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THE CHARACTERISTIC ANALYSIS OF A HYBRID MULTIFLUID TURBULENT-MIX MODEL

BAOLIAN CHENG AND CHARLES W. CRANFILL

ABSTRACT. A thorough analysis of the characteristics of a multifluid turbulent mix model in the case of one-dimensional two phase flows is presented under various physical circumstances. It has been found that the new hybrid multifluid turbulent mix model has all real characteristics if either real or turbulent viscosity is present. When real viscosity vanishes, the model still has all real characteristics for zero relative motion between fluids. For nonzero relative motions between fluids, the model will have all real characteristics if the disordered motions and turbulent viscosity together are generated with the nonzero relative motions simultaneously. The implications of the results are further discussed.

1. INTRODUCTION

It is well known that small asymmetries or disturbances in fluid systems containing multiple materials produce bouyancy- and shear- driven instabilities at the material interfaces. These instabilities lead to turbulent mixing of the materials. In the past three decades, various models of such mixing, from the simplest phenomenological model to the different variants of semiempirical models, were proposed. Among them, there were two different types of models describing such processes, one was the purely diffusive single fluid turbulence model introduced by Andronov et al^[1,2] and by Besnard, Harlow, and Rauen Zahn^[3,4], the other was the purely convective multifluid interpenetration model developed by Youngs^[5]. However, the flow motions in a real mixing process are early-time convective and late-time diffusive. A correct description of the fundamental mixing processes, therefore, should embody both motions. A few mix models which combine the single fluid and multifluid characterizations have been constructed^[6-8]. In particular, a novel hybrid multifluid turbulent mix model was recently proposed by Cranfill^[9,10]. This model innovatively provides a more natural formalism for deriving the combined multifluid turbulent mix equations by dividing the turbulent contributions to the bulk fluid properties into ordered convective and disordered diffusive parts. In this model, the ordered contributions are explicitly given as the sums of the multifluid drift motions, and summing the multifluid interpenetration equations over the mixture constituents uniquely produces the bulk fluid equations with turbulent contributions. The resulting energy cascade in this model is from bulk kinetic to ordered drift kinetic to disordered turbulent kinetic to thermal internal energy. Such exclusive features intrigue us to further examine this model mathematically by utilizing the method of characteristics.

The method of characteristics has been widely used among analysts of one dimensional unsteady flows due to its relative simplicity and the ease with which boundary conditions can be incorporated. In the view of mathematics, a model with complete physics should

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have all real characteristics, which insures that the predictions from its governing equations are meaningful; any imaginary characteristics are associated with unphysical instabilities that frequently cause the numerical evaluation of the equations to give senseless predictions for the dynamics of the system. The occurrence and significance of complex characteristics in two-phase flow equations have been discussed extensively over the past decades. It has been realized that the imaginary characteristics are directly due to the assumptions of pressure equilibrium and no dissipation in inviscid flows. The completion of physics of the two-phase flow equations and the justification of the assumption of zero dissipation in two inviscid flows with non-zero relative velocity between phases has never been examined. Now we propose that the addition of disordered diffusive motions to the two-phase flow equations would consummate the physics missed in previous models and provide a natural dissipation for the system which will make the equations more realistic. In this report, we will perform a thorough analysis of the characteristics of the new hybrid multifluid turbulent mix model in the case of one dimensional two-phase flows under various physical situations. We will show that the physical origin of the complex characteristics in previous models is not the single pressure assumption, but rather the incomplete and improper physics incorporated in the governing equations, in particular, the neglect of the disordered motions in the dynamical equations. It is evident that if the fluids are inviscid, any relative velocity between them should inevitably generate instabilities that set up disordered motions simultaneously. However this physical phenomenon has often been omitted in the traditional so called ill-posed one pressure two phase flow equations. This report will show that as long as this effect is taken into account properly, the one pressure hybrid multifluid turbulence model will have real characteristics even in the inviscid limit.

This report is organized as follows: Section 2 gives a summary of the hybrid multifluid turbulent mix model. In section 3 we reduce the model to one dimensional two-phase flow as an example. A thorough analysis of the characteristics for the one-dimensional equations under various physical situations is displayed in Section 4. Finally, in Section 5 we will discuss the physical implications of our results and identify possible constraints to the model. The physical origin of the complex characteristics in the traditional one-pressure inviscid two-phase flow equations is also discussed.

2. THE HYBRID TURBULENT MIX MODEL

The new hybrid multifluid turbulent mix model proposed and developed by Cranfill^[9,10] is based on the usual decomposition of the fluid properties into mean and fluctuating parts whose evolution equations are obtained from the Navier-Stokes equations. Correlations among the fluctuating parts produce turbulent contributions to the bulk fluid properties. The turbulent contributions in the model are uniquely divided into ordered and disordered parts, where the ordered parts are obtained from the average drift motions produced by a set of multifluid interpenetration equations, while the disordered parts are obtained from a set of single fluid turbulence equations. In this scenario, the multifluid and the single fluid turbulence equations are naturally coupled together, which enables the model to correctly describe both the early-time convective and the late-time diffusive drift motions seen in numerical and experimental investigations of the evolution of interfacial instabilities. This provides a justifiable description of the turbulent mixing in many fluid flows of interest. The mathematical description of the model is given in the remainder of this section.

2.1 Nomenclature.

$\langle \rangle$	= averaging operator,
c_j	= concentration function of j -th material,
f_j	= fraction by volume of j -th material,
x	= ratio of instantaneous mass density to average mass density,
x_j	= fraction by mass of j -th material,
m	= local mass density,
m_j	= local mass density of j -th material,
ρ	= average mass density of fluid ,
ρ_j	= average mass density of j -th material,
$\bar{u}$	= average velocity of bulk flow,
$\bar{v}'$	= local velocity fluctuation of bulk flow,
$\bar{w}$	= average value of $\bar{v}'$,
$\bar{w}_j$	= average value of $\bar{v}'$ in j -th material,
δ	= unit dyadic,
R	= total Reynold stress of bulk flow,
R_o, R_d	= ordered and disordered components of R ,
R_{oj}, R_{dj}	= values of R_o and R_d in j -th material,
Ψ	= local thermal stress of bulk flow,
P	= average value of Ψ ,
P	= average fluid pressure,
P_j	= value of P in j -th material,
P'	= fluid pressure fluctuation,
P'_j	= value of P' in j -th material,
Π	= local viscous stress of bulk flow,
Π_o	= average value of Π ,
Π'	= local viscous stress fluctuation of bulk flow due to $\bar{v}'$,
Q	= average value of Π' ,
Π_j, Π'_j	= average values of Π and Π' in j -th material,
Γ_d	= average turbulent viscous stress of bulk flow,
Γ_{dj}	= value of Γ_d in j -th material,
D_d, D'_d	= diffusion tensor and correction to it due to P' ,
D_{dj}, D'_{dj}	= values of D_d and D'_d in j -th material,
T, T'	= average fluid temperature and temperature fluctuation,
T_j, T'_j	= values of T and T' in j -th material,
k	= turbulent kinetic energy per unit mass of bulk flow,
k_o, k_d	= ordered and disordered components of k ,
k_{oj}, k_{dj}	= values of k_o and k_d in j -th material,
θ	= local internal energy per unit mass of bulk flow,
i	= average value of θ ,
θ_j	= average value of θ in j -th material,
θ'	= internal energy fluctuation due to T' ,
$\bar{s}, \bar{\sigma}, \bar{\pi}$	= energy fluxes associated with $\bar{v}'$,
$\bar{s}_{o(d)}, \bar{\sigma}_{o(d)}, \bar{\pi}_{o(d)}$	= ordered (disordered) components of $\bar{s}, \bar{\sigma}$, and $\bar{\pi}$,

$\vec{s}_{o(d)j}, \vec{\sigma}_{o(d)j}, \vec{\pi}_{o(d)j}$	= values of $\vec{s}_{o(d)}$, $\vec{\sigma}_{o(d)}$, and $\vec{\pi}_{o(d)}$ in j -th materials,
$\vec{s}'$	= correction to $\vec{s}$ due to T' ,
$\vec{p}'$	= energy flux associated with P' and $\vec{v}'$,
$\vec{p}'_o, \vec{p}'_d$	= ordered and disordered components of $\vec{p}'$,
$\vec{p}'_j, \vec{p}'_{oj}, \vec{p}'_{dj}$	= values of $\vec{p}'$, $\vec{p}'_o$, and $\vec{p}'_d$ in j -th material,
$\vec{\chi}$	= local heat flux of bulk flow,
$\vec{q}$	= average value of χ ,
$\vec{q}'$	= correction to $\vec{q}$ due to T' ,
ϵ	= turbulent energy dissipation rate per unit mass of bulk flow,
ϵ_o, ϵ_d	= ordered and disordered components of ϵ ,
$\epsilon_{oj}, \epsilon_{dj}$	= values of ϵ_o and ϵ_d in j -th material,
ϵ'	= correction to ϵ due to P' ,
ϵ'_o, ϵ'_d	= ordered and disordered components of ϵ' ,
$\epsilon'_{oj}, \epsilon'_{dj}$	= values of ϵ'_o and ϵ'_d in j -th material,
l	= turbulent dissipation length scale,
l_o, l_d	= values of l associated with ordered and disordered motions,
ω	= phenomenological drag frequency,
ω_o, ω_d	= values of ω associated with ordered and disordered motions,
ω'	= correction to ω due to P' ,
ϕ_o, ϕ_d	= sources for k_o and k_d ,
ϕ_{oj}, ϕ_{dj}	= values of ϕ_o and ϕ_d in j -th material,
η, η_d	= viscosities of bulk flow and turbulence,
η_j, η_{dj}	= values of η and η_d in j -th material,
κ	= local conductivity of bulk flow,
κ_j	= average value of κ in j -th material,
$\vec{\nabla}$	= gradient,
$\vec{\nabla} \cdot$	= divergence,
$\vec{\nabla}_o$	= symmetric gradient,
$\frac{d}{dt}$	= $\frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla}$.

2.2 Definitions.

$$\rho = \langle m \rangle, \quad (1)$$

$$\rho \vec{u} = \langle m \vec{v} \rangle, \quad (2)$$

$$\vec{v}' = \vec{v} - \vec{u}, \quad \vec{w} = \langle \vec{v}' \rangle, \quad (3)$$

$$\rho i = \langle m \theta \rangle, \quad (4)$$

$$\Psi = P\delta + \Pi, \quad \mathbf{P} = \langle \Psi \rangle, \quad (5)$$

$$\vec{q} = \langle \vec{\chi} \rangle, \quad (6)$$

$$f_j = \langle c_j \rangle, \quad (7)$$

$$x_j = \langle c_j x \rangle, \quad (8)$$

$$\rho_j = x_j \rho, \quad (9)$$

$$x_j \vec{w}_j = \langle c_j x \vec{v}' \rangle, \quad (10)$$

$$x_j \theta_j = \langle c_j x \theta \rangle, \quad (11)$$

$$f_j P_j = \langle c_j P \rangle, \quad (12)$$

$$f_j T_j = \langle c_j T \rangle, \quad (13)$$

$$f_j \Pi_j = \langle c_j \Pi \rangle, \quad f_j \Pi'_j = \langle c_j \Pi' \rangle, \quad (14)$$

$$f_j \eta_j = \langle c_j \eta \rangle, \quad (15)$$

$$f_j \kappa_j = \langle c_j \kappa \rangle, \quad (16)$$

$$\sum_j f_j = \sum_j x_j = 1, \quad (17)$$

$$\sum_j x_j \vec{w}_j = 0, \quad \sum_j f_j \vec{w}_j = \vec{w}, \quad (18)$$

$$\sum_j x_j \theta_j = i, \quad (19)$$

$$\sum_j f_j \eta_j = \langle \eta \rangle, \quad (20)$$

$$\sum_j f_j \kappa_j = \langle \kappa \rangle. \quad (21)$$

$$\mathbf{R} = \langle m \vec{v}'^2 \rangle = \mathbf{R}_o + \mathbf{R}_d, \quad (22)$$

$$\mathbf{R}_o = \sum_j f_j \mathbf{R}_{oj}, \quad \mathbf{R}_{oj} = f_j^{-1} \rho x_j \vec{w}_j^2, \quad (23)$$

$$\mathbf{R}_d = \sum_j f_j \mathbf{R}_{dj}, \quad \mathbf{R}_{dj} = f_j^{-1} \langle c_j \rho x (\vec{v}' - \vec{w}_j)^2 \rangle = \frac{2}{3} \rho k_d \delta + \Gamma_{dj}, \quad (24)$$

$$\Gamma_{dj} = -\eta_d [\vec{\nabla} \circ (\vec{u} + \vec{w}_j) - \frac{2}{3} \vec{\nabla} \cdot (\vec{u} + \vec{w}_j) \delta], \quad \eta_d = C_{\eta d} \omega^{-1} (\frac{2}{3} \rho k_d), \quad (25)$$

$$\mathbf{P} = \sum_j f_j (P_j \delta + \Pi_j) = P \delta + \langle \Pi \rangle, \quad (26)$$

$$\langle \Pi \rangle \equiv \sum_j f_j \Pi_j \equiv \Pi_o = - \langle \eta \rangle (\vec{\nabla} \circ \vec{u} - \frac{2}{3} \vec{\nabla} \cdot \vec{u}) + \mathbf{Q}, \quad (27)$$

$$\Pi_j \equiv -\eta_j [\vec{\nabla} \circ (\vec{u} + \vec{w}_j) - \frac{2}{3} \vec{\nabla} \cdot (\vec{u} + \vec{w}_j) \delta], \quad (28)$$

$$\mathbf{Q} = \langle \mathbf{\Pi}' \rangle = \sum_j f_j \mathbf{\Pi}'_j, \quad \mathbf{\Pi}'_j = -\eta_j (\vec{\nabla} \circ \vec{w}_j - \frac{2}{3} \vec{\nabla} \cdot \vec{w}_j \delta), \quad (29)$$

$$\mathbf{D}_d \equiv \omega^{-1}(\mathbf{R}_d + \mathbf{\Pi}_o) \equiv \sum_j f_j \mathbf{D}_{dj}, \quad \mathbf{D}_{dj} \equiv \omega^{-1}(\mathbf{R}_{dj} + \mathbf{\Pi}_j), \quad (30)$$

$$\omega \rho x_j \vec{w}_j \equiv - \langle \mathbf{\Pi} \cdot \vec{\nabla} c_j \rangle, \quad \omega = \frac{2}{3} C_\omega \sqrt{k_o + k_d} / l, \quad (31)$$

$$\vec{s} = \langle m \theta \vec{v}' \rangle = \vec{s}_o + \vec{s}_d, \quad \vec{s}_o = \sum_j \rho x_j \theta_j \vec{w}_j, \quad (32)$$

$$\vec{s}_d = \sum_j f_j \vec{s}_{dj} = \sum_j \langle c_j \rho x \theta (\vec{v}' - \vec{w}_j) \rangle, \quad (33)$$

$$f_j \vec{s}_{dj} = -f_j C_{id} \left(\frac{\partial i}{\partial T} \right)_\rho \omega^{-1} (\mathbf{R}_{dj} + \mathbf{\Pi}_j) \cdot \vec{\nabla} T = -f_j C_{id} \left(\frac{\partial i}{\partial T} \right)_\rho \mathbf{D}_{dj} \cdot \vec{\nabla} T, \quad (34)$$

$$\vec{q} = - \langle \kappa \rangle \vec{\nabla} T, \quad (35)$$

$$\vec{\sigma} = \langle \frac{1}{2} m v'^2 \vec{v}' \rangle = \vec{\sigma}_o + \vec{\sigma}_d, \quad \vec{\sigma}_o = \sum_j \frac{1}{2} \rho x_j w_j^2 \vec{w}_j, \quad (36)$$

$$\vec{\sigma}_d = \sum_j (\rho x_j k_{dj} \vec{w}_j + f_j \mathbf{R}_{dj} \cdot \vec{w}_j + f_j \vec{\sigma}_{dj}), \quad f_j \vec{\sigma}_{dj} = \langle c_j \frac{1}{2} \rho x |\vec{v}' - \vec{w}_j|^2 (\vec{v}' - \vec{w}_j) \rangle, \quad (37)$$

$$\vec{\pi} = \langle \mathbf{\Pi} \cdot \vec{v}' \rangle = \vec{\pi}_o + \vec{\pi}_d, \quad \vec{\pi}_o = \sum_j f_j \mathbf{\Pi}_j \cdot \vec{w}_j, \quad (38)$$

$$\vec{\pi}_d = \sum_j f_j \vec{\pi}_{dj}, \quad f_j \vec{\pi}_{dj} = \langle c_j \mathbf{\Pi} \cdot (\vec{v}' - \vec{w}_j) \rangle = \langle c_j \mathbf{\Pi} \cdot \vec{v}' \rangle - f_j \mathbf{\Pi}_j \cdot \vec{w}_j, \quad (39)$$

$$\vec{\sigma}_d + \vec{\pi}_d = \sum_j (\rho x_j k_{dj} \vec{w}_j + f_j \mathbf{R}_{dj} \cdot \vec{w}_j) + \sum_j f_j (\vec{\sigma}_{dj} + \vec{\pi}_{dj}), \quad (40)$$

$$\vec{\sigma}_{dj} + \vec{\pi}_{dj} = -C_{kd} \omega^{-1} (\mathbf{R}_{dj} + \mathbf{\Pi}_j) \cdot \vec{\nabla} k_d = -C_{kd} \mathbf{D}_{dj} \cdot \vec{\nabla} k_d, \quad (41)$$

$$\rho k \equiv \langle \frac{1}{2} m v'^2 \rangle = \rho k_o + \rho k_d, \quad \rho k_o = \sum_j \frac{1}{2} \rho x_j w_j^2, \quad (42)$$

$$\rho k_d = \sum_j \rho x_j k_{dj} \equiv \sum_j \langle c_j \frac{1}{2} \rho x |\vec{v}' - \vec{w}_j|^2 \rangle = \sum_j f_j \rho k_d, \quad (43)$$

$$\rho \epsilon = \rho \epsilon_o + \rho \epsilon_d, \quad \rho \epsilon_o = - \sum_j f_j \mathbf{\Pi}'_j : \vec{\nabla} \vec{w}_j, \quad (44)$$

$$\rho \epsilon_d = \sum_j \rho x_j \epsilon_{dj} \equiv - \sum_j \langle c_j \mathbf{\Pi}' : \vec{\nabla} (\vec{v}' - \vec{w}_j) \rangle = \sum_j f_j \rho \epsilon_d, \quad (45)$$

$$l \equiv k_d \sqrt{k_o + k_d} / \epsilon_d, \quad (46)$$

$$\rho\phi_o = \sum_j \vec{\nabla} \cdot (f_j \mathbf{R}_{dj}) \cdot \vec{w}_j, \quad \rho\phi_d = \sum_j f_j \mathbf{R}_{dj} : \vec{\nabla}(\vec{u} + \vec{w}_j), \quad (47)$$

C_{xy} ($xy = \eta d, id, kd, ld$, and ω) are constants adjusted to fit experiments. Subscripts “o” and “d” represent the respective ordered-convective and disordered-diffusive components of a quantity.

2.3 Model Equations. The mathematical equations defining the new multifluid turbulent mix model are summarized as follows:

- Bulk fluid equations:

$$0 = \frac{d}{dt}\rho + \rho \vec{\nabla} \cdot \vec{u}, \quad (48)$$

$$0 = \rho \frac{d}{dt}\vec{u} + \vec{\nabla} \cdot (\mathbf{R} + \mathbf{P}), \quad (49)$$

$$0 = \rho \frac{d}{dt}\vec{i} + \vec{\nabla} \cdot (\vec{s} + \vec{q}) + (\mathbf{P} + \mathbf{Q}) : \vec{\nabla}\vec{u} + P \vec{\nabla} \cdot \vec{w} - \rho\epsilon. \quad (50)$$

- Multifluid interpenetration equations:

$$0 = \rho \frac{d}{dt}x_j + \vec{\nabla} \cdot (\rho x_j \vec{w}_j), \quad (51)$$

$$0 = \rho \frac{d}{dt}(x_j \vec{w}_j) + \rho x_j \left(\frac{d}{dt} + \vec{w}_j \cdot \vec{\nabla} \right) \vec{u} + \vec{\nabla} \cdot (\rho x_j \vec{w}_j^2 + f_j \mathbf{R}_{dj} + f_j \mathbf{\Pi}_j) + f_j \vec{\nabla} P + \omega \rho x_j \vec{w}_j, \quad (52)$$

$$P_j(m_j, \theta_j) = P(\rho, i, x_1, x_2, \dots), \quad T_j(m_j, \theta_j) = T(\rho, i, x_1, x_2, \dots), \quad m_j = \rho x_j / f_j. \quad (53)$$

- Single-fluid turbulence equations:

$$0 = \rho \frac{d}{dt}k + \vec{\nabla} \cdot (\vec{\sigma} + \vec{\pi}) + (\mathbf{R} - \mathbf{Q}) : \vec{\nabla}\vec{u} + (\vec{\nabla}P) \cdot \vec{w} + \rho\epsilon, \quad (54)$$

$$0 = \rho \frac{d}{dt}k_o + \vec{\nabla} \cdot (\vec{\sigma}_o + \vec{\pi}_o) + (\mathbf{R}_o - \mathbf{Q}) : \vec{\nabla}\vec{u} + (\vec{\nabla}P) \cdot \vec{w} + \rho\phi_o + 2\omega\rho k_o + \rho\epsilon_o, \quad (55)$$

$$0 = \rho \frac{d}{dt}k_d + \vec{\nabla} \cdot (\rho k_d \vec{w} - C_{kd} \mathbf{D}_d \cdot \vec{\nabla}k_d) + \rho\phi_d - 2\omega\rho k_o + \rho\epsilon_d, \quad (56)$$

$$0 = \rho \frac{d}{dt}l + \vec{\nabla} \cdot (\rho l \vec{w} - C_{ld} \mathbf{D}_d \cdot \vec{\nabla}l) - C_{lo\rho}(l \vec{\nabla} \cdot \vec{u} + \sqrt{k_o + k_d}). \quad (57)$$

The mixture here is assumed to be in an equilibrium with all materials having the same pressure, $P_j = P$, and temperature, $T_j = T$, so the f_j and θ_j can be derived from the equations of state Eq.(53) without solving separate evolution equations for them. For given values of ρ , i and the x_j , the f_j , θ_j , P , and T will be uniquely determined. Thus the model has evolution equations for all the independent variables ρ , $\vec{u}$, i , x_j , $\vec{w}_j$, k_d , and l which describe the dynamics of the fluid system. Therefore this new hybrid multifluid turbulent mix model is mathematically complete.

If the mixture has a minute departure from both pressure and temperature equilibrium, then different materials would have different pressure and temperature fluctuations, P'_j and T'_j . Under this physical circumstance, the evolution equations of the fluid system need to be modified as discussed in Appendix A.

It is worthwhile to point out that this model only contains one drag frequency and one turbulence length scale for simplicity. In addition, it does not include mass diffusion which would allow the materials to mix at the atomic level. A real mixing process probably involves a spectrum of both ω and l , and the materials might mix at a microscopic scale in most fluid systems. One important advantage of this model is that its evolution equations are symmetrical in all materials and are formulated with vectors. This distinctive character allows the model to be applied to systems that contain any numbers of materials and spatial dimensions. We will apply this model to a one-dimensional two-fluid system as an example in the next section.

3. ONE-DIMENSIONAL TWO-FLUID SYSTEM

3.1 General theory. In a one-dimensional system consisting of only two fluids, the evolution equations are obtained directly from the new multifluid turbulent mix model presented in Section 2 by setting the material index $j = 1$ and 2. For convenience, we replace the two material values of each physical quantity with their volume average and difference as follows:

$$w \equiv f_1 w_1 + f_2 w_2 = (f_1 - x_1) \delta w, \quad \delta w \equiv w_1 - w_2, \quad w_1 = x_2 \delta w, \quad w_2 = -x_1 \delta w, \quad (58)$$

$$\eta \equiv f_1 \eta_1 + f_2 \eta_2, \quad \delta \eta = \eta_1 - \eta_2, \quad \eta_1 = \eta + f_2 \delta \eta, \quad \eta_2 = \eta - f_1 \delta \eta, \quad (59)$$

$$\Gamma_d = f_1 \Gamma_{d1} + f_2 \Gamma_{d2}, \quad \delta \Gamma_d = \Gamma_{d1} - \Gamma_{d2}, \quad \Gamma_{d1} = \Gamma_d + f_2 \delta \Gamma_d, \quad \Gamma_{d2} = \Gamma_d - f_1 \delta \Gamma_d, \quad (60)$$

$$\Pi_o = f_1 \Pi_1 + f_2 \Pi_2, \quad \delta \Pi_o = \Pi_1 - \Pi_2, \quad \Pi_1 = \Pi_o + f_2 \delta \Pi_o, \quad \Pi_2 = \Pi_o - f_1 \delta \Pi_o, \quad (61)$$

$$\mathbf{Q} = f_1 \Pi'_1 + f_2 \Pi'_2, \quad \delta \mathbf{Q} = \Pi'_1 - \Pi'_2, \quad \Pi'_1 = \mathbf{Q} + f_2 \delta \mathbf{Q}, \quad \Pi'_2 = \mathbf{Q} - f_1 \delta \mathbf{Q}, \quad (62)$$

$$\Phi = \Pi_o + \Gamma_d, \quad \delta \Phi = \delta \Pi_o + \delta \Gamma_d, \quad (63)$$

$$\mathbf{R}_d = f_1 \mathbf{R}_{d1} + f_2 \mathbf{R}_{d2}, \quad \delta \mathbf{R}_d = \mathbf{R}_{d1} - \mathbf{R}_{d2}, \quad \mathbf{R}_{d1} = \mathbf{R}_d + f_2 \delta \mathbf{R}_d, \quad \mathbf{R}_{d2} = \mathbf{R}_d - f_1 \delta \mathbf{R}_d, \quad (64)$$

$$i = x_1 \theta_1 + x_2 \theta_2, \quad \delta \theta = \theta_1 - \theta_2, \quad \theta_1 = i + x_2 \delta \theta, \quad \theta_2 = i - x_1 \delta \theta, \quad (65)$$

where we have used the notation $\Pi_o \equiv \langle \Pi \rangle$ and $\mathbf{Q} = \langle \Pi' \rangle$. Assume that the mean fluid velocity is along the x direction, and let $\hat{x}$ represent the corresponding unit vector, then the one-dimensional dynamical variables in the model become

$$\frac{\partial w_1}{\partial x} = -\delta w \frac{\partial x_1}{\partial x} + x_2 \frac{\partial \delta w}{\partial x}, \quad \frac{\partial w_1}{\partial t} = -\delta w \frac{\partial x_1}{\partial t} + x_2 \frac{\partial \delta w}{\partial t}, \quad (66)$$

$$\frac{\partial w_2}{\partial x} = -\delta w \frac{\partial x_1}{\partial x} - x_1 \frac{\partial \delta w}{\partial x}, \quad \frac{\partial w_2}{\partial t} = -\delta w \frac{\partial x_1}{\partial t} - x_1 \frac{\partial \delta w}{\partial t}, \quad (67)$$

$$\frac{\partial w}{\partial x} = \delta w \frac{\partial f_1}{\partial x} - \delta w \frac{\partial x_1}{\partial x} + (f_1 - x_1) \frac{\partial \delta w}{\partial x}, \quad \frac{\partial w}{\partial t} = \delta w \frac{\partial f_1}{\partial t} - \delta w \frac{\partial x_1}{\partial t} + (f_1 - x_1) \frac{\partial \delta w}{\partial t}, \quad (68)$$

$$\Gamma_d = \frac{2}{3} \Gamma_d (\delta - 3\hat{x}^2), \quad \Gamma_d = \eta_d \left[\frac{\partial u}{\partial x} - \delta w \frac{\partial x_1}{\partial x} + (f_1 - x_1) \frac{\partial \delta w}{\partial x} \right], \quad (69)$$

$$\delta\Gamma_d = \frac{2}{3}\delta\Gamma_d(\delta - 3\hat{x}^2), \quad \delta\Gamma_d = \eta_d \frac{\partial\delta w}{\partial x}, \quad (70)$$

$$\Gamma_{d1} = \frac{2}{3}\Gamma_{d1}(\delta - 3\hat{x}^2), \quad \Gamma_{d1} = \Gamma_d + f_2\delta\Gamma_d, \quad (71)$$

$$\Gamma_{d2} = \frac{2}{3}\Gamma_{d2}(\delta - 3\hat{x}^2), \quad \Gamma_{d2} = \Gamma_d - f_1\delta\Gamma_d, \quad (72)$$

$$\frac{\partial\Gamma_{d1}}{\partial x} = \frac{\partial\Gamma_d}{\partial x} - \delta\Gamma_d \frac{\partial f_1}{\partial x} + f_2 \frac{\partial\delta\Gamma_d}{\partial x}, \quad \frac{\partial\Gamma_{d2}}{\partial x} = \frac{\partial\Gamma_d}{\partial x} - \delta\Gamma_d \frac{\partial f_1}{\partial x} - f_1 \frac{\partial\delta\Gamma_d}{\partial x}, \quad (73)$$

$$\mathbf{R}_o = R_o \hat{x}^2, \quad R_o \equiv \delta : \mathbf{R}_o = \rho x_1 x_2 \delta w^2, \quad \rho k_o = \frac{1}{2} R_o, \quad (74)$$

$$\mathbf{R}_d = \frac{2}{3}\rho k_d \delta + \frac{2}{3}\Gamma_d(\delta - 3\hat{x}^2), \quad (75)$$

$$\mathbf{R}_{d1} = \frac{2}{3}\rho k_d \delta + \frac{2}{3}\Gamma_{d1}(\delta - 3\hat{x}^2), \quad \mathbf{R}_{d2} = \frac{2}{3}\rho k_d \delta + \frac{2}{3}\Gamma_{d2}(\delta - 3\hat{x}^2), \quad (76)$$

$$\begin{aligned} \rho\phi_d &= \mathbf{R}_d : \hat{x}^2 \left(\frac{\partial u}{\partial x} - \delta w \frac{\partial x_1}{\partial x} \right) + (f_1 - x_1) \mathbf{R}_{d1} : \hat{x}^2 \frac{\partial\delta w}{\partial x} \\ &= \left(\frac{2}{3}\rho k_d - \frac{4}{3}\Gamma_d \right) \left(\frac{\partial u}{\partial x} - \delta w \frac{\partial x_1}{\partial x} \right) + [(f_1 - x_1) \left(\frac{2}{3}\rho k_d - \frac{4}{3}\Gamma_d \right) - \frac{4}{3}f_1 f_2 \delta\Gamma_d] \frac{\partial\delta w}{\partial x} \\ &= \left(\frac{2}{3}\rho k_d - \frac{4}{3}\Gamma_d \right) \left[\frac{\partial u}{\partial x} - \delta w \frac{\partial x_1}{\partial x} + (f_1 - x_1 - f_1 f_2 \frac{\delta\Gamma_d}{\rho k_d/2 - \Gamma_d}) \frac{\partial\delta w}{\partial x} \right]. \end{aligned} \quad (77)$$

$$\Pi_o = \frac{2}{3}\Pi_o(\delta - 3\hat{x}^2), \quad \delta\Pi_o = \frac{2}{3}\delta\Pi_o(\delta - 3\hat{x}^2), \quad (78)$$

$$\Pi_o = \eta \left[\frac{\partial u}{\partial x} - \delta w \frac{\partial x_1}{\partial x} + (f_1 - x_1 + f_1 f_2 \frac{\delta\eta}{\eta}) \frac{\partial\delta w}{\partial x} \right], \quad (79)$$

$$\delta\Pi_o = \delta\eta \left[\frac{\partial u}{\partial x} - \delta w \frac{\partial x_1}{\partial x} + (f_2 - x_1 + \frac{\eta}{\delta\eta}) \frac{\partial\delta w}{\partial x} \right], \quad (80)$$

$$\Pi_1 = \frac{2}{3}\Pi_1(\delta - 3\hat{x}^2), \quad \Pi_2 = \frac{2}{3}\Pi_2(\delta - 3\hat{x}^2), \quad (81)$$

$$\Pi_1 = (\eta + f_2\delta\eta) \left(\frac{\partial u}{\partial x} - \delta w \frac{\partial x_1}{\partial x} + x_2 \frac{\partial\delta w}{\partial x} \right), \quad (82)$$

$$\Pi_2 = (\eta - f_1\delta\eta) \left(\frac{\partial u}{\partial x} - \delta w \frac{\partial x_1}{\partial x} - x_1 \frac{\partial\delta w}{\partial x} \right), \quad (83)$$

$$\mathbf{Q} = \frac{2}{3}Q(\delta - 3\hat{x}^2), \quad \delta\mathbf{Q} = \frac{2}{3}\delta Q(\delta - 3\hat{x}^2), \quad (84)$$

$$Q = \eta \left[-\delta w \frac{\partial x_1}{\partial x} + (f_1 - x_1 + f_1 f_2 \frac{\delta\eta}{\eta}) \frac{\partial\delta w}{\partial x} \right] = \Pi_o - \eta \frac{\partial u}{\partial x}, \quad (85)$$

$$\delta Q = \delta\eta \left[-\delta w \frac{\partial x_1}{\partial x} + (f_2 - x_1 + \frac{\eta}{\delta\eta}) \frac{\partial\delta w}{\partial x} \right] = \delta\Pi_o - \delta\eta \frac{\partial u}{\partial x}, \quad (86)$$

$$\Pi'_1 = \frac{2}{3}\Pi'_1(\delta - 3\hat{x}^2), \quad \Pi'_2 = \frac{2}{3}\Pi'_2(\delta - 3\hat{x}^2), \quad (87)$$

$$\Pi'_1 = (\eta + f_2\delta\eta)(-\delta w \frac{\partial x_1}{\partial x} + x_2 \frac{\partial \delta w}{\partial x}) = \Pi_1 - \eta_1 \frac{\partial u}{\partial x}, \quad (88)$$

$$\Pi'_2 = (\eta - f_1\delta\eta)(-\delta w \frac{\partial x_1}{\partial x} - x_1 \frac{\partial \delta w}{\partial x}) = \Pi_2 - \eta_2 \frac{\partial u}{\partial x}, \quad (89)$$

$$\Phi = \frac{2}{3}\Phi(\delta - 3\hat{x}^2), \quad \delta\Phi = \frac{2}{3}\delta\Phi(\delta - 3\hat{x}^2), \quad (90)$$

$$\Phi = \Pi_o + \Gamma_d = (\eta + \eta_d)[\frac{\partial u}{\partial x} - \delta w \frac{\partial x_1}{\partial x} + (f_1 - x_1 + f_1 f_2 \frac{\delta\eta}{\eta + \eta_d}) \frac{\partial \delta w}{\partial x}], \quad (91)$$

$$\delta\Phi = \delta\Pi_o + \delta\Gamma_d = \delta\eta[\frac{\partial u}{\partial x} - \delta w \frac{\partial x_1}{\partial x} + (f_2 - x_1 + \frac{\eta + \eta_d}{\delta\eta}) \frac{\partial \delta w}{\partial x}], \quad (92)$$

$$\mathbf{P} = P\delta + \frac{2}{3}\Pi_o(\delta - 3\hat{x}^2), \quad (93)$$

$$\vec{s}_o = (\rho x_1 w_1 \theta_1 + \rho x_2 w_2 \theta_2)\hat{x} = \rho x_1 x_2 \delta w \delta \theta \hat{x}, \quad (94)$$

$$\vec{s}_d = -C_{id}(\frac{\partial i}{\partial T})_\rho \vec{D}_{dx} \frac{\partial T}{\partial x}, \quad (95)$$

$$\vec{D}_{dx} \equiv \mathbf{D}_d \cdot \hat{x} = D_{dx} \hat{x}, \quad D_{dx} = \omega^{-1}(\frac{2}{3}\rho k_d - \frac{4}{3}\Phi), \quad (96)$$

$$\mathbf{D}_d = \omega^{-1}(\mathbf{R}_d + \Pi_o) = \omega^{-1}[\frac{2}{3}\rho k_d \delta + \frac{2}{3}\Phi(\delta - 3\hat{x}^2)], \quad (97)$$

$$\vec{q} = -(f_1 \kappa_1 + f_2 \kappa_2) \frac{\partial T}{\partial x} \hat{x} = -\kappa \frac{\partial T}{\partial x} \hat{x}, \quad (98)$$

$$\begin{aligned} \rho \epsilon_o &= \mathbf{Q} : \hat{x}^2 \delta w \frac{\partial x_1}{\partial x} + (x_1 \mathbf{Q} - f_1 \Pi'_1) : \hat{x}^2 \frac{\partial \delta w}{\partial x} \\ &= \frac{4}{3}Q[-\delta w \frac{\partial x_1}{\partial x} + (f_1 - x_1 + f_1 f_2 \frac{\delta Q}{Q}) \frac{\partial \delta w}{\partial x}]. \end{aligned} \quad (99)$$

Note that $\delta : \hat{x}^2 = 1$, $f_2 = 1 - f_1$, and $x_2 = 1 - x_1$. For simplicity, we introduce $a_1 = x_1 x_2 = x_1(1 - x_1)$, $a_2 = x_2 - x_1 = 1 - 2x_1$, and $a_3 = f_1 - x_1$. With these expressions, in the case of both pressure and temperature equilibrium, the one-dimensional evolution equations for the two-fluid system reduce to

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0, \quad (100)$$

$$\begin{aligned} \rho \frac{\partial u}{\partial t} + (\frac{2}{3}k_d + a_1 \delta w^2) \frac{\partial \rho}{\partial x} + \rho u \frac{\partial u}{\partial x} + \rho a_2 \delta w^2 \frac{\partial x_1}{\partial x} + 2\rho a_1 \delta w \frac{\partial \delta w}{\partial x} + \\ \frac{2}{3}\rho \frac{\partial k_d}{\partial x} - \frac{4}{3}\frac{\partial \Phi}{\partial x} + \frac{\partial P}{\partial x} = 0, \end{aligned} \quad (101)$$

$$\rho \frac{\partial x_1}{\partial t} + a_1 \delta w \frac{\partial \rho}{\partial x} + \rho(u + a_2 \delta w) \frac{\partial x_1}{\partial x} + \rho a_1 \frac{\partial \delta w}{\partial x} = 0, \quad (102)$$

$$\begin{aligned}
& \rho x_1 \frac{\partial u}{\partial t} + \rho a_2 \delta w \frac{\partial x_1}{\partial t} + \rho a_1 \frac{\partial \delta w}{\partial t} + \left[\frac{2}{3} f_1 k_d + a_1 (1 - x_1) \delta w^2 \right] \frac{\partial \rho}{\partial x} + \\
& \rho x_1 [u + (1 - x_1) \delta w] \frac{\partial u}{\partial x} + \rho [a_2 u + (1 - x_1) (1 - 3x_1) \delta w] \delta w \frac{\partial x_1}{\partial x} + \rho a_1 [u + 2(1 - x_1) \delta w] \frac{\partial \delta w}{\partial x} + \\
& \frac{2}{3} f_1 \rho \frac{\partial k_d}{\partial x} - \frac{4}{3} f_1 \left[\frac{\partial \Phi}{\partial x} + (1 - f_1) \frac{\partial \delta \Phi}{\partial x} \right] + f_1 \frac{\partial P}{\partial x} + \\
& \left[\frac{2}{3} \rho k_d - \frac{4}{3} (\Phi + (1 - 2f_1) \delta \Phi) \right] \frac{\partial f_1}{\partial x} + \omega a_1 \rho \delta w = 0,
\end{aligned} \tag{103}$$

$$\begin{aligned}
& \rho \frac{\partial i}{\partial t} + a_1 \delta \theta \delta w \frac{\partial \rho}{\partial x} + \left[P - \frac{4}{3} (\Pi_o + Q) \right] \frac{\partial u}{\partial x} + (a_2 \rho \delta \theta + \frac{4}{3} Q - P) \delta w \frac{\partial x_1}{\partial x} + \\
& [a_1 \rho \delta \theta + a_3 P - \frac{4}{3} Q a_3 - \frac{4}{3} \delta Q f_1 (1 - f_1)] \frac{\partial \delta w}{\partial x} + \rho u \frac{\partial i}{\partial x} + \\
& \frac{\partial s_d}{\partial x} + \frac{\partial q}{\partial x} + P \delta w \frac{\partial f_1}{\partial x} + \rho a_1 \delta w \frac{\partial \delta \theta}{\partial x} - \rho \epsilon_d = 0,
\end{aligned} \tag{104}$$

$$\begin{aligned}
& \rho \frac{\partial k_d}{\partial t} + k_d a_3 \delta w \frac{\partial \rho}{\partial x} + \left(\frac{2}{3} \rho k_d - \frac{4}{3} \Gamma_d \right) \frac{\partial u}{\partial x} - \left(\frac{5}{3} \rho k_d - \frac{4}{3} \Gamma_d \right) \delta w \frac{\partial x_1}{\partial x} + \\
& \left[\frac{5}{3} \rho k_d a_3 - \frac{4}{3} \Gamma_d a_3 - \frac{4}{3} f_1 (1 - f_1) \delta \Gamma_d \right] \frac{\partial \delta w}{\partial x} + \rho (u + a_3 \delta w) \frac{\partial k_d}{\partial x} + \frac{\partial J_k}{\partial x} + \\
& \rho k_d \delta w \frac{\partial f_1}{\partial x} - \omega \rho a_1 \delta w^2 + \rho \epsilon_d = 0,
\end{aligned} \tag{105}$$

$$\begin{aligned}
& \rho \frac{\partial l}{\partial t} + a_3 l \delta w \frac{\partial \rho}{\partial x} - C_{lo} \rho l \frac{\partial u}{\partial x} - \rho l \delta w \frac{\partial x_1}{\partial x} + a_3 \rho l \frac{\partial \delta w}{\partial x} + \rho (u + a_3 \delta w) \frac{\partial l}{\partial x} + \\
& \frac{\partial J_l}{\partial x} + \rho l \delta w \frac{\partial f_1}{\partial x} - C_{lo} \rho \sqrt{k_o + k_d} = 0.
\end{aligned} \tag{106}$$

The total viscous stress Φ and its variation $\delta\Phi$ are given by Eqs.(91) and (92), and the fluxes of internal energy, $s_d \hat{x}$, heat, $q \hat{x}$, turbulent kinetic energy, $J_k \hat{x}$, and turbulent length scale, $J_l \hat{x}$, satisfy the following auxiliary equations

$$s_d + D_s \frac{\partial T}{\partial x} = 0, \quad D_s = C_{id} \left(\frac{\partial i}{\partial T} \right) \rho D_{dx}, \tag{107}$$

$$q + D_q \frac{\partial T}{\partial x} = 0, \quad D_q = \kappa, \tag{108}$$

$$J_k + D_k \frac{\partial k_d}{\partial x} = 0, \quad D_k = C_{kd} D_{dx}, \tag{109}$$

$$J_l + D_l \frac{\partial l}{\partial x} = 0, \quad D_l = C_{ld} D_{dx}. \tag{110}$$

Here the introduction of the fluxes is purely for the characteristic analysis of the system, that is, to avoid second derivatives of the dynamic quantities appearing in the equations. The f_j , $\delta\theta$, P , and T will be determined from the equations of the state of the system. For ideal gases, the relationships among them are much simpler than for real fluids.

It is clear that the one-dimensional evolution equations presented above in the case of pressure equilibrium along with the equations of state form a complete and general description of the system.

3.2 Ideal gases. For simplicity, we now apply the general one-dimensional evolution equations to a system with two ideal gases.

The equation of state for each ideal gas is described by

$$P_j = \rho_j \theta_j (\gamma_j - 1) / f_j, \quad \text{or} \quad P_j = \frac{N_j}{V_j} k_B T_j, \quad (111)$$

where γ_j is the ratio of specific heats of the j -th material, k_B is Boltzmann's constant, $N_j/V_j = \Gamma_j \rho_j / f_j$, $\Gamma_j \equiv N_{Av}/A_j$, $N_{Av} = 6.02 \times 10^{23}$ is Avogadro's number, $V_j \equiv f_j V$ and A_j respectively express the volume and mass number of the j -th material, and V is the total volume of the system. In the situation of both pressure and temperature equilibrium, the volume fraction (f_j) of the j -th material is determined directly from the equations of state

$$f_j = \frac{\Gamma_j \rho_j}{\sum_{j'} \Gamma_{j'} \rho_{j'}}.$$

For two materials ($j = 1, 2$), this gives

$$f_j = \frac{\Gamma_j \rho_j}{\Gamma_1 \rho_1 + \Gamma_2 \rho_2} = \frac{\Gamma_j x_j}{\Gamma_1 x_1 + \Gamma_2 x_2}. \quad (112)$$

This leads to useful identities:

$$\frac{\Gamma_1 x_1}{f_1} = \frac{\Gamma_2 x_2}{f_2} \quad \text{or} \quad \frac{\Gamma_2}{\Gamma_1} = \frac{f_2 x_1}{x_2 f_1}, \quad (113)$$

$$\Gamma_1 - \Gamma_2 = \frac{\Gamma_1}{f_1} - \frac{\Gamma_2}{x_1} = \frac{\Gamma_1}{x_2} - \frac{\Gamma_2}{f_2}. \quad (114)$$

With these expressions, we can write the following thermodynamical quantities and their derivatives explicitly,

$$P_1 = P_2 = P = (\Gamma_1 x_1 + \Gamma_2 x_2) \rho k_B T, \quad (115)$$

$$\theta_j = \frac{\Gamma_j}{\gamma_j - 1} k_B T, \quad (116)$$

$$i = x_1 \theta_1 + x_2 \theta_2 = \frac{P}{\rho(\gamma - 1)}, \quad \frac{1}{\gamma - 1} = \frac{f_1}{\gamma_1 - 1} + \frac{f_2}{\gamma_2 - 1}, \quad (117)$$

$$\delta \theta \equiv \theta_1 - \theta_2 = k_B T \left(\frac{\Gamma_1}{\gamma_1 - 1} - \frac{\Gamma_2}{\gamma_2 - 1} \right), \quad (118)$$

$$\frac{\partial f_1}{\partial t} = \frac{\Gamma_1 \Gamma_2}{(\Gamma_1 x_1 + \Gamma_2 x_2)^2} \frac{\partial x_1}{\partial t} = \frac{f_1 f_2}{x_1 x_2} \frac{\partial x_1}{\partial t}, \quad (119)$$

$$\frac{\partial f_1}{\partial x} = \frac{\Gamma_1 \Gamma_2}{(\Gamma_1 x_1 + \Gamma_2 x_2)^2} \frac{\partial x_1}{\partial x} = \frac{f_1 f_2}{x_1 x_2} \frac{\partial x_1}{\partial x}, \quad (120)$$

$$-\frac{1}{(\gamma-1)^2} \frac{\partial \gamma}{\partial t} = \left(\frac{1}{\gamma_1-1} - \frac{1}{\gamma_2-1} \right) \frac{\partial f_1}{\partial t}, \quad (121)$$

$$-\frac{1}{(\gamma-1)^2} \frac{\partial \gamma}{\partial x} = \left(\frac{1}{\gamma_1-1} - \frac{1}{\gamma_2-1} \right) \frac{\partial f_1}{\partial x}, \quad (122)$$

$$\begin{aligned} \frac{\partial P}{\partial x} &= (\Gamma_1 - \Gamma_2) \rho k_B T \frac{\partial x_1}{\partial x} + \frac{\Gamma_1 x_1}{f_1} k_B T \frac{\partial \rho}{\partial x} + \frac{\Gamma_1 x_1}{f_1} \rho k_B \frac{\partial T}{\partial x} \\ &= \left(\frac{f_1}{x_1} - \frac{f_2}{x_2} \right) P \frac{\partial x_1}{\partial x} + \frac{P}{\rho} \frac{\partial \rho}{\partial x} + \frac{P}{T} \frac{\partial T}{\partial x}, \end{aligned} \quad (123)$$

$$\frac{\partial i}{\partial t} = \frac{i}{T} \frac{\partial T}{\partial t} + \delta \theta \frac{\partial x_1}{\partial t}, \quad \frac{\partial i}{\partial x} = \frac{i}{T} \frac{\partial T}{\partial x} + \delta \theta \frac{\partial x_1}{\partial x}, \quad (124)$$

$$\frac{\partial \delta \theta}{\partial t} = k_B \left(\frac{\Gamma_1}{\gamma_1-1} - \frac{\Gamma_2}{\gamma_2-1} \right) \frac{\partial T}{\partial t}, \quad (125)$$

$$\frac{\partial \delta \theta}{\partial x} = k_B \left(\frac{\Gamma_1}{\gamma_1-1} - \frac{\Gamma_2}{\gamma_2-1} \right) \frac{\partial T}{\partial x}. \quad (126)$$

Here γ is regarded as the average ratio of specific heats of the system (i.e., the total material). Substituting these into the general one-dimensional evolution equations presented in the previous subsection gives the equations for two ideal gases

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0, \quad (127)$$

$$\begin{aligned} \rho \frac{\partial u}{\partial t} + \left(\frac{2}{3} k_d + a_1 \delta w^2 + \frac{P}{\rho} \right) \frac{\partial \rho}{\partial x} + \rho u \frac{\partial u}{\partial x} + \rho [a_2 \delta w^2 + (\Gamma_1 - \Gamma_2) k_B T] \frac{\partial x_1}{\partial x} + \\ 2 \rho a_1 \delta w \frac{\partial \delta w}{\partial x} + \frac{P}{T} \frac{\partial T}{\partial x} + \frac{2}{3} \rho \frac{\partial k_d}{\partial x} - \frac{4}{3} \frac{\partial \Phi}{\partial x} = 0, \end{aligned} \quad (128)$$

$$\rho \frac{\partial x_1}{\partial t} + a_1 \delta w \frac{\partial \rho}{\partial x} + \rho (u + a_2 \delta w) \frac{\partial x_1}{\partial x} + \rho a_1 \frac{\partial \delta w}{\partial x} = 0, \quad (129)$$

$$\begin{aligned} \rho x_1 \frac{\partial u}{\partial t} + \rho a_2 \delta w \frac{\partial x_1}{\partial t} + \rho a_1 \frac{\partial \delta w}{\partial t} + \left[\frac{2}{3} f_1 k_d + a_1 (1 - x_1) \delta w^2 + f_1 \frac{P}{\rho} \right] \frac{\partial \rho}{\partial x} + \\ \rho x_1 [u + (1 - x_1) \delta w] \frac{\partial u}{\partial x} + \rho [a_2 u \delta w + (1 - x_1) (1 - 3x_1) \delta w^2 + f_1 \rho (\Gamma_1 - \Gamma_2) k_B T + \\ \frac{2}{3} \rho k_d \frac{f_1 f_2}{x_1 x_2} - \frac{4}{3} (\Phi + (1 - 2f_1) \delta \Phi) \frac{f_1 f_2}{x_1 x_2}] \frac{\partial x_1}{\partial x} + \rho a_1 [u + 2(1 - x_1) \delta w] \frac{\partial \delta w}{\partial x} + \\ f_1 \frac{P}{T} \frac{\partial T}{\partial x} + \frac{2}{3} f_1 \rho \frac{\partial k_d}{\partial x} - \frac{4}{3} f_1 \left[\frac{\partial \Phi}{\partial x} + (1 - f_1) \frac{\partial \delta \Phi}{\partial x} \right] + \omega a_1 \rho \delta w = 0, \end{aligned} \quad (130)$$

$$\begin{aligned}
& \rho \delta \theta \frac{\partial x_1}{\partial t} + \rho \frac{i}{T} \frac{\partial T}{\partial t} + a_1 \delta \theta \delta w \frac{\partial \rho}{\partial x} + [P - \frac{4}{3}(\Pi_o + Q)] \frac{\partial u}{\partial x} + \\
& [a_2 \rho \delta \theta \delta w + (\frac{4}{3}Q - P) \delta w + \rho u \delta \theta] \frac{\partial x_1}{\partial x} + (a_1 \rho \delta \theta + a_3 P - \frac{4}{3}Q a_3 - \frac{4}{3} \delta Q f_1 f_2) \frac{\partial \delta w}{\partial x} + \\
& \frac{\rho}{T} (iu + a_1 \delta \theta \delta w) \frac{\partial T}{\partial x} + \frac{\partial s_d}{\partial x} + \frac{\partial q}{\partial x} - \rho \epsilon_d = 0,
\end{aligned} \tag{131}$$

$$\begin{aligned}
& \rho \frac{\partial k_d}{\partial t} + k_d a_3 \delta w \frac{\partial \rho}{\partial x} + (\frac{2}{3} \rho k_d - \frac{4}{3} \Gamma_d) \frac{\partial u}{\partial x} - [(\frac{5}{3} - \frac{f_1 f_2}{x_1 x_2}) \rho k_d - \frac{4}{3} \Gamma_d] \delta w \frac{\partial x_1}{\partial x} + \\
& (\frac{5}{3} \rho k_d a_3 - \frac{4}{3} \Gamma_d a_3 - \frac{4}{3} f_1 f_2 \delta \Gamma_d) \frac{\partial \delta w}{\partial x} + \rho (u + a_3 \delta w) \frac{\partial k_d}{\partial x} + \\
& \frac{\partial J_k}{\partial x} - \omega \rho a_1 \delta w^2 + \rho \epsilon_d = 0,
\end{aligned} \tag{132}$$

$$\begin{aligned}
& \rho \frac{\partial l}{\partial t} + a_3 l \delta w \frac{\partial \rho}{\partial x} - C_{lo} \rho l \frac{\partial u}{\partial x} - \rho l \delta w (1 - \frac{f_1 f_2}{x_1 x_2}) \frac{\partial x_1}{\partial x} + a_3 \rho l \frac{\partial \delta w}{\partial x} + \rho (u + a_3 \delta w) \frac{\partial l}{\partial x} + \\
& \frac{\partial J_l}{\partial x} - C_{lo} \rho \sqrt{k_o + k_d} = 0.
\end{aligned} \tag{133}$$

$$\Phi - (\eta + \eta_d) \left[\frac{\partial u}{\partial x} - \delta w \frac{\partial x_1}{\partial x} + (a_3 + f_1 f_2 \frac{\delta \eta}{\eta + \eta_d}) \frac{\partial \delta w}{\partial x} \right] = 0, \tag{134}$$

$$\delta \Phi - \delta \eta \left[\frac{\partial u}{\partial x} - \delta w \frac{\partial x_1}{\partial x} + (1 - f_1 - x_1 + \frac{\eta + \eta_d}{\delta \eta}) \frac{\partial \delta w}{\partial x} \right] = 0, \tag{135}$$

$$s_d + D_s \frac{\partial T}{\partial x} = 0, \quad D_s = C_{id} \left(\frac{\partial i}{\partial T} \right)_\rho D_{dx}, \tag{136}$$

$$q + D_q \frac{\partial T}{\partial x} = 0, \quad D_q = \kappa, \tag{137}$$

$$J_k + D_k \frac{\partial k_d}{\partial x} = 0, \quad D_k = C_{kd} D_{dx}, \tag{138}$$

$$J_l + D_l \frac{\partial l}{\partial x} = 0, \quad D_l = C_{ld} D_{dx}. \tag{139}$$

We have expressed the dynamical variables i , $\delta \theta$, and P as functions of mass fraction x_1 , mass density ρ , and temperature T by using the equations of state. For convenience, we introduce the following parameters

$$\begin{aligned}
a_4 & \equiv 1 - \frac{f_1 f_2}{x_1 x_2}, \\
b_1 & \equiv \frac{P}{\rho} = \frac{\Gamma_1 x_1 k_B T}{f_1}, \quad b_2 \equiv (\Gamma_1 - \Gamma_2) k_B T, \\
c_1 & \equiv \frac{2}{3} k_d - \frac{4}{3 \rho} \Gamma_d, \quad c_2 \equiv \left[\frac{2}{3} k_d - \frac{4}{3 \rho} (\Phi + (1 - 2 f_1) \delta \Phi) \right] \frac{f_1 f_2}{x_1 x_2}.
\end{aligned} \tag{140}$$

Moreover, in order to do the characteristic analysis of the system in the next section, we prefer to eliminate $\frac{\partial u}{\partial t}$ and $\frac{\partial x_1}{\partial t}$ from Eqs.(130) and (131) using their evolution equations. Thus the above equations for ideal gases are finally simplified into

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0, \quad (141)$$

$$\begin{aligned} \frac{\partial u}{\partial t} + \left(\frac{2}{3}k_d + a_1\delta w^2 + b_1\right)\frac{1}{\rho}\frac{\partial \rho}{\partial x} + u\frac{\partial u}{\partial x} + (a_2\delta w^2 + b_2)\frac{\partial x_1}{\partial x} + 2a_1\delta w\frac{\partial \delta w}{\partial x} + \\ \frac{b_1}{T}\frac{\partial T}{\partial x} + \frac{2}{3}\frac{\partial k_d}{\partial x} - \frac{4}{3\rho}\frac{\partial \Phi}{\partial x} = 0, \end{aligned} \quad (142)$$

$$\frac{\partial x_1}{\partial t} + \frac{a_1\delta w}{\rho}\frac{\partial \rho}{\partial x} + (u + a_2\delta w)\frac{\partial x_1}{\partial x} + a_1\frac{\partial \delta w}{\partial x} = 0, \quad (143)$$

$$\begin{aligned} \frac{\partial \delta w}{\partial t} + \left(\frac{2}{3}\frac{a_3}{a_1}k_d + \frac{a_3}{a_1}b_1\right)\frac{1}{\rho}\frac{\partial \rho}{\partial x} + \delta w\frac{\partial u}{\partial x} + (-\delta w^2 + \frac{a_3}{a_1}b_2 + \frac{c_2}{a_1})\frac{\partial x_1}{\partial x} + (u + a_2\delta w)\frac{\partial \delta w}{\partial x} + \\ \frac{a_3}{a_1}\frac{b_1}{T}\frac{\partial T}{\partial x} - \frac{2}{3}\frac{a_3}{a_1}\frac{\partial k_d}{\partial x} - \frac{4}{3\rho}\frac{a_3}{a_1}\frac{\partial \Phi}{\partial x} + \frac{4}{3\rho a_1}f_1f_2\frac{\partial \delta \Phi}{\partial x} + \omega\delta w = 0, \end{aligned} \quad (144)$$

$$\begin{aligned} \frac{\partial T}{\partial t} + \frac{(\gamma-1)T}{b_1\rho}\left[P - \frac{4}{3}(\Pi_o + Q)\right]\frac{\partial u}{\partial x} + \frac{(\gamma-1)T}{b_1}\left(\frac{4}{3\rho}Q - a_4b_1\right)\delta w\frac{\partial x_1}{\partial x} + \\ \left[a_3\left(P - \frac{4}{3}Q\right) + \frac{4}{3}\delta Q f_1f_2\right]\frac{(\gamma-1)T}{b_1\rho}\frac{\partial \delta w}{\partial x} + \left(u + \frac{(\gamma-1)\delta\theta}{b_1}a_1\delta w\right)\frac{\partial T}{\partial x} + \\ \frac{(\gamma-1)T}{b_1\rho}\frac{\partial}{\partial x}(s_d + q) - \frac{(\gamma-1)T}{b_1}\epsilon_d = 0, \end{aligned} \quad (145)$$

$$\begin{aligned} \frac{\partial k_d}{\partial t} + \frac{k_d a_3 \delta w}{\rho}\frac{\partial \rho}{\partial x} + c_1\frac{\partial u}{\partial x} - (c_1 + a_4k_d)\delta w\frac{\partial x_1}{\partial x} + [a_3(c_1 + k_d) - \frac{4}{3\rho}f_1f_2\delta\Gamma_d]\frac{\partial \delta w}{\partial x} + \\ (u + a_3\delta w)\frac{\partial k_d}{\partial x} + \frac{1}{\rho}\frac{\partial J_k}{\partial x} - \omega a_1\delta w^2 + \epsilon_d = 0, \end{aligned} \quad (146)$$

$$\begin{aligned} \frac{\partial l}{\partial t} + \frac{a_3 l \delta w}{\rho}\frac{\partial \rho}{\partial x} - C_{lo}l\frac{\partial u}{\partial x} - a_4 l \delta w\frac{\partial x_1}{\partial x} + a_3 l \frac{\partial \delta w}{\partial x} + (u + a_3\delta w)\frac{\partial l}{\partial x} + \\ \frac{1}{\rho}\frac{\partial J_l}{\partial x} - C_{lo}\sqrt{k_o + k_d} = 0. \end{aligned} \quad (147)$$

$$\Phi - \eta^*\left[\frac{\partial u}{\partial x} - \delta w\frac{\partial x_1}{\partial x} + (a_3 + f_1f_2\frac{\delta\eta}{\eta^*})\frac{\partial \delta w}{\partial x}\right] = 0, \quad \eta^* = \eta + \eta_d, \quad (148)$$

$$\delta\Phi - \delta\eta\left[\frac{\partial u}{\partial x} - \delta w\frac{\partial x_1}{\partial x} + (a_2 - a_3 + \frac{\eta^*}{\delta\eta})\frac{\partial \delta w}{\partial x}\right] = 0, \quad (149)$$

$$S + D_T\frac{\partial T}{\partial x} = 0, \quad S \equiv s_d + q, \quad D_T \equiv D_s + D_q = C_{id}\left(\frac{\partial i}{\partial T}\right)_\rho D_{dx} + \kappa, \quad (150)$$

$$J_k + D_k \frac{\partial k_d}{\partial x} = 0, \quad D_k = C_{kd} D_{dx}, \quad (151)$$

$$J_l + D_l \frac{\partial l}{\partial x} = 0, \quad D_l = C_{ld} D_{dx}. \quad (152)$$

Here we have introduced the total viscosity of the system, $\eta^* = \eta + \eta_d$, i.e., the sum of the real and turbulent viscosities. It is apparent that as long as one of the viscosities (η or η_d) does not vanish, η^* will remain non-zero, so will Φ and $\delta\Phi$.

Manifestly, now the above set of equations can be cast into a matrix form as follows:

$$\mathcal{A} \frac{\partial \mathcal{U}}{\partial t} + \mathcal{B} \frac{\partial \mathcal{U}}{\partial x} + \mathcal{C} = 0, \quad (153)$$

where $\mathcal{A}$ and $\mathcal{B}$ are square matrices consisting of the coefficients of the derivatives of the dynamical variables in the equations. $\mathcal{C}$ is a “source” column vector which contains the non-derivative terms in the equations. $\mathcal{U}$ is a column vector whose components are the dynamical variables. The transpose of $\mathcal{U}$ is given by

$$\mathcal{U}^T \equiv (\rho, u, x_1, \delta w, T, k_d, l, \Phi, \delta\Phi, S, J_k, J_l), \quad (154)$$

We shall see that this expression is helpful throughout the characteristic analysis.

4. CHARACTERISTICS

Now we consider the one-dimensional two-fluid ideal-gas system discussed in the last section as an example.

It is well known that the condition of compatibility for the linear homogeneous system of Eq.(153) to have non-trivial solutions is determined by the characteristic values λ satisfying the equation

$$|\mathcal{B} - \lambda \mathcal{A}| = 0, \quad (155)$$

where $|\mathcal{M}|$ expresses the determinant of matrix $\mathcal{M}$. Note that the source vector $\mathcal{C}$ is by definition not included in the characteristic analysis. Referring to Eq.(153), $\mathcal{A}$ and $\mathcal{B}$ are 12×12 matrices that have the form

$$\mathcal{A} = (a_i \delta_{ij}), \quad \mathcal{B} = (b_{ij}),$$

where $a_i = 1$, for $i = 1, 2, \dots, 7$; $a_i = 0$ for $i = 8, 9, \dots, 12$; δ_{ij} is the Kronecker symbol;

$$b_{11} = u, \quad b_{12} = \rho, \quad b_{1i} = 0, \quad i = 2, 3, \dots, 12; \quad (156)$$

$$b_{21} = \left(\frac{2}{3}k_d + a_1\delta w^2 + b_1\right)/\rho, \quad b_{22} = u, \quad b_{23} = a_2\delta w^2 + b_2, \quad b_{24} = 2a_1\delta w, \\ b_{25} = \frac{b_1}{T}, \quad b_{26} = \frac{2}{3}, \quad b_{27} = 0, \quad b_{28} = -\frac{4}{3\rho}, \quad b_{2i} = 0, \quad i = 9, 10, 11, 12; \quad (157)$$

$$b_{31} = a_1\delta w/\rho, \quad b_{32} = 0, \quad b_{33} = u + a_2\delta w, \quad b_{34} = a_1, \quad b_{3i} = 0, \quad i = 5, 6, \dots, 12; \quad (158)$$

$$b_{41} = \left(\frac{2}{3}k_d + b_1\right)\frac{a_3}{a_1\rho}, \quad b_{42} = \delta w, \quad b_{43} = -\delta w^2 + \frac{a_3}{a_1}b_2 + \frac{c_2}{a_1}, \quad b_{44} = u + a_2\delta w, \\ b_{45} = \frac{a_3b_1}{a_1T}, \quad b_{46} = \frac{2a_3}{3a_1}, \quad b_{47} = 0, \quad b_{48} = -\frac{4}{3\rho}\frac{a_3}{a_1}, \\ b_{49} = -\frac{4}{3\rho}\frac{f_1f_2}{a_1}, \quad b_{4i} = 0, \quad i = 10, 11, 12; \quad (159)$$

$$b_{51} = 0, \quad b_{52} = \frac{(\gamma-1)T}{b_1\rho}\left[P - \frac{4}{3}(\Pi_o + Q)\right], \quad b_{53} = \frac{(\gamma-1)T}{b_1\rho}\left(\frac{4}{3}Q - a_4P\right)\delta w, \\ b_{54} = \frac{\gamma-1}{b_1\rho}\left[a_3\left(P - \frac{4}{3}\Pi_o\right) + \frac{4}{3}\delta Q f_1f_2\right], \quad b_{55} = u + \frac{(\gamma-1)\delta\theta}{b_1}a_1\delta w, \\ b_{5i} = 0, \quad i = 6, 7, 8, 9, 11, 12, \quad b_{510} = \frac{(\gamma-1)T}{b_1\rho}; \quad (160)$$

$$b_{61} = \frac{a_3k_d\delta w}{\rho}, \quad b_{62} = c_1, \quad b_{63} = -(c_1 + a_4k_d)\delta w, \quad b_{64} = (c_1 + k_d)a_3 - \frac{4}{3\rho}f_1f_2\delta\Gamma_d, \\ b_{65} = 0, \quad b_{66} = u + a_3\delta w, \quad b_{6i} = 0, \quad i = 7, 8, 9, 10, 12, \quad b_{611} = \frac{1}{\rho}; \quad (161)$$

$$b_{71} = \frac{a_3l\delta w}{\rho}, \quad b_{72} = -C_{lo}l, \quad b_{73} = -a_4l\delta w, \quad b_{74} = a_3l, \quad b_{75} = b_{76} = 0, \\ b_{77} = u + a_3\delta w, \quad b_{7i} = 0, \quad i = 8, 9, 10, 11, \quad b_{712} = 1/\rho; \quad (162)$$

$$b_{81} = 0, \quad b_{82} = -\eta^*, \quad b_{83} = \eta^*\delta w, \quad b_{84} = -\eta^*(a_3 + f_1f_2\frac{\delta\eta}{\eta^*}), \\ b_{8i} = 0, \quad i = 5, 6, \dots, 12; \quad (163)$$

$$b_{91} = 0, \quad b_{92} = -\delta\eta, \quad b_{93} = \delta w\delta\eta, \quad b_{94} = -(a_2 - a_3)\delta\eta - \eta^*, \\ b_{9i} = 0, \quad i = 5, 6, \dots, 12; \quad (164)$$

$$b_{10i} = 0, \quad i \neq 5, \quad b_{105} = D_T; \quad (165)$$

$$b_{11i} = 0, \quad i \neq 6, \quad b_{116} = D_k; \quad (166)$$

$$b_{12i} = 0, \quad i \neq 7, \quad b_{127} = D_l. \quad (167)$$

The source vector $\mathcal{C}$ is given by

$$\mathcal{C}^T \equiv \left(0, 0, 0, \omega\delta w, -\frac{(\gamma-1)T}{b_1}\epsilon_d, -\omega a_1\delta w^2 + \epsilon_d, -C_{lo}\sqrt{k_o + k_d}, \Phi, \delta\Phi, S, J_k, J_l\right). \quad (168)$$

Thus, the characteristic equation (155) can be expressed as

$$\begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} - \lambda & b_{23} & b_{24} & b_{25} & b_{26} & 0 & b_{28} & 0 & 0 & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} - \lambda & b_{45} & b_{46} & 0 & b_{48} & b_{49} & 0 & 0 & 0 \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} - \lambda & 0 & 0 & 0 & 0 & b_{510} & 0 & 0 \\ b_{61} & b_{62} & b_{63} & b_{64} & 0 & b_{66} - \lambda & 0 & 0 & 0 & 0 & b_{611} & 0 \\ b_{71} & b_{72} & b_{73} & b_{74} & 0 & 0 & b_{77} - \lambda & 0 & 0 & 0 & 0 & b_{712} \\ 0 & b_{82} & b_{83} & b_{84} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{92} & b_{93} & b_{94} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & D_l & 0 & 0 & 0 & 0 & 0 \end{vmatrix} = 0. \quad (169)$$

In the remainder of this section, we will analyze the characteristics, λ , of the model by solving this equation under various physical situations. In order to see the individual effects of the dissipations on the characteristics of the model, we divide our analysis into two classes: A. in the situations with viscosity and with or without various diffusions; B. in the situations without viscosity and with or without various diffusions.

A. Non-zero viscosity ($\eta^* \neq 0$).

Case 1 Two-phase flow with all diffusions ($D_T \neq 0$, $D_k \neq 0$, $D_l \neq 0$). In this case, the characteristic equation is

$$b_{28}b_{49}b_{510}b_{611}b_{712}D_TD_kD_l \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ 0 & b_{82} & b_{83} & b_{84} \\ 0 & b_{92} & b_{93} & b_{94} \end{vmatrix} = 0, \quad (170)$$

i.e.,

$$\frac{16}{9} \frac{1}{\rho^5} \frac{f_1 f_2}{a_1} \frac{(\gamma - 1)T}{b_1} D_TD_kD_l \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ 0 & b_{82} & b_{83} & b_{84} \\ 0 & b_{92} & b_{93} & b_{94} \end{vmatrix} = 0. \quad (171)$$

For non-zero D_T , D_k , and D_l , this equation requires

$$\begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ 0 & b_{82} & b_{83} & b_{84} \\ 0 & b_{92} & b_{93} & b_{94} \end{vmatrix} = 0. \quad (172)$$

Utilizing the properties of matrices, do (column 2) $\times \frac{b_{11}-\lambda}{-b_{12}} + (\text{Column 1})$, then this equation is simplified into

$$b_{12} \begin{vmatrix} b_{31} & b_{33} - \lambda & b_{34} \\ -b_{82}(b_{11} - \lambda)/b_{12} & b_{83} & b_{84} \\ -b_{92}(b_{11} - \lambda)/b_{12} & b_{93} & b_{94} \end{vmatrix} = 0. \quad (173)$$

Since $b_{12} = \rho \neq 0$, we must have

$$\begin{vmatrix} b_{31} & b_{33} - \lambda & b_{34} \\ -b_{82}(b_{11} - \lambda)/b_{12} & b_{83} & b_{84} \\ -b_{92}(b_{11} - \lambda)/b_{12} & b_{93} & b_{94} \end{vmatrix} = 0. \quad (174)$$

Expand this equation and note that

$$b_{83}b_{94} - b_{93}b_{84} = \delta w(b_{92}b_{84} - b_{82}b_{94}) \quad \text{and} \quad b_{82}b_{93} = b_{83}b_{92}$$

for all values of η and η_d , we then further have

$$[b_{31}\delta w - (b_{33} - \lambda)(b_{11} - \lambda)\frac{1}{b_{12}}](b_{82}b_{94} - b_{92}b_{84}) = 0.$$

Substituting the b_{ij} s into this equation gives

$$\frac{1}{\rho}[a_1\delta w^2 - (u + a_2\delta w - \lambda)(u - \lambda)][\eta^{*2} + (f_2 - f_1)\eta^*\delta\eta - f_1f_2\delta\eta^2] = 0. \quad (175)$$

For non-zero η^* , the second bracketed expression does not vanish in general, so this leads to

$$a_1\delta w^2 - (u + a_2\delta w - \lambda)(u - \lambda) = 0,$$

and two real roots λ are obtained:

$$\begin{aligned} \lambda_1 &= u + x_2\delta w = u + w_1, \\ \lambda_2 &= u - x_1\delta w = u + w_2. \end{aligned} \quad (176)$$

Therefore in the presence of all dissipations, the new hybrid turbulent mix model, like other two-phase flow models, has all real characteristics.

Case 2 Two-phase flow without temperature diffusion ($D_T = 0$, $D_k \neq 0$, $D_l \neq 0$). In this case, the characteristic equation is

$$b_{28}b_{49}b_{611}b_{712}D_kD_l(b_{55} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ 0 & b_{82} & b_{83} & b_{84} \\ 0 & b_{92} & b_{93} & b_{94} \end{vmatrix} = 0, \quad (177)$$

i.e.,

$$\frac{16}{9} \frac{1}{\rho^4} \frac{f_1 f_2}{a_1} D_k D_l (b_{55} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ 0 & b_{82} & b_{83} & b_{84} \\ 0 & b_{92} & b_{93} & b_{94} \end{vmatrix} = 0. \quad (178)$$

For non-zero D_k and D_l , three real roots λ are obtained:

$$\begin{aligned}\lambda_1 &= u + x_2 \delta w = u + w_1, \\ \lambda_2 &= u - x_1 \delta w = u + w_2, \\ \lambda_3 &= b_{55} = u + \frac{(\gamma - 1) \delta \theta}{b_1} a_1 \delta w.\end{aligned}\quad (179)$$

Again the model has all real characteristics.

Case 3 Two-phase flow without turbulent kinetic energy diffusion ($D_k = 0$, $D_T \neq 0$, $D_l \neq 0$).
In this case, the characteristic equation is

$$b_{28} b_{49} b_{510} b_{712} D_T D_l (b_{66} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ 0 & b_{82} & b_{83} & b_{84} \\ 0 & b_{92} & b_{93} & b_{94} \end{vmatrix} = 0, \quad (180)$$

i.e.,

$$\frac{16}{9} \frac{1}{\rho^4} \frac{f_1 f_2}{a_1} \frac{(\gamma - 1) T}{b_1} D_T D_l (b_{66} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ 0 & b_{82} & b_{83} & b_{84} \\ 0 & b_{92} & b_{93} & b_{94} \end{vmatrix} = 0. \quad (181)$$

For non-zero D_T and D_l , three real roots λ are obtained:

$$\begin{aligned}\lambda_1 &= u + x_2 \delta w = u + w_1, \\ \lambda_2 &= u - x_1 \delta w = u + w_2, \\ \lambda_3 &= b_{66} = u + (f_1 - x_1) \delta w = u + w.\end{aligned}\quad (182)$$

Case 4 Two-phase flow without turbulent length scale diffusion ($D_l = 0$, $D_T \neq 0$, $D_k \neq 0$).
In this case, the characteristic is

$$b_{28} b_{49} b_{510} b_{611} D_T D_k (b_{77} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ 0 & b_{82} & b_{83} & b_{84} \\ 0 & b_{92} & b_{93} & b_{94} \end{vmatrix} = 0, \quad (183)$$

i.e.,

$$\frac{16}{9} \frac{1}{\rho^4} \frac{f_1 f_2}{a_1} \frac{(\gamma - 1) T}{b_1} D_T D_k (b_{77} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ 0 & b_{82} & b_{83} & b_{84} \\ 0 & b_{92} & b_{93} & b_{94} \end{vmatrix} = 0. \quad (184)$$

It is evident that the roots λ are the same as in Case 3:

$$\begin{aligned}\lambda_1 &= u + x_2 \delta w = u + w_1, \\ \lambda_2 &= u - x_1 \delta w = u + w_2, \\ \lambda_3 &= b_{77} = u + (f_1 - x_1) \delta w = u + w.\end{aligned}\quad (185)$$

Case 5 Two-phase flow without turbulent diffusions ($D_k = 0 = D_l$, $D_T \neq 0$). In this case, the characteristic equation is

$$b_{28}b_{49}b_{510}D_T(b_{66} - \lambda)(b_{77} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ 0 & b_{82} & b_{83} & b_{84} \\ 0 & b_{92} & b_{93} & b_{94} \end{vmatrix} = 0, \quad (186)$$

i.e.,

$$\frac{16}{9} \frac{1}{\rho^3} \frac{f_1 f_2}{a_1} \frac{(\gamma - 1)T}{b_1} (b_{66} - \lambda)(b_{77} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ 0 & b_{82} & b_{83} & b_{84} \\ 0 & b_{92} & b_{93} & b_{94} \end{vmatrix} = 0. \quad (187)$$

Four real roots λ are obtained:

$$\begin{aligned} \lambda_1 &= u + x_2 \delta w = u + w_1, \\ \lambda_2 &= u - x_1 \delta w = u + w_2, \\ \lambda_3 &= b_{66} = u + (f_1 - x_1) \delta w = u + w, \\ \lambda_4 &= b_{77} = u + (f_1 - x_1) \delta w = u + w. \end{aligned} \quad (188)$$

Case 6 Two-phase flow without temperature and turbulent kinetic energy diffusions ($D_T = 0 = D_k$, $D_l \neq 0$). In this case, the characteristic equation is

$$b_{28}b_{49}b_{712}D_l(b_{55} - \lambda)(b_{66} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ 0 & b_{82} & b_{83} & b_{84} \\ 0 & b_{92} & b_{93} & b_{94} \end{vmatrix} = 0, \quad (189)$$

i.e.,

$$\frac{16}{9} \frac{1}{\rho^3} \frac{f_1 f_2}{a_1} D_l (b_{55} - \lambda)(b_{66} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ 0 & b_{82} & b_{83} & b_{84} \\ 0 & b_{92} & b_{93} & b_{94} \end{vmatrix} = 0. \quad (190)$$

For non-zero D_l , four real roots λ are obtained:

$$\begin{aligned} \lambda_1 &= u + x_2 \delta w = u + w_1, \\ \lambda_2 &= u - x_1 \delta w = u + w_2, \\ \lambda_3 &= b_{55} = u + \frac{(\gamma - 1) \delta \theta}{b_1} a_1 \delta w, \\ \lambda_4 &= b_{66} = u + (f_1 - x_1) \delta w = u + w. \end{aligned} \quad (191)$$

Case 7 Two-phase flow without temperature and turbulent length scale diffusions ($D_T = 0 = D_l$, $D_k \neq 0$). In this case, the characteristic equation is

$$b_{28}b_{49}b_{611}D_k(b_{55} - \lambda)(b_{77} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ 0 & b_{82} & b_{83} & b_{84} \\ 0 & b_{92} & b_{93} & b_{94} \end{vmatrix} = 0, \quad (192)$$

i.e.,

$$\frac{16}{9} \frac{1}{\rho^3} \frac{f_1 f_2}{a_1} D_k(b_{55} - \lambda)(b_{77} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ 0 & b_{82} & b_{83} & b_{84} \\ 0 & b_{92} & b_{93} & b_{94} \end{vmatrix} = 0. \quad (193)$$

If $D_k \neq 0$, then the roots λ are the same as in *Case 6*:

$$\begin{aligned} \lambda_1 &= u + x_2 \delta w = u + w_1, \\ \lambda_2 &= u - x_1 \delta w = u + w_2, \\ \lambda_3 &= b_{55} = u + \frac{(\gamma - 1)\delta\theta}{b_1} a_1 \delta w, \\ \lambda_4 &= b_{77} = u + (f_1 - x_1) \delta w = u + w. \end{aligned} \quad (194)$$

Case 8 Two-phase flow without any diffusions ($D_T = D_l = D_k = 0$). In this case, the characteristic equation is

$$b_{28}b_{49}(b_{55} - \lambda)(b_{66} - \lambda)(b_{77} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ 0 & b_{82} & b_{83} & b_{84} \\ 0 & b_{92} & b_{93} & b_{94} \end{vmatrix} = 0, \quad (195)$$

i.e.,

$$\frac{1}{\rho^2} (b_{55} - \lambda)(b_{66} - \lambda)(b_{77} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ 0 & b_{82} & b_{83} & b_{84} \\ 0 & b_{92} & b_{93} & b_{94} \end{vmatrix} = 0. \quad (196)$$

For nonzero η^* , five real roots λ are obtained:

$$\begin{aligned} \lambda_1 &= u + x_2 \delta w = u + w_1, \\ \lambda_2 &= u - x_1 \delta w = u + w_2, \\ \lambda_3 &= b_{55} = u + \frac{(\gamma - 1)\delta\theta}{b_1} a_1 \delta w, \\ \lambda_4 &= b_{66} = u + (f_1 - x_1) \delta w = u + w, \\ \lambda_5 &= b_{77} = u + (f_1 - x_1) \delta w = u + w. \end{aligned} \quad (197)$$

Up to now we found that the turbulent kinetic energy and length scale diffusions (D_k and D_l) have the same impacts on the characteristics of the model, the existence or vanishing of either one results in the same characteristics. This same behaviour is also found for the real and turbulent viscosities (η and η_d), i.e., the vanishing of either one results in the same

characteristics. Thus $\eta^* \neq 0$ always gives two real roots: $\lambda_1 = u + w_1$ and $\lambda_2 = u + w_2$. $D_T = 0$ produces a third real root: $\lambda_3 = u + \frac{(\gamma-1)\delta\theta}{b_1}a_1\delta w$. $D_k = 0$ and $D_l = 0$ each produce a fourth real root: $\lambda_4 = u + w$.

B. Zero viscosity ($\eta^* = 0$, so $\eta = 0$, $\delta\eta = 0$, and $\eta_d = 0$).

Case 1 Two-phase flow with all diffusions ($D_T \neq 0$, $D_l \neq 0$, $D_k \neq 0$). In this case, the characteristic equation is

$$b_{510}b_{611}b_{712}D_TD_kD_l \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{21} & b_{22} - \lambda & b_{23} & b_{24} \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} - \lambda \end{vmatrix} = 0, \quad (198)$$

i.e.,

$$\frac{1}{\rho^3} \frac{(\gamma-1)T}{b_1} D_TD_kD_l \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{21} & b_{22} - \lambda & b_{23} & b_{24} \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} - \lambda \end{vmatrix} = 0, \quad (199)$$

For non-zero D_T , D_k , and D_l , this equation requires

$$\begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{21} & b_{22} - \lambda & b_{23} & b_{24} \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} - \lambda \end{vmatrix} = 0. \quad (200)$$

Substituting the b_{ij} s into this equation gives

$$\begin{vmatrix} u - \lambda & \rho & 0 & 0 \\ (\frac{2}{3}k_d + a_1\delta w^2 + b_1)/\rho & u - \lambda & a_2\delta w^2 + b_2 & 2a_1\delta w \\ a_1\delta w/\rho & 0 & u + a_2\delta w - \lambda & a_1 \\ a_3(\frac{2}{3}k_d + b_1)/(a_1\rho) & \delta w & -\delta w^2 + (a_3/a_1)b_2 + (c_2/a_1) & u + a_2\delta w - \lambda \end{vmatrix} = 0. \quad (201)$$

By doing $(\text{Row } 1) \times (-\frac{\delta w}{\rho}) + (\text{Row } 4)$, this equation can be reduced to

$$\begin{vmatrix} u - \lambda & \rho & 0 & 0 \\ \frac{1}{\rho}(\frac{2}{3}k_d + a_1\delta w^2 + b_1) & u - \lambda & a_2\delta w^2 + b_2 & 2a_1\delta w \\ \frac{1}{\rho}a_1\delta w & 0 & u + a_2\delta w - \lambda & a_1 \\ \frac{1}{\rho}[(\frac{2}{3}k_d + b_1)\frac{a_3}{a_1} - (u - \lambda)\delta w] & 0 & -\delta w^2 + \frac{a_3}{a_1}b_2 + \frac{c_2}{a_1} & u + a_2\delta w - \lambda \end{vmatrix} = 0. \quad (202)$$

For convenience, let $y = u - \lambda$, and add $(\text{Row } 3) \times (-2\delta w)$ to $(\text{Row } 2)$ giving

$$\begin{vmatrix} y & \rho & 0 & 0 \\ \frac{1}{\rho}(\frac{2}{3}k_d + a_1\delta w^2 + b_1 - 2a_1\delta w^2) & y & a_2\delta w^2 + b_2 - 2(a_2\delta w + y)\delta w & 0 \\ \frac{1}{\rho}a_1\delta w & 0 & a_2\delta w + y & a_1 \\ \frac{1}{\rho}[(\frac{2}{3}k_d + b_1)\frac{a_3}{a_1} - y\delta w] & 0 & -\delta w^2 + \frac{a_3}{a_1}b_2 + \frac{c_2}{a_1} & a_2\delta w + y \end{vmatrix} = 0. \quad (203)$$

Add $(\text{Col } 4) \times (-\delta w/\rho)$ to $(\text{Col } 1)$ giving

$$\begin{vmatrix} \frac{1}{\rho}(\frac{2}{3}k_d + a_1\delta w^2 + b_1 - 2a_1\delta w^2) & \rho & 0 & 0 \\ y & a_2\delta w^2 + b_2 - 2(a_2\delta w + y)\delta w & 0 & 0 \\ 0 & 0 & a_2\delta w + y & a_1 \\ \frac{1}{\rho}[\frac{a_3}{a_1}(\frac{2}{3}k_d + b_1) - 2y\delta w - a_2\delta w^2] & 0 & -\delta w^2 + \frac{a_3}{a_1}b_2 + \frac{c_2}{a_1} & a_2\delta w + y \end{vmatrix} = 0, \quad (204)$$

i.e.,

$$\begin{vmatrix} \frac{1}{\rho}(\frac{2}{3}k_d + b_1 - a_1\delta w^2) & \rho & 0 & 0 \\ y & b_2 - 2y\delta w - a_2\delta w^2 & 0 & 0 \\ 0 & 0 & a_2\delta w + y & a_1 \\ \frac{1}{\rho}[\frac{a_3}{a_1}(\frac{2}{3}k_d + b_1) - 2y\delta w - a_2\delta w^2] & 0 & -\delta w^2 + \frac{a_3}{a_1}b_2 + \frac{c_2}{a_1} & a_2\delta w + y \end{vmatrix} = 0. \quad (205)$$

Expanding according to the second row then gives

$$\begin{aligned} & [y^2 - (\frac{2}{3}k_d + b_1 - a_1\delta w^2)] \times \begin{vmatrix} a_2\delta w + y & a_1 \\ -\delta w^2 + \frac{a_3}{a_1}b_2 + \frac{c_2}{a_1} & a_2\delta w + y \end{vmatrix} - \\ & a_1[b_2 - 2y\delta w - a_2\delta w^2] \times \begin{vmatrix} \frac{1}{\rho}[\frac{a_3}{a_1}(\frac{2}{3}k_d + b_1) - 2y\delta w - a_2\delta w^2] & \rho \end{vmatrix} = 0, \end{aligned} \quad (206)$$

i.e.,

$$\begin{aligned} & [y^2 - (\frac{2}{3}k_d + b_1 - a_1\delta w^2)] \times [(a_2\delta w + y)^2 + a_1\delta w^2 - a_3b_2 - c_2] - \\ & (b_2 - 2y\delta w - a_2\delta w^2) \times [\frac{2}{3}a_3k_d + a_3b_1 - 2a_1y\delta w - a_1a_2\delta w^2] = 0. \end{aligned} \quad (207)$$

This is a forth order polynomial in y :

$$y^4 + A_3y^3 + A_2y^2 + A_1y + A_0 = 0, \quad (208)$$

where

$$A_3 = 2a_2\delta w;$$

$$A_2 = (a_2^2 - 2a_1)\delta w^2 - a_3b_2 - c_2 - b_1 - \frac{2}{3}k_d;$$

$$A_1 = \frac{4}{3}\delta w(a_3 - a_2)k_d + 2(a_3b_1 - a_2b_1 + a_1b_2)\delta w - 2a_1a_2\delta w^3;$$

$$A_0 = -(\frac{2}{3}k_d - a_1\delta w^2 + b_1)[(a_1 + a_2^2)\delta w^2 - a_3b_2 - c_2] - (\frac{2}{3}a_3k_d + a_3b_1 - a_1a_2\delta w^2)(b_2 - a_2\delta w^2).$$

In order to determine the properties (either real or complex) of the roots of this quartic, we use the following useful lemma (i.e., the W-lemma):

Lemma 1 (W). *Consider the real quartic*

$$P(y) = y^4 + A_3y^3 + A_2y^2 + A_1y + A_0.$$

Necessary and sufficient conditions for P to have four real roots are that dP/dy have three real roots, $y_1 \leq y_2 \leq y_3$, such that the following three inequalities are satisfied

$$P(y_1) \leq 0, \quad P(y_2) \geq 0, \quad \text{and} \quad P(y_3) \leq 0.$$

The plan of the proof becomes evident when the graph of a typical quartic is considered: roughly the graph has the shape of a "W" with the three local extrema occurring at y_1, y_2, y_3 . The proof can be found in [11,12].

To apply the *Lemma* to equation (208), we must find the roots of the following cubic equation:

$$P' \equiv \frac{1}{4} \frac{dP}{dy} = y^3 + B_2 y^2 + B_1 y + B_0, \quad (209)$$

where

$$\begin{aligned} B_2 &= \frac{3}{4} A_3 = \frac{3}{2} a_2 \delta w, \\ B_1 &= \frac{1}{2} A_2 = (a_2^2 - 2a_1) \delta w^2 - \frac{1}{2} (a_3 b_2 + c_2 + b_1) - \frac{1}{3} k_d \\ &= -\frac{1}{2} (a_3 b_2 + b_1) (1 + \alpha), \\ B_0 &= \frac{1}{4} A_1 = \frac{1}{3} \delta w (a_3 - a_2) k_d + \frac{1}{2} (a_3 b_1 - a_2 b_1 + a_1 b_2 - a_1 a_2 \delta w^2) \delta w \\ &= \frac{1}{2} (a_3 b_1 - a_2 b_1 + a_1 b_2) \delta w (1 + \beta). \end{aligned} \quad (210)$$

Here

$$\begin{aligned} \alpha &\equiv \left[\frac{2}{3} k_d + c_2 - (a_2^2 - 2a_1) \delta w^2 \right] / (a_3 b_2 + b_1) \\ &= \frac{a_1}{a_1 + a_3^2} \left[\frac{2}{3} \left(1 + \frac{f_1 f_2}{x_1 x_2} \right) \frac{k_d}{b_1} - (a_2^2 - 2a_1) \frac{\delta w^2}{b_1} \right], \\ \beta &\equiv \left[\frac{2}{3} (a_3 - a_2) k_d - a_1 a_2 \delta w^2 \right] / (a_3 b_1 - a_2 b_1 + a_1 b_2) \\ &= \frac{2(a_3 - a_2)}{3(2a_3 - a_2)} \frac{k_d}{b_1} - \frac{a_1 a_2}{2a_3 - a_2} \frac{\delta w^2}{b_1}. \end{aligned} \quad (211)$$

Let $y = x - \frac{1}{3} B_2$, then Eq.(209) becomes

$$x^3 + C_1 x + C_0 = 0, \quad (212)$$

where

$$\begin{aligned} C_1 &= B_1 - \frac{1}{3} B_2^2 \\ &= -\frac{1}{2} \frac{a_3^2 + a_1}{a_1} b_1 (1 + \alpha + \frac{3}{2} \frac{a_1 a_2}{a_1 + a_3^2} \frac{\delta w^2}{b_1}) \\ &= -\frac{1}{2} \frac{a_3^2 + a_1}{a_1} b_1 \left[1 + \frac{a_1}{a_1 + a_3^2} \left(\frac{2}{3} \left(1 + \frac{f_1 f_2}{x_1 x_2} \right) \frac{k_d}{b_1} + \frac{1}{2} \frac{\delta w^2}{b_1} \right) \right] \\ &= -\frac{1}{2} \frac{a_3^2 + a_1}{a_1} b_1 (1 + \gamma_1), \end{aligned} \quad (213)$$

and

$$\begin{aligned}
C_0 &= \frac{1}{27}(2B_2^3 - 9B_2B_1 + 27B_0) \\
&= \frac{1}{4}a_2^3\delta w^3 + \left[\frac{1}{4}\frac{a_3^2 - a_1}{a_1}a_2 + a_3 + \frac{1}{4}\frac{a_3^2}{a_1}a_2\alpha - \frac{1}{2}a_2\left(\beta - \frac{1}{2}\alpha\right) + a_3\beta\right]\delta w b_1 \\
&= \left(\frac{1}{4}\frac{a_3^2 - a_1}{a_1}a_2 + a_3\right)\delta w b_1(1 + \gamma_2),
\end{aligned} \tag{214}$$

in which γ_1 and γ_2 are respectively defined as

$$\gamma_1 = \frac{a_1}{a_1 + a_3^2} \left[\frac{2}{3} \left(1 + \frac{f_1 f_2}{x_1 x_2} \right) \frac{k_d}{b_1} + \frac{1}{2} \frac{\delta w^2}{b_1} \right], \tag{215}$$

$$\begin{aligned}
\gamma_2 &= \frac{1}{\frac{1}{4}\frac{a_3^2 - a_1}{a_1}a_2 + a_3} \left[\frac{1}{4}a_2^3 \frac{\delta w^2}{b_1} + \frac{1}{4}\frac{a_3^2}{a_1}a_2\alpha - \frac{1}{2}a_2\left(\beta - \frac{1}{2}\alpha\right) + a_3\beta \right] \\
&= \frac{1}{\frac{1}{4}\frac{a_3^2 - a_1}{a_1}a_2 + a_3} \left[\frac{1}{4}a_2^3 \frac{\delta w^2}{b_1} + \frac{1}{4}a_2 \frac{a_3^2 + a_1}{a_1} \alpha + \frac{1}{2}(2a_3 - a_2)\beta \right] \\
&= \frac{1}{\frac{1}{4}\frac{a_3^2 - a_1}{a_1}a_2 + a_3} \cdot \frac{1}{6} \cdot \left[a_2 \left(1 + \frac{f_1 f_2}{x_1 x_2} \right) + 2(a_3 - a_2) \right] \frac{k_d}{b_1}.
\end{aligned} \tag{216}$$

For solution let

$$E = \left[-\frac{C_0}{2} + \left(\frac{C_0^2}{4} + \frac{C_1^3}{27} \right)^{1/2} \right]^{1/3} \quad \text{and} \quad \tilde{E} = \left[-\frac{C_0}{2} - \left(\frac{C_0^2}{4} + \frac{C_1^3}{27} \right)^{1/2} \right]^{1/3},$$

then the values of x are given by

$$x = E + \tilde{E}, \quad -\frac{E + \tilde{E}}{2} + \frac{E - \tilde{E}}{2}\sqrt{-3}, \quad -\frac{E + \tilde{E}}{2} - \frac{E - \tilde{E}}{2}\sqrt{-3}. \tag{217}$$

Thus, the three roots y of Eq.(209) will be

$$y_3 = -\frac{1}{2}a_2\delta w + E + \tilde{E}, \tag{218}$$

$$y_2 = -\frac{1}{2}a_2\delta w - \frac{E + \tilde{E}}{2} + \frac{E - \tilde{E}}{2}\sqrt{-3}, \tag{219}$$

$$y_1 = -\frac{1}{2}a_2\delta w - \frac{E + \tilde{E}}{2} - \frac{E - \tilde{E}}{2}\sqrt{-3}. \tag{220}$$

For real B_2 , B_1 , and B_0 , these three roots are all real *when and only when*

$$\frac{C_0^2}{4} + \frac{C_1^3}{27} \leq 0, \tag{221}$$

since E and $\bar{E}$ are then complex conjugates. Substituting the expressions for C_0 and C_1 into the above condition gives

$$\begin{aligned}
& \frac{C_0^2}{4} + \frac{C_1^3}{27} \\
&= \frac{1}{4} \left(\frac{1}{4} \frac{a_3^2 - a_1}{a_1} a_2 + a_3 \right)^2 \delta w^2 b_1^2 (1 + \gamma_2)^2 + \frac{1}{27} \left(-\frac{1}{2} \frac{a_3^2 + a_1}{a_1} b_1 \right)^3 (1 + \gamma_1)^3 \\
&= -\frac{1}{6^3} \left(\frac{a_3^2 + a_1}{a_1} b_1 \right)^3 \left\{ 1 + 3\gamma_1 + 3\gamma_1^2 + \gamma_1^3 - 54a_1 \frac{[\frac{1}{4}(a_3^2 - a_1)a_2 + a_1 a_3]^2 (1 + 2\gamma_2 + \gamma_2^2)}{(a_3^2 + a_1)^3} \frac{\delta w^2}{b_1} \right\} \\
&= -\frac{1}{6^3} \left(\frac{a_3^2 + a_1}{a_1} b_1 \right)^3 G,
\end{aligned} \tag{222}$$

where

$$G = 1 + 3\gamma_1 + 3\gamma_1^2 + \gamma_1^3 - 54a_1 \frac{[\frac{1}{4}(a_3^2 - a_1)a_2 + a_1 a_3]^2 (1 + 2\gamma_2 + \gamma_2^2)}{(a_3^2 + a_1)^3} \frac{\delta w^2}{b_1}. \tag{223}$$

For simplification, we introduce the following parameters:

$$\xi \equiv \frac{a_1}{a_3^2 + a_1} \frac{k_d}{b_1}, \quad \zeta \equiv \frac{a_1}{a_3^2 + a_1} \frac{\delta w^2}{b_1}, \quad \theta \equiv \frac{\xi}{\zeta} = \frac{k_d}{\delta w^2}, \quad \theta' \equiv \left(1 + \frac{f_1 f_2}{x_1 x_2}\right) \theta. \tag{224}$$

Then Eqs.(215)-(216) become

$$\begin{aligned}
\gamma_1 &= \frac{2}{3} \left(1 + \frac{f_1 f_2}{x_1 x_2}\right) \xi + \frac{1}{2} \zeta = \zeta \left(\frac{2}{3} \theta' + \frac{1}{2}\right), \\
\gamma_2 &= \frac{2}{3} \frac{a_3^2 + a_1}{(a_3^2 - a_1)a_2 + 4a_1 a_3} \left(a_2 \frac{f_1 f_2}{x_1 x_2} + 2f_1 - 1\right) \frac{k_d}{b_1} = \frac{2}{3} \frac{1}{\bar{F}} \frac{a_2 \frac{f_1 f_2}{x_1 x_2} + 2f_1 - 1}{1 + \frac{f_1 f_2}{x_1 x_2}} \zeta \theta'.
\end{aligned}$$

Here

$$\bar{F} = \frac{(a_3^2 - a_1)a_2 + 4a_1 a_3}{a_3^2 + a_1} = \frac{\Gamma_1^2 x_1 - \Gamma_2^2 x_2}{\Gamma_1^2 x_1 + \Gamma_2^2 x_2} = \frac{x_1 - d^2 x_2}{x_1 + d^2 x_2} \tag{225}$$

is a function of x_1 and Γ_i ($i = 1, 2$), and we have used the expressions:

$$\begin{aligned}
a_3 = f_1 - x_1 &= \frac{(\Gamma_1 - \Gamma_2)x_1 x_2}{\Gamma_1 x_1 + \Gamma_2 x_2}, \quad \frac{a_1}{a_3} = \frac{\Gamma_1 x_1 + \Gamma_2 x_2}{\Gamma_1 - \Gamma_2}, \\
\frac{a_1}{a_3^2} &= \frac{(\Gamma_1 x_1 + \Gamma_2 x_2)^2}{(\Gamma_1 - \Gamma_2)^2 x_1 x_2}, \quad \text{and} \quad d \equiv \frac{\Gamma_2}{\Gamma_1}.
\end{aligned}$$

Incorporating these expressions into the function G gives

$$G = 1 + H_1 \left(\frac{\delta w^2}{b_1}\right) + H_2 \left(\frac{\delta w^4}{b_1^2}\right) + H_3 \left(\frac{\delta w^6}{b_1^3}\right), \tag{226}$$

where

$$H_1 = 3\gamma_1 - \frac{54}{16} \bar{F}^2 \frac{\delta w^2}{b_1} = \zeta \left(2\theta' + \frac{3}{2} - \frac{54}{16} \bar{F}^2\right), \tag{227}$$

$$\begin{aligned}
H_2 &= 3\gamma_1^2 - \frac{54}{16} \cdot 2\gamma_2\zeta = \zeta^2 \left[\frac{4}{3}\theta'^2 + \left(2 - \frac{9}{2}\tilde{F} \frac{a_2 \frac{f_1 f_2}{x_1 x_2} + 2f_1 - 1}{1 + \frac{f_1 f_2}{x_1 x_2}} \right) \theta' + \frac{3}{4} \right] \\
&= \zeta^2 \left(\frac{4}{3}\theta'^2 + g_1\theta' + \frac{3}{4} \right),
\end{aligned} \tag{228}$$

$$\begin{aligned}
H_3 &= \gamma_1^3 - \frac{54}{16} \gamma_2^2 \zeta = \zeta^3 \left\{ \frac{8}{27}\theta'^3 + \left[\frac{2}{3} - \frac{3}{2}a_2^2 - 6(a_3 - a_2) \frac{a_2 \frac{f_1 f_2}{x_1 x_2} + a_3}{(1 + \frac{f_1 f_2}{x_1 x_2})^2} \right] \theta'^2 + \frac{1}{2}\theta' + \frac{1}{8} \right\} \\
&= \zeta^3 \left(\frac{8}{27}\theta'^3 + g_2\theta'^2 + \frac{1}{2}\theta' + \frac{1}{8} \right),
\end{aligned} \tag{229}$$

in which g_1 and g_2 are respectively defined as

$$\begin{aligned}
g_1 &= 2 - \frac{9}{2}\tilde{F} \frac{a_2 \frac{f_1 f_2}{x_1 x_2} + 2f_1 - 1}{1 + \frac{f_1 f_2}{x_1 x_2}} = 2 - \frac{9}{2}\tilde{F} \left(a_2 + 2 \cdot \frac{a_3 - a_2}{1 + \frac{f_1 f_2}{x_1 x_2}} \right) \\
&= 2 - \frac{9}{2}\tilde{F}\tilde{G}_1, \quad \tilde{G}_1 \equiv a_2 + 2 \cdot \frac{a_3 - a_2}{1 + \frac{f_1 f_2}{x_1 x_2}}, \\
g_2 &= \frac{2}{3} - \frac{3}{2}a_2^2 - 6(a_3 - a_2) \frac{a_2 \frac{f_1 f_2}{x_1 x_2} + a_3}{(1 + \frac{f_1 f_2}{x_1 x_2})^2} \\
&= \frac{2}{3} - \frac{3}{2}a_2^2 - 6(a_3 - a_2)\tilde{G}_2, \quad \tilde{G}_2 \equiv \frac{a_2 \frac{f_1 f_2}{x_1 x_2} + a_3}{(1 + \frac{f_1 f_2}{x_1 x_2})^2}.
\end{aligned} \tag{230}$$

Defining

$$H_1 \equiv \zeta \cdot H'_1, \quad H_2 \equiv \zeta^2 \cdot H'_2, \quad H_3 \equiv \zeta^3 \cdot H'_3, \tag{231}$$

further gives

$$G = 1 + \zeta H'_1 + \zeta^2 H'_2 + \zeta^3 H'_3, \tag{232}$$

in which the H'_i ($i = 1, 2, 3$) are functions only of θ' :

$$\begin{aligned}
H'_1 &= 2\theta' + \frac{3}{2} - \frac{54}{16}\tilde{F}^2, \\
H'_2 &= \frac{4}{3}\theta'^2 + g_1\theta' + \frac{3}{4}, \\
H'_3 &= \frac{8}{27}\theta'^3 + g_2\theta'^2 + \frac{1}{2}\theta' + \frac{1}{8}.
\end{aligned} \tag{233}$$

It is clear that the factor $\frac{a_2^2 + a_1}{a_1} b_1$ in equation (222) is always positive for any values of x_1 and f_1 , so inequality (221) will be satisfied if and only if $G(\zeta, \theta') \geq 0$ for all values of x_1 , f_1 , ζ , and θ' . According to the definitions (224), ξ , ζ , θ , and θ' are all nonnegative parameters.

Statement 1. $G \geq 0$ for any $\zeta \geq 0$ and $\theta' \geq 0$.

Proof:

(i) $G = G_{\min}$ at $d = 0$.

If and only if there is a point at which $\tilde{F}$, $\tilde{G}_1$, and $\tilde{G}_2$ all reach their maxima, then the H'_i s ($i = 1, 2, 3$) will attain their minima, and so does G .

Since

$$\frac{\partial \tilde{F}}{\partial x_1} = \frac{2d^2}{(x_1 + d^2x_2)^2} \quad \text{and} \quad \frac{\partial \tilde{F}}{\partial d} = -\frac{4dx_1x_2}{(x_1 + d^2x_2)^2},$$

both $\frac{\partial \tilde{F}}{\partial x_1} = 0$ and $\frac{\partial \tilde{F}}{\partial d} = 0$ produce $\tilde{F} = \tilde{F}_{max} = 1$ when and only when $d = 0$, and $\tilde{F} = \tilde{F}_{min} = -1$ when and only when $d \rightarrow \infty$. Therefore at $d = 0$, H'_1 reaches its minimum:

$$H'_{1min} = H'_1(d = 0) = 2\theta' - \frac{15}{8}.$$

Similarly, it is straightforward to show that the functions

$$a_3 - a_2 = \frac{x_1^2 - dx_2^2}{x_1 + dx_2}, \quad \text{and} \quad \frac{f_1f_2}{x_1x_2} = \frac{d}{(x_1 + dx_2)^2}$$

will both have their extreme values at $d = 0$ as follows:

$$(a_3 - a_2)_{max} = (a_3 - a_2)|_{d=0} = x_1, \quad \text{and} \quad \left(\frac{f_1f_2}{x_1x_2}\right)_{min} = \frac{f_1f_2}{x_1x_2}|_{d=0} = 0,$$

which results in a maximum of $\tilde{G}_1$ or a minimum of g_1 , and a minimum of H'_2 :

$$\tilde{G}_{1max} = \tilde{G}_1|_{d=0} = x_2 + x_1 = 1,$$

$$g_{1min} = g_1(d = 0) = -\frac{5}{2},$$

$$H'_{2min} = H'_2(d = 0) = \frac{4}{3}\theta'^2 - \frac{5}{2}\theta' + \frac{3}{4}.$$

The same arguments apply to

$$\tilde{G}_2 = \frac{a_2 \frac{f_1f_2}{x_1x_2} + a_3}{(1 + \frac{f_1f_2}{x_1x_2})^2} = \frac{a_2}{1 + \frac{f_1f_2}{x_1x_2}} + \frac{a_3 - a_2}{(1 + \frac{f_1f_2}{x_1x_2})^2}$$

and g_2 , i.e., at $d = 0$, $\tilde{G}_2$ attains its maximum and g_2 its minimum:

$$\tilde{G}_{2max} = \tilde{G}_2|_{d=0} = x_2 \quad \text{and} \quad g_{2min} = g_2(d = 0) = -\frac{5}{6}.$$

This gives

$$H'_{3min} = H'_3(d = 0) = \frac{8}{27}\theta'^3 - \frac{5}{6}\theta'^2 + \frac{1}{2}\theta' + \frac{1}{8}.$$

Therefore, we conclude that for any $\zeta \geq 0$ and $\theta' \geq 0$, the following is always true

$$G \geq G_{min} \equiv G(d = 0) = 1 + \zeta H'_{1min} + \zeta^2 H'_{2min} + \zeta^3 H'_{3min}, \quad (234)$$

i.e.,

$$G \geq G_{min},$$

$$G_{min} = 1 + \zeta(2\theta' - \frac{15}{8}) + \zeta^2(\frac{4}{3}\theta'^2 - \frac{5}{2}\theta' + \frac{3}{4}) + \zeta^3(\frac{8}{27}\theta'^3 - \frac{5}{6}\theta'^2 + \frac{1}{2}\theta' + \frac{1}{8}). \quad (235)$$

(ii) $G_{min} \geq 0$ for any $\zeta \geq 0$ and $\theta' \geq 0$.

We rewrite G_{min} into the convenient form

$$G_{min} = 1 + 2(\theta' - \frac{15}{16})\zeta + \frac{4}{3}(\theta' - \frac{3}{2})(\theta' - \frac{3}{8})\zeta^2 + \frac{8}{27}(\theta' - \frac{3}{2})^2(\theta' + \frac{3}{16})\zeta^3. \quad (236)$$

From this expression, we see that $\theta' \geq \frac{3}{2}$ gives all coefficients ≥ 0 , so $G_{min} \geq 0$ for all $\zeta \geq 0$. Thus the only case we need to consider is $\theta' \leq \frac{3}{2}$. To see the lower limit of G_{min} , perform

$$\frac{\partial G_{min}}{\partial \zeta} = 2(\theta' - \frac{15}{16}) + \frac{8}{3}(\theta' - \frac{3}{2})(\theta' - \frac{3}{8})\zeta + \frac{8}{9}(\theta' - \frac{3}{2})^2(\theta' + \frac{3}{16})\zeta^2 = 0. \quad (237)$$

This is a quadratic equation in ζ , i.e., $a\zeta^2 + b\zeta + c = 0$. Because

$$b^2 - 4ac = \frac{9}{4}\theta'^2 - \frac{27}{4}\theta' + \frac{81}{16} = (\frac{3}{2}\theta' - \frac{9}{4})^2 \geq 0$$

for any θ' , Eq.(237) will have two real roots:

$$\zeta_+ = -\frac{3}{2} \frac{\theta' - \frac{15}{16}}{(\theta' - \frac{3}{2})(\theta' + \frac{3}{16})} \quad \text{and} \quad \zeta_- = -\frac{3}{2} \frac{1}{\theta' - \frac{3}{2}}, \quad (238)$$

at which the second derivative of G_{min} with respect to ζ

$$\frac{\partial^2 G_{min}}{\partial \zeta^2} = \frac{8}{3}(\theta' - \frac{3}{2})(\theta' - \frac{3}{8}) + \frac{16}{9}(\theta' - \frac{3}{2})^2(\theta' + \frac{3}{16})\zeta$$

has the respective values

$$\left. \frac{\partial^2 G_{min}}{\partial \zeta^2} \right|_{\zeta=\zeta_+} = \frac{3}{2}(\theta' - \frac{3}{2}) \quad \text{and} \quad \left. \frac{\partial^2 G_{min}}{\partial \zeta^2} \right|_{\zeta=\zeta_-} = -\frac{3}{2}(\theta' - \frac{3}{2}). \quad (239)$$

For $\theta' < \frac{3}{2}$,

$$\left. \frac{\partial^2 G_{min}}{\partial \zeta^2} \right|_{\zeta=\zeta_+} < 0 \quad \text{and} \quad \left. \frac{\partial^2 G_{min}}{\partial \zeta^2} \right|_{\zeta=\zeta_-} > 0,$$

so ζ_+ gives the maximum of G_{min} , and ζ_- gives the minimum of the G_{min} . In this situation the minimum of G_{min} is

$$\begin{aligned} G_{min}^{min}(\zeta_-) &= 1 + 2(\theta' - \frac{15}{16})\zeta_- + \frac{4}{3}(\theta' - \frac{3}{2})(\theta' - \frac{3}{8})\zeta_-^2 + \frac{8}{27}(\theta' - \frac{3}{2})^2(\theta' + \frac{3}{16})\zeta_-^3 \\ &= 1 + \frac{\frac{3}{2} - \theta'}{\theta' - \frac{3}{2}} = 0. \end{aligned} \quad (240)$$

Thus, for any $\zeta \geq 0$ and $\theta' \geq 0$, the following is always true

$$G \geq G_{min} \geq 0.$$

□

Therefore the necessary and sufficient condition (221) for equation (209) to have three real roots is satisfied unconditionally. Next we will examine the signs of the polynomial P at the three real roots, i.e., Eqs.(218)-(220).

To simplify our calculation, we consider

$$\begin{aligned} & P(y) - P'(y) \cdot y - \frac{A_3}{4} P'(y) \\ &= \left(\frac{1}{2}A_2 - \frac{3}{16}A_3^2\right)y^2 + \left(\frac{3}{4}A_1 - \frac{1}{8}A_2A_3\right)y + A_0 - \frac{1}{16}A_1A_3. \end{aligned} \quad (241)$$

For the roots $\{y_i\}$ of $P'(y)$, $P'(y_i) \equiv 0$, and we have

$$P(y_i) - P'(y_i) \cdot y_i - \frac{A_3}{4} P'(y_i) \equiv P(y_i),$$

i.e.,

$$P(y_i) \equiv \left(\frac{1}{2}A_2 - \frac{3}{16}A_3^2\right)y_i^2 + \left(\frac{3}{4}A_1 - \frac{1}{8}A_2A_3\right)y_i + A_0 - \frac{1}{16}A_1A_3, \quad (242)$$

in which the coefficients are calculated as follows:

$$\frac{1}{2}A_2 - \frac{3}{16}A_3^2 = -\frac{1}{2}\left(1 + \frac{a_3^2}{a_1}\right)b_1\left[1 + \zeta\left(\frac{2}{3}\theta' + \frac{1}{2}\right)\right], \quad (243)$$

$$\frac{3}{4}A_1 - \frac{1}{8}A_2A_3 = \delta w b_1 \left\{ 3a_3 - \frac{3}{2}a_2 + \left(1 + \frac{a_3^2}{a_1}\right) \left[\zeta\theta' \left(\frac{a_3 - a_2}{1 + \frac{f_1 f_2}{x_1 x_2}} + \frac{1}{6}a_2 \right) + \frac{1}{4}a_2(1 - \zeta) \right] \right\}, \quad (244)$$

$$\begin{aligned} A_0 - \frac{1}{16}A_1A_3 &= \left(1 + \frac{a_3^2}{a_1}\right)b_1^2\zeta \left\{ \zeta\theta' \frac{a_3^2 + a_1}{a_1} \left[\frac{1}{2} \frac{a_2(a_3 - a_2)}{1 + \frac{f_1 f_2}{x_1 x_2}} - \frac{2}{3}a_1 \right] + \right. \\ &\left. \frac{2}{3}\theta' \frac{\frac{f_1 f_2}{x_1 x_2}}{1 + \frac{f_1 f_2}{x_1 x_2}} \left[\frac{2}{3} \frac{a_1 + a_3^2}{a_1(1 + \frac{f_1 f_2}{x_1 x_2})} \zeta\theta' + 1 \right] + \frac{1}{4}(a_1 + a_3^2)\zeta - [(a_3 - a_2)^2 + \frac{1}{2}a_2a_3 + 2a_1 - \frac{1}{4}] \right\}, \end{aligned} \quad (245)$$

where we have used

$$\begin{aligned} A_3 &= 2a_2\delta w, \\ A_2 &= -(1 + \frac{a_3^2}{a_1})b_1\left[1 + \zeta\left(\frac{2}{3}\theta' - a_2^2 + 2a_1\right)\right], \\ A_1 &= 2\delta w b_1(2a_3 - a_2)\left[1 + \frac{1}{2a_3 - a_2}\left(1 + \frac{a_3^2}{a_1}\right)\zeta\left(\frac{2}{3}\frac{a_3 - a_2}{1 + \frac{f_1 f_2}{x_1 x_2}}\theta' - a_1a_2\right)\right], \\ A_0 &= \left(\frac{a_1 + a_3^2}{a_1}b_1\right)^2\zeta\left[\frac{2}{3}\zeta\theta'\left(\frac{a_2(a_3 - a_2)}{1 + \frac{f_1 f_2}{x_1 x_2}} - a_1\right) + \right. \\ &\left. \frac{2}{3}\theta' \frac{\frac{f_1 f_2}{x_1 x_2}}{1 + \frac{f_1 f_2}{x_1 x_2}} \left(\frac{2}{3} \frac{\zeta\theta'}{1 + \frac{f_1 f_2}{x_1 x_2}} + \frac{a_1}{a_1 + a_3^2}\right) + a_1^2\zeta - (a_3 - a_2)^2 - a_1\right]. \end{aligned}$$

For our analysis, we introduce

$$\begin{aligned} D &\equiv \left| \sqrt{\frac{C_0^2}{4} + \frac{C_1^3}{27}} \right| = \frac{|\sqrt{6}|}{36} \left(\frac{a_1 + a_3^2}{a_1} b_1 \right)^{3/2} G^{1/2}, \\ D' &\equiv \left| \sqrt{\frac{C_0^2}{4} + D^2} \right| = \frac{|\sqrt{6}|}{36} \left(\frac{a_1 + a_3^2}{a_1} b_1 \right)^{3/2} \left(1 + \frac{1}{2}\zeta + \frac{2}{3}\zeta\theta' \right)^{3/2}, \end{aligned}$$

and

$$\phi \equiv \arctan\left(\frac{D}{-C_0/2}\right),$$

or

$$\tan \phi \equiv \frac{D}{-C_0/2} = -\frac{2\sqrt{6} \left[\frac{(1+\gamma_1)^3}{\zeta} - \frac{27}{8} \tilde{F}^2 (1+\gamma_2)^2 \right]^{1/2}}{\tilde{F} (1+\gamma_2) \aleph},$$

$$\aleph = \begin{cases} +1 & \text{if } \delta w > 0 \\ -1 & \text{if } \delta w < 0. \end{cases}$$

Then E and $\tilde{E}$ become

$$E = \left(-\frac{C_0}{2} + iD\right)^{1/3} = D^{1/3} e^{i\phi/3},$$

$$\tilde{E} = \left(-\frac{C_0}{2} - iD\right)^{1/3} = D^{1/3} e^{-i\phi/3},$$

where $i = \sqrt{-1}$. With these expressions, the roots (218)-(220), which are the extrema of $P(y)$, are given by

$$\begin{aligned} y_3 &= -\frac{1}{2}a_2\delta w + 2D^{1/3} \cos \frac{\phi}{3}, \\ y_2 &= -\frac{1}{2}a_2\delta w - D^{1/3} \left(\cos \frac{\phi}{3} + \sqrt{3} \sin \frac{\phi}{3} \right) = -\frac{1}{2}a_2\delta w - 2D^{1/3} \cos \left(\frac{\phi}{3} - \frac{\pi}{3} \right), \\ y_1 &= -\frac{1}{2}a_2\delta w - D^{1/3} \left(\cos \frac{\phi}{3} - \sqrt{3} \sin \frac{\phi}{3} \right) = -\frac{1}{2}a_2\delta w - 2D^{1/3} \cos \left(\frac{\phi}{3} + \frac{\pi}{3} \right). \end{aligned} \quad (246)$$

If we define

$$\tan \phi' \equiv \frac{D}{C_0/2} \quad \text{or} \quad \cos \phi' \equiv \frac{C_0/2}{D'} = \frac{3\sqrt{6}}{4} \frac{\tilde{F}(1+\gamma_2)\zeta^{1/2}\aleph}{(1 + \frac{1}{2}\zeta + \frac{2}{3}\xi')^{3/2}},$$

then $\cos \phi' = -\cos \phi$ and $\phi = \phi' \pm \pi$. Therefore for the choice of $\phi = \phi' - \pi$, the above three roots can be rewritten

$$\begin{aligned} y_1 &= -\frac{1}{2}a_2\delta w - 2D^{1/3} \cos \frac{\phi'}{3}, \\ y_2 &= -\frac{1}{2}a_2\delta w + 2D^{1/3} \cos \left(\frac{\phi'}{3} + \frac{\pi}{3} \right), \\ y_3 &= -\frac{1}{2}a_2\delta w + 2D^{1/3} \cos \left(\frac{\phi'}{3} - \frac{\pi}{3} \right). \end{aligned} \quad (247)$$

In order to see the range of ϕ' , we let

$$\frac{\partial \cos \phi'}{\partial \zeta} = 0.$$

This gives the equation

$$\tilde{F} \left[\frac{1}{2} \frac{\zeta^{-1/2}}{(1 + \frac{1}{2}\zeta + \frac{2}{3}\xi')^{3/2}} - \frac{3}{4} \frac{\zeta^{1/2}}{(1 + \frac{1}{2}\zeta + \frac{2}{3}\xi')^{5/2}} \right] = 0.$$

For nonzero $\tilde{F}$, the solutions are

$$\zeta_c = 1 + \frac{2}{3}\xi' \quad \text{or} \quad \zeta_c = \infty,$$

at which $\cos \phi'$ reaches its extrema:

$$(\cos \phi')_c = \frac{3\sqrt{6}}{4} \frac{\tilde{F}(1 + \gamma_2)\zeta_c^{1/2}\aleph}{(1 + \frac{1}{2}\zeta_c + \frac{2}{3}\xi')^{3/2}} \quad \text{or} \quad (\cos \phi')_c = 0, \quad \phi'_c = \frac{(n+1)\pi}{2}, \quad n = 0, 1, 2, \dots$$

In the case of $k_d = 0$,

$$\zeta_c = 1 \quad \text{and} \quad (\cos \phi')_c = \tilde{F}\aleph.$$

For $\delta w > 0$, $\aleph = 1$, $(\cos \phi')_c = \tilde{F}$. If $\delta w < 0$, $\aleph = -1$, then $(\cos \phi')_c = -\tilde{F}$. Assuming that material 2 is the heavy one, so $0 \leq d \leq 1$, in this case, $\tilde{F}$ varies from 0 ($d=1$) to 1 ($d=0$). Then whether the extreme of $\cos \phi'$ is maximum or minimum only depends on the sign of δw . Noting that $\partial \phi' / \partial \zeta < 0$, so for $\aleph = 1$, $(\cos \phi')_c \equiv (\cos \phi')_{max} = \tilde{F}$ and $\phi'_c = \phi_{min}$. For $\aleph = -1$, $(\cos \phi')_c \equiv (\cos \phi')_{min} = -\tilde{F}$ and $\phi'_c = \phi_{max}$. Thereupon for all signs of δw , ϕ' varies from $-\tilde{F}$ to $\tilde{F}$. In the extreme case: $\tilde{F}_{max(min)} = 1$ (0), we can estimate the possible range for ϕ' :

$$0 \leq \phi' \leq \frac{\pi}{2} \quad \text{for} \quad \delta w > 0$$

and

$$\frac{\pi}{2} \leq \phi' \leq \pi \quad \text{for} \quad \delta w < 0.$$

If $\phi = \phi' - \pi$, then for both $\aleph = 1$ and $\aleph = -1$, $y_1 \leq y_2 \leq y_3$, while if $\phi = \phi' + \pi$, then roots (247) will be permuted from (y_1, y_2, y_3) to (y_3, y_1, y_2) .

Now we examine the signs of the polynomial P at these three extrema.

A. $\delta w = 0$. In this situation, there is no relative motion between the two fluids. Setting $\zeta = 0$ reduces Eqs.(243-245) to

$$\begin{aligned} \frac{1}{2}A_2 - \frac{3}{16}A_3^2 &= -\frac{1}{2}\left(\frac{a_1 + a_3^2}{a_1}b_1\right)\left(1 + \frac{2}{3}\xi'\right), \\ \frac{3}{4}A_1 - \frac{1}{8}A_2A_3 &= 0, \\ A_0 - \frac{1}{16}A_1A_2 &= \frac{2}{3}\left(\frac{a_1 + a_3^2}{a_1}b_1^2\right)\frac{f_1f_2}{x_1x_2}\xi\left(\frac{2(a_1 + a_3^2)}{3a_1}\xi + 1\right), \\ D^{1/3} &= \frac{\sqrt{6}}{6}\left(\frac{a_1 + a_3^2}{a_1}b_1\right)^{1/2}\left(1 + \frac{2}{3}\xi'\right)^{1/2}, \quad \phi' \rightarrow \frac{\pi}{2}, \end{aligned}$$

where $\xi' = (1 + \frac{f_1f_2}{x_1x_2})\xi$ and ξ is defined by Eq.(224). The three roots are then

$$\begin{aligned} y_1 &= -\frac{\sqrt{2}}{2}\left(\frac{a_1 + a_3^2}{a_1}b_1\right)^{1/2}\left(1 + \frac{2}{3}\xi'\right)^{1/2}, \\ y_2 &= 0, \\ y_3 &= \frac{\sqrt{2}}{2}\left(\frac{a_1 + a_3^2}{a_1}b_1\right)^{1/2}\left(1 + \frac{2}{3}\xi'\right)^{1/2}, \end{aligned} \tag{248}$$

and Eq.(242) becomes

$$P(y_i) = \frac{a_1 + a_3^2}{a_1} b_1 \left[-\frac{1}{2} \left(1 + \frac{2}{3} \xi' \right) y_i^2 + \frac{2}{3} b_1 \frac{f_1 f_2}{x_1 x_2} \xi \left(\frac{2(a_1 + a_3^2)}{3a_1} \xi + 1 \right) \right].$$

In particular,

$$\begin{aligned} P(y_1) &= P(y_3) = \frac{a_1 + a_3^2}{a_1} b_1 \left[-\frac{1}{2} \left(1 + \frac{2}{3} \xi' \right) y_1^2 + \frac{2}{3} b_1 \frac{f_1 f_2}{x_1 x_2} \xi \left(\frac{2(a_1 + a_3^2)}{3a_1} \xi + 1 \right) \right] \\ &= \frac{a_1 + a_3^2}{a_1} b_1 \left[-\frac{1}{4} \frac{a_1 + a_3^2}{a_1} b_1 \left(1 + \frac{2}{3} \xi' \right)^2 + \frac{4}{9} b_1 \frac{f_1 f_2}{x_1 x_2} \frac{a_1 + a_3^2}{a_1} \xi^2 + \frac{2}{3} b_1 \frac{f_1 f_2}{x_1 x_2} \xi \right] \\ &= \left(\frac{a_1 + a_3^2}{a_1} b_1 \right)^2 \left[-\frac{1}{4} - \frac{\xi}{3} \left(1 + \frac{f_1 f_2}{x_1 x_2} - \frac{2a_1}{a_1 + a_3^2} \frac{f_1 f_2}{x_1 x_2} \right) - \frac{\xi^2}{9} \left(1 - \frac{f_1 f_2}{x_1 x_2} \right)^2 \right] \\ &= - \left(\frac{a_1 + a_3^2}{a_1} b_1 \right)^2 \left[\frac{1}{4} + \frac{\xi}{3} \left(1 + \frac{a_3^2 - a_1}{a_3^2 + a_1} \frac{f_1 f_2}{x_1 x_2} \right) + \frac{\xi^2}{9} \left(1 - \frac{f_1 f_2}{x_1 x_2} \right)^2 \right], \\ P(y_2) &= \frac{2}{3} \left(\frac{a_1 + a_3^2}{a_1} b_1^2 \right) \frac{f_1 f_2}{x_1 x_2} \xi \left(\frac{2(a_1 + a_3^2)}{3a_1} \xi + 1 \right). \end{aligned} \quad (249)$$

Clearly,

$$P(y_2) \geq 0 \quad \forall \quad \xi \geq 0$$

and since

$$\frac{a_3^2 - a_1}{a_3^2 + a_1} \frac{f_1 f_2}{x_1 x_2} \geq \frac{-a_1}{a_3^2 + a_1} \frac{f_1 f_2}{x_1 x_2} \geq -\frac{f_1 f_2}{x_1 x_2},$$

i.e.,

$$\frac{\xi}{3} \left(1 + \frac{a_3^2 - a_1}{a_3^2 + a_1} \frac{f_1 f_2}{x_1 x_2} \right) \geq \frac{\xi}{3} \left(1 - \frac{f_1 f_2}{x_1 x_2} \right),$$

we also have

$$P(y_1) = P(y_3) \leq - \left(\frac{a_1 + a_3^2}{a_1} b_1 \right)^2 \left[\frac{1}{2} - \frac{\xi}{3} \left(1 - \frac{f_1 f_2}{x_1 x_2} \right) \right]^2 \leq 0 \quad \forall \quad \xi \geq 0.$$

Therefore according to *Lemma 1*, Eq.(208) has all real roots for $\delta w = 0$.

B. $\delta w \neq 0$. In this situation, there are relative motions between the fluids. Physically, any relative motion in inviscid flow will inevitably produce turbulence(k_d); for example, the Kelvin-Helmholtz (KH) instability will be generated simultaneously at the fluid interface with an exponential growth rate $\sim \frac{\sqrt{\rho_1 \rho_2}}{\rho_1 + \rho_2} \delta w$ and will directly lead to turbulence and result in nonzero turbulent viscosity $\eta_d \sim l \delta w$, where l is the length scale of the turbulence. Thus, in principle, we will be back to the same situation as in *Case 1* for $\eta^* \neq 0$, which we have shown always gives real characteristics.

However, if the generation process for turbulent viscosity or turbulent kinetic energy is slow, then at an early time, for a tiny η_d , we would think that a good approximation should be $\eta_d = 0$. Therefore, in order to see quantitatively the effects of the turbulent kinetic energy on the characteristics of the model, we would need to consider a more generalized situation, i.e., $\delta w \neq 0$, $k_d \neq 0$, but $\eta_d = 0$, although this might not be physically realistic.

In this situation, for convenience, we let

$$D_2 = \frac{1}{2} A_2 - \frac{3}{16} A_3^2, \quad D_1 = \frac{3}{4} A_1 - \frac{1}{8} A_2 A_3, \quad D_0 = A_0 - \frac{1}{16} A_1 A_2,$$

then polynomial $P(y_i)$ becomes

$$P(y_i) = D_2 y_i^2 + D_1 y_i + D_0. \quad (250)$$

This has maximum value

$$P(y_i)_{max} = -\frac{D_1^2}{4D_2} + D_0 \quad (251)$$

at $y_{ic} = -\frac{D_1}{2D_2}$. It is clear that as $\delta w \rightarrow \infty$, $k_d \rightarrow \infty$, $\eta_d \neq 0$, Eq.(208) will always have four real roots. Thus for nonzero δw , physically, the infinitesimal η_d only can happen if δw and k_d are tiny. Therefore the possible region for unreal characteristics of the model will be in the neighborhood of $\delta w = 0$ where η_d is negligible. Hence, the behaviour of the fluid under the sufficiently small nonzero δw will impose significant constraints to the model for having all real characteristics in all values of δw .

To get the constraint on the model in inviscid flow, we now consider the neighborhood of small δw , i.e., $0 < \zeta \ll 1$. In this region,

$$\phi' \rightarrow \frac{\pi}{2}, \quad \cos \frac{\phi'}{3} \rightarrow \frac{\sqrt{3}}{2}, \quad \cos(\frac{\phi'}{3} + \frac{\pi}{3}) \rightarrow 0, \quad \cos(\frac{\phi'}{3} - \frac{\pi}{3}) \rightarrow \frac{\sqrt{3}}{2}.$$

Thus the three roots Eq.(247) become

$$y_1 = -\frac{1}{2}a_2\delta w - D'^{1/3}\sqrt{3}, \quad y_2 = -\frac{1}{2}a_2\delta w, \quad y_3 = -\frac{1}{2}a_2\delta w + D'^{1/3}\sqrt{3}$$

Substituting D' into above gives

$$y_1 = \sqrt{\frac{a_1 + a_3^2}{a_1}} b_1 \left[-\frac{1}{2}a_2\zeta^{1/2} - \left(\frac{1}{2} + \zeta\left(\frac{1}{3}\theta' + \frac{1}{4}\right)\right)^{1/2} \right], \quad (252)$$

$$y_2 = -\sqrt{\frac{a_1 + a_3^2}{a_1}} b_1 \cdot \frac{1}{2}a_2\zeta^{1/2}, \quad (253)$$

$$y_3 = \sqrt{\frac{a_1 + a_3^2}{a_1}} b_1 \left[-\frac{1}{2}a_2\zeta^{1/2} + \left(\frac{1}{2} + \zeta\left(\frac{1}{3}\theta' + \frac{1}{4}\right)\right)^{1/2} \right]. \quad (254)$$

Note that $y_1 < y_2 < y_3$. As $\delta w \rightarrow 0$ (or $\zeta \rightarrow 0$),

$$D_2 \rightarrow -\frac{1}{2}\left(\frac{a_1 + a_3^2}{a_1}b_1\right) + O(\zeta),$$

$$D_1 \rightarrow \left(\frac{a_1 + a_3^2}{a_1}b_1\right)^{3/2}\zeta^{1/2}\left(3a_3 - \frac{5}{4}a_2 + \frac{a_2a_3^2}{4a_1}\right) + O(\zeta^{3/2}),$$

$$D_0 \rightarrow O(\zeta^k), \quad k \geq 1.$$

Thus, for sufficiently tiny δw , the first term in Eq.(250) dominates, which leads to

$$P(y_1) \sim P(y_3) \rightarrow -\frac{1}{4}\left(\frac{a_1 + a_3^2}{a_1}b_1\right)^2 + O(\zeta^k) < 0.$$

According to *Lemma 1*, if $P(y_1)$ and $P(y_3)$ are negative, then the sufficient condition for $P(y)$ to have four real roots is

$$P(y_2) \geq 0. \quad (255)$$

Substituting y_2 into Eq.(250), we have the constraint

$$\begin{aligned}
 P(y_2) = & \left(\frac{a_1 + a_3^2}{a_1} b_1 \right)^2 \zeta \left\{ -\frac{a_2^2}{8} \left(1 + \frac{1}{2} \zeta + \frac{2}{3} \zeta \theta' \right) \right. \\
 & - \frac{a_2 a_1}{2(a_1 + a_3^2)} \left[3a_3 - \frac{3}{2} a_2 + \frac{a_1 + a_3^2}{a_1} \zeta \theta' \left(\frac{1}{6} a_2 + \frac{a_3 - a_2}{1 + \frac{f_1 f_2}{x_1 x_2}} \right) + \frac{a_1 + a_3^2}{4a_1} a_2 (1 - \zeta) \right] \\
 & + \zeta \theta' \left[\frac{a_2(a_3 - a_2)}{2(1 + \frac{f_1 f_2}{x_1 x_2})} - \frac{2}{3} a_1 \right] + \frac{2}{3} \theta \frac{f_1 f_2}{x_1 x_2} \left(\frac{2}{3} \zeta \theta + \frac{a_1}{a_1 + a_3^2} \right) \\
 & \left. + \frac{1}{4} a_1 \zeta - \frac{a_1}{a_1 + a_3^2} \left[(a_3 - a_2)^2 + \frac{1}{2} a_2 a_3 + 2a_1 - \frac{1}{4} \right] \right\} \geq 0. \quad (256)
 \end{aligned}$$

Combining the same terms in the constraint gives

$$\begin{aligned}
 & \zeta \left[-\frac{a_2^2}{8} \left(1 + \frac{1}{2} \zeta + \frac{2}{3} \zeta \theta' \right) - \frac{3 a_1 a_2 (2a_3 - a_2)}{4(a_1 + a_3^2)} - \frac{1}{12} a_2^2 \zeta \theta' - \frac{1}{8} a_2^2 (1 - \zeta) - \frac{2}{3} a_1 \zeta \theta' \right. \\
 & \left. + \frac{4}{9} \frac{f_1 f_2}{x_1 x_2} \zeta \theta^2 + \frac{2}{3} \frac{f_1 f_2}{x_1 x_2} \frac{a_1}{a_1 + a_3^2} \theta + \frac{1}{4} a_1 \zeta - \frac{a_1}{a_1 + a_3^2} \left(a_3^2 + a_2^2 - \frac{3}{2} a_2 a_3 + 2a_1 - \frac{1}{4} \right) \right] \geq 0. \quad (257)
 \end{aligned}$$

Noting that

$$4a_1 + a_2^2 = 1, \quad (258)$$

and doing further expansion and combination, we have

$$\begin{aligned}
 & \zeta \left[-\frac{1}{4} a_2^2 - \frac{3 a_1 a_2 (2a_3 - a_2)}{4(a_1 + a_3^2)} + \frac{1}{16} \zeta - \frac{1}{6} \zeta \theta' + \frac{4}{9} \frac{f_1 f_2}{(1 + \frac{f_1 f_2}{x_1 x_2})^2} \zeta \theta'^2 \right. \\
 & \left. + \frac{2}{3} \frac{f_1 f_2}{1 + \frac{f_1 f_2}{x_1 x_2}} \frac{a_1}{a_1 + a_3^2} \theta' - \frac{a_1}{a_1 + a_3^2} \left(a_3^2 + a_2^2 - \frac{3}{2} a_2 a_3 + 2a_1 - \frac{1}{4} \right) \right] \geq 0, \quad (259)
 \end{aligned}$$

where we have used $\theta' = (1 + \frac{f_1 f_2}{x_1 x_2}) \theta$. Combining the second and last terms in the above constraint gives

$$-\frac{a_1}{a_1 + a_3^2} \left(\frac{3}{4} a_2 (2a_3 - a_2) + a_3^2 + a_2^2 - \frac{3}{2} a_2 a_3 + 2a_1 - \frac{1}{4} \right) = -\frac{a_1}{a_1 + a_3^2} \frac{1}{4} (a_1 + a_3^2) = -\frac{1}{4} a_1.$$

This further simplifies Eq.(259) into

$$\zeta \left[-\frac{1}{4} + \frac{1}{16} \zeta - \frac{1}{6} \zeta \theta' + \frac{2}{3} \frac{f_1 f_2}{1 + \frac{f_1 f_2}{x_1 x_2}} \frac{a_1}{a_1 + a_3^2} \theta' + \frac{4}{9} \frac{f_1 f_2}{(1 + \frac{f_1 f_2}{x_1 x_2})^2} \zeta \theta'^2 \right] \geq 0, \quad (260)$$

or

$$\zeta \left\{ \frac{4}{9} \frac{f_1 f_2}{x_1 x_2} \zeta \theta^2 + \left[\frac{2}{3} \frac{a_1}{a_1 + a_3^2} \frac{f_1 f_2}{x_1 x_2} - \frac{1}{6} \left(1 + \frac{f_1 f_2}{x_1 x_2} \right) \zeta \right] \theta + \frac{1}{16} \zeta - \frac{1}{4} \right\} \geq 0. \quad (261)$$

For $\zeta > 0$, this requires that

$$\frac{4}{9} \frac{f_1 f_2}{x_1 x_2} \zeta \theta^2 + \left[\frac{2}{3} \frac{a_1}{a_1 + a_3^2} \frac{f_1 f_2}{x_1 x_2} - \frac{1}{6} \left(1 + \frac{f_1 f_2}{x_1 x_2} \right) \zeta \right] \theta + \frac{1}{16} \zeta - \frac{1}{4} \geq 0. \quad (262)$$

In order to see the fundamental physics of this constraint, we now consider two special circumstances:

(i) $k_d = 0$. If we ignore the contribution from the turbulent motions of the fluid and set $\theta = 0$, then the constraint becomes

$$\frac{1}{16}\zeta - \frac{1}{4} \geq 0,$$

that is

$$\zeta \geq 4 \quad \text{or} \quad \frac{\delta w^2}{b_1} \geq 4 \frac{a_1 + a_3^2}{a_1}. \quad (263)$$

This implies that in two-phase inviscid flow, if the turbulent kinetic energy is not taken into account, then there would exist a forbidden region for the velocity difference of two fluids, $0 < \delta w^2 < 4 \frac{a_1 + a_3^2}{a_1} b_1$, in which the model has complex characteristics and becomes illposed. This is consistent with the results obtained by Stewart and Wendroff^[13].

(ii) $k_d \neq 0$. For nonzero and nonnegative k_d , it is straightforward to solve inequality (262) for θ :

$$\theta_- \geq \theta \geq \theta_+, \quad (264)$$

$$\begin{aligned} \theta_{\pm} &= \frac{3}{4\zeta} \left\{ -\frac{a_1}{a_1 + a_3^2} + \frac{1 + \frac{f_1 f_2}{x_1 x_2}}{4 \frac{f_1 f_2}{x_1 x_2}} \zeta \pm \left[\left(\frac{a_1}{a_1 + a_3^2} - \frac{1 + \frac{f_1 f_2}{x_1 x_2}}{4 \frac{f_1 f_2}{x_1 x_2}} \zeta \right)^2 - \frac{\zeta(\frac{1}{4}\zeta - 1)}{\frac{f_1 f_2}{x_1 x_2}} \right]^{1/2} \right\} \\ &= \frac{3b_1}{4\delta w^2} \left\{ -1 + \frac{1 + \frac{f_1 f_2}{x_1 x_2}}{4 \frac{f_1 f_2}{x_1 x_2}} \frac{\delta w^2}{b_1} \pm \left[\left(1 - \frac{1 + \frac{f_1 f_2}{x_1 x_2}}{4 \frac{f_1 f_2}{x_1 x_2}} \frac{\delta w^2}{b_1} \right)^2 - \frac{\frac{\delta w^2}{b_1} \left(\frac{1}{4} \frac{\delta w^2}{b_1} - \frac{a_1 + a_3^2}{a_1} \right)}{\frac{f_1 f_2}{x_1 x_2}} \right]^{1/2} \right\} \\ &= \frac{3b_1}{4\delta w^2} \left[-1 + \frac{1 + \frac{f_1 f_2}{x_1 x_2}}{4 \frac{f_1 f_2}{x_1 x_2}} \frac{\delta w^2}{b_1} \pm \left(1 - \frac{1 + \frac{f_1 f_2}{x_1 x_2}}{4 \frac{f_1 f_2}{x_1 x_2}} \cdot \frac{\delta w^2}{b_1} \right) \Delta \right], \end{aligned} \quad (265)$$

where

$$\Delta = \left[1 + \frac{\frac{a_1 + a_3^2}{a_1} \frac{x_1 x_2}{f_1 f_2} \frac{\delta w^2}{b_1}}{\left(1 - \frac{1}{4} \left(1 + \frac{x_1 x_2}{f_1 f_2} \right) \frac{\delta w^2}{b_1} \right)^2} \right]^{1/2}, \quad (266)$$

and the high order term $O(\frac{\delta w^4}{b_1^2})$ has been neglected. For small δw ,

$$\Delta \sim 1 + \frac{1}{2} \frac{\frac{a_1 + a_3^2}{a_1} \frac{x_1 x_2}{f_1 f_2} \frac{\delta w^2}{b_1}}{\left(1 - \frac{1}{4} \left(1 + \frac{x_1 x_2}{f_1 f_2} \right) \frac{\delta w^2}{b_1} \right)^2} \sim 1 + \frac{1}{2} \frac{(a_1 + a_3^2) x_1 x_2 \delta w^2}{a_1 f_1 f_2 b_1}.$$

Substituting $\theta_{\pm}$ into solution (264), we obtain that for $\theta \leq \theta_-$,

$$\frac{k_d}{b_1} \leq \frac{3}{4} \left[-1 + \frac{1 + \frac{f_1 f_2}{x_1 x_2}}{4 \frac{f_1 f_2}{x_1 x_2}} \frac{\delta w^2}{b_1} - \left(1 - \frac{1 + \frac{f_1 f_2}{x_1 x_2}}{4 \frac{f_1 f_2}{x_1 x_2}} \cdot \frac{\delta w^2}{b_1} \right) \Delta \right], \quad (267)$$

and for $\theta \geq \theta_+$,

$$\frac{k_d}{b_1} \geq \frac{3}{4} \left[-1 + \frac{1 + \frac{f_1 f_2}{x_1 x_2}}{4 \frac{f_1 f_2}{x_1 x_2}} \frac{\delta w^2}{b_1} + \left(1 - \frac{1 + \frac{f_1 f_2}{x_1 x_2}}{4 \frac{f_1 f_2}{x_1 x_2}} \cdot \frac{\delta w^2}{b_1} \right) \Delta \right]. \quad (268)$$

If δw is small, neglecting the $\frac{\delta w^2}{b_1^4}$ term, Eq.(267) becomes

$$\frac{k_d}{b_1} \leq \frac{3}{4} \left[-2 + \frac{1}{2} \left(1 - \frac{a_3^2}{a_1} \frac{x_1 x_2}{f_1 f_2} \right) \frac{\delta w^2}{b_1} \right]. \quad (269)$$

It is evident that for any nonnegative k_d , this constraint is unphysical because the second term is much less than the first term if $\zeta \ll 1$. Therefore the real physical constraint to the model will be given by Eq.(268), i.e.,

$$\frac{k_d}{b_1} \geq \frac{3}{8} \cdot \frac{a_1 + a_3^2}{a_1} \cdot \frac{x_1 x_2}{f_1 f_2} \cdot \frac{\delta w^2}{b_1} \quad \text{or} \quad k_d \geq \frac{3}{8} \cdot \frac{a_1 + a_3^2}{a_1} \cdot \frac{x_1 x_2}{f_1 f_2} \cdot \delta w^2, \quad (270)$$

if δw is small. This tells us that at very early times when the turbulent viscosity is negligible, the model has all real characteristics for all physical ranges of δw when and only when the turbulent kinetic energy satisfies

$$k_d \geq \frac{3}{4} \cdot \frac{a_1 + a_3^2}{a_1} \cdot \frac{k_o}{f_1 f_2}.$$

This imposes an initial constraint on the model. This constraint physically means that the disordered motion is generated simultaneously with the ordered motion. If the two fluids are the same ($w_1 = w_2$), or if one of the fluids is vacuum ($w_1 = w_2 = 0$), then $k_o = 0$ and condition (270) is apparently satisfied. Therefore the model has all real characteristics.

When δw is not small, physically the inevitable Kelvin-Helmholtz instability generated due to δw will result in nonnegligible k_d and η_d , in which case the model always gives real characteristics as shown in *Case 1* of $\eta_* \neq 0$. A more accurate constraint on the model for general δw is discussed in Appendix B.

Therefore we now conclude that in inviscid flow, for zero relative motion between the fluids, the hybrid multifluid turbulent mix model always has all real characteristics. If there are non zero relative motions between the fluids and if turbulence (k_d), turbulent viscosity (η_d), and relative motion (δw) between fluids are generated simultaneously, then the hybrid multifluid turbulent mix model will also have real characteristics in all ranges of δw . If at early times, δw is very small ($\zeta \ll 1$) and η_d is negligible, then the model will possess complex characteristics unless the turbulent kinetic energy is equal or greater than $\frac{3}{4} \cdot \frac{a_1 + a_3^2}{a_1} \cdot \frac{k_o}{f_1 f_2}$. Moreover, if the turbulent contribution is completely ignored ($k_d = 0$), then there always exists an unremovable range for δw , $0 < \delta w^2 < \frac{4(a_1 + a_3^2)}{a_1} b_1$, in which the model has complex roots and becomes illposed as displayed in all traditional pure multiphase flow models. It is apparent that taking into account the contributions of turbulence may dismiss the well known illposedness of the pure multiphase flow model in the inviscid case if and only if condition (270) is satisfied forever. Otherwise, the introduction of k_d only narrows the illposed region and improves the traditional multiphase flow model but cannot eradicate the illposedness. So taking into account the contributions of turbulence is not enough to complete the physics of turbulent mixing processes — some fundamental physics is still missing. For example, molecular diffusion is known to play an important role at late times of mixing as observed in recent linear electric motor (LEM) experiments^[14]. Whence, for a correct and complete discription of the dynamics of turbulent mixing, additional physical processes such as molecular diffusion have to be taken into account so as to bridge the

scales from macro to micro smoothly and to allow all parameters to take all values without generating illposedness.

The dynamic evolution of k_d is examined in Appendix C, which shows that for inviscid flow with small δw , the ratio k_d/k_o always increases with time, so the hybrid turbulent mix model will have all real characteristics if the constraint (270) is satisfied initially.

In the remainder of this section, we will list the characteristic equations for all other cases in the limit of inviscid flow, although most of these equations can only be evaluated numerically. Apparently, *Case 1* is the simplest and only one which can be analysed analytically in this family.

Case 2 Two-phase flow without temperature diffusion ($D_T = 0$, $D_k \neq 0$, $D_l \neq 0$). In this case, the characteristic equation is

$$b_{712}b_{611}D_kD_l \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 & 0 \\ b_{21} & b_{22} - \lambda & b_{23} & b_{24} & b_{25} \\ b_{31} & 0 & b_{33} - \lambda & b_{34} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} - \lambda & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} - \lambda \end{vmatrix} = 0, \quad (271)$$

i.e.,

$$\frac{1}{\rho^2}D_kD_l \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 & 0 \\ b_{21} & b_{22} - \lambda & b_{23} & b_{24} & b_{25} \\ b_{31} & 0 & b_{33} - \lambda & b_{34} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} - \lambda & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} - \lambda \end{vmatrix} = 0. \quad (272)$$

Case 3 Two-phase flow without turbulent kinetic energy diffusion ($D_k = 0$, $D_T \neq 0$, $D_l \neq 0$). In this case, the characteristic equation is

$$b_{510}b_{712}D_TD_l \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 & 0 \\ b_{21} & b_{22} - \lambda & b_{23} & b_{24} & b_{26} \\ b_{31} & 0 & b_{33} - \lambda & b_{34} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} - \lambda & b_{46} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{66} - \lambda \end{vmatrix} = 0, \quad (273)$$

i.e.,

$$\frac{1}{\rho^2} \frac{(\gamma - 1)T}{b_1} D_TD_l \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 & 0 \\ b_{21} & b_{22} - \lambda & b_{23} & b_{24} & b_{26} \\ b_{31} & 0 & b_{33} - \lambda & b_{34} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} - \lambda & b_{46} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{66} - \lambda \end{vmatrix} = 0. \quad (274)$$

Case 4 Two-phase flow without turbulent length scale diffusion ($D_l = 0$, $D_T \neq 0$, $D_k \neq 0$). In this case, the characteristic is

$$b_{510}b_{611}D_TD_k(b_{77} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{21} & b_{22} - \lambda & b_{23} & b_{24} \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} - \lambda \end{vmatrix} = 0, \quad (275)$$

i.e.,

$$\frac{1}{\rho^2} \frac{(\gamma-1)T}{b_1} D_T D_k (b_{77} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 \\ b_{21} & b_{22} - \lambda & b_{23} & b_{24} \\ b_{31} & 0 & b_{33} - \lambda & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} - \lambda \end{vmatrix} = 0. \quad (276)$$

For non-zero D_T and D_k , one of the roots is given by

$$\lambda_1 = b_{77} = u + (f_1 - x_1)\delta w = u + w,$$

the other roots are as same as in *Case 1* of this class.

Case 5 Two-phase flow without turbulent diffusions ($D_k = 0 = D_l$, $D_T \neq 0$). In this case, the characteristic equation is

$$b_{510} D_T (b_{77} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 & 0 \\ b_{21} & b_{22} - \lambda & b_{23} & b_{24} & b_{26} \\ b_{31} & 0 & b_{33} - \lambda & b_{34} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} - \lambda & b_{46} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{66} - \lambda \end{vmatrix} = 0, \quad (277)$$

i.e.,

$$\frac{(\gamma-1)T}{\rho b_1} (b_{77} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 & 0 \\ b_{21} & b_{22} - \lambda & b_{23} & b_{24} & b_{26} \\ b_{31} & 0 & b_{33} - \lambda & b_{34} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} - \lambda & b_{46} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{66} - \lambda \end{vmatrix} = 0. \quad (278)$$

It has one real root

$$\lambda_1 = b_{77} = u + (f_1 - x_1)\delta w = u + w$$

and the others are as same as in *Case 3* of the class.

Case 6 Two-phase flow without temperature and turbulent kinetic energy diffusions ($D_T = 0 = D_k$, $D_l \neq 0$). In this case, the characteristic equation is

$$b_{712} D_l \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} - \lambda & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & 0 & b_{33} - \lambda & b_{34} & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} - \lambda & b_{45} & b_{46} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} - \lambda & 0 \\ b_{61} & b_{62} & b_{63} & b_{64} & 0 & b_{66} - \lambda \end{vmatrix} = 0, \quad (279)$$

i.e.,

$$\frac{D_l}{\rho} \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} - \lambda & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & 0 & b_{33} - \lambda & b_{34} & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} - \lambda & b_{45} & b_{46} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} - \lambda & 0 \\ b_{61} & b_{62} & b_{63} & b_{64} & 0 & b_{66} - \lambda \end{vmatrix} = 0. \quad (280)$$

Case 7 Two-phase flow without temperature and turbulent length scale diffusions ($D_T = 0 = D_l, D_k \neq 0$). In this case, the characteristic equation is

$$b_{611}D_k(b_{77} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 & 0 \\ b_{21} & b_{22} - \lambda & b_{23} & b_{24} & b_{25} \\ b_{31} & 0 & b_{33} - \lambda & b_{34} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} - \lambda & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} - \lambda \end{vmatrix} = 0, \quad (281)$$

i.e.,

$$\frac{D_k}{\rho} \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 & 0 \\ b_{21} & b_{22} - \lambda & b_{23} & b_{24} & b_{25} \\ b_{31} & 0 & b_{33} - \lambda & b_{34} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} - \lambda & b_{45} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} - \lambda \end{vmatrix} = 0. \quad (282)$$

Clearly, besides the real root

$$\lambda_1 = b_{77} = u + (f_1 - x_1)\delta w = u + w,$$

all other roots are the same as in *Case 2* of the class.

Case 8 Two-phase flow without any diffusions ($D_T = D_l = D_k = 0$). In this case, the characteristic equation is

$$(b_{77} - \lambda) \begin{vmatrix} b_{11} - \lambda & b_{12} & 0 & 0 & 0 & 0 \\ b_{21} & b_{22} - \lambda & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & 0 & b_{33} - \lambda & b_{34} & 0 & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} - \lambda & b_{45} & b_{46} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} - \lambda & 0 \\ b_{61} & b_{62} & b_{63} & b_{64} & 0 & b_{66} - \lambda \end{vmatrix} = 0. \quad (283)$$

It has seven roots, one is

$$\lambda_1 = b_{77} = u + (f_1 - x_1)\delta w = u + w,$$

and the other six are the same as in it *Case 6*.

From above, we find that the characteristics of the model are affected by the turbulent kinetic energy and temperature diffusions. The presence or absence of both or either one will change the characteristics of the model. However the turbulence length scale diffusion has no effect on the characteristics of the model in the limit of inviscid flow.

5. DISCUSSION

In this report, we have studied the characteristics of the new hybrid multifluid turbulent mix model. In particular, we have performed a thorough analysis of the characteristics of this model in the case of one dimensional two-phase flows under various physical situations. Through our studies, we find that the new hybrid multifluid turbulent mix model under the assumption of pressure and temperature equilibrium has all real characteristics if either real or turbulent viscosity is present. The characteristic values of the model are affected by the turbulent kinetic energy and temperature diffusions, but are unaffected by the turbulent length scale diffusion. In inviscid flow where the real viscosity vanishes, the hybrid multifluid turbulent mix model still has real characteristic values if the relative motion between the

fluids is zero. For nonzero relative motions between the fluids, if the disordered motions (turbulence) and turbulent viscosity together are generated simultaneously with the nonzero relative motions, then this model again has all real characteristics for all values of δw .

However, if at early times δw is very small and η_d is negligible, then the model will possess complex characteristics in a certain physical range of δw unless the turbulent kinetic energy is equal to or greater than $\frac{3}{4} \cdot \frac{a_1 + a_2^2}{a_1} \cdot \frac{k_o}{f_1 f_2}$. This constitutes a constraint on the model. In the extreme case where the disordered turbulent contribution is completely ignored ($k_d = 0$), there is an unremovable range for δw , $0 < \delta w^2 < \frac{4(a_1 + a_2^2)}{a_1} b_1$, in which the model has complex characteristics and becomes illposed as do traditional single-pressure multiphase flow models in the absence of dissipation.

It is clear that the addition of disordered diffusive motion to the two-phase flow equations provides a natural dissipation which reduces the well-known illposedness of the single-pressure two-phase flow equations in the inviscid limit. However, the illposedness is only removed if constraint (270) is satisfied, a condition we have not been able to prove. Thus the hybrid multifluid turbulent mix model must still be missing some crucial physical processes. Molecular diffusion is one such process. Nevertheless, this model in its present form is well-posed even for inviscid fluids if both k_d and η_d are kept nonzero when δw is not zero.

In the past decades, the physical origin of the complex characteristics in inviscid single-pressure two-phase flow models has been discussed extensively. It is known that this illposedness can be removed by the addition of viscosity or by letting the pressures of the two phases differ. Now we have shown that the addition of disordered turbulent kinetic energy k_d with its associated turbulent viscosity η_d will also remove the illposedness. In fact, even when $\eta_d = 0$, the illposedness is removed if k_d satisfies condition (270). Note that the disordered turbulent motions are physically necessary as the late-time decay products of the ordered drift motions even if they aren't needed to produce well-posed equations. For example, it is not adequate to obtain well-posed equations by simply letting the phase pressures differ since, in the absence of cohesive forces such as surface tension, a pressure imbalance can only be sustained by the ram pressures of the drift motions, and these drift motions decay away at late time.

Finally we would like to point out that this model only contains one drag frequency and one turbulent length scale for simplicity. A real mixing process probably involves a spectrum of both ω and l , and the materials eventually must mix at a microscopic scale. Therefore continuing improvements to this model are still ongoing. Nevertheless, the important advantage of this model is that its evolution equations are symmetrical in all materials and are formulated with vectors. This distinctive character allows the model to be applied to systems that contain any numbers of materials and spatial dimensions.

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APPENDIX A. PRESSURE NONEQUILIBRIUM SITUATION

For mixtures not in pressure and temperature equilibrium, the different materials have different pressure and temperature fluctuations, P'_j and T'_j . Under this physical circumstance, the evolution equations of the fluid system become

- Bulk fluid equations:

$$0 = \frac{d}{dt}\rho + \rho \vec{\nabla} \cdot \vec{u}, \quad (284)$$

$$0 = \rho \frac{d}{dt}\vec{u} + \vec{\nabla} \cdot (\mathbf{R} + \mathbf{P}), \quad (285)$$

$$0 = \rho \frac{d}{dt}i + \vec{\nabla} \cdot (\vec{s} + \vec{q} + \vec{s}' + \vec{q}') + (\mathbf{P} + \mathbf{Q}) : \vec{\nabla}\vec{u} + P\vec{\nabla} \cdot \vec{w} - \rho(\epsilon + \epsilon'). \quad (286)$$

- Multifluid interpenetration equations:

$$0 = \rho \frac{d}{dt}\mathbf{x}_j + \vec{\nabla} \cdot (\rho \mathbf{x}_j \vec{w}_j), \quad (287)$$

$$\begin{aligned} 0 = & \rho \frac{d}{dt}(\mathbf{x}_j \vec{w}_j) + \rho \mathbf{x}_j \vec{w}_j \cdot \vec{\nabla}\vec{u} + \vec{\nabla} \cdot (\rho \mathbf{x}_j \vec{w}_j^2 + f_j \mathbf{R}_{dj} + f_j \mathbf{\Pi}_j + f_j P'_j \delta) \\ & + f_j \vec{\nabla} P + (\omega + \omega') \rho \mathbf{x}_j \vec{w}_j + \rho \mathbf{x}_j \frac{d}{dt}\vec{u}, \end{aligned} \quad (288)$$

- Single-fluid turbulence equations:

$$0 = \rho \frac{d}{dt}k + \vec{\nabla} \cdot (\vec{\sigma} + \vec{\pi} + \vec{p}') + (\mathbf{R} - \mathbf{Q}) : \vec{\nabla}\vec{u} + (\vec{\nabla} P) \cdot \vec{w} + \rho(\epsilon + \epsilon'), \quad (289)$$

$$\begin{aligned} 0 = & \rho \frac{d}{dt}k_o + \vec{\nabla} \cdot (\vec{\sigma}_o + \vec{\pi}_o + \vec{p}'_o) + (\mathbf{R}_o - \mathbf{Q}) : \vec{\nabla}\vec{u} + (\vec{\nabla} P) \cdot \vec{w} \\ & + \rho \phi_o + 2(\omega + \omega') \rho k_o + \rho(\epsilon_o + \epsilon'_o), \end{aligned} \quad (290)$$

$$\begin{aligned} 0 = & \rho \frac{d}{dt}k_d + \vec{\nabla} \cdot [\rho k_d \vec{w} - C_{kd}(\mathbf{D}_d + \mathbf{D}'_d) \cdot \vec{\nabla} k_d + \vec{p}'_d] \\ & + \rho \phi_d - 2(\omega + \omega') \rho k_o + \rho(\epsilon_d + \epsilon'_d), \end{aligned} \quad (291)$$

$$0 = \rho \frac{d}{dt}l + \vec{\nabla} \cdot [\rho l \vec{w} - C_{ld}(\mathbf{D}_d + \mathbf{D}'_d) \cdot \vec{\nabla} l] - C_{lo} \rho (l \vec{\nabla} \cdot \vec{u} + \sqrt{k_o + k_d}). \quad (292)$$

Where

$$f_j P'_j = \langle c_j P' \rangle, \quad \langle P' \rangle = \sum_j f_j P'_j = 0, \quad (293)$$

$$\omega' \rho \mathbf{x}_j \vec{w}_j \equiv - \langle P' \delta \cdot \vec{\nabla} c_j \rangle, \quad (294)$$

$$\mathbf{D}_d + \mathbf{D}'_d \equiv (\omega + \omega')^{-1} (\mathbf{R}_d + \mathbf{\Pi}_o), \quad (295)$$

$$\mathbf{D}'_d \simeq -\frac{\omega'}{\omega} \mathbf{D}_d = \sum_j f_j \mathbf{D}'_{dj}, \quad \mathbf{D}'_{dj} = -\frac{\omega'}{\omega} \mathbf{D}_{dj}, \quad \text{when } |\omega'| \ll \omega, \quad (296)$$

$$\vec{p}' \equiv \langle P' \delta \cdot \vec{v}' \rangle = \vec{p}'_o + \vec{p}'_d, \quad \vec{p}'_o \equiv \sum_j f_j P'_j \delta \cdot \vec{w}_j, \quad (297)$$

$$\vec{p}'_d \equiv \sum_j f_j \vec{p}'_{dj}, \quad f_j \vec{p}'_{dj} = \langle c_j P' \delta \cdot (\vec{v}' - \vec{w}_j) \rangle, \quad (298)$$

$$\vec{\sigma}_d + \vec{\pi}_d + \vec{p}'_d = \rho k_d \vec{w} + \sum_j f_j \mathbf{R}_{dj} \cdot \vec{w}_j + \sum_j f_j (\vec{\sigma}_{dj} + \vec{\pi}_{dj} + \vec{p}'_{dj}), \quad (299)$$

$$\vec{s}' \equiv \langle \rho \mathbf{x} \theta' \vec{v}' \rangle \equiv \sum_j \rho \mathbf{x}_j \theta'_j \vec{w}_j + \sum_j \langle c_j \rho \mathbf{x} \theta' (\vec{v}' - \vec{w}_j) \rangle, \quad (300)$$

$$\rho \mathbf{x}_j \theta'_j = \langle c_j \rho \mathbf{x} \theta' \rangle, \quad \vec{q}' = - \langle \kappa \rangle \vec{\nabla} T', \quad (301)$$

$$\rho \epsilon' = \rho \epsilon'_o + \rho \epsilon'_d, \quad \rho \epsilon'_o \equiv - \sum_j f_j P'_j \delta : \vec{\nabla} \vec{w}_j = \sum_j \rho \mathbf{x}_j \epsilon'_{oj}, \quad (302)$$

$$\rho \epsilon'_d \equiv - \sum_j \langle c_j P' \delta : \vec{\nabla} (\vec{v}' - \vec{w}_j) \rangle = \sum_j \rho \mathbf{x}_j \epsilon'_{dj}, \quad (303)$$

$$\vec{\nabla} \cdot \vec{p}'_o = \sum_j \vec{w}_j \cdot \vec{\nabla} (f_j P'_j) - \rho \epsilon'_o, \quad (304)$$

$$\vec{\nabla} \cdot \vec{p}'_d = \sum_j \langle (\vec{v}' - \vec{w}_j) \cdot \vec{\nabla} (c_j P') \rangle - \rho \epsilon'_d, \quad (305)$$

and the new turbulence length scale is defined as

$$l \equiv k_d \sqrt{k_o + k_d} / (\epsilon_d + \epsilon'_d).$$

The pressure and temperature fluctuations, P'_j and T'_j , are determined by the equations of state of each material, as are the internal energy fluctuations. If the fluid system is highly conductive, then T'_j is negligible, and the system is nearly in temperature equilibrium, so the pressure is only a function of the mass density:

$$P + P' = \psi(m).$$

The average pressure in each material accordingly satisfies

$$f_j P_j = f_j P + f_j P'_j = f_j \psi_j(m_j), \quad (306)$$

where $f_j \psi_j(m_j) \equiv \langle c_j \psi(m) \rangle$, and ψ_j is given by the equation of state of the j -th material. For a small mass density fluctuation, the pressure fluctuations can be approximated by a Taylor expansion about the mean mass density ρ :

$$P + P' = \psi(\rho) + \psi_m(\rho)(m - \rho) + \frac{1}{2} \psi_{mm}(\rho)(m - \rho)^2 + \dots, \quad (307)$$

where $\psi_m(\rho) = (\partial \psi / \partial m)_{m=\rho}$, $\psi_{mm}(\rho) = (\partial^2 \psi / \partial m^2)_{m=\rho}$, and

$$P = \psi(\rho) + \frac{1}{2} \psi_{mm}(\rho) \langle (m - \rho)^2 \rangle + \dots$$

To a second order approximation $O[(m - \rho)^2]$, P' is given by

$$P' \simeq \psi_m(\rho)(m - \rho) + \frac{1}{2} \psi_{mm}(\rho)[(m - \rho)^2 - \langle (m - \rho)^2 \rangle], \quad m = \rho \mathbf{x}, \quad (308)$$

and the individual pressure fluctuations in each material are

$$\begin{aligned}
 f_j P'_j &= \langle c_j P' \rangle \\
 &= \psi_m(\rho) \langle c_j(m - \rho) \rangle + \frac{1}{2} \psi_{mm}(\rho) \langle (c_j - f_j)(m - \rho)^2 \rangle \\
 &= \psi_m(\rho)(x_j - f_j)\rho + \\
 &\quad \frac{1}{2} \psi_{mm}(\rho)[-2(1 - x_j) \langle (c_j - f_j)(x - x_j) \rangle + \langle (c_j - f_j)(x - x_j)^2 \rangle] \rho^2 \\
 &= \psi_m(\rho)(x_j - f_j)\rho + \psi_{mm}(\rho)[(f_j - x_j)(1 - x_j) + \frac{1}{2} \langle (c_j - f_j)(x - x_j)^2 \rangle] \rho^2.
 \end{aligned} \tag{309}$$

This further gives the expression for the total pressure of each material up to the second order approximation:

$$\begin{aligned}
 f_j P_j &\equiv f_j \psi_j(m_j) \\
 &\simeq f_j P + \psi_m(\rho)(x_j - f_j)\rho + \\
 &\quad \psi_{mm}(\rho)[(f_j - x_j)(1 - x_j) + \frac{1}{2} \langle (c_j - f_j)(x - x_j)^2 \rangle] \rho^2.
 \end{aligned} \tag{310}$$

In general, since $\langle (c_j - f_j)(x - x_j)^2 \rangle / 2$ is negligible compared to $(f_j - x_j)(1 - x_j)$ in a fluid system with small fluctuations, this will further simplify P_j to

$$P_j \simeq P + f_j^{-1} \psi_m(\rho)(x_j - f_j)\rho + f_j^{-1} \psi_{mm}(\rho)(f_j - x_j)(1 - x_j)\rho^2. \tag{311}$$

This expression is valid for most materials. In some simple gaseous systems, e.g. ideal gases, the high-order terms will be exactly zero ($O[\geq (m - \rho)^2] = 0$), so we are able to obtain an exact expression for P_j

$$P_j = \psi_j(m_j) = P + f_j^{-1} \psi_m(\rho)(x_j - f_j)\rho. \tag{312}$$

Either of these expressions for P_j together with the equations of the state of each material will uniquely determine the volume fraction f_j .

Thus, in the physical situation of $P'_j \neq 0$ but $T'_j = 0$, the model has equations for all the independent dynamical variables ($\rho, \vec{u}, i, x_j, \vec{w}_j, k_d, l$, and P'_j). Therefore this new hybrid multifluid turbulent mix model is also mathematically closed up to the second-order Taylor expansion even if the fluid systems have a small departure from pressure equilibrium.

Similar to the pressure equilibrium case, applying the model to a one-dimensional system containing only two fluids gives the following evolution equations:

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0, \tag{313}$$

$$\begin{aligned}
 \rho \frac{\partial u}{\partial t} + \left(\frac{2}{3} k_d + a_1 \delta w^2\right) \frac{\partial \rho}{\partial x} + \rho u \frac{\partial u}{\partial x} + \rho a_2 \delta w^2 \frac{\partial x_1}{\partial x} + 2 \rho a_1 \delta w \frac{\partial \delta w}{\partial x} + \\
 \frac{2}{3} \rho \frac{\partial k_d}{\partial x} - \frac{4}{3} \frac{\partial \Phi}{\partial x} + \frac{\partial P}{\partial x} = 0,
 \end{aligned} \tag{314}$$

$$\rho \frac{\partial x_1}{\partial t} + a_1 \delta w \frac{\partial \rho}{\partial x} + \rho(u + a_2 \delta w) \frac{\partial x_1}{\partial x} + \rho a_1 \frac{\partial \delta w}{\partial x} = 0, \tag{315}$$

$$\begin{aligned}
& \rho x_1 \frac{\partial u}{\partial t} + \rho a_2 \delta w \frac{\partial x_1}{\partial t} + \rho a_1 \frac{\partial \delta w}{\partial t} + \left[\frac{2}{3} f_1 k_d + a_1 (1 - x_1) \delta w^2 \right] \frac{\partial \rho}{\partial x} + \rho x_1 [u + (1 - x_1) \delta w] \frac{\partial u}{\partial x} + \\
& \rho [a_2 u + (1 - x_1) (1 - 3x_1) \delta w] \delta w \frac{\partial x_1}{\partial x} + \rho a_1 [u + 2(1 - x_1) \delta w] \frac{\partial \delta w}{\partial x} + \\
& \frac{2}{3} f_1 \rho \frac{\partial k_d}{\partial x} - \frac{4}{3} f_1 \left[\frac{\partial \Phi}{\partial x} + (1 - f_1) \frac{\partial \delta \Phi}{\partial x} \right] + f_1 \frac{\partial P}{\partial x} + \frac{\partial}{\partial x} (f_1 P'_1) + \\
& \left[\frac{2}{3} \rho k_d - \frac{4}{3} (\Phi + (1 - 2f_1) \delta \Phi) \right] \frac{\partial f_1}{\partial x} + (\omega + \omega') a_1 \rho \delta w = 0, \tag{316}
\end{aligned}$$

$$\begin{aligned}
& \rho \frac{\partial i}{\partial t} + a_1 \delta \theta \delta w \frac{\partial \rho}{\partial x} + \left[P - \frac{4}{3} (\Pi_o + Q) \right] \frac{\partial u}{\partial x} + (a_2 \rho \delta \theta + \frac{4}{3} Q - P) \delta w \frac{\partial x_1}{\partial x} + \\
& [a_1 \rho \delta \theta + a_3 P + f_1 P'_1 - \frac{4}{3} Q a_3 - \frac{4}{3} \delta Q f_1 (1 - f_1)] \frac{\partial \delta w}{\partial x} + \rho u \frac{\partial i}{\partial x} + \\
& \frac{\partial s_d}{\partial x} + \frac{\partial q}{\partial x} + P \delta w \frac{\partial f_1}{\partial x} + \rho a_1 \delta w \frac{\partial \delta \theta}{\partial x} - \rho (\epsilon_d + \epsilon'_d) = 0, \tag{317}
\end{aligned}$$

$$\begin{aligned}
& \rho \frac{\partial k_d}{\partial t} + k_d a_3 \delta w \frac{\partial \rho}{\partial x} + \left(\frac{2}{3} \rho k_d - \frac{4}{3} \Gamma_d \right) \frac{\partial u}{\partial x} - \left(\frac{5}{3} \rho k_d - \frac{4}{3} \Gamma_d \right) \delta w \frac{\partial x_1}{\partial x} + \\
& \left[\frac{5}{3} \rho k_d a_3 - \frac{4}{3} \Gamma_d a_3 - \frac{4}{3} f_1 (1 - f_1) \delta \Gamma_d \right] \frac{\partial \delta w}{\partial x} + \rho (u + a_3 \delta w) \frac{\partial k_d}{\partial x} + \\
& \frac{\partial J_k}{\partial x} + \rho k_d \delta w \frac{\partial f_1}{\partial x} - (\omega + \omega') \rho a_1 \delta w^2 + \rho \epsilon_d = 0, \tag{318}
\end{aligned}$$

$$\begin{aligned}
& \rho \frac{\partial l}{\partial t} + a_3 l \delta w \frac{\partial \rho}{\partial x} - C_{lo} \rho l \frac{\partial u}{\partial x} - \rho l \delta w \frac{\partial x_1}{\partial x} + a_3 \rho l \frac{\partial \delta w}{\partial x} + \rho (u + a_3 \delta w) \frac{\partial l}{\partial x} + \\
& \frac{\partial J_l}{\partial x} + \rho l \delta w \frac{\partial f_1}{\partial x} - C_{lo} \rho \sqrt{k_o + k_d} = 0. \tag{319}
\end{aligned}$$

Here we have used $\rho \epsilon'_o = -f_1 P'_1 \partial \delta w / \partial x$. The auxiliary equations for the new fluxes including P'_j are given by

$$\Phi - (\eta + \eta_d) \left[\frac{\partial u}{\partial x} - \delta w \frac{\partial x_1}{\partial x} + (a_3 + f_1 f_2 \frac{\delta \eta}{\eta + \eta_d}) \frac{\partial \delta w}{\partial x} \right] = 0, \tag{320}$$

$$\delta \Phi - \delta \eta \left[\frac{\partial u}{\partial x} - \delta w \frac{\partial x_1}{\partial x} + (1 - f_1 - x_1 + \frac{\eta + \eta_d}{\delta \eta}) \frac{\partial \delta w}{\partial x} \right] = 0, \tag{321}$$

$$s_d + (D_s + D'_s) \frac{\partial T}{\partial x} = 0, \quad D'_s = C_{sd} \left(\frac{\partial i}{\partial T} \right)_\rho D'_{dx}, \tag{322}$$

$$q + D_q \frac{\partial T}{\partial x} = 0, \quad D_q = \kappa, \tag{323}$$

$$J_k + (D_k + D'_k) \frac{\partial k_d}{\partial x} = 0, \quad D'_k = C_{kd} D'_{dx}, \tag{324}$$

$$J_l + (D_l + D'_l) \frac{\partial l}{\partial x} = 0, \quad D'_l = C_{ld} D'_{dx}. \tag{325}$$

The coefficients D'_k , D'_l , and D'_s are, respectively, the corrections to the diffusion coefficients D_k , D_l , and D_s due to the effects of P'_j .

Up to the second-order approximation for P' , we have

$$\begin{aligned} f_1 P'_1 &\simeq \psi_m(\rho)(x_1 - f_1)\rho + \psi_{mm}(\rho)(f_1 - x_1)x_2\rho^2 \\ &= a_3\rho[-\psi_m(\rho) + \psi_{mm}(\rho)\rho x_2] \end{aligned} \quad (326)$$

and

$$\begin{aligned} \frac{\partial}{\partial x}(f_1 P'_1) &= \rho(-\psi_m + \rho x_2 \psi_{mm}) \frac{\partial f_1}{\partial x} + \rho[\psi_m - \rho(x_2 + a_3)\psi_{mm}] \frac{\partial x_1}{\partial x} - \\ &\quad a_3[\rho\psi_{m\rho} + \psi_m - x_2\rho(\rho\psi_{mm\rho} + 2\psi_{mm})] \frac{\partial \rho}{\partial x}, \end{aligned} \quad (327)$$

where we have used the notations that

$$\psi_m \equiv \psi_m(\rho), \quad \psi_{m\rho} \equiv \partial\psi_m(\rho)/\partial\rho, \quad \psi_{mm} \equiv \psi_{mm}(\rho), \quad \psi_{mm\rho} \equiv \partial\psi_{mm}(\rho)/\partial\rho.$$

Incorporating these approximations into the equations for δw and i gives

$$\begin{aligned} &\rho x_1 \frac{\partial u}{\partial t} + \rho a_2 \delta w \frac{\partial x_1}{\partial t} + \rho a_1 \frac{\partial \delta w}{\partial t} + \\ &\left[\frac{2}{3} f_1 k_d + a_1(1 - x_1)\delta w^2 - a_3(\rho\psi_{m\rho} + \psi_m) + a_3 x_2 \rho(\rho\psi_{mm\rho} + 2\psi_{mm}) \right] \frac{\partial \rho}{\partial x} + \\ &\rho x_1 [u + (1 - x_1)\delta w] \frac{\partial u}{\partial x} + \rho [a_2 u \delta w + (1 - x_1)(1 - 3x_1)\delta w^2 + \rho\psi_m - (x_2 + a_3)\rho^2\psi_{mm}] \frac{\partial x_1}{\partial x} + \\ &\rho a_1 [u + 2(1 - x_1)\delta w] \frac{\partial \delta w}{\partial x} + \frac{2}{3} f_1 \rho \frac{\partial k_d}{\partial x} - \frac{4}{3} f_1 \left(\frac{\partial \Phi}{\partial x} + f_2 \frac{\partial \delta \Phi}{\partial x} \right) + f_1 \frac{\partial P}{\partial x} + \\ &\left[\frac{2}{3} \rho k_d - \frac{4}{3} (\Phi + (1 - 2f_1)\delta \Phi) - \rho\psi_m + x_2 \rho^2 \psi_{mm} \right] \frac{\partial f_1}{\partial x} + \\ &(\omega + \omega') a_1 \rho \delta w = 0, \end{aligned} \quad (328)$$

$$\begin{aligned} &\rho \frac{\partial i}{\partial t} + a_1 \delta \theta \delta w \frac{\partial \rho}{\partial x} + \left[P - \frac{4}{3} (\Pi_o + Q) \right] \frac{\partial u}{\partial x} + (a_2 \rho \delta \theta + \frac{4}{3} Q - P) \delta w \frac{\partial x_1}{\partial x} + \\ &[a_1 \rho \delta \theta + a_3 P - a_3 \rho \psi_m + a_3 x_2 \rho^2 \psi_{mm} - \frac{4}{3} Q a_3 - \frac{4}{3} \delta Q f_1 f_2] \frac{\partial \delta w}{\partial x} + \rho u \frac{\partial i}{\partial x} + \\ &\frac{\partial s_d}{\partial x} + \frac{\partial q}{\partial x} + P \delta w \frac{\partial f_1}{\partial x} + \rho a_1 \delta w \frac{\partial \delta \theta}{\partial x} - \rho(\epsilon_d + \epsilon'_d) = 0. \end{aligned} \quad (329)$$

For ideal gases, $\psi_{mm} = 0 = \psi_{mm\rho}$, so these two equations will further reduce to

$$\begin{aligned} &\rho x_1 \frac{\partial u}{\partial t} + \rho a_2 \delta w \frac{\partial x_1}{\partial t} + \rho a_1 \frac{\partial \delta w}{\partial t} + \left[\frac{2}{3} f_1 k_d + a_1(1 - x_1)\delta w^2 - a_3(\rho\psi_{m\rho} + \psi_m) \right] \frac{\partial \rho}{\partial x} + \\ &\rho x_1 [u + (1 - x_1)\delta w] \frac{\partial u}{\partial x} + \rho [a_2 u \delta w + (1 - x_1)(1 - 3x_1)\delta w^2 + \rho\psi_m] \frac{\partial x_1}{\partial x} + \\ &\rho a_1 [u + 2(1 - x_1)\delta w] \frac{\partial \delta w}{\partial x} + \frac{2}{3} f_1 \rho \frac{\partial k_d}{\partial x} - \frac{4}{3} f_1 \left(\frac{\partial \Phi}{\partial x} + f_2 \frac{\partial \delta \Phi}{\partial x} \right) + f_1 \frac{\partial P}{\partial x} + \\ &\left[\frac{2}{3} \rho k_d - \frac{4}{3} (\Phi + (1 - 2f_1)\delta \Phi) - \rho\psi_m \right] \frac{\partial f_1}{\partial x} + (\omega + \omega') a_1 \rho \delta w = 0, \end{aligned} \quad (330)$$

$$\begin{aligned}
& \rho \frac{\partial i}{\partial t} + a_1 \delta \theta \delta w \frac{\partial \rho}{\partial x} + [P - \frac{4}{3}(\Pi_o + Q)] \frac{\partial u}{\partial x} + (a_2 \rho \delta \theta + \frac{4}{3}Q - P) \delta w \frac{\partial x_1}{\partial x} + \\
& (a_1 \rho \delta \theta + a_3 P - a_3 \rho \psi_m - \frac{4}{3}Q a_3 - \frac{4}{3} \delta Q f_1 f_2) \frac{\partial \delta w}{\partial x} + \rho u \frac{\partial i}{\partial x} + \frac{\partial s_d}{\partial x} + \\
& \frac{\partial q}{\partial x} + P \delta w \frac{\partial f_1}{\partial x} + \rho a_1 \delta w \frac{\partial \delta \theta}{\partial x} - \rho(\epsilon_d + \epsilon'_d) = 0.
\end{aligned} \tag{331}$$

APPENDIX B. GENERAL CONSTRAINT ON THE MODEL IN INVISCID FLOW

The initial constraint (270) to the model can be relaxed with the increase of δw . For general δw , the sufficient condition for W shape $P(y)$ having all real roots is given by

$$P(y_2) \geq 0.$$

For convenience, we rewrite y_2 into

$$y_2 = \sqrt{\frac{a_1 + a_3^2}{a_1}} b_1 \left[-\frac{1}{2} a_2 \zeta^{1/2} + 2 \left(\frac{1 + \gamma_1}{6} \right)^{1/2} \cos \Psi \right], \quad (332)$$

where γ_1 is given by Eq.(215) and $\Psi \equiv \frac{1}{3}(\phi' + \pi)$. Thus

$$y_2^2 = \frac{a_1 + a_3^2}{a_1} b_1 \left[\frac{1}{4} a_2^2 \zeta + \frac{2}{3} (1 + \gamma_1) \cos^2 \Psi - a_2 \sqrt{\frac{2}{3}} (1 + \gamma_1) \zeta \cos \Psi \right]. \quad (333)$$

Substituting these expressions into Eq.(248) gives

$$\begin{aligned} P(y_2) = & -\frac{1}{2} \left(\frac{a_1 + a_3^2}{a_1} b_1 \right)^2 (1 + \gamma_1) \left[\frac{1}{4} a_2^2 \zeta + \frac{2}{3} (1 + \gamma_1) \cos^2 \Psi - a_2 \sqrt{\frac{2}{3}} (1 + \gamma_1) \zeta \cos \Psi \right] + \\ & + \left(\frac{a_1 + a_3^2}{a_1} b_1^2 \right) \left[-\frac{1}{2} a_2 \zeta^{1/2} + 2 \left(\frac{1 + \gamma_1}{6} \right)^{1/2} \cos \Psi \right] \left\{ 3a_3 - \frac{3}{2} a_2 + \left(1 + \frac{a_3^2}{a_1} \right) \times \right. \\ & \left. \left[\zeta \theta' \left(\frac{a_3 - a_2}{1 + \frac{f_1 f_2}{x_1 x_2}} + \frac{1}{6} a_2 \right) + \frac{1}{4} a_2 (1 - \zeta) \right] \right\} + \left(1 + \frac{a_3^2}{a_1} \right) b_1^2 \zeta \left\{ \zeta \theta' \frac{a_3^2 + a_1}{a_1} \times \right. \\ & \left. \left[\frac{a_2 (a_3 - a_2)}{2(1 + \frac{f_1 f_2}{x_1 x_2})} - \frac{2}{3} a_1 \right] + \frac{2}{3} \theta' \frac{f_1 f_2}{1 + \frac{f_1 f_2}{x_1 x_2}} \left[\frac{2}{3} \frac{a_1 + a_3^2}{a_1 (1 + \frac{f_1 f_2}{x_1 x_2})} \zeta \theta' + 1 \right] + \right. \\ & \left. \left. \frac{1}{4} (a_1 + a_3^2) \zeta - \left[(a_3 - a_2)^2 + \frac{1}{2} a_2 a_3 + 2a_1 - \frac{1}{4} \right] \right\} \right\} \\ = & \left(\frac{a_1 + a_3^2}{a_1} b_1 \right)^2 \left\{ -\frac{1}{8} a_2^2 \zeta (1 + \gamma_1) - \frac{1}{3} (1 + \gamma_1)^2 \cos^2 \Psi + (1 + \gamma_1)^{3/2} a_2 \sqrt{\frac{\zeta}{6}} \cos \Psi + \right. \\ & \frac{a_1 \zeta^{1/2}}{a_1 + a_3^2} \left[-\frac{1}{2} a_2 \zeta^{1/2} + 2 \left(\frac{1 + \gamma_1}{6} \right)^{1/2} \cos \Psi \right] \left\{ 3a_3 - \frac{3}{2} a_2 + \left(1 + \frac{a_3^2}{a_1} \right) \times \right. \\ & \left. \left[\zeta \theta' \left(\frac{a_3 - a_2}{1 + \frac{f_1 f_2}{x_1 x_2}} + \frac{1}{6} a_2 \right) + \frac{1}{4} a_2 (1 - \zeta) \right] \right\} + \zeta^2 \theta' \left\{ \left[\frac{a_2 (a_3 - a_2)}{2(1 + \frac{f_1 f_2}{x_1 x_2})} - \frac{2}{3} a_1 \right] + \right. \\ & \frac{2a_1}{3(a_1 + a_3^2)} \frac{f_1 f_2}{x_1 x_2} \zeta \theta' \left[\frac{2}{3} \frac{a_1 + a_3^2}{a_1 (1 + \frac{f_1 f_2}{x_1 x_2})} \zeta \theta' + 1 \right] + \\ & \left. \left. \frac{a_1}{a_1 + a_3^2} \zeta \left\{ \frac{1}{4} (a_1 + a_3^2) \zeta - \left[(a_3 - a_2)^2 + \frac{1}{2} a_2 a_3 + 2a_1 - \frac{1}{4} \right] \right\} \right\} \right\} \geq 0. \end{aligned} \quad (334)$$

Since

$$\left(\frac{a_1 + a_3^2}{a_1} b_1 \right)^2 \geq 0,$$

Eq.(321) thus implies that

$$\begin{aligned}
& -\frac{1}{8}a_2^2\zeta(1+\gamma_1) - \frac{1}{3}(1+\gamma_1)^2\cos^2\Psi + (1+\gamma_1)^{3/2}a_2\sqrt{\frac{\zeta}{6}}\cos\Psi - \frac{3}{4}\frac{a_1a_2}{a_1+a_3^2}\zeta(2a_3-a_2) - \\
& \frac{1}{2}a_2\zeta^2\theta'\left(\frac{a_3-a_2}{1+\frac{f_1f_2}{x_1x_2}} + \frac{1}{6}a_2\right) - \frac{1}{8}a_2^2\zeta(\zeta-1) + \frac{3a_1(2a_3-a_2)}{a_1+a_3^2}\left(\frac{1+\gamma_1}{6}\zeta\right)^{1/2}\cos\Psi + \\
& 2\left(\frac{1+\gamma_1}{6}\zeta\right)^{1/2}\left(\frac{a_3-a_2}{1+\frac{f_1f_2}{x_1x_2}} + \frac{1}{6}a_2\right)\zeta\theta'\cos\Psi + \frac{a_2(1-\zeta)}{2}\left(\frac{1+\gamma_1}{6}\zeta\right)^{1/2}\cos\Psi + \\
& \left(\frac{a_2(a_3-a_2)}{2(1+\frac{f_1f_2}{x_1x_2})} - \frac{2}{3}a_1\right)\zeta^2\theta' + \frac{4}{9}\frac{f_1f_2}{x_1x_2}\zeta^2\theta^2 + \frac{2a_1}{3(a_1+a_3^2)}\frac{f_1f_2}{x_1x_2}\zeta\theta + \\
& \frac{a_1}{a_1+a_3^2}\zeta\left[\frac{1}{4}(a_1+a_3^2)\zeta - a_3^2 - a_2^2 + \frac{3}{2}a_2a_3 - 2a_1 + \frac{1}{4}\right] \geq 0. \tag{335}
\end{aligned}$$

Combining the same terms together in Eq.(322) reduces it to

$$\begin{aligned}
& -\frac{1}{4}a_2^2\zeta - \frac{1}{8}a_2^2\zeta(\gamma_1-\zeta) - \frac{1}{3}(1+\gamma_1)^2\cos^2\Psi + (1+\gamma_1)^{3/2}a_2\sqrt{\frac{\zeta}{6}}\cos\Psi - \left(\frac{1}{12}a_2^2 + \frac{2}{3}a_1\right)\eta^2\theta' \\
& \frac{3a_1(2a_3-a_2)}{a_1+a_3^2}\left(\frac{1+\gamma_1}{6}\zeta\right)^{1/2}\cos\Psi + 2\left(\frac{1+\gamma_1}{6}\zeta\right)^{1/2}\left(\frac{a_3-a_2}{1+\frac{f_1f_2}{x_1x_2}} + \frac{1}{6}a_2\right)\zeta\theta'\cos\Psi + \\
& \frac{a_2(1-\zeta)}{2}\left(\frac{1+\gamma_1}{6}\zeta\right)^{1/2}\cos\Psi + \frac{4}{9}\frac{f_1f_2}{x_1x_2}\zeta^2\theta^2 + \frac{2a_1}{3(a_1+a_3^2)}\frac{f_1f_2}{x_1x_2}\zeta\theta + \\
& \frac{a_1}{a_1+a_3^2}\zeta\left[\frac{3}{4}a_2^2 + \frac{1}{4}(a_1+a_3^2)\zeta - (a_3^2+a_2^2+2a_1-\frac{1}{4})\right] \geq 0. \tag{336}
\end{aligned}$$

Using identity Eq.(258), the last term in above equation becomes

$$\frac{a_1}{a_1+a_3^2}\zeta\left[\frac{3}{4}a_2^2\zeta + \frac{1}{4}(a_1+a_3^2)\zeta - (a_3^2+a_2^2+2a_1-\frac{1}{4})\zeta\right] = \frac{1}{4}a_1\zeta^2 - a_1\zeta,$$

this together with γ_1 's expression further simplify Eq.(323) into

$$\begin{aligned}
& -\frac{1}{4}a_2^2\zeta - \frac{1}{6}a_2^2\zeta^2\theta' + \frac{1}{16}\zeta^2 - \frac{1}{3}(1+\gamma_1)^2\cos^2\Psi + \left(\frac{1+\gamma_1}{6}\zeta\right)^{1/2}\cos\Psi\left[a_2(1+\gamma_1) + \right. \\
& \left. \frac{3a_1(2a_3-a_2)}{a_1+a_3^2} + \left(\frac{2(a_3-a_2)}{1+\frac{f_1f_2}{x_1x_2}} + \frac{1}{3}a_2\right)\zeta\theta' + \frac{1}{2}a_2(1-\zeta)\right] + \frac{4}{9}\frac{f_1f_2}{x_1x_2}\zeta^2\theta^2 + \\
& \left. \frac{2a_1}{3(a_1+a_3^2)}\frac{f_1f_2}{x_1x_2}\zeta\theta + \frac{1}{4}a_1\zeta^2 - a_1\zeta \geq 0. \tag{337}
\end{aligned}$$

Again applying Eq.(258) and Eq.(225), we are able to further reduce the constraint equation to

$$\begin{aligned}
& \frac{1}{16}\zeta^2 - \frac{1}{4}\zeta - \frac{1}{6}\zeta^2\theta' - \frac{1}{3}(1+\gamma_1)^2\cos^2\Psi + \left(\frac{1+\gamma_1}{6}\zeta\right)^{1/2}\cos\Psi\left[\frac{3}{2}\bar{F} + \right. \\
& \left. \left(\frac{2(a_3-a_2)}{1+\frac{f_1f_2}{x_1x_2}} + a_2\right)\zeta\theta'\right] + \frac{4}{9}\frac{f_1f_2}{x_1x_2}\zeta^2\theta^2 + \frac{2a_1}{3(a_1+a_3^2)}\frac{f_1f_2}{x_1x_2}\zeta\theta \geq 0. \tag{338}
\end{aligned}$$

Substituting γ_1 into Eq.(338) gives

$$\begin{aligned} & \frac{1}{16}\zeta^2 - \frac{1}{4}\zeta - \frac{1}{6}\zeta^2\theta' - \frac{1}{3}(1 + \zeta + \frac{1}{4}\zeta^2 + \frac{4}{3}\zeta\theta' + \frac{1}{3}\zeta^2\theta' + \frac{4}{9}\zeta^2\theta'^2) \cos^2 \Psi + \\ & + (\frac{1}{6}\zeta + \frac{1}{12}\zeta^2 + \frac{1}{9}\zeta^2\theta')^{1/2} \cos \Psi [\frac{3}{2}\tilde{F} + (\frac{2(a_3 - a_2)}{1 + \frac{f_1 f_2}{x_1 x_2}} + a_2)\zeta\theta'] + \\ & \frac{4}{9} \frac{f_1 f_2}{x_1 x_2} \zeta^2 \theta^2 + \frac{2a_1}{3(a_1 + a_3^2)} \frac{f_1 f_2}{x_1 x_2} \zeta \theta \geq 0. \end{aligned} \quad (339)$$

Noticing that

$$\zeta\theta' = \xi' = (1 + \frac{f_1 f_2}{x_1 x_2})\xi, \quad \zeta\theta = \xi, \quad \zeta^2\theta' = \zeta\xi', \quad \zeta^2\theta = \zeta\xi,$$

then Eq.(339) can be finally written as

$$\begin{aligned} & \frac{1}{16}\zeta^2 - \frac{1}{4}\zeta - \frac{1}{6}\zeta\xi' - \frac{1}{3}(1 + \zeta + \frac{1}{4}\zeta^2 + \frac{4}{3}\xi' + \frac{1}{3}\zeta\xi' + \frac{4}{9}\xi'^2) \cos^2 \Psi + \\ & (\frac{1}{6}\zeta + \frac{1}{12}\zeta^2 + \frac{1}{9}\zeta\xi')^{1/2} [\frac{3}{2}\tilde{F} + 2(a_3 - a_2)\xi + a_2\xi'] \cos \Psi + \\ & \frac{4}{9} \frac{f_1 f_2}{x_1 x_2} \xi^2 + \frac{2a_1}{3(a_1 + a_3^2)} \frac{f_1 f_2}{x_1 x_2} \xi \geq 0. \end{aligned} \quad (340)$$

This equation provides a general constraint for the model to have all real characteristics even in inviscid flow. Clearly, at both $\delta w = 0$ and $\delta w \rightarrow \infty$, Eq.(340) is satisfied because $\cos \Psi = 0$ in both cases, this tells us that the model does have real characteristics in both situations. These are well consistent with the results that we obtained earlier. Meanwhile, we see that a large ξ (i.e., large turbulent kinetic energy k_d) always assures the satisfaction of Eq.(340). However, it is obvious that condition (340) will break down if both δw and k_d are small especially in the early stage of the dynamic evolution of the system when the second term in the equation is dominant. For zero ξ ($k_d = 0$), Eq.(340) becomes

$$\frac{1}{16}\zeta^2 - \frac{1}{4}\zeta - \frac{1}{3}(1 + \frac{\zeta}{2})^2 \cos^2 \Psi + (\frac{1}{6}\zeta + \frac{1}{12}\zeta^2)^{1/2} \frac{3}{2}\tilde{F} \cos \Psi \geq 0. \quad (341)$$

Neglecting the contributions of the last two terms, we have

$$\frac{\zeta}{4}(\frac{1}{4}\zeta - 1) \geq 0,$$

which gives that

$$\zeta = 0 \quad \text{or} \quad \zeta \geq 4, \quad \text{i.e.,} \quad \delta w = 0 \quad \text{or} \quad \frac{\delta w^2}{b_1} \geq 4 \frac{a_1 + a_3^2}{a_1},$$

these are identical to Eq.(263).

These results reveal that in two-phase inviscid flow with non zero relative motions, if the contributions from turbulence are completely ignored, then the model has real characteristics when and only when

$$\frac{\delta w^2}{b_1} \geq 4 \frac{a_1 + a_3^2}{a_1}.$$

The region

$$0 < \delta w^2 < 4 \frac{a_1 + a_3^2}{a_1} b_1$$

in which the model possesses complex characteristics will be reduced with the increase of k_d and finally disappears when k_d is large enough. The required k_d that model will have all real characteristics for all $\delta w \geq 0$ can be estimated by solving equation (340).

APPENDIX C. THE EVOLUTION OF TURBULENT KINETIC ENERGY

The evolution equation of ordered kinetic energy k_o of a fluid mixture is given by Eq. (55):

$$0 = \rho \frac{d}{dt} k_o + \vec{\nabla} \cdot (\vec{\sigma}_o + \vec{\pi}_o) + (\mathbf{R}_o - \mathbf{Q}) : \vec{\nabla} \vec{u} + (\vec{\nabla} P) \cdot \vec{w} + \rho \phi_o + 2\omega \rho k_o + \rho \epsilon_o. \quad (342)$$

In a one-dimensional system consisting of only two fluids,

$$\vec{\sigma}_o = \sum_{j=1}^2 \frac{1}{2} \rho x_j w_j^3 \hat{x} = \rho k_o a_2 \delta w \hat{x}, \quad (343)$$

$$\vec{\nabla} \cdot \vec{\sigma}_o = k_o a_2 \delta w \frac{\partial \rho}{\partial x} + \frac{a_2^2 - 2a_1}{a_1} \rho k_o \delta w \frac{\partial x_1}{\partial x} + 3a_2 \rho k_o \frac{\partial \delta w}{\partial x}, \quad (344)$$

and

$$\begin{aligned} \rho \phi_o &= \frac{2}{3} \frac{\partial(f_1 \rho k_d)}{\partial x} w_1 - \frac{4}{3} \frac{\partial(f_1 \Gamma_{d1})}{\partial x} w_1 + \frac{2}{3} \frac{\partial(f_2 \rho k_d)}{\partial x} w_2 - \frac{4}{3} \frac{\partial(f_2 \Gamma_{d2})}{\partial x} w_2 \\ &= \frac{2}{3} \frac{\partial(\rho k_d)}{\partial x} w + \frac{2}{3} \rho k_d \delta w \frac{\partial f_1}{\partial x} - \frac{4}{3} \left[w_1 \frac{\partial(f_1 \Gamma_{d1})}{\partial x} + w_2 \frac{\partial(f_2 \Gamma_{d2})}{\partial x} \right] \\ &= \frac{2}{3} k_d w \frac{\partial \rho}{\partial x} + \frac{2}{3} \rho w \frac{\partial k_d}{\partial x} + \frac{2}{3} \rho k_d \delta w \frac{\partial f_1}{\partial x} - \frac{4}{3} \left[w_1 \frac{\partial(f_1 \Gamma_{d1})}{\partial x} + w_2 \frac{\partial(f_2 \Gamma_{d2})}{\partial x} \right]. \end{aligned} \quad (345)$$

For our purpose, we now consider inviscid flow, i.e., $\eta = 0$, $\rho \epsilon_o = 0$, and further assume that the turbulent viscosity is negligible ($\eta_d = 0$ and $\Gamma_{dj} = 0$). Thus the one-dimensional k_o equation becomes

$$\begin{aligned} \rho \frac{dk_o}{dt} + (a_2 k_o + \frac{2}{3} a_3 k_d) \delta w \frac{\partial \rho}{\partial x} + 2 \rho k_o \frac{\partial u}{\partial x} + \frac{a_2^2 - 2a_1}{a_1} \rho k_o \delta w \frac{\partial x_1}{\partial x} + 3a_2 \rho k_o \frac{\partial \delta w}{\partial x} \\ + \frac{2}{3} \rho a_3 \delta w \frac{\partial k_d}{\partial x} + a_3 \delta w \frac{\partial P}{\partial x} + \frac{2}{3} \rho k_d \delta w \frac{\partial f_1}{\partial x} + 2\omega \rho k_o = 0, \end{aligned} \quad (346)$$

where we have used $w = a_3 \delta w$ and $\vec{\pi}_o = 0$. Combining this equation with the one-dimensional k_d equation (105) for the case $\eta = 0 = \eta_d$,

$$\begin{aligned} \rho \frac{dk_d}{dt} + k_d a_3 \delta w \frac{\partial \rho}{\partial x} + \frac{2}{3} \rho k_d \frac{\partial u}{\partial x} - \frac{5}{3} \rho k_d \delta w \frac{\partial x_1}{\partial x} + \frac{5}{3} \rho k_d a_3 \frac{\partial \delta w}{\partial x} + \\ \rho a_3 \delta w \frac{\partial k_d}{\partial x} + \frac{\partial J_k}{\partial x} + \rho k_d \delta w \frac{\partial f_1}{\partial x} - \omega \rho a_1 \delta w^2 + \rho \epsilon_d = 0, \end{aligned} \quad (347)$$

and defining

$$\psi \equiv \frac{k_d}{k_o},$$

gives

$$\frac{\rho}{\psi} \frac{d\psi}{dt} = \frac{\rho}{k_d} \frac{dk_d}{dt} - \frac{\rho}{k_o} \frac{dk_o}{dt},$$

that is

$$\begin{aligned}
\frac{\rho}{\psi} \frac{d\psi}{dt} &= -\delta w [a_3(1 - \frac{2}{3}\psi) - a_2] \frac{\partial \rho}{\partial x} + \frac{4}{3} \rho \frac{\partial u}{\partial x} + \rho \delta w (\frac{5}{3} + \frac{a_2^2 - 2a_1}{a_1}) \frac{\partial x_1}{\partial x} - \\
&\quad \rho (\frac{5}{3} a_3 - 3a_2) \frac{\partial \delta w}{\partial x} - \rho (1 - \frac{2}{3}\psi) \frac{a_3 \delta w}{k_d} \frac{\partial k_d}{\partial x} - \frac{1}{k_d} \frac{\partial J_k}{\partial x} + \\
&\quad \frac{a_3 \delta w}{k_o} \frac{\partial P}{\partial x} - \rho \delta w (1 - \frac{2}{3}\psi) \frac{\partial f_1}{\partial x} + 2\omega \rho (1 + \frac{1}{\psi}) - \rho \frac{\epsilon_d}{k_d} \\
&= -\delta w [a_3(1 - \frac{2}{3}\psi) - a_2] \frac{\partial \rho}{\partial x} + \frac{4}{3} \rho \frac{\partial u}{\partial x} + \rho \delta w (\frac{5}{3} + \frac{a_2^2 - 2a_1}{a_1}) \frac{\partial x_1}{\partial x} - \\
&\quad \rho (\frac{5}{3} a_3 - 3a_2) \frac{\partial \delta w}{\partial x} - \rho a_3 \delta w (1 - \frac{2}{3}\psi) \frac{\partial \ln k_d}{\partial x} + \frac{1}{k_d} \frac{\partial}{\partial x} (D_k \frac{\partial k_d}{\partial x}) + \\
&\quad \frac{a_3 \delta w}{k_o} \frac{\partial P}{\partial x} - \rho \delta w (1 - \frac{2}{3}\psi) \frac{\partial f_1}{\partial x} + \omega \rho (1 + \frac{2}{\psi}),
\end{aligned} \tag{348}$$

where we have used $\epsilon_d = \omega k_d$. For tiny δw , Eq.(348) becomes

$$\frac{\rho}{\psi} \frac{d\psi}{dt} \sim \frac{4}{3} \rho \frac{\partial u}{\partial x} + \frac{1}{k_d} \frac{\partial}{\partial x} (D_k \frac{\partial k_d}{\partial x}) + \omega \rho (1 + \frac{2}{\psi}). \tag{349}$$

From equation (349), we see that for both δw and ψ initially tiny, $\frac{d\psi}{dt} = 2\omega > 0$, so ψ grows with time unless $\omega = 0$ or both k_o and k_d vanish at the same time. Note that $D_k \sim \eta_d \approx 0$ in our discussion, while $\frac{4}{3} \frac{\partial u}{\partial x} = -\frac{d}{dt}(\ln \rho^{4/3})$, thus equation (349) can be rewritten as

$$\frac{d}{dt} \ln(\psi \rho^{4/3}) \sim \omega (1 + \frac{2}{\psi}) \geq 0 \quad \text{when } \delta w \text{ is tiny.}$$

Therefore we conclude that k_d and k_o in inviscid flows are generated simultaneously by any nonzero relative motion between the fluids. As long as k_d and η_d are kept nonzero, as shown in Section 4, the hybrid turbulent mix model will always have real characteristics.

Meanwhile, it is worthwhile to point out that $\psi = \text{constant}$ or a function of only x_1 and x_2 are not solutions to equation (348). Therefore the relationship between k_d and k_o is much more complicated than the one recently proposed by Ramshaw^[15]. The real relationship between them only can be determined by solving Eq.(348).