactor were determined by fixing these parameters: triangularity, $\delta = 0.3$, elongation, $\kappa = 2.3$, H-factor, $H = 2.39$, profile shape factors, $\alpha_n = 0.7$, $\alpha_T = 1.0$, effective charge, $Z = 2.1$, effective mass, $M_i = 2.0$, minor radius, $a = 1.01\text{ m}$, major radius, $R =$

1.42 m, toroidal field, $B_\phi = 3.6 T$, dilution factor, $d = 0.7$, and atomic mass of working gas, $A_i = 2$. The plasma current was assumed to be $I_p = 29 MA$, and the auxiliary heating power set to $P_{aux} = 240 MW$.

3.1.3 ITER

The values given for ITER were determined by fixing these parameters: triangularity, $\delta = 0.4$, elongation, $\kappa = 2.0$, H-factor, $H = 1.7$, profile shape factors, $\alpha_n = 0.5$, $\alpha_T = 1.0$, effective charge, $Z = 1.5$, effective mass, $M_i = 2.0$, minor radius, $a = 2.0 m$, major radius, $R = 5.7 m$, toroidal field, $B_\phi = 5.2 T$, dilution factor, $d = 0.7$, and atomic mass of working gas, $A_i = 2$. The plasma current was assumed to be $I_p = 20 MA$, and the auxiliary heating power set to $P_{aux} = 300 MW$.

3.1.4 START

The values given for START were determined by fixing these parameters: triangularity, $\delta = 0.3$, elongation, $\kappa = 1.6$, H-factor, $H = 1.6$, profile shape factors, $\alpha_n = 0.7$, $\alpha_T = 1.0$, effective charge, $Z = 2.1$, effective mass, $M_i = 2.0$, minor radius, $a = 0.15 m$, major radius, $R = 0.2 m$, toroidal field, $B_\phi = 0.51 T$, dilution factor, $d = 0.7$, and atomic mass of working gas, $A_i = 2$. The plasma current was assumed to be $I_p = 100 kA$, and the auxiliary heating power set to $P_{aux} = 0 MW$.

3.2 Results

The plasma beta, β , is an important indicator of any device's performance (the higher the better). The simulation for ETG shows that the attainable plasma beta ($I_p = 1.0 MA$) is approximately $\beta \simeq 0.31$. These values are well into the operating range necessary for reactor physics (see Figure 1).

The ranges of allowable scaled Debye length, λ , are depicted in Figure 2. This parameter will only be important if the transport processes are not well described by a plasma model based on quasineutrality. This seems an unlikely state of affairs [5], so the fact that ETG may not encompass the same range of Debye lengths as a reactor is probably not important. On the other hand, the values for ETG are closer to those of a reactor than are the values for the other machines shown.

Figure 1: Range of plasma beta, β , for various experimental devices. Note that the most optimistic conditions for ETG yield similar β values to those of a reactor.

Figure 2: Range of scaled Debye length, λ , for various experimental devices. Note that the ETG range is closest to that of a reactor.

Figure 3: Collisionality ranges of different experimental devices. The range for ETG more closely approximates that of a reactor than START. This is mainly a function of the higher temperatures possible in ETG.

Figure 3 displays the ranges of collisionality, ν_* , attainable by the various devices. Although ETG is more collisional than every machine except START (due to the relatively cool temperatures primarily), it would overlap a reactor for lower density values, since the collisionality is linear in electron density.

The most serious departure of ETG parameters from reactor parameters occurs for the scaled ion gyroradius, ρ_* . The ρ_* values for ETG are significantly higher than those for a proposed reactor, owing mostly to the relatively small toroidal field in ETG ($B_\phi = 0.3 T$) as compared to that of a reactor ($B_\phi = 3.6 T$). However, this large difference allows ETG to “fill in” data points in an unexplored region of parameter space. The large variation in ρ_* should give ETG the chance to answer the question: “Does confinement scale like Bohm or gyro-Bohm?”.

4 Dimensionless parameter scans

The first question to be answered is: “Is it possible to perform a scan over dimensionless variables such that three are held fixed (say ν_* , q_ψ and β) while the fourth, ρ_* , is varied?”. One way of answering this question is by writing

Figure 4: Range of scaled ion gyroradius values, ρ_* , attainable in ETG versus other experimental devices. The significant fact is that the wide range available for ETG should help in discriminating between Bohm or gyro-Bohm scaling.

each of the three dimensionless variables in terms of I_p , P_{aux} , B_ϕ , and $\bar{n}_{e20}$. Once this is done, one constructs the function

$$f(I_p, B_\phi, \bar{n}_{e20}, P_{aux}) = (\nu_* - \nu_{*o}, q_\psi - q_{\psi o}, \beta - \beta_o) \quad (12)$$

for an arbitrary (but attainable) set of values ν_{*o} , $q_{\psi o}$, and β_o . One approach is to fix one of the engineering variables, say B_ϕ . Next, one then searches for a triplet (I_p, n_e, P_{aux}) such that $f(I_p, B_\phi, n_e, P_{aux}) = 0$. Then one steps to the next value of B_ϕ and so on. These series of points can then be used to determine the resulting ρ_* values. Figures 5, 6, 7 and 8 show the electron density, plasma current, auxiliary power values and toroidal field required as a function of ρ_* . These four plots were made with the assumptions that $\nu_{*o} = 0.01$, $q_\psi = 5$, and $\beta_o = 0.1$. Also, the various constraints mentioned in Appendix A were enforced. The most tenuous assumption in the model is that the confinement time was considered to be Lackner-Gottardi in form. In this sense, the operational points generated here should be regarded as a first order attempt to generate a sequence of discharges that are identical except for ρ_* . Of course, if the scaling is different from Lackner-Gottardi, these points will be slightly off. However, one way to counteract this imperfection

suggests itself. If one is not too far wrong from the desired operational parameters, then one might try to measure the following gradients at each scan point: $\nabla\beta$, ∇q , $\nabla\nu_*$. Here we mean that

$$\nabla = \left(\frac{\partial}{\partial I_p}, \frac{\partial}{\partial P_{aux}}, \frac{\partial}{\partial n_e} \right) \quad (13)$$

Then one would try to find the "right" discharge condition by the taking the jump:

$$(\delta I_p, \delta n_e, \delta P_{aux}) = (\nabla f^{-1}) \cdot f. \quad (14)$$

When applied iteratively, this process is analogous to the Newton-Raphson method for finding the root of an equation.

4.1 ρ_* scan: will ETG determine the difference between Bohm and gyro-Bohm ?

ETG will be useful in sorting out whether transport is Bohm-like, gyro-Bohm-like, or neither only if a ρ_* scan provides sufficient dynamic range to discern between the two possibilities. It turns out that if energy confinement is Bohm-like, then the confinement time should scale as ρ_*^{-2} , and if it is gyro-Bohm-like, it should scale as ρ_*^{-3} . Figure 5 shows that in ETG we can expect to cover about a factor of two in ρ_* . This leads to variations in the confinement time by factors of 4 and 8 for Bohm and gyro-Bohm, respectively. This is well above the experimental uncertainties in τ_E .

One caveat is worth mentioning here. In DIII-D, another high- β machine, it was found that the electron and ion energy loss channels behaved differently. If the ion channel dominated the power balance (due to strong NBI heating), the transport was Goldston-like; if the electron channel dominated (strong RF heating), the transport was gyro-Bohm in nature [19]. The main lesson to learn from this is that a simple power law interpretation may be impossible if the electron and ion channels contribute roughly equally to the transport (in fact even if one dominates there is no guarantee).

Based on these ideas, one concludes that ETG will indeed be able to contribute meaningfully to the Bohm versus gyro-Bohm debate. It will be able to achieve this by exploring a new region of ρ_* space. It should be possible to create a sequence of discharges with a factor of two variation in the value of ρ_* , which provides an ample dynamic range for distinguishing

Figure 5: Electron density, $\bar{n}_{e20}$, for ρ_* scaling sequence as a function of ρ_* .

Figure 6: Plasma current, I_p , (in MA) for ρ_* scaling sequence as a function of ρ_* .

Figure 7: Auxiliary power input, P_{aux} , (in MW) for ρ_* scaling sequence as a function of ρ_* .

Figure 8: Toroidal field, B_ϕ , (in Tesla) for ρ_* scaling sequence as a function of ρ_* .

between Bohm or gyro-Bohm confinement scaling. If it becomes apparent that either scaling is manifestly true, then the confinement at much lower ρ_* (but the same ν_* , q and β) can be predicted with greater confidence. In this way ETG could provide valuable insight into the predicted confinement of a Spherical Tokamak Reactor.

4.2 q_ψ scan

4.3 ν_* scan

4.4 β scan

5 Experimental strategy for determining the confinement function

If one assumes that $\omega_i \tau_E = f(\beta, \nu_*, \rho_*, q)$ for a fixed set of geometric parameters (A, κ, δ) , then one might wish to vary the independent parameters I_p , n_e ,

P_{aux} , and B_ϕ to try to map out the function $\omega_i \tau_E$ as completely as possible. If each of these parameters were allowed to vary on a mesh of size N , and M measurements were taken at each mesh point to reduce experimental noise, then a total of $\sim (NM)^4$ total measurements are required to map out the space as completely as possible. For $M = N = 10$, the required amount of data becomes extremely large. Perhaps scale invariance techniques would be useful in reducing the dimensionality of the operating space that must be explored.

The first step that should be undertaken is to try to attain identical values for the dimensionless parameters β , ν_* , ρ_* , and q on several different devices. The resultant values of the scaled confinement, $\omega_i \tau_E$, should then be compared to see if this technique has any chance of working. If the $\omega_i \tau_E$ values are significantly different, then one is forced to conclude that there are other important dimensionless quantities that have been left out of the model. If the values are close, then perhaps one can apply the technique of varying a single dimensionless parameter at a time with success.

6 Energy confinement power laws and dimensionless quantities

It is a common practice to perform numerical fits of the energy confinement time to power law forms involving the independent engineering quantities I_p , B_ϕ , n_e and P . Typically, one tries to fit functions of the form

$$\omega \tau_E = C_o I_p^{\alpha_1} B_\phi^{\alpha_2} n_e^{\alpha_3} P^{\alpha_4}, \quad (15)$$

where C_o , α_1 , α_2 , α_3 , and α_4 are regarded as fit parameters. Unfortunately, fits of this form are somewhat less than illuminating from a physics standpoint since they are functions of engineering variables rather than physics variables. It is difficult to look at a power law of this form and determine what the driving mechanism is for energy transport. It is also true that the power law form itself is arbitrary, rather than physically motivated. This having been said, one might still wish to ask the question, "If I have a power law fit to engineering parameter, can I express the same scaling law as a power law in the fundamental dimensionless quantities β , q , ρ_* , and ν_* ?" In other words, given a fit to the engineering parameters mentioned above,

one would like to uniquely express the scaled confinement as a power law of the form

$$\omega\tau_E = C'_0 \beta^{\gamma_1} \rho_*^{\gamma_2} q^{\gamma_3} \nu_*^{\gamma_4}. \quad (16)$$

The exponents γ_1 , γ_2 , γ_3 and γ_4 are determined by requiring that the confinement time scale with the engineering variables in the same way as in the engineering fit. The strategy is to try to express each of the dimensionless variables individually as power laws in the engineering variables, plus perhaps various powers of $\omega\tau_E$. We then have an expression of the form

$$\omega\tau_E = g(\beta, \rho_*, q, \nu_*, \omega\tau_E) \quad (17)$$

which is an implicit equation for $\omega\tau_E$. We then explicitly solve for $\omega\tau_E$ in this equation. At this point we equate the two expressions for $\omega\tau_E$; the resultant equation relates γ_1 , γ_2 , γ_3 and γ_4 to α_1 , α_2 , α_3 , and α_4 . There are, however, a few difficulties with this approach.

The main difficulty arises from the fact that the collisionality, ν_* , is a function of the electron temperature, and the scaled ion gyroradius, ρ_* , depends upon the ion temperature. This would be alright if each of these quantities could be expressed as simple power laws involving only the engineering parameters and $\omega\tau_E$. However, the (density weighted, volume averaged) ion temperature is given by the expression

$$T_i = \frac{P\tau_E}{\frac{3}{2}n_e \left(\frac{n_i}{n_e} + \frac{T_e}{T_i} \right)} \quad (18)$$

This expression will constitute a power law only if T_i/T_e and n_i/n_e are constant. So for conditions at high density we should have $T_e \rightarrow T_i$, and therefore perhaps we can simply write

$$T_i \sim T_e \sim P\tau_E n_e^{-1}. \quad (19)$$

One heuristic approach to this problem is to set $T_i = T_e = (P\tau_E)/n_e$. Only ν_* and ρ_* will be affected by this redefinition. To indicate that these slightly different definitions are being used, we will use the symbols $\bar{\nu}_*$ and $\bar{\rho}_*$ instead of ν_* and ρ_* . If T_i/T_e is in fact a constant, then $\bar{\nu}_*$ and $\bar{\rho}_*$ can be replaced by ν_* and ρ_* , respectively in the confinement scaling laws. The definition for β is

$$\beta = \frac{\mu_o P\tau_E}{\pi^2 R a^2 \kappa B_\phi^2} \sim P\tau_E B_\phi^{-2}. \quad (20)$$

Adopting the new definition for $\bar{\rho}_*$, we find that

$$\bar{\rho}_* = \frac{M_i v_T}{Z_i e B_\phi} \sim P^{1/2} \tau_E^{1/2} B_\phi^{-1} n_e^{-1/2}. \quad (21)$$

The definition for q yields the following relation:

$$q = \frac{2\pi \kappa a B_\phi}{\mu_o A I_p} \sim B_\phi I_p^{-1}. \quad (22)$$

The modified collisionality is

$$\bar{\nu}_* = \frac{A^{3/2} R q \nu_*(v_T)}{v_T} \sim n_e q T e^{-2} \sim B_\phi I_p^{-1} n_e^3 \tau_E^{-2} P^{-2}. \quad (23)$$

Näively one might wish to make the equivalence

$$\omega \tau_E = C_o I_p^{\alpha_1} B_\phi^{\alpha_2} n_e^{\alpha_3} P^{\alpha_4} = C'_o \beta^n \bar{\rho}_*^{\gamma_1} q^{\gamma_2} \bar{\nu}_*^{\gamma_3}, \quad (24)$$

however one must first explicitly solve for $\omega \tau_E$ as a function of the engineering variables in the scaling law for the dimensionless variables. Once this is done, the powers of the N engineering variables are equated and then the N resultant equations can be solved for the M dimensionless variable exponents.

6.1 General method for converting power laws

Suppose one has a power law fit for the energy confinement time, $\omega \tau_E$, as a function of the engineering variables, $x_1, x_2, \dots, x_N$:

$$\omega \tau_E \sim \prod_{i=1}^N x_i^{\alpha_i}. \quad (25)$$

One wishes to express this power law in terms of another set of dimensionless variables, y_i :

$$\omega \tau_E \sim \prod_{i=1}^N y_i^{\gamma_i}. \quad (26)$$

Each of the dimensionless variables is in general a function of the engineering variables and the scaled confinement time:

$$y_i \sim (\omega \tau_E)^{\delta_i} \prod_{j=1}^N x_j^{\epsilon_{ji}}. \quad (27)$$

The power law involving the dimensionless variables can be written in the form:

$$\omega\tau_E \sim \prod_{i=1}^N \left((\omega\tau_E)^{\delta_i} \prod_{j=1}^N x_j^{\epsilon_{ji}} \right)^{\gamma_i}. \quad (28)$$

To obtain an explicit relation for $\omega\tau_E$ we factor out the $\omega\tau_E$ dependence:

$$\omega\tau_E \sim (\omega\tau_E)^{\sum_{i=1}^N \delta_i \gamma_i} \prod_{i=1}^N \prod_{j=1}^N x_j^{\epsilon_{ji} \gamma_i}. \quad (29)$$

This expression inverts to the explicit relation

$$\omega\tau_E \sim \prod_{i=1}^N \left(\prod_{j=1}^N x_j^{\epsilon_{ji} \gamma_i} \right)^{1/(1 - \sum_{i=1}^N \gamma_i \delta_i)}. \quad (30)$$

Finally, comparison with the original power law in the engineering variables shows that the exponent of x_j is given by

$$\sum_{i=1}^N \frac{\epsilon_{ji} \gamma_i}{1 - \sum_{i=1}^N \delta_i \gamma_i} = \alpha_j. \quad (31)$$

The equations for the exponents constitute a system of N equations. The j^{th} equation is

$$\sum_{i=1}^N (\epsilon_{ji} + \alpha_j \delta_i) \gamma_i = \alpha_j. \quad (32)$$

This system of equations may be expressed in a compact matrix form:

$$(\epsilon + \vec{\alpha} \times \vec{\delta}^T) \cdot \vec{\gamma} = \vec{\alpha}. \quad (33)$$

The vectors $\vec{\alpha}$ and $\vec{\delta}^T$ are given by

$$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \quad (34)$$

and

$$\vec{\delta}^T = (\delta_1, \delta_2, \dots, \delta_n) \quad (35)$$

Here each element of the matrix ϵ is given by

$$\epsilon_{ij} = \frac{x_i}{y_j} \frac{\partial y_j}{\partial x_i}. \quad (36)$$

The inverse process is also easy to effect. Given a power law in the dimensionless variables ($\vec{\gamma}$ is specified), one can show that the exponents to the equivalent engineering fit are:

$$\vec{\alpha} = \left(\frac{1}{1 - \delta \cdot \gamma} \right) \epsilon \cdot \vec{\gamma} \quad (37)$$

Note that all that is required for a solution to exist is that $\delta \cdot \gamma \neq 1$. This simply means that if the scaling law is equivalent to $\omega\tau \sim \omega\tau$, it tells you nothing !

Note that if the y_j were independent of the confinement time, then the requirement that the system be uniquely invertible is simply $|\epsilon| \neq 0$. That is, the solubility of the system depends only upon the choice of variables. However, if any of the y_j involve powers of $\omega\tau_E$, then the invertibility depends both upon the choice of variables and the fit coefficients. Also, it is allowable for the number of dimensionless variables, M , to be less than or equal to the number of engineering variables. However, if $M < N$, one must have

$$R(\epsilon + \vec{\alpha} \times \vec{\delta}^T) = M \quad (38)$$

where the operation $R()$ denotes taking the rank of a matrix.

6.2 Example

As an example, consider the following choice of engineering variables: $x_1 = I_p$, $x_2 = B_\phi$, $x_3 = n_e$, and $x_4 = P$. If we choose $y_1 = \beta$, then we find

$$y_1 = \beta \sim P\tau_E B_\phi^{-2} \sim (\omega\tau_E) x_2^{-3} x_4.$$

Therefore we find

$$\delta_1 = 1, \epsilon_{11} = 0, \epsilon_{21} = -3, \epsilon_{31} = 0, \epsilon_{41} = 1.$$

With $y_2 = \rho_*$ one can write

$$y_2 \sim (\omega\tau_E)^{1/2} P^{1/2} B_\phi^{-3/2} n_e^{-1/2} \sim (\omega\tau_E)^{1/2} x_2^{-3/2} x_3^{-1/2} x_4^{1/2}.$$

Therefore

$$\delta_2 = \frac{1}{2}, \epsilon_{12} = 0, \epsilon_{22} = -\frac{3}{2}, \epsilon_{32} = -\frac{1}{2}, \epsilon_{42} = \frac{1}{2}.$$

Choosing $y_3 = q$ leads to the relation

$$y_3 = q \sim B_\phi I_p^{-1} \sim x_1^{-1} x_2.$$

The associated coefficients for y_3 are

$$\delta_3 = 0, \epsilon_{13} = -1, \epsilon_{23} = 1, \epsilon_{33} = 0, \epsilon_{43} = 0.$$

Setting $y_4 = \bar{\nu}_*$, we find

$$y_4 = \bar{\nu}_* \sim (\omega\tau_E)^{-2} I_p^{-1} n_e^3 P^{-2} B_\phi^3 \sim (\omega\tau_E)^{-2} x_1^{-1} x_2^3 x_3^3 x_4^{-2}.$$

The associated coefficients are

$$\delta_4 = -2, \epsilon_{14} = -1, \epsilon_{24} = 3, \epsilon_{34} = 3, \epsilon_{44} = -2.$$

For the particular set of engineering variables and dimensionless variables chosen, the matrices ϵ and $\bar{\delta}^T$ are given by

$$\epsilon = \begin{bmatrix} 0 & 0 & -1 & -1 \\ -3 & -\frac{3}{2} & 1 & 3 \\ 0 & -\frac{1}{2} & 0 & 3 \\ 1 & \frac{1}{2} & 0 & -2 \end{bmatrix} \quad (39)$$

and

$$\bar{\delta}^T = \begin{pmatrix} 1 & \frac{1}{2} & 0 & -2 \end{pmatrix}. \quad (40)$$

As a concrete example, consider the Lackner-Gottardi scaling for confinement time. One has

$$\omega\tau_E \sim I_p^{2/5} B_\phi^{7/5} n_e^{3/5} P^{-3/5}. \quad (41)$$

This scaling law yields the relation

$$\omega\tau_E = \beta^0 \bar{\rho}_*^{-3} q^{-1} \bar{\nu}_*^0. \quad (42)$$

For constant geometric and atomic parameters, Lackner-Gottardi scaling predicts a strong dependence on $\bar{\rho}_*$ and q only.

For the particular set of engineering variables (I_p , B_ϕ , n_e , and P) and dimensionless variables (β , $\bar{\rho}_*$, q and $\bar{v}_*$) chosen one can show that the dimensionless variable exponents are given by:

$$\gamma_1 = \frac{\alpha_1 + \alpha_2 + 4\alpha_3 + 7\alpha_4}{4(1 + \alpha_4)} \quad (43)$$

$$\gamma_2 = \frac{-(3\alpha_1 + 3\alpha_2 + 4\alpha_3 + 9\alpha_4)}{2(1 + \alpha_4)} \quad (44)$$

$$\gamma_3 = \frac{-3\alpha_1 + \alpha_2 + 3\alpha_4}{4(1 + \alpha_4)} \quad (45)$$

$$\gamma_4 = \frac{-(\alpha_1 + \alpha_2 + 3\alpha_4)}{4(1 + \alpha_4)} \quad (46)$$

Notice that all that is necessary for these equations to have a solution is that the confinement time *not* be inversely proportional to the input power! There is no particular significance to this fact; it merely is a consequence of the choice of dimensionless variables.

It can be shown that if the factors

$$\frac{1}{\frac{3}{2} \left(1 + \frac{n_i T_i}{n_e T_e} \right)} \sim I_p^{\theta_1} B_\phi^{\theta_2} n_e^{\theta_3} P^{\theta_4} \quad (47)$$

and

$$\frac{1}{\frac{3}{2} \left(\frac{n_i}{n_e} + \frac{T_e}{T_i} \right)} \sim I_p^{\xi_1} B_\phi^{\xi_2} n_e^{\xi_3} P^{\xi_4} \quad (48)$$

then the solution is given by the series of equations:

$$\begin{aligned} \gamma_1 \chi = & \alpha_1(-2 + 3\theta_3 + 3\theta_4 - 4\xi_3 - 3\theta_4\xi_3 - 6\xi_4 + 3\theta_3\xi_4) + \\ & \alpha_2(-2 + 3\theta_3 + 3\theta_4 - 4\xi_3 - 3\theta_4\xi_3 - 6\xi_4 + 3\theta_3\xi_4) + \\ & \alpha_3(-8 - 3\theta_1 - 3\theta_2 - 9\theta_4 + 4\xi_1 + 3\theta_4\xi_1 + \\ & 4\xi_2 + 3\theta_4\xi_2 + 4\xi_4 - 3\theta_1\xi_4 - 3\theta_2\xi_4) + \\ & \alpha_4(-14 - 3\theta_1 - 3\theta_2 + 9\theta_3 + 6\xi_1 - 3\theta_3\xi_1 + \\ & 6\xi_2 - 3\theta_3\xi_2 - 4\xi_3 + 3\theta_1\xi_3 + 3\theta_2\xi_3) \end{aligned}$$

$$\gamma_2 \chi = \alpha_1(12 - 6\theta_3) + \alpha_2(12 - 6\theta_3) + \alpha_3(16 + 6\theta_1 + 6\theta_2 + 18\theta_4) + \alpha_4(36 - 18\theta_3)$$

$$\begin{aligned}
\gamma_3 \chi = & \alpha_1(-6 - 3\theta_2 - 9\theta_4 + 6\xi_2 - 3\theta_3\xi_2 + 6\xi_3 + 3\theta_2\xi_3 \\
& + 9\theta_4\xi_3 + 18\xi_4 - 9\theta_3\xi_4) + \\
& \alpha_2(2 + 3\theta_1 - 6\xi_1 + 3\theta_3\xi_1 - 2\xi_3 - 3\theta_1\xi_3) + \\
& \alpha_3(-6\xi_1 - 3\theta_2\xi_1 - 9\theta_4\xi_1 + 2\xi_2 + 3\theta_1\xi_2 + 6\xi_4 + 9\theta_1\xi_4) + \\
& \alpha_4(6 + 9\theta_1 - 18\xi_1 + 9\theta_3\xi_1 - 6\xi_3 - 9\theta_1\xi_3) + \\
\gamma_4 \chi = & \alpha_1(2 - 2\xi_3) + \alpha_2(2 - 2\xi_3) + \alpha_3(2\xi_1 + 2\xi_2 + 6\xi_4) + \alpha_4(6 - 6\xi_3)
\end{aligned}$$

and the denominator χ is given by

$$\begin{aligned}
\chi = & \alpha_1(3\theta_4 - 3\theta_4\xi_3 - 6\xi_4 + 3\theta_3\xi_4) + \\
& \alpha_2(3\theta_4 - 3\theta_4\xi_3 - 6\xi_4 + 3\theta_3\xi_4) + \\
& \alpha_3(3\theta_4\xi_1 + 3\theta_4\xi_2 - 8\xi_4 - 3\theta_1\xi_4 - 3\theta_2\xi_4) + \\
& \alpha_4(-8 - 3\theta_1 - 3\theta_2 + 6\xi_1 - 3\theta_3\xi_1 + 6\xi_2 - 3\theta_3\xi_2 + 8\xi_3 + 3\theta_1\xi_3 + 3\theta_2\xi_3) + \\
& -8 - 3\theta_1 - 3\theta_2 - 9\theta_4 + 6\xi_1 - 3\theta_3\xi_1 + 6\xi_2 - 3\theta_3\xi_2 + \\
& 8\xi_3 + 3\theta_1\xi_3 + 3\theta_2\xi_3 + 9\theta_4\xi_3 + 18\xi_4 - 9\theta_3\xi_4
\end{aligned}$$

For the purposes of comparison we have compiled a table of empirical scaling laws, their associated engineering variable exponents, and the equivalent exponents of β , $\bar{p}_*$, q and $\bar{v}_*$ (see Table 1).

Here we have recast all of the scaling laws in terms of $q \sim a^2 B_\phi / (RI_p)$ rather than q_{cyl} so that the dependence upon κ would be more explicit. That is, we set $q_{cyl} \sim \kappa^2 q$ everywhere. The exponents of a , R , and κ for the various power laws are shown in Table 2.

6.3 Bohm and gyro-Bohm power laws

If one restricts the physics of confinement to be either Bohm or gyro-Bohm in nature, then the form of any candidate power law for confinement is highly constrained. For instance, consider the Bohm diffusion coefficient:

$$D^B \sim \rho_*^2 v_T \quad (49)$$

If one can consider the ratios n_e/n_i and T_e/T_i to be constant, then one can show that

$$v_T \sim \beta^{1/2} B_\phi n_e^{-1/2}. \quad (50)$$

Since the scaled confinement is roughly (assuming $\chi \sim D$)

$$\omega\tau \sim B_\phi a^2 / D \quad (51)$$

Scaling Law	I_p	B_ϕ	n_e	P_{aux}	β	$\bar{p}_*$	q	$\bar{\nu}_*$
Lackner-Gottardi	$\frac{2}{5}$	$\frac{7}{5}$	$\frac{3}{5}$	$-\frac{3}{5}$	0	-3	-1	0
DTEM	1	1	$\frac{3}{5}$	$-\frac{3}{5}$	$\frac{1}{8}$	$-\frac{15}{4}$	$-\frac{19}{8}$	$-\frac{1}{8}$
Goldston (L-mode)	1	1	0	$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{3}{2}$	$-\frac{7}{4}$	$-\frac{1}{4}$
Goldston Quad (Ohmic)	$-\frac{1}{2}$	$\frac{3}{2}$	1	0	$\frac{5}{4}$	$-\frac{7}{2}$	$\frac{3}{4}$	$-\frac{1}{4}$
Goldston Quad (Auxiliary)	1	1	1	0	$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{3}{2}$	$-\frac{7}{4}$
DIID-D/JET (Nevins)	$-\frac{1}{2}$	$\frac{3}{2}$	1	0	$\frac{5}{4}$	$-\frac{7}{2}$	$\frac{3}{4}$	$-\frac{1}{4}$
Kaye-Goldston (L-mode)	$\frac{5}{4}$	1	$\frac{1}{4}$	$-\frac{3}{5}$	$\frac{5}{4}$	$-\frac{7}{2}$	$\frac{3}{4}$	$-\frac{1}{4}$
Kaye-All Complex	$\frac{17}{20}$	$\frac{13}{10}$	0	$-\frac{1}{2}$	$-\frac{27}{40}$	$-\frac{39}{20}$	$-\frac{11}{8}$	$-\frac{13}{40}$
ITER89-P	$\frac{17}{20}$	$\frac{6}{5}$	0	$-\frac{1}{2}$	$-\frac{29}{40}$	$-\frac{33}{20}$	$-\frac{57}{40}$	$-\frac{11}{40}$
Neo-Alcator	-1	2	1	0	$\frac{5}{4}$	$-\frac{7}{2}$	$\frac{5}{4}$	$-\frac{1}{4}$

Table 1: Exponents of engineering parameters from various empirical power laws and the equivalent exponents of the dimensionless variables β , $\bar{p}_*$, q and $\bar{\nu}_*$.

Scaling Law	a	R	κ
Lackner-Gottardi	2	$\frac{5}{2}$	$\frac{5}{2}$
DTEM	$\frac{37}{16}$	$\frac{35}{16}$	$\frac{1}{4}$
Goldston (L-mode)	$\frac{57}{40}$	$\frac{11}{8}$	$-\frac{1}{2}$
Goldston Quad (Ohmic)	$-\frac{3}{8}$	$\frac{27}{8}$	$\frac{5}{2}$
Goldston Quad (Auxiliary)	$-\frac{7}{40}$	$\frac{11}{8}$	$\frac{1}{2}$
DIII-D/JET (Nevins)	$\frac{5}{8}$	$\frac{7}{8}$	$-\frac{1}{2}$
Kaye-Goldston (L-mode)	$\frac{189}{64}$	$\frac{71}{64}$	$\frac{7}{16}$
Kaye-All Complex	$\frac{17}{80}$	$\frac{27}{80}$	$\frac{3}{20}$
ITER89-P	$\frac{33}{80}$	$\frac{53}{80}$	$\frac{11}{20}$
Neo-Alcator	$-\frac{3}{8}$	$\frac{31}{8}$	$\frac{3}{2}$

Table 2: Exponents of a , R , and κ for various empirical power laws.

one can easily show that for Bohm diffusion one gets

$$\omega\tau_E^B \sim \beta^{-1/2} \rho_*^{-2} n_e^{1/2}. \quad (52)$$

This equation forces several relationships between the engineering variable exponents. Using the techniques of the previous section, one finds that

$$\omega\tau_E^B \sim I_p^{\frac{1-8\gamma_3}{26}} B_\phi^{\frac{57+8\gamma_3}{26}} n_e^{\frac{1}{2}} P^{-\frac{9}{13}} \quad (53)$$

Here γ_3 is the exponent of q in the dimensionless variable power law (also the exponent of the collisionality, ν_* , is $\gamma_4 = -1/8$). Therefore, if one wishes to test whether data from a particular machine is Bohm-like, then all one needs to do is to vary γ_3 to find the best fit to the data. If the fit is a good one, the data is Bohm-like. A similar analysis for gyro-Bohm diffusion yields the scaling law

$$\omega\tau_E^{g-B} \sim I_p^{\frac{3-8\gamma_3}{30}} B_\phi^{\frac{75+8\gamma_3}{30}} n_e^{\frac{1}{2}} P^{-\frac{11}{15}} \quad (54)$$

with $\gamma_4 = -3/8$.

At this point it is perhaps appropriate to revisit the motivation for using power law forms in fits to confinement data. One way of looking at the

coefficients derived from power law fits is in connection with the Taylor series expansion of a multivariate function. If a function f is a power law in N variables $(x_1, x_2, \dots, x_N)$, then the exponents of the power law will be the same as the coefficients of the first order Taylor series expansion in relative quantities $\delta x_k/x_k$. That is, any function expressible as a Taylor series (and with $f_o \neq 0$) has the expansion:

$$\frac{\delta f}{f_o} = \left(\frac{x_o \nabla f|_{x=x_o}}{f_o} \right) \cdot \left(\frac{\delta x}{x_o} \right) + \left(\mathcal{D} \cdot \frac{\delta x}{x_o} \right) \cdot \frac{\delta x}{x_o} + \mathcal{O}(\delta x^3) \quad (55)$$

Here $\mathcal{D}$ is the matrix of logarithmic partial derivatives:

$$\mathcal{D}_{ij} = \frac{x_i x_j}{f_o} \frac{\partial^2 f}{\partial x_i \partial x_j}. \quad (56)$$

The first order approximation to the series has coefficients given by

$$c_j = \frac{x_{jo}}{f_o} \frac{\partial f}{\partial x_j} \quad (57)$$

These coefficients would be identical to the exponents of the x_j if f were a power law. If the confinement were not expressible as a power law, then the coefficients would differ from the exponents. The approach of fitting to a Taylor series form in the relative variables has the advantage that it is a linear least-squares problem (just like fitting exponents). The added benefit is that non-power law theories can be compared to the fit parameters since no ansatz is used for the functional form of the confinement law. It is worth mentioning that since confinement times may span a order of magnitude or more in a given machine, that this approach may fail if the dimensionless parameters have too great a dynamic range.

7 Conclusions

In summary, ETG will be useful in resolving several physics issues relevant to Spherical Tokamak Reactor concepts. First, it will provide a test of whether transport is Bohm or gyro-Bohm in nature. ETG is well suited for this task, as it should be able to span a factor of two in ρ_* operating space while the other dimensionless variables ν_* , q , and β are held constant. This should

provide ample dynamic range to discriminate between Bohm or gyro-Bohm scaling of $\omega_i \tau_E$. The second point is that ETG will operate in a completely different range of ρ_* space from other high performance machines, opening up a previously inaccessible region of parameter space. ETG is also a (very) high- β machine. It would be the only device that would have all of its parameters except ρ_* similar to those of a Spherical Tokamak Reactor. If it turns out that the transport scales definitively as either Bohm or gyro-Bohm, then extrapolation to reactor conditions with significantly lower values of ρ_* would become more credible.

A Spreadsheet Formulary

A.1 Basic formulas

The following formulas were used to calculate the dimensionless variables and other figures of merit described earlier in the text. The stored energy is defined to be

$$W_T = W_e + W_i + W_B \quad (58)$$

where

$$W_e = \frac{3}{2} \int n_e T_e dV = \frac{3}{2} \langle n_e \rangle \langle T_e \rangle_n 2\pi^2 \kappa a^2 R \quad (59)$$

is the electron thermal energy. In calculating β , we neglect the beam ion energy. That is, we use $W = W_e + W_i$. The beam parameters (besides the absorbed beam power) are only important in estimating a correction to W_i due to dilution of the working gas ions by the beam ions. The correction is on the order of a few percent. The volume-averaged density is

$$\langle n_e \rangle = \frac{n_{eo}}{\alpha_n + 1} \quad (60)$$

where α_n is the density shape factor. The volume-averaged, density-weighted electron temperature is

$$\langle T_e \rangle_n = T_{eo} \left(\frac{\alpha_n + 1}{\alpha_n + \alpha_T + 1} \right). \quad (61)$$

The working ion thermal energy is

$$W_i = \frac{3}{2} \int n_i T_i dV = \frac{n_i}{n_e} \frac{T_i}{T_e} W_e \quad (62)$$

Here we define the "energy dilution factor" as being

$$f_i \equiv \frac{n_i}{n_e} = \left(1 - \sum_{imp.} Z_j f_j - f_B Z_B \right) / Z_i \quad (63)$$

Here $\sum_{imp.} Z_j f_j$ is the sum over all of the impurity species of the ion charge times the fraction impurity densities ($f_j = n_j/n_e$). Z_B is the neutral beam

ion charge, Z_i is the working ion charge, and f_B is the fractional population of beam ions:

$$f_B = \min \left(1, \frac{P_B^{abs} \tau_B}{\langle E_B \rangle \langle n_e \rangle 2\pi^2 \kappa a^2 R} \right) \quad (64)$$

P_B^{abs} is the absorbed beam power, τ_B is the beam ion confinement time (we approximated this as a slowing-down time), and $\langle E_B \rangle$ is the beam ion mean energy:

$$\langle E_B \rangle = E_B F_1 + \frac{1}{2} E_B F_2 + \frac{1}{3} E_B F_3 \quad (65)$$

Here $F_1 = 0.728$, $F_2 = 0.131$, and $F_3 = 0.141$ are the primary (E_B), secondary ($E_B/2$), and tertiary ($E_B/3$) beam power fractions [16]. The absorbed beam power was approximated for ETG parameters as [18]:

$$P_B^{abs} = (1 - \exp(-3.5 n_{e0} \ell)) P_B \quad (66)$$

Here n_{e0} is the central density in units of $10^{20} m^{-3}$, and ℓ is the chord length in m of the beam path through the plasma (we assume midplane injection tangent to the geometric axis). The ion/electron temperature ratio was approximated by a fit to TFTR, DIII-D and JT-60 neutral beam L-mode discharge data. The form used in the calculations is [17]

$$\frac{T_i}{T_e} = 0.425 + 0.252 \frac{R^{0.376} (W = P_{tot} \tau_{sc})^{0.445}}{\bar{n}_{e20}^{0.625} a^{1.373} P_{OH}^{0.089}} \quad (67)$$

Here W , P_{OH} , R and a are in MJ , MW , m and m , respectively. τ_{sc} is the confinement time given by the scaling law of choice (we have used Lackner-Gottardi). The beam ion contribution to the stored energy is

$$W_B = P_B^{abs} \tau'_B \quad (68)$$

The effective slowing-down time τ'_B is given by

$$\tau'_B = \frac{0.1 A_B T_e^{3/2}}{Z_B^2 \bar{n}_{e20} \log \Lambda_{ie}} (F_1 p_1 + F_2 p_2 + F_3 p_3) \quad (69)$$

The ion-electron Coulomb logarithm is given by the formula:

$$\log \Lambda_{ie} = 17 + \log(0.1 \langle T_e \rangle_n \langle \bar{n}_{e20} \rangle^{-1/2}) \quad (70)$$

The beam pressure-weighting factors are given by:

$$p_j = \text{Max} \left(0, 1 + \frac{1}{3x_j^2} \log \left(\frac{x_j^2 + 2x_j + 1}{x_j^2 - x_j + 1} - \frac{2}{\sqrt{3}x_j^2} \tan^{-1} \left(\frac{2x_j - 1}{\sqrt{3}} + 0.5236 \right) \right) \right) \quad (71)$$

and the parameter x_j is defined as

$$x_j = (E_j/E_c)^{1/2}. \quad (72)$$

Here $E_j = E_B/j$ and

$$E_c = 14.8 A_B T_{eo} \left(\frac{f_i Z_i^2 A_i^{-1} \log \Lambda_{Bi} + \sum_{imp} f_j Z_j^2 A_j^{-1} \log \Lambda_{Bj}}{\log \Lambda_{ie}} \right)^{2/3} \quad (73)$$

and if $E_B > 100 A_B Z_B^2 Z_j^2$, then

$$\log \Lambda_{Bj} = 19.1 + \log \left(\frac{A_j}{A_j + A_B} (T_{eo} A_B E_B \bar{n}_{e20}^{-1})^{1/2} \right) \quad (74)$$

else

$$\log \Lambda_{Bj} = 17.3 + \log \left(\frac{A_j E_B}{Z_B Z_j (A_j + A_B)} (T_{eo} \bar{n}_{e20}^{-1})^{1/2} \right). \quad (75)$$

It is worth mentioning a practical point here. The dilution factor, f_i , must be calculated in order to determine the p_j . This requires calculation of a slowing-down time (which depends on the p_j). In order to avoid a circular algorithm, we will approximate τ_B for the purposes of calculating the dilution factor as

$$\tau_B = \tau'_B / (F_1 p_1 + F_2 p_2 + F_3 p_3). \quad (76)$$

The quantity in parentheses is of order unity, and no great error in the dilution factor is made by neglecting it.

The electron temperature was determined for a fixed input beam power P_B , line-averaged electron density $\bar{n}_{e20}$ and plasma current I_p by equating the stored energy W given by the sum of the electron, ion, and beam ion contributions to the stored energy definition given by

$$W_{conf} = P_{tot} \tau_E \quad (77)$$

and

$$P_{tot} = P_B^{abs} + P_{OH} \quad (78)$$

The resultant equation is an implicit relation for T_e which must be solved numerically. For our calculations we chose the Lackner-Gottardi confinement scaling given by:

$$\tau_E = 0.12H (M_i/2)^{1/2} a^{0.4} R^{1.8} \kappa I_p^{0.8} q_{cyl}^{0.4} (\bar{n}_{e20}/P_{tot})^{0.6} (1 + \kappa)^{-0.8}, \quad (79)$$

The Ohmic heating power is

$$P_{OH} = I_p^2 R_p \quad (80)$$

where

$$R_p = \frac{0.08 Z_{eff} R}{\kappa a^2 < T_e^{3/2} >} \quad (81)$$

is a Spitzer-like form scaled to START values, and

$$< T_e^{3/2} > = \frac{T_{eo}^{3/2}}{1 + \frac{3}{2} \alpha_T}. \quad (82)$$

The definition for Z_{eff} is

$$Z_{eff} = \sum f_j Z_j^2 (keV) \quad (83)$$

where the sum extends over all ion species. For ETG, we assumed that $f_O = 0.01$, $Z_O = 8$, $f_C = 0.02$, $Z_C = 6$, $f_{Fe} = 0.0001$, and $Z_{Fe} = 26$ [10]. Although it is true that Z_{eff} depends upon the dilution factor, the dependence is very weak. For this reason it may be acceptable to calculate Z_{eff} assuming $f_i = 1$.

The density and temperature profiles were assumed to have the form

$$n_e(\rho) = n_{eo} (1 - \rho^2)^{\alpha_n} \quad (84)$$

$$T_e(\rho) = T_{eo} (1 - \rho^2)^{\alpha_T} \quad (85)$$

A.2 Constraints

In the calculations for determining the maximum and minimum possible values for the various dimensionless parameters, we incorporated several reasonable constraints. First, we set a Troyon-like β_N limit:

$$\beta_N = < \beta > \frac{a B_\phi}{I_p} < 5 \quad (86)$$

Here $\langle \beta \rangle$ is in %, a is in m , B_ϕ is in T and I_p is in MA . Operationally this is a minimum current, maximum density constraint. We also limit the density on the low side by allowing no more than 15 % beam shine-through. For ETG with a density shape factor of $\alpha_n = 0.7$, this constraint may be expressed as:

$$n_{e0} > 0.24 \quad (87)$$

The density is also bounded on the high side by the Greenwald limit. We conservatively took the upper density to be 80 % of the Greenwald limit. That is,

$$\bar{n}_{e20} < 0.8 \frac{I_p (MA)}{\pi a^2} \quad (88)$$

Another current-limiting constraint is that we demand

$$q_\psi > 3 \quad (89)$$

Here the definition of q_ψ that we use is

$$q_\psi = \frac{5B_\phi a^2 (1.22 - 0.68\epsilon)(1 + \kappa^2(1 + 2\delta^2 - 1.2\delta^3))}{2RI_p (1 - \epsilon^2)^2} \quad (90)$$

B Confinement scaling laws

The following section is a listing of the scaling laws for energy confinement time (in *ms*) used in calculating the scalings of confinement time with various dimensionless variables. The exponents were rounded to nearby fractions.

Lackner-Gottardi [11]

$$\tau_E = 120 H I_p^{4/5} R^{9/5} a^{2/5} \bar{n}_{e20}^{3/5} q_{cyl}^{2/5} \kappa (1 + \kappa)^{-4/5} \quad (91)$$

Goldston (L-mode) [12]

$$\tau_E = 30 H I_p R^{7/4} \kappa^{1/2} a^{-2/5} P_{net}^{-1/2} M_{eff}^{1/2} \quad (92)$$

Kaye-Goldston [12]

$$\tau_E = 44 H I_p^{5/4} R^{8/5} \kappa^{3/10} a^{-1/2} P_{net}^{-3/5} B_\phi^{-1/10} M_{eff}^{1/2} \quad (93)$$

DIID-D/JET [13]

$$\tau_E = 53 H I_p R^{3/2} P_{net}^{-1/2} \quad (94)$$

ITER89-P [13]

$$\tau_E = 48 H I_p^{17/20} R^{6/5} \kappa^{1/2} a^{3/10} P_{net}^{-1/2} B_\phi^{1/5} M_{eff}^{1/2} \bar{n}_{e20}^{1/10} \quad (95)$$

DTEM (L-mode) [14]

$$\tau_E = 128 \alpha_n^{-2/5} I_p^{9/10} R^{17/10} P_{net}^{-3/5} B_\phi^{1/10} B_\phi^{1/10} \quad (96)$$

Neo-Alcator [12]

$$\tau_E = 70 < n_{e20} > a R^2 q_{cyl} \quad (97)$$

Kaye-All Complex [12]

$$\tau_E = 67 H I_p^{17/20} R^{17/20} \kappa^{1/4} a^{3/10} P_{net}^{-1/2} B_\phi^{3/10} M_{eff}^{1/2} \bar{n}_{e20}^{1/10} \quad (98)$$

Goldston Quadrature - Ohmic [15]

$$\tau_E = 102.6 R^2 a \bar{n}_{e20} q_{cyl}^{1/2} \quad (99)$$

Goldston Quadrature - Auxiliary [15]

$$\tau_E = 30.2 H I_p R^{7/4} \kappa^{1/2} a^{-2/5} P_{net}^{-1/2} M_{eff}^{1/2} \quad (100)$$

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