ORNL/TM-12820

**OAK RIDGE
NATIONAL
LABORATORY**

MARTIN MARIETTA

***A Posteriori* Pointwise Error
Estimates for the Boundary
Element Method**

Glaucio H. Paulino
L. J. Gray
Vrej Zarikian

MANAGED BY
MARTIN MARIETTA ENERGY SYSTEMS, INC.
FOR THE UNITED STATES
DEPARTMENT OF ENERGY

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

RECEIVED

MAR 14 1995

OSTI

This report has been reproduced directly from the best available copy.

Available to DOE and DOE contractors from the Office of Scientific and Technical Information, P.O. Box 62, Oak Ridge, TN 37831; prices available from (615) 576-8401, FTS 626-8401.

Available to the public from the National Technical Information Service, U.S. Department of Commerce, 5285 Port Royal Rd., Springfield, VA 22161.

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

Engineering Physics and Mathematics Division

Mathematical Sciences Section

**A POSTERIORI POINTWISE ERROR ESTIMATES FOR THE
BOUNDARY ELEMENT METHOD**

Glaucio H. Paulino [†]

L. J. Gray [‡]

Vrej Zarikian [§]

[†] School of Civil and Environmental Engineering
Cornell University
Ithaca, New York 14853

[‡] Mathematical Sciences Section
Engineering Physics and Mathematics Division
Oak Ridge National Laboratory
P. O. Box 2008
Oak Ridge, Tennessee 37831-6367

[§] Department of Mathematics
University of Central Florida
Orlando, Florida 32816

Date Published: January 1995

Research was supported by the Applied Mathematical Sciences Subprogram, Office of Basic Energy Sciences, and the GO-NII project, U.S. Department of Energy. The first author acknowledges partial financial support provided by the CNPq Brazilian Agency. V. Zarikian's work was part of the the Research Experiences for Undergraduates Program, funded by National Science Foundation and the University of Tennessee Science Alliance.

Prepared by the
Oak Ridge National Laboratory
Oak Ridge, Tennessee 37831
managed by
Martin Marietta Energy Systems, Inc.
for the
U.S. DEPARTMENT OF ENERGY
under Contract No. DE-AC05-84OR21400

MASTER

Contents

1	Introduction	1
2	Local Error Estimates	2
2.1	Boundary Integral Equations	2
2.2	Boundary Error Estimate	3
2.2.1	Relationship Between the Exact and the Estimated Errors	4
2.3	Interior Error Estimates	5
2.3.1	Error Estimate for $\phi(P)$, $P \in B$	6
2.3.2	Error Estimate for $\phi_D(P)$, $P \in B$	7
2.4	Remarks	7
3	Numerical Aspects	9
4	Computational Results	10
4.1	Dirichlet Problems	11
4.1.1	Circle	11
4.1.2	Ellipses	12
4.2	Mixed Boundary Conditions	12
4.2.1	Simple Modeling	15
4.2.2	"Double Node" Modeling for Change in Boundary Conditions	15
4.3	Interior Error Estimates	17
4.3.1	Error for $\phi(P)$, $P \in B$	17
4.3.2	Error for $\phi_D(P)$, $P \in B$	18
4.4	Verification of the "Symmetry" of the Method with Respect to Equations (1) and (3)	22
4.5	Generic Example: L-Shaped Object	22
5	Analytical Sensitivity Analyses of the Error Estimates on the Boundary and in the Interior	25
6	Conclusions and Extensions	27
7	Appendix: Error Estimation Method in Elasticity	30
8	References	32

List of Tables

1	Element connectivity for "double node" modeling using Overhauser elements.	17
2	Response of L-shaped object.	24

List of Figures

1	Interchange of Eqs. (1) and (3).	8
2	Exact (E) versus estimated ($\mathcal{E}$) errors for two mesh discretizations on the boundary of a circle with unit radius and Dirichlet boundary conditions.	13
3	Exact (E) versus estimated ($\mathcal{E}$) errors for the circle with unit radius (36 nodes) and Dirichlet boundary conditions. The estimated errors have been obtained with $\mathbf{D} \equiv \mathbf{T}$ in Eq. (4).	13
4	Exact (E) versus estimated ($\mathcal{E}$) errors for two ellipses (100 nodes each) with different aspect ratios a/b and Dirichlet boundary conditions.	14
5	Exact (E) versus estimated ($\mathcal{E}$) errors for the elliptical geometry (100 nodes) with $a/b = 2$ and Dirichlet boundary conditions. The mesh has been discretized with equal arc length and the estimated errors have been obtained with $\mathbf{D} \equiv \mathbf{T}$ in Eq. (4).	14
6	Exact (E) versus estimated ($\mathcal{E}$) errors for the elliptical geometry (36 nodes) with $a/b = 2$ and Dirichlet boundary conditions.	15
7	Exact (E) versus estimated ($\mathcal{E}$) errors for the elliptical geometry (36 nodes) with $a/b = 2$ and mixed boundary conditions.	16
8	Exact (E) versus estimated ($\mathcal{E}$) errors for the elliptical geometry (36 nodes) with $a/b = 2$ and mixed boundary conditions. The estimated errors have been obtained with $\mathbf{D} \equiv \mathbf{T}$ in Eq. (4).	16
9	Exact (E) versus estimated ($\mathcal{E}$) errors for the elliptical geometry (38 nodes and 36 elements) using double node modeling for change in boundary conditions. Nodes 1 and 19 on the Dirichlet endpoints, and nodes 37 and 38 on the Neumann endpoints of the boundary.	18
10	Exact (E_ϕ) versus estimated ($\mathcal{E}_\phi$) errors for the potential on the circle $x^2 + y^2 = (0.4)^2$ interior to the ellipse $x^2 + y^2/(0.5)^2 = 1$. There are 36 equally spaced interior points.	19
11	Exact (E_ϕ) versus estimated ($\mathcal{E}_\phi$) errors for the potential on the straight segment from $y = -0.4$ to $y = 0.4$. There are 17 equally spaced interior points.	19
12	Exact ($E_{\phi\mathbf{D}}$) versus estimated ($\mathcal{E}_{\phi\mathbf{D}}$) errors for the directional derivatives of the potential on the circle $x^2 + y^2 = (0.4)^2$ interior to the ellipse $x^2 + y^2/(0.5)^2 = 1$. For each interior point, the direction $\mathbf{D}$ is the outward unit normal to the circle specified in a counter-clockwise sense. There are 36 equally spaced interior points.	20

13	Exact ($E_\phi \mathbf{D}$) versus estimated ($\mathcal{E}_\phi \mathbf{D}$) errors for the directional derivatives of the potential on the straight segment from $y = -0.4$ to $y = 0.4$, which is interior to the ellipse $x^2 + y^2/(0.5)^2 = 1$. There are 17 equally spaced interior points.	21
14	Exact ($E_\phi \mathbf{D}$) versus estimated ($\mathcal{E}_\phi \mathbf{D}$) errors for the directional derivatives of the potential on the straight segment $x = y$ from $x = -0.35$ to $x = 0.35$, which is interior to the ellipse $x^2 + y^2/(0.5)^2 = 1$. There are 15 equally spaced interior points.	21
15	Exact (E) versus estimated ($\mathcal{E}$) errors for the elliptical geometry (36 nodes) with $a/b = 2$, and Dirichlet boundary conditions. Equation (22) has been used for error estimation.	23
16	Exact (E) versus estimated ($\mathcal{E}$) normalized errors for the elliptical geometry (36 nodes) with $a/b = 2$ and Dirichlet boundary conditions. Equation (22) has been used for error estimation.	23
17	L-shaped object.	24
18	Estimated ($\mathcal{E}$) errors for the L-shaped domain (97 nodes and 92 elements) with mixed boundary conditions. The numbers above the points denote nodal numbers.	25

A POSTERIORI POINTWISE ERROR ESTIMATES FOR THE BOUNDARY ELEMENT METHOD

Glaucio H. Paulino

L. J. Gray

Vrej Zarikian

Abstract

This report presents a new approach for a *posteriori* pointwise error estimation in the boundary element method. The estimator relies upon the evaluation of hypersingular integral equations, and is therefore intrinsic to the boundary integral equation approach. This property allows some theoretical justification by mathematically correlating the exact and estimated errors. A methodology is developed for approximating the error on the boundary as well as in the interior of the domain. In the interior, error estimates for both the function and its derivatives (e.g. potential and interior gradients for potential problems, displacements and stresses for elasticity problems) are presented. Extensive computational experiments have been performed for the two dimensional Laplace equation on interior domains, employing Dirichlet and mixed boundary conditions. The results indicate that the error estimates successfully track the form of the exact error curve. Moreover, a reasonable estimate of the magnitude of the actual error is also obtained.

Key words: error estimates, boundary element method, boundary integral equation, singular integrals, hypersingular integrals

1. Introduction

Whenever a numerical method is utilized to solve the governing differential equations of a problem, error is introduced by the discretization process which reduces the continuous mathematical model to one having a finite number of degrees of freedom. The discretization errors are defined as the difference between the actual solution of the differential equation and its numerical approximation. Reliable estimation of these errors is essential to guarantee a certain level of accuracy of the numerical solution, and is a key component of adaptive procedures. Estimation of the discretization error in the Boundary Element Method (BEM) is the focus of this work.

There are several recent surveys of the literature on error estimation and adaptivity, and the reader is directed to the appropriate references. KITA and KAMIYA (1994) have reviewed some of the recent studies on error estimation and adaptive mesh refinement in the BEM. The volumes edited by BREBBIA and ALIABADI (1993) and by BABUŠKA *et al.* (1986) review adaptive techniques for the Finite Element Method (FEM) and the BEM. Recent surveys of the literature in FEM include the articles by NOOR and BABUŠKA (1987), ODEN and DEMKOWICZ (1989), BABUŠKA and GUO (1992), STROUBOULIS and HAQUE (1992a, 1992b), and BABUŠKA and SURI (1994). MACKERLE (1993, 1994) has compiled a long listing of references on mesh generation, refinement, error analysis and adaptive techniques for FEM and BEM that were published from 1990 to 1993. The volume edited by BABUŠKA *et al.* (1983) presents adaptive techniques for the FEM and the Finite Difference Method (FDM). A general *a posteriori* error estimation method, which can be applied to the FEM, BEM, and FDM has been presented by KELLY *et al.* (1987). The above references indicate the importance of the field of error estimation and adaptivity in the numerical analysis of partial differential equations.

Recent textbooks in the FEM emphasize the field of adaptive solution techniques. The book by ZIENKIEWICZ and TAYLOR (1989) includes a chapter on "error estimation and adaptivity" (Chapter 14), which can be supplemented by References [58, 59, 60]. The recent book by SZABÓ and BABUŠKA (1991) is primarily concerned with this subject. Due to the importance of this topic, textbooks on the BEM will likely follow the same trend.

Quite naturally, the BEM literature on error estimation and adaptivity shows a significant influence of the FEM. Although it is certainly possible to benefit from the FEM technology, a simple translation of concepts from one method to the other may not be the most appropriate. GUIGGIANI (1990) has commented that more attention should be devoted to techniques intrinsic to the BEM, and within this spirit, the method proposed herein is rooted in the boundary integral methodology.

A new local error estimate, based upon hypersingular integral equations, is introduced in this report. These equations have proven to be highly useful in a variety of situations, and their definition and numerical evaluation are now well understood (see, for example, GRAY 1989; GRAY *et al.* 1990; KRISHNASAMY *et al.* 1992a; GUIGGIANI *et al.* 1992; ROSEN and CORMACK 1992, 1993, 1994). The measure of the error on the boundary of a body is taken as the amount by which the solution to the standard Boundary Integral Equation (BIE) fails to satisfy the equally valid hypersingular BIE.

Distinctive advantages of this approach are that the estimator does not require boundary values away from the nodal points, it does not involve any adjustable parameters, it is defined by an analytical formula, and it is intrinsic to the BEM. This allows the derivation of a relationship between the exact and estimated errors, and provides a natural extension of the procedure to interior error estimates. It is worth mentioning that, despite its importance, interior error estimation has not received attention in the BEM literature. Furthermore, a primary goal of this report is to present extensive test calculations which validate the error method and provide a reasonable basis for claiming that the proposed error estimates are reliable and will perform well in most situations.

The remaining sections of this report are organized as follows. First, the basic BIEs are given, and the method for estimating the local error on the boundary is presented, including the derivation of a relationship between the exact and the estimated errors. Next, the method for interior estimates is developed. This includes error estimates for both the function (e.g. potential) and its derivatives. Subsequent to the theoretical part of this work, some aspects concerning the computational implementation are discussed, and numerical examples comparing the exact and estimated errors are presented. Afterwards, a framework for sensitivity analysis of the errors on the boundary and in the interior is introduced. Finally, conclusions are stated and promising directions for future research are discussed.

2. Local Error Estimates

A method for *a posteriori* evaluation of the error on the boundary, ∂B , and in the interior, B , of the domain $\Omega = B \cup \partial B$ is presented in this section. For interior error estimation, the formulation is developed for both the field variable and its derivatives (e.g. potential and interior gradients for potential problems). For simplicity, the error estimates will be defined in the context of the two dimensional Laplace equation $\nabla^2 \phi = 0$. However, the method is general, and applicable to any boundary integral formulation (see Appendix).

2.1. Boundary Integral Equations

For simplicity, the subscript notation

$$\phi_D \equiv \frac{\partial}{\partial D} \phi \quad \text{and} \quad \phi_d \equiv \frac{\partial}{\partial d} \phi$$

will be employed for differentiation, with the capital/lower case letters utilized to distinguish differentiation with respect to the collocation point P and the field point Q (integration variable), respectively, in a BIE. Here, D and d refer to generic unit direction vectors, and ϕ denotes the potential.

With the above notation, the boundary integral formulation for the two dimensional Laplace equation can be written as

$$\phi(P) + \int_{\partial B} \phi(Q) \frac{\partial}{\partial n} G(P, Q) ds_Q = \int_{\partial B} \phi n(Q) G(P, Q) ds_Q, \quad (1)$$

where ϕ_n is the boundary flux (normal derivative of ϕ), ds denotes a differential length element, and the Green's function is taken as the point source potential

$$G(P, Q) = -\frac{1}{2\pi} \log \|Q - P\| \quad (2)$$

Equation (1) is valid for points P in the interior B (thus the coefficient 1 for the leading term). However, as discussed by LUTZ and GRAY (1993), it is also valid for P on the boundary, provided the singular integrals are defined as a *limit-to-the-boundary*. Let ∂B_d and ∂B_n denote the portions of the boundary having Dirichlet (ϕ prescribed) and Neumann (ϕ_n prescribed) boundary conditions, respectively. Thus, Eq. (1) can be employed to solve for the unknowns ϕ on ∂B_n and ϕ_n on ∂B_d , and $\partial B = \partial B_d + \partial B_n$.

For $P \in B$, the integrands in Eq. (1) are not singular, and the derivative of this equation with respect to P can be computed by interchanging the order of differentiation and integration. The resulting gradient BIE can therefore be expressed as

$$(\nabla \phi \cdot \mathbf{D})(P) + \int_{\partial B} \phi(Q) \nabla \frac{\partial}{\partial \mathbf{n}} G(P, Q) \cdot \mathbf{D} \, ds_Q = \int_{\partial B} \phi_n(Q) \nabla G(P, Q) \cdot \mathbf{D} \, ds_Q, \quad (3)$$

where $\mathbf{D} = (D_1, D_2)$ is any specified unit direction vector. Once again, a well defined integral equation on the boundary, $P \in \partial B$, results by defining the singular integrals as a *limit* in which P approaches ∂B (GRAY 1989, 1993).

2.2. Boundary Error Estimate

Assume now that a particular problem has been solved using Eq. (1), resulting in approximate solutions

$$\tilde{\phi} \quad \text{and} \quad \tilde{\phi}_n$$

for the unknown boundary values of potential and flux, respectively. These approximate solutions, together with the specified boundary conditions, determine these boundary functions on the approximate geometry. Although these functions have been determined to satisfy Eq. (1), in whatever sense this equation was approximated (e.g. collocation, Galerkin), these values are not necessarily consistent with Eq. (3).

The proposed error estimate $\mathcal{E}(P)$ is defined as the error which arises when the approximate solution is substituted into Eq. (3). Thus,

$$\begin{aligned} \mathcal{E}(P) = & -\tilde{\phi} \mathbf{D}(P) - \int_{\widetilde{\partial B}} \tilde{\phi}(Q) \frac{\partial}{\partial \mathbf{D}} \frac{\partial}{\partial \mathbf{n}} G(P, Q) \, ds_Q \\ & + \int_{\widetilde{\partial B}} \tilde{\phi}_n(Q) \frac{\partial}{\partial \mathbf{D}} G(P, Q) \, ds_Q, \end{aligned} \quad (4)$$

where $\widetilde{\partial B}$ represents the approximated boundary geometry. Therefore, the estimator $\mathcal{E}(P)$ is quite simply the amount by which the solution to the standard BIE (Eq. (1)) fails to satisfy the hypersingular BIE (Eq. (3)). The specific choice of the vector $\mathbf{D}$ in the definition of the error estimate will be given later on this report.

The estimator could have been defined with the sign reversed. The decision to employ Eq. (4) has been guided by the results of computational experiments with the interior error estimates (see Section 4). Moreover, if the boundary error estimates are to be used for a self adaptive mesh refinement procedure, an appropriate error norm is generally used in the refinement process (e.g. GUIGGIANI 1990; SZABÓ and BABUŠKA 1991). In this case, only the magnitude of the local error is needed.

Note that the estimator defined by Eq. (4) does not require boundary values away from the nodal points, it does not involve adjustable parameters, it is defined by an analytical formula, and it is intrinsic to the BEM. These properties are important for the derivations presented in the remainder of this report.

2.2.1. Relationship Between the Exact and the Estimated Errors

Most previous definitions of error estimates for the BEM have been essentially heuristic in nature, and thus it would be difficult to derive a mathematical relationship between the exact and estimated errors. The error estimate presented in this work has a concrete mathematical formulation, rooted in the BEM, and consequently such a relationship is naturally obtained, as shown below.

If $\phi(Q)$ and $(\phi_{\mathbf{n}})(Q)$ represent the exact solution, Eq. (3) is valid everywhere on ∂B and can be rewritten as

$$0 = -\phi_{\mathbf{D}}(P) - \int_{\partial B} \phi(Q) \frac{\partial}{\partial \mathbf{D}} \frac{\partial}{\partial \mathbf{n}} G(P, Q) ds_Q + \int_{\partial B} \phi_{\mathbf{n}}(Q) \frac{\partial}{\partial \mathbf{D}} G(P, Q) ds_Q \quad (5)$$

Subtracting this equation from Eq. (4), one obtains

$$\begin{aligned} \mathcal{E}(P) = & - \left[\phi_{\mathbf{D}} - \tilde{\phi}_{\mathbf{D}} \right] (P) - \int_{\partial B} \left[\phi - \tilde{\phi} \right] (Q) \frac{\partial}{\partial \mathbf{D}} \frac{\partial}{\partial \mathbf{n}} G(P, Q) ds_Q \\ & + \int_{\partial B} \left[\phi_{\mathbf{n}} - \tilde{\phi}_{\mathbf{n}} \right] (Q) \frac{\partial}{\partial \mathbf{D}} G(P, Q) ds_Q \quad , \end{aligned} \quad (6)$$

where the error in approximating the geometry has been neglected in the present analysis (i.e. it is assumed that $\partial B \equiv \partial \tilde{B}$). Defining the notation

$$\begin{aligned} E_{\phi_{\mathbf{D}}}(P) &= \left[\phi_{\mathbf{D}} - \tilde{\phi}_{\mathbf{D}} \right] (P) \\ E_{\phi}(Q) &= \left[\phi - \tilde{\phi} \right] (Q) \\ E_{\phi_{\mathbf{n}}}(Q) &= \left[\phi_{\mathbf{n}} - \tilde{\phi}_{\mathbf{n}} \right] (Q) \end{aligned} \quad (7)$$

for the exact errors, one obtains

$$\begin{aligned} \mathcal{E}(P) = & -E_{\phi_{\mathbf{D}}}(P) - \int_{\partial B} E_{\phi}(Q) \frac{\partial}{\partial \mathbf{D}} \frac{\partial}{\partial \mathbf{n}} G(P, Q) ds_Q \\ & + \int_{\partial B} E_{\phi_{\mathbf{n}}}(Q) \frac{\partial}{\partial \mathbf{D}} G(P, Q) ds_Q \quad , \end{aligned} \quad (8)$$

which establishes the relationship between the exact error E and the estimated error function $\mathcal{E}$ on the boundary.

For a collocation point P with Dirichlet boundary conditions, the estimator is a measure of the error in the normal derivative, and $\mathcal{E}(P) \equiv \mathcal{E}_{\phi\mathbf{N}}(P)$, where $\mathbf{N} = \mathbf{n}(P)$. In addition, $E_{\phi}(Q) = 0$, $Q \in \partial B_d$, and $E_{\phi\mathbf{n}}(Q) = 0$, $Q \in \partial B_n$, and thus Eq. (8) becomes

$$\begin{aligned} \mathcal{E}_{\phi\mathbf{N}}(P) = & -E_{\phi\mathbf{D}}(P) - \int_{\partial B} E_{\phi}(Q) \frac{\partial}{\partial \mathbf{D}} \frac{\partial}{\partial \mathbf{n}} G(P, Q) ds_Q \\ & + \int_{\partial B} E_{\phi\mathbf{n}}(Q) \frac{\partial}{\partial \mathbf{D}} G(P, Q) ds_Q \end{aligned} \quad (9)$$

Similarly, for a collocation point P with Neumann boundary conditions, $\mathcal{E}(P) \equiv \mathcal{E}_{\phi}(P)$, and Eq. (8) becomes

$$\mathcal{E}_{\phi}(P) = - \int_{\partial B} E_{\phi}(Q) \frac{\partial}{\partial \mathbf{D}} \frac{\partial}{\partial \mathbf{n}} G(P, Q) ds_Q + \int_{\partial B} E_{\phi\mathbf{n}}(Q) \frac{\partial}{\partial \mathbf{D}} G(P, Q) ds_Q \quad (10)$$

The relationship between $\mathcal{E}(P)$ and $E(P)$ gives rise to an interesting possibility for obtaining an even better estimate of the discretization errors. As the left hand side in Eq. (8) (or equivalently, Eqs. (9) and (10)) is known, the unknown values $E_{\phi}(P)$, $P \in \partial B_n$ and $E_{\phi\mathbf{n}}(P)$, $P \in \partial B_d$, can be determined. Of course, the exact error will not be computed because it has been assumed that $\partial B \equiv \partial \bar{B}$, and numerical approximation will be required to solve for E . Nevertheless, this procedure might yield a more accurate error estimate than $\mathcal{E}$. This is a fruitful area for further investigation.

2.3. Interior Error Estimates

As pointed out previously, interior error estimation is a subject that has not received attention in the BEM literature. Assessment of the interior error is important to investigate the behavior of field variables, especially in the neighborhood of corners, kinks, or locations with high boundary curvature. Moreover, it may have important applications in at least two areas of nonlinear BEM analysis, the Dual Reciprocity Method (DRM) and material nonlinearities. The DRM has been applied to a range of nonlinear problems (see, for example, the book by PARTRIDGE *et al.* 1991). This method leads to a formulation involving boundary integrals only, however, interior nodal points are also included in the formulation. Therefore, error estimation for the DRM should involve both boundary and interior error estimates.

WEI *et al.* (1994) have recently performed elasto-viscoplastic shape optimization of two dimensional solids by the BEM. They have stated that "The choice of mesh, especially the internal cells, is crucial for the solution of this class of problems. The best approach is to use adaptive meshing during the iterative optimization process." Therefore, numerical analysis of materially nonlinear problems by the BEM (for further information, see the book by CHANDRA and MUKHERJEE 1995) is a field where interior error estimation is of fundamental importance. In this case, domain integrals must also be added in the error formulation.

As will be demonstrated by the numerical examples in Section 4, $\mathcal{E}(P)$ provides an effective estimation of the error *everywhere* on the boundary. As a consequence, it is possible to develop an interior error estimate based upon the boundary error estimates.

This formulation is developed below, and error estimates are presented for both the potential ϕ and its directional derivatives $\phi_{\mathbf{D}}$.

2.3.1. Error Estimate for $\phi(P)$, $P \in B$

For an interior point $P \in B$, the exact solution satisfies

$$\phi(P) = - \int_{\partial B} \phi(Q) \frac{\partial}{\partial \mathbf{n}} G(P, Q) ds_Q + \int_{\partial B} \phi_{\mathbf{n}}(Q) G(P, Q) ds_Q \quad , \quad (11)$$

whereas the approximate solution at P is given by

$$\tilde{\phi}(P) = - \int_{\partial B} \tilde{\phi}(Q) \frac{\partial}{\partial \mathbf{n}} G(P, Q) ds_Q + \int_{\partial B} \tilde{\phi}_{\mathbf{n}}(Q) G(P, Q) ds_Q \quad , \quad (12)$$

the boundary values $\tilde{\phi}(Q)$ and $\tilde{\phi}_{\mathbf{n}}(Q)$ having been computed. Subtraction of Eq. (12) from Eq. (11) gives

$$\begin{aligned} [\phi - \tilde{\phi}](P) = & - \int_{\partial B} [\phi - \tilde{\phi}](Q) \frac{\partial}{\partial \mathbf{n}} G(P, Q) ds_Q \\ & + \int_{\partial B} [\phi_{\mathbf{n}} - \tilde{\phi}_{\mathbf{n}}](Q) G(P, Q) ds_Q \quad , \end{aligned} \quad (13)$$

where, as before, the error in approximating the geometry has been neglected. To obtain an estimate of the error at P , replace the exact errors (between square braces) on the boundary by the estimated values,

$$\begin{aligned} [\phi - \tilde{\phi}](Q) & \approx \mathcal{E}_{\phi}(Q) \\ [\phi_{\mathbf{n}} - \tilde{\phi}_{\mathbf{n}}](Q) & \approx \mathcal{E}_{\phi_{\mathbf{n}}}(Q) \quad . \end{aligned} \quad (14)$$

For Dirichlet portions of the boundary, $\mathcal{E}_{\phi_{\mathbf{n}}}(Q) \equiv \mathcal{E}(Q)$, and assume

$$\mathcal{E}_{\phi}(Q) \equiv 0 \quad . \quad (15)$$

Similarly, for Neumann portions of the boundary, $\mathcal{E}_{\phi}(Q) \equiv \mathcal{E}(Q)$, and assume

$$\mathcal{E}_{\phi_{\mathbf{n}}}(Q) \equiv 0 \quad . \quad (16)$$

Substituting Eqs. (14) in Eq. (13), one obtains

$$\mathcal{E}_{\phi}(P) = - \int_{\partial B} \mathcal{E}_{\phi}(Q) \frac{\partial}{\partial \mathbf{n}} G(P, Q) ds_Q + \int_{\partial B} \mathcal{E}_{\phi_{\mathbf{n}}}(Q) G(P, Q) ds_Q \quad , \quad (17)$$

where

$$\mathcal{E}_{\phi}(P) \approx [\phi - \tilde{\phi}](P) \quad . \quad (18)$$

As all quantities on the right hand side of Eq. (17) are known, $\mathcal{E}_{\phi}(P)$ can be calculated.

2.3.2. Error Estimate for $\phi_D(P)$, $P \in B$

Equation (13) is the starting point for the derivation of an expression for $\mathcal{E}_{\phi_D}(P)$. Differentiating Eq. (13) with respect to D ($\|D\| = 1$), one obtains

$$\begin{aligned} [\phi_D - \tilde{\phi}_D](P) = & - \int_{\partial B} [\phi - \tilde{\phi}](Q) \frac{\partial}{\partial D} \frac{\partial}{\partial n} G(P, Q) ds_Q \\ & + \int_{\partial B} [\phi n - \tilde{\phi} n](Q) \frac{\partial}{\partial D} G(P, Q) ds_Q \quad . \end{aligned} \quad (19)$$

As before (Section 2.3.1), the exact errors (between square braces) on the boundary are approximated by the estimated ones, and thus the error estimate for the directional derivatives of the potential at P is given by

$$\mathcal{E}_{\phi_D}(P) = - \int_{\partial B} \mathcal{E}_{\phi}(Q) \frac{\partial}{\partial D} \frac{\partial}{\partial n} G(P, Q) ds_Q + \int_{\partial B} \mathcal{E}_{\phi n}(Q) \frac{\partial}{\partial D} G(P, Q) ds_Q \quad , \quad (20)$$

where

$$\mathcal{E}_{\phi_D}(P) \approx [\phi_D - \tilde{\phi}_D](P) \quad . \quad (21)$$

Again, all the quantities on the right hand side of Eq. (20) are known and $\mathcal{E}_{\phi_D}(P)$ can be calculated.

2.4. Remarks

Of the four remarks presented below, the first three concern relevant characteristics of the proposed *a posteriori* error estimates. The last remark discusses differences between this work and a previous error estimation technique which relies on hypersingular integral equations.

Remark 1. *Directional derivatives in the hypersingular BIEs.*

The vector D in Eqs. (3) and (4) has not been specified yet. One would like to choose a direction D for which the error estimate $\mathcal{E}$ best tracks the exact error in the problem. Computational tests, involving two dimensional problems, have been performed with both

$$D \equiv N \quad \text{and} \quad D \equiv T \quad ,$$

where N and T denote the normal and tangential unit vectors, respectively, on the boundary ∂B at the source point P . Our tests indicate that using $D \equiv N$ in Eq. (4) provides more reliable error estimates than using $D \equiv T$. However, the use of $D \equiv T$ in Eq. (4) might be useful in other situations, such as fracture mechanics problems (see Section 6).

An advantage of the error method presented here is that it can be readily extended to three dimensional problems. Note that the functional form of the error equations remain the same, i.e. Eqs. (4), (17) and (20) hold for both two and three dimensional problems. For three dimensional problems, using Eq. (4) with the directional derivatives taken normal to the boundary is simpler and more efficient than using this equation with the directional derivatives taken tangent to the boundary. The former

alternative requires one evaluation of Eq. (4) while the later requires two evaluations of this equation.

Remark 2. “Symmetry” of the method with respect to equations (1) and (3).

The argument which led to Eq. (4) is symmetric, i.e. if the hypersingular integral equation is used to solve the boundary value problem (INGBER and MONDY 1993; GRAY *et al.* 1994), then the standard BEM equation can be used for error estimation. In other words, with respect to the error estimation method, Eqs. (1) and (3) are complementary to each other. This is illustrated by Figure 1.

Figure 1: Interchange of Eqs. (1) and (3).

Specifically, assume that a particular problem has been solved using Eq. (3), resulting in approximate solutions $\tilde{\phi}$ and $\tilde{\phi}_{\mathbf{n}}$. The error estimate $\mathcal{E}(P)$ is now defined as the error which arises when the approximate solution is substituted into Eq. (1). Thus,

$$\mathcal{E}(P) = -\tilde{\phi}(P) - \int_{\partial B} \tilde{\phi}(Q) \frac{\partial}{\partial \mathbf{n}} G(P, Q) ds_Q + \int_{\partial B} \tilde{\phi}_{\mathbf{n}}(Q) G(P, Q) ds_Q. \quad (22)$$

Similarly to Eq. (4), the sign in Eq. (22) is also arbitrary. For the error in the interior, equations analogous to (17) and (20) can be readily obtained.

This “symmetry” property is also important and necessary for estimating the error in the Symmetric-Galerkin BEM (e.g. HARTMAN *et al.* 1985; MAIER *et al.* 1990; SIRTORI *et al.* 1992). This approach potentially combines the accuracy of the Galerkin method with the speed of collocation calculations (BALAKRISHNA *et al.* 1994a, 1994b). In this method, both the standard and the hypersingular BIEs are employed, the choice being dictated by the prescribed boundary condition. Thus, the ability to interchange the role of the two equations is essential for applying the error estimation approach to this new, and potentially important, approximation method. Test error estimation calculations using the Symmetric-Galerkin BEM are in progress.

Remark 3. Continuity to the boundary.

The formulation for the interior error estimates, presented in Section 2.3, is not continuous to the boundary. Continuity at the boundary is clearly a desirable property, and at the expense of additional computation, is achievable. The basic idea is to take $\mathcal{E}_{\phi}$ on ∂B_n and $\mathcal{E}_{\phi_{\mathbf{n}}}$ on ∂B_d , obtained from Eq. (4), as “boundary conditions” to Eq. (1),

thereby solving for $\mathcal{E}_\phi$ on ∂B_d and $\mathcal{E}_{\phi\mathbf{n}}$ on ∂B_n , rather than assuming that they are zero. This procedure will lead to a set of boundary errors which provide a continuous limit to the boundary of the interior errors. Although the assumptions of zero error, Eqs. (15) and (16), are reasonable, they are of an heuristic nature and do not have theoretical justification. The reason for first examining the method as described above is that it requires less computational effort, and will probably suffice for most applications.

Remark 4. Related work.

An error indicator which also relies upon the ability to evaluate hypersingular integrals was recently proposed by INGBER and MITRA (1992). The approach suggested in the present report, for evaluation of the error on the boundary, is therefore related to that discussed by these authors, but there is a fundamental difference in the two techniques. The idea motivating the procedure by INGBER and MITRA (1992) is to measure the discrepancy between the imposed boundary conditions and the boundary element solution. They define the Error Indicator $EI(j)$ for a boundary element ∂B_j as:

$$EI(j) = \begin{cases} \left\{ \int_{\partial B_j} [\phi - \tilde{\phi}]^2 ds \right\}^{\frac{1}{2}} & \partial B_j \in \partial B_d \\ \left\{ \int_{\partial B_j} [\phi\mathbf{n} - \tilde{\phi}\mathbf{n}]^2 ds \right\}^{\frac{1}{2}} & \partial B_j \in \partial B_n \end{cases} \quad (23)$$

As discussed above, the indicator $\mathcal{E}(P)$ proposed herein (Eq. (4)) is the amount by which the solution to the standard boundary integral equation fails to satisfy the hypersingular integral equation. While Eq. (23) requires less computational work than Eq. (4), INGBER and MITRA (1992) have not presented any justification for their procedure. Moreover, it does not appear that their error method can be extended in a natural fashion into the interior.

3. Numerical Aspects

This section briefly describes some aspects of the numerical implementation employed in the computational tests. Isoparametric Overhauser elements are used in the approximation of Eqs. (1) and (3). For the standard BEM, these cubic elements provide a higher approximation than the traditional linear and quadratic elements (e.g. CAMP and GIPSON 1991). For the hypersingular BEM, the Overhauser approximation provides the required smoothness of the density functions at the collocation points (GRAY 1994), i.e. ϕ is $C^{1,\alpha}$ (Hölder continuous). For a discussion of continuity requirements for density functions in the BEM, see KRISHNASAMY *et al.* (1992b).

In two dimensions, the Overhauser element utilizes four consecutive boundary nodes, $\{S_k\}_{k=1}^4$, to define the interval between the second and third nodes $[S_2, S_3]$. The explicit parametric form is:

$$Q(t) = \left(\sum_{k=1}^{k=4} N_k(t) x_k, \sum_{k=1}^{k=4} N_k(t) y_k \right) \quad , \quad (24)$$

where $t \in [0, 1]$ and $S_k = (x_k, y_k)$ are the coordinates of the four points defining the element. The shape functions N_k ($k = 1, \dots, 4$) are:

$$\begin{aligned} N_1(t) &= -(t - 2t^2 + t^3)/2 \\ N_2(t) &= (2 - 5t^2 + 3t^3)/2 \\ N_3(t) &= (t + 4t^2 - 3t^3)/2 \\ N_4(t) &= -(t^2 - t^3)/2 \end{aligned} \quad (25)$$

As indicated above, Overhauser elements are used in the discretization of both potential and flux in Eqs. (1) and (3). For example, the corresponding isoparametric approximation for ϕ , denoted by $\Phi(t)$ is then

$$\Phi(t) = \sum_{i=1}^4 N_i(t) \phi(S_i) \quad (26)$$

It is a straightforward exercise to verify that the approximations for $Q(t)$ and ϕ are differentiable everywhere on the boundary.

Corners have been handled here by a "double node" approach. It consists of defining one of the endpoint nodes (e.g. $k = 1$) and its adjacent node (e.g. $k = 2$) at the same location on the Overhauser element. At this location, the element does not provide C^1 continuity for the boundary geometry and the functional representation. The other endpoint of the element maintains the continuity conditions provided it is connected with another Overhauser element. Recently, a special treatment of Overhauser elements for approximating strongly singular integrals at a boundary corner has been presented by GRAY (1994).

Two integration techniques have been employed here: numerical and analytical. Numerical integrations, evaluated with four point Gauss quadrature, have been employed for non-singular integrals. Analytical integrations have been used to compute the singular integrals. These integrals have been evaluated by means of the *limit-to-the-boundary* definition (GRAY 1993).

Note that if $\mathbf{D} \equiv \mathbf{N} \equiv \mathbf{n}(P)$, then all quantities on the right hand side of Eq. (4) are known, and $\mathcal{E}(P)$ can be readily calculated. If $\mathbf{D} \neq \mathbf{N}$, then the first term on the right hand side of Eq. (4) can be obtained by differentiating the shape functions to compute the required tangential derivatives.

4. Computational Results

A primary goal of this report is to examine, by means of computational experiments, the performance of the new error estimates. Numerical results have been obtained for the two dimensional Laplace equation on interior domains, employing Dirichlet and mixed boundary conditions. Additional examples can be found in Reference [53].

In order to assess various features of the proposed error estimates, the following examples are presented:

(1) Dirichlet problems

(1.1) Circle

- (1.2) Ellipses
- (2) Mixed boundary conditions
 - (2.1) Simple modeling
 - (2.2) "Double node" modeling for change in boundary conditions
- (3) Interior Error Estimates
 - (3.1) Error for $\phi(P)$, $P \in B$
 - (3.2) Error for $\phi_D(P)$, $P \in B$
- (4) Verification of the "symmetry" of the method with respect to Eqs. (1) and (3)
- (5) Generic example: L-shaped domain

For all but the last test calculation, the problems possess a simple analytical solution, allowing a comparison between the exact and estimated errors. The potential field is specified as

$$\phi(x, y) = \Re(z^4) = x^4 - 6x^2y^2 + y^4, \quad (27)$$

where $z = x + iy$, $i = \sqrt{-1}$, and $\Re(\cdot)$ denotes the real part of the complex function. As the goal with this procedure is to assess the accuracy of the proposed error estimates, a fourth order potential field is employed to ensure that the Overhauser cubics introduce error. The corresponding gradient of the potential in Eq. (27) is

$$\nabla\phi(x, y) = (4x^3 - 12xy^2, 4y^3 - 12x^2y) \quad (28)$$

Unless otherwise stated, $D \equiv N$ has been used for the error estimate in Eq. (4), and the meshes for the elliptical geometries, $x^2/a^2 + y^2/b^2 = 1$, have been discretized with equal angles. This discretization procedure is likely to introduce more error variation over the boundary than using a discretization with equal arc length. The last example represents a more practical problem geometry and boundary conditions for which there is no analytical solution available.

4.1. Dirichlet Problems

Two types of problems are considered here. The first is a circle of unit radius with two discretizations, and the second are ellipses with different aspect ratios, $a/b = 2.00$ and $a/b = 1.25$.

4.1.1. Circle

The unit circle $x^2 + y^2 = 1$ is subjected to Dirichlet boundary conditions. In the present example, $N = (x, y)$ and, from Eq. (28), $\phi_N(x, y) = 4x^4 - 24x^2y^2 + 4y^4$. Note that both polynomials ϕ and ϕ_N are of fourth order, while the Overhauser elements are cubic.

Two mesh discretizations have been considered, one with nodes at 10° intervals (36 nodes and elements) and the other one with nodes at 5° intervals (72 nodes and

elements). Figure 2 shows the comparison between the exact and estimated errors for the two discretizations. The main feature of this graph is that the estimated errors follow the same trend of the exact errors for both discretizations. This graph also shows that (1) for each discretization, the estimator given by Eq. (4) overestimates the error; (2) as expected, the magnitude of the errors for the finer mesh (72 nodes) are smaller than the corresponding ones for the coarser mesh (36 nodes); (3) the difference between the estimated and exact errors are smaller for the finer mesh (72 nodes) than for the coarser one (36 nodes); and (4) the symmetry of the problem is captured by both the BEM solution (Eq. (1)) and the hypersingular BIE for the boundary error estimate (Eq. (4)).

To illustrate the influence of the direction vector $\mathbf{D}$ and to justify our choice for $\mathbf{D} \equiv \mathbf{N}$ in Eq. (4), the circle with 36 nodes has been considered again. Figure 3 shows a comparison between the exact and estimated errors using $\mathbf{D} \equiv \mathbf{T}$ in Eq. (4). Note that in this case, the estimated and exact errors are out of phase. Therefore, the use of $\mathbf{D} \equiv \mathbf{N}$ is apparently more reliable than the use of $\mathbf{D} \equiv \mathbf{T}$.

4.1.2. Ellipses

Figure 4 shows a comparison between the exact and estimated errors for the ellipses $x^2 + y^2/(0.5)^2 = 1.0$ (i.e. $a/b = 2.00$) and $x^2 + y^2/(0.8)^2 = 1.0$ (i.e. $a/b = 1.25$) discretized with 100 nodes each. Globally, the exact errors for the ellipse with $a/b = 2.00$ are less than the ones for the ellipse with $a/b = 1.25$, except at the $y = 0$ regions of the boundary. The estimated errors for the ellipse with $a/b = 2.00$ are also less than the ones for the ellipse with $a/b = 1.25$, but at the $y = 0$ regions of the boundary, where one would expect the opposite. However, the overall error prediction for each case is still very good and the estimator does capture the trend of the actual error curve in both cases.

Figure 5 shows a comparison between the exact and estimated errors for the ellipse with $a/b = 2.00$. The mesh has been discretized with equal arc length, and the estimated errors have been obtained with $\mathbf{D} \equiv \mathbf{T}$ in Eq. (4). In this case, the estimator predicts reasonably well the shape of the actual error curve, but seriously overestimates the errors at the $y = 0$ regions of the boundary.

Figure 6 shows a comparison between the exact and estimated errors for the ellipse with $a/b = 2.00$ and 36 nodes. The errors estimated by Eq. (4) predict reasonably well the shape and magnitude of the actual errors, even for this coarse mesh discretization.

4.2. Mixed Boundary Conditions

Again, the elliptical geometry with $a/b = 2$ is considered here, and the coarser mesh discretization with 36 elements is employed. The mixed boundary conditions are Dirichlet on the top ($y \geq 0$), and Neumann on the bottom ($y < 0$) of the ellipse (Eqs. (27) and (28), respectively). Two modeling strategies are presented below.

Figure 2: Exact (E) versus estimated ($\mathcal{E}$) errors for two mesh discretizations on the boundary of a circle with unit radius and Dirichlet boundary conditions.

Figure 3: Exact (E) versus estimated ($\mathcal{E}$) errors for the circle with unit radius (36 nodes) and Dirichlet boundary conditions. The estimated errors have been obtained with $D \equiv T$ in Eq. (4).

Figure 4: Exact (E) versus estimated ($\mathcal{E}$) errors for two ellipses (100 nodes each) with different aspect ratios a/b and Dirichlet boundary conditions.

Figure 5: Exact (E) versus estimated ($\mathcal{E}$) errors for the elliptical geometry (100 nodes) with $a/b = 2$ and Dirichlet boundary conditions. The mesh has been discretized with equal arc length and the estimated errors have been obtained with $D \equiv T$ in Eq. (4).

Figure 6: Exact (E) versus estimated ($\mathcal{E}$) errors for the elliptical geometry (36 nodes) with $a/b = 2$ and Dirichlet boundary conditions.

4.2.1. Simple Modeling

The element connectivity for this example has followed the standard approach employed in the previous examples, that is, nothing special is done at the $y = 0$ junctures of ∂B_d and ∂B_n . As the change in boundary conditions occurs at a smooth part of the boundary, this is appropriate. This same boundary value problem and discretization are used in the next two sections. Figures 7 and 8 show a comparison between the exact and estimated errors considering $D \equiv N$ and $D \equiv T$, respectively, in Eq. (4). Note that a much better prediction of the discretization errors is achieved with $D \equiv N$ than with $D \equiv T$.

The influence of boundary conditions on the errors can be seen by comparing Figures 6 and 7. It is interesting to note the similarity on the shape of the estimated error curves in either case.

4.2.2. "Double Node" Modeling for Change in Boundary Conditions

To introduce large errors, and therefore to test whether the error estimate $\mathcal{E}$ can respond to significant errors, the modeling near $y = 0$ was altered. Two extra nodes (37 and 38) have been added at the end of the nodal coordinates list. The "double nodes" are 1 and 37, and 19 and 38. Nodes 1 and 19 are on the Dirichlet endpoints, and nodes 37 and 38 on the Neumann endpoints of the boundary. The element connectivity for this example is illustrated by Table 1. Aside from being unnecessary, this procedure introduces errors by altering the Overhauser approximation to the geometry and functions in the

Figure 7: Exact (E) versus estimated ($\mathcal{E}$) errors for the elliptical geometry (36 nodes) with $a/b = 2$ and mixed boundary conditions.

Figure 8: Exact (E) versus estimated ($\mathcal{E}$) errors for the elliptical geometry (36 nodes) with $a/b = 2$ and mixed boundary conditions. The estimated errors have been obtained with $D \equiv T$ in Eq. (4).

Table 1: Element connectivity for “double node” modeling using Overhauser elements.

El	El_Id	S_1	S_2	S_3	S_4
1	1	1	1	2	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	1	$i - 1$	i	$i + 1$	$i + 2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
18	1	17	18	19	19
19	2	38	38	20	21
20	2	38	20	21	22
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j	2	$j - 1$	j	$j + 1$	$j + 2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
36	2	35	36	37	37

El $\equiv$ Element number; El_Id $\equiv$ Element Identification (El_Id = 1 for Dirichlet and El_Id = 2 for Neumann boundary conditions); $1 < i < 18$; $20 < j < 36$.

neighborhood of the $y = 0$ regions of the boundary. Figure 9 shows a comparison between the exact and estimated errors for this example. This graph shows that, as expected, the errors are concentrated on the double nodes, i.e. 1, 19, 37 and 38, and on their immediate neighbor nodes, while for the rest of the boundary the errors are much smaller. The error estimate given by Eq. (4) captures the trend of the actual errors, and gives a clear indication of serious problems in the “double node” region.

4.3. Interior Error Estimates

As mentioned previously, interior error estimation is an important area of investigation. The purpose of this section is to examine the reliability of the formulation presented in Section 2.3. The calculation of the interior error makes use of previously calculated boundary error estimates. Numerical results are presented for both the potential ϕ and its directional derivatives ϕ_D .

4.3.1. Error for $\phi(P)$, $P \in B$

Figure 10 shows a comparison between the exact and estimated errors for the potential on the circle $x^2 + y^2 = (0.4)^2$ interior to the ellipse $x^2 + y^2/(0.5)^2 = 1$. There are 36 interior points at 10° intervals. The graph in Figure 10 shows that (1) the estimator given by Eq. (17) overestimates the magnitude of the interior errors; (2) the estimator provides a good prediction of the inflection points, shape, and magnitude of the actual error curve; and (3) the difference between the exact and estimated errors is smaller on

Figure 9: Exact (E) versus estimated ($\mathcal{E}$) errors for the elliptical geometry (38 nodes and 36 elements) using double node modeling for change in boundary conditions. Nodes 1 and 19 on the Dirichlet endpoints, and nodes 37 and 38 on the Neumann endpoints of the boundary.

the top than on the bottom part of the circle. This last observation is quite reasonable, as Dirichlet boundary conditions are specified on the top of the ellipse, and the error is being calculated for the potential ϕ .

Figure 11 shows a comparison between the exact and estimated errors for the potential on the straight segment from $y = -0.4$ to $y = 0.4$. From Eq. (27), $\phi = y^4$ on this segment. There are 17 equally spaced interior points, one at each 0.05 length interval. For this example, the estimator given by Eq. (17) gives a good prediction for the shape and magnitude of the actual errors.

4.3.2. Error for $\phi_D(P)$, $P \in B$

Figure 12 shows a comparison between the exact and estimated errors for the directional derivatives of the potential on the same interior circle employed above. For each interior point, the direction D is the outward unit normal to the circle specified in a counter-clockwise sense. Therefore, from Eq. (28), $\phi_D = (4x^4 - 24x^2y^2 + 4y^4)/0.4$. There are 36 interior points at 10° intervals. The graph in Figure 12 shows that (1) the estimator given by Eq. (20) overestimates the magnitude of the interior errors; and (2) the estimator provides a good prediction of the inflection points, shape, and magnitude of the exact error curve. Note that there are two kinks in the exact error curve, one at the top and one at the bottom of the circle. This is probably due to the proximity of these points to the boundary. The distance between these points and the ellipse is 0.1 and the element length on the top or bottom of the ellipse is 0.1738. In this case, the techniques for interior point evaluation by ZHAO *et al.* (1994) could be employed for accurate computation of interior potential gradients.

Figure 10: Exact (E_ϕ) versus estimated ($\mathcal{E}_\phi$) errors for the potential on the circle $x^2 + y^2 = (0.4)^2$ interior to the ellipse $x^2 + y^2 / (0.5)^2 = 1$. There are 36 equally spaced interior points.

Figure 11: Exact (E_ϕ) versus estimated ($\mathcal{E}_\phi$) errors for the potential on the straight segment from $y = -0.4$ to $y = 0.4$. There are 17 equally spaced interior points.

Figure 12: Exact (E_{ϕ_D}) versus estimated ($\mathcal{E}_{\phi_D}$) errors for the directional derivatives of the potential on the circle $x^2 + y^2 = (0.4)^2$ interior to the ellipse $x^2 + y^2/(0.5)^2 = 1$. For each interior point, the direction D is the outward unit normal to the circle specified in a counter-clockwise sense. There are 36 equally spaced interior points.

It is interesting to note the similarity of the shape of the error curves in Figures 10 and 12, which have been obtained for two different quantities, i.e. the potential and its directional derivatives, respectively. This similarity is expected because, for the points on the circle $x^2 + y^2 = (0.4)^2$, $\phi_D = 10\phi$, where ϕ is given by Eq. (27). As might be expected, the magnitude of the errors for the potential (see Figure 10) is less than those for its derivatives (see Figure 12). However, this is not always the case, as can be seen from the next example.

Figure 13 shows a comparison between the exact and estimated errors for the directional derivatives of the potential on the straight segment from $y = -0.4$ to $y = 0.4$, which is interior to the ellipse $x^2 + y^2/(0.5)^2 = 1$. The derivatives are evaluated in the direction $D = (0, 1)$. Therefore, from Eq. (28), $\phi_D = 4y^3$. There are 17 equally spaced interior points, one at each 0.05 length interval. For this example, the estimator given by Eq. (20) gives a good prediction for the shape and magnitude of the actual errors. Comparing Figures 11 and 13, one can verify that, in this case, the overall magnitude of the errors for the potential is larger than those for the potential derivatives.

Finally, Figure 14 shows a comparison between the exact and estimated errors for the directional derivatives of the potential on the straight segment $x = y$ from $x = -0.35$ to $x = 0.35$, which is interior to the ellipse $x^2 + y^2/0.5^2 = 1$. The derivatives are evaluated in the direction $D = (1, 1)/\sqrt{2}$. Therefore, from Eq. (28), $\phi_D = -16x^3/\sqrt{2}$. There are 15 equally spaced interior points, one at each 0.05 length interval. For this example, the estimator given by Eq. (20) once again gives a good prediction for the shape and magnitude of the actual errors.

Figure 13: Exact ($E_{\phi \mathbf{D}}$) versus estimated ($\mathcal{E}_{\phi \mathbf{D}}$) errors for the directional derivatives of the potential on the straight segment from $y = -0.4$ to $y = 0.4$, which is interior to the ellipse $x^2 + y^2/(0.5)^2 = 1$. There are 17 equally spaced interior points.

Figure 14: Exact ($E_{\phi \mathbf{D}}$) versus estimated ($\mathcal{E}_{\phi \mathbf{D}}$) errors for the directional derivatives of the potential on the straight segment $x = y$ from $x = -0.35$ to $x = 0.35$, which is interior to the ellipse $x^2 + y^2/(0.5)^2 = 1$. There are 15 equally spaced interior points.

4.4. Verification of the “Symmetry” of the Method with Respect to Equations (1) and (3)

The “symmetry” of the two BIEs, one for solution of the problem and one for error estimation, has been discussed in Section 2.4 (Remark 2). To support this argument, an example with Dirichlet boundary conditions is presented here. It is worth mentioning that there is a rationale for solving Dirichlet problems using the hypersingular BIE. INGBER and MONDY (1993) and GRAY *et al.* (1994) have pointed out that a stable second kind integral equation is obtained by employing a hypersingular equation for this type of problem, while the standard boundary integral formulation leads to a potentially unstable first kind integral.

Figure 15 shows the exact versus estimated errors for the elliptical geometry (36 nodes) with $a/b = 2$ and Dirichlet boundary conditions. The boundary value problem has been solved by Eq. (3), and the errors have been estimated by Eq. (22). In this example, the estimator underestimates the error. However, the shape of the actual error curve is well predicted. This can be easily verified by normalizing both the exact and the estimated errors with respect to the maximum error for each quantity. This is shown in Figure 16. The trend of this graph shows that the normalized estimated errors overestimate the normalized actual errors for this example.

4.5. Generic Example: L-Shaped Object

The last example, extracted from Reference [5], is reasonably representative of geometry and boundary conditions that arise in practice, and it does not have an analytical solution. The geometry is shown in Figure 17. The left vertical face of the object is maintained at $\phi = 1000$ units, while the right vertical face is forced to be at $\phi = 0$ units. All the other boundaries are perfectly insulated, i.e. $\phi_{\mathbf{n}} = 0$ units.

The discretization adopted for the L-shaped object consists of 97 nodes (92 geometrically distinct nodes and 5 double nodes at the corners) and 92 Overhauser elements. Table 2 compares the results obtained with the present implementation and those reported in Reference [5] for the symmetric Galerkin boundary element analysis with analytical evaluation of singular integrals. The discretization used in Reference [5] consists of 92 distinct nodes and 46 isoparametric quadratic elements. The present discretization and the one in Reference [5] have the same set of nodal points in the boundary element mesh. From Table 2, it is observed that a good agreement between the two boundary element solutions is achieved. As noted in Section 2.4 (Remark 2), it would be very interesting to estimate the error in a symmetric Galerkin context using the ideas presented in this report.

Figure 18 shows the estimated errors for the L-shaped object discretized with 97 nodes and 92 Overhauser elements. The two more pronounced peaks are at nodes 47 ($\mathbf{n} = (0, 1)$) and 55 ($\mathbf{n} = (1, 0)$), which correspond to the endpoints of the circular arc. These error peaks are expected because regions with high curvature are expected to have a large error. Moreover, the error at node 47 is bigger than the error at node 55. This too is reasonable, as node 47 is located at the long thin part of the L-shaped object.

Figure 15: Exact (E) versus estimated ($\mathcal{E}$) errors for the elliptical geometry (36 nodes) with $a/b = 2$, and Dirichlet boundary conditions. Equation (22) has been used for error estimation.

Figure 16: Exact (E) versus estimated ($\mathcal{E}$) normalized errors for the elliptical geometry (36 nodes) with $a/b = 2$ and Dirichlet boundary conditions. Equation (22) has been used for error estimation.

Figure 17: L-shaped object.

Table 2: Response of L-shaped object.

Node	Sample point		Overhauser		Galerkin BEM	
	x	y	ϕ	ϕn	ϕ	ϕn
6	25.0	0.0	926.14	0.0	926.10	0.0
18	85.0	0.0	379.50	0.0	379.37	0.0
27	120.0	3.33	0.0	-10.827	0.0	-10.841
29	120.0	6.67	0.0	-10.827	0.0	-10.841
34	105.0	10.0	162.64	0.0	162.57	0.0
45	55.0	10.0	705.37	0.0	705.07	0.0
49	46.17	10.76	814.09	0.0	813.88	0.0
51	42.93	12.93	860.38	0.0	860.34	0.0
58	40.0	27.5	945.43	0.0	945.45	0.0
66	40.0	65.0	984.94	0.0	984.93	0.0
72	25.0	80.0	989.41	0.0	989.40	0.0
82	0.0	55.0	1000.0	0.748	1000.0	0.748
92	0.0	5.0	1000.0	2.537	1000.0	2.536

Figure 18: Estimated ($\mathcal{E}$) errors for the L-shaped domain (97 nodes and 92 elements) with mixed boundary conditions. The numbers above the points denote nodal numbers.

There are smaller peaks at nodes 1, 25, 31, 69, and 77. These are the corner nodes. The double nodes, corresponding to each of these corner nodes are located at the end of nodal list, i.e. nodes 93 to 97. There is a perturbation on the error curve involving nodes 8 to 14. This is an interesting phenomenon. It is probably due to the influence of change of curvature on the part of the boundary which is immediately above the segment from nodes 8 to 14. Finally, this example shows that the estimator given by Equation 22 also gives a very good indication of the actual error for this complicated example.

5. Analytical Sensitivity Analyses of the Error Estimates on the Boundary and in the Interior

The explicit form of the error estimates (Eqs. (4), (17) and (20)) allows their direct use in sensitivity analysis. The goal here is to obtain analytical expressions for the sensitivities of the errors on the boundary and in the interior. Information on sensitivity analysis and optimization in computational mechanics can be found in the books by HAUG *et al.* (1986), and HAFTKA and GÜRDAL (1992).

Evaluation of the sensitivity of the errors is important in applications such as shape optimization using adaptive refinement (BUGEDA and OLIVER 1993). For example, by means of sensitivities (error and boundary mesh), the error estimates on ∂B can be projected from one design to next one in the iterative loop of the optimization process. Note that, in this case, an *a posteriori* error estimate provides, by projection, an *a priori* error estimate for the new design.

A procedure for obtaining the sensitivities of the errors can be outlined as follows. First, Eq. (1) is solved numerically and the unknowns $\tilde{\phi}$ and $\tilde{\phi}_n$ on ∂B are determined. Second, the sensitivity BIE is obtained from Eq. (1) by means of the Direct Differentiation Approach (DDA) (see, for example, BARONE and YANG 1988; ZHANG and MUKHERJEE 1991a). After the numerical solution of the sensitivity BIE, the unknown sensitivities $\tilde{\phi}^*$ and $\tilde{\phi}_n^*$ on ∂B are determined. Here, a superscribed * denotes a derivative with respect to a component $b_i \equiv b$ of a design variable vector $\mathbf{b}$, i.e.

$$(\cdot)^* = \frac{\partial}{\partial b} (\cdot) \quad (29)$$

Finally, the sensitivity analysis of the error estimates can be performed.

The equation for the sensitivity of the error on ∂B , obtained by applying the DDA to Eq. (4) is

$$\begin{aligned} \mathcal{E}^*(\mathbf{b}; P) = & -\phi_D^*(\mathbf{b}; P) \\ & + \int_{\partial B} \left[-\phi^*(\mathbf{b}; Q) \frac{\partial}{\partial D} \frac{\partial}{\partial \mathbf{n}} G(\mathbf{b}; P, Q) + \phi_n^*(\mathbf{b}; Q) \frac{\partial}{\partial D} G(\mathbf{b}; P, Q) \right] ds(\mathbf{b}; Q) \\ & + \int_{\partial B} \left[-\phi(\mathbf{b}; Q) \left(\frac{\partial}{\partial D} \frac{\partial}{\partial \mathbf{n}} G \right)^*(\mathbf{b}; P, Q) + \phi_n(\mathbf{b}; Q) \left(\frac{\partial}{\partial D} G \right)^*(\mathbf{b}; P, Q) \right] ds(\mathbf{b}; Q) \\ & + \int_{\partial B} \left[-\phi(\mathbf{b}; Q) \frac{\partial}{\partial D} \frac{\partial}{\partial \mathbf{n}} G(\mathbf{b}; P, Q) + \phi_n(\mathbf{b}; Q) \frac{\partial}{\partial D} G(\mathbf{b}; P, Q) \right] ds^*(\mathbf{b}; Q) \quad (30) \end{aligned}$$

Following BARONE and YANG (1988),

$$ds^*(\mathbf{b}; Q) = \left[\dot{x}_{k,k}(Q) - n_i n_j \dot{x}_{i,j}(Q) \right] ds(\mathbf{b}; Q) \quad , \quad (31)$$

where $\dot{x}$ denotes the design velocity. An alternative formula for ds^* can be found in ZHANG and MUKHERJEE (1991a).

Let $K(\mathbf{b}, P, Q)$ denote either the singular or the hypersingular kernel in Eq. (4). The differentiated kernel $\dot{K}(\mathbf{b}, P, Q)$ in Eq. (30) has the same order of singularity as the original kernel $K(\mathbf{b}, P, Q)$ because

$$\dot{K}(\mathbf{b}; P, Q) = K_{,k}(\mathbf{b}; P, Q) \left[\dot{x}_k(Q) - \dot{x}_k(P) \right] \quad , \quad (32)$$

and

$$\dot{x}_k(Q) - \dot{x}_k(P) \sim O(\|Q - P\|) \quad . \quad (33)$$

Thus, Eq. (30) is a hypersingular BIE for the sensitivities of the errors on ∂B .

The equation for the sensitivity of the error estimate for $\phi(P)$, $P \in B$, obtained by applying the DDA to Eq. (17) is

$$\begin{aligned}
 \mathcal{E}_\phi^*(b; P) = & \\
 & + \int_{\partial B} \left[-\mathcal{E}_\phi^*(b; Q) \frac{\partial}{\partial n} G(b; P, Q) + \mathcal{E}_{\phi n}^*(b; Q) G(b; P, Q) \right] ds(b; Q) \\
 & + \int_{\partial B} \left[-\mathcal{E}_\phi(b; Q) \left(\frac{\partial}{\partial n} G \right)^*(b; P, Q) + \mathcal{E}_{\phi n}(b; Q) G^*(b; P, Q) \right] ds(b; Q) \\
 & + \int_{\partial B} \left[-\mathcal{E}_\phi(b; Q) \frac{\partial}{\partial n} G(b; P, Q) + \mathcal{E}_{\phi n}(b; Q) G(b; P, Q) \right] ds^*(b; Q) \quad (34)
 \end{aligned}$$

Note that the sensitivities $\mathcal{E}_\phi^*(b; Q)$ and $\mathcal{E}_{\phi n}^*(b; Q)$ are functions of $\mathcal{E}^*(b; Q)$, which is obtained from Eq. (30). Specifically, for the error method used in the present work, these sensitivities are calculated as follows. For $Q \in \partial B_d$, $\mathcal{E}_\phi^*(b; Q) \equiv 0$ (from Eq. (15)) and $\mathcal{E}_{\phi n}^*(b; Q) \equiv \mathcal{E}^*(b; Q)$ (from Eq. (30)). For $Q \in \partial B_n$, $\mathcal{E}_{\phi n}^*(b; Q) \equiv 0$ (from Eq. (16)) and $\mathcal{E}_\phi^*(b; Q) \equiv \mathcal{E}^*(b; Q)$ (from Eq. (30)).

The equation for the sensitivity of the error estimate for $\phi_D(P)$, $P \in B$, obtained by applying the DDA to Eq. (17) is

$$\begin{aligned}
 \mathcal{E}_{\phi D}^*(b; P) = & \\
 & + \int_{\partial B} \left[-\mathcal{E}_\phi^*(b; Q) \frac{\partial}{\partial D} \frac{\partial}{\partial n} G(b; P, Q) + \mathcal{E}_{\phi n}^*(b; Q) \frac{\partial}{\partial D} G(b; P, Q) \right] ds(b; Q) \\
 & + \int_{\partial B} \left[-\mathcal{E}_\phi(b; Q) \left(\frac{\partial}{\partial D} \frac{\partial}{\partial n} G \right)^*(b; P, Q) + \mathcal{E}_{\phi n}(b; Q) \left(\frac{\partial}{\partial D} G \right)^*(b; P, Q) \right] ds(b; Q) \\
 & + \int_{\partial B} \left[-\mathcal{E}_\phi(b; Q) \frac{\partial}{\partial D} \frac{\partial}{\partial n} G(b; P, Q) + \mathcal{E}_{\phi n}(b; Q) \frac{\partial}{\partial D} G(b; P, Q) \right] ds^*(b; Q) \quad (35)
 \end{aligned}$$

The sensitivities $\mathcal{E}_\phi^*(b; Q)$ and $\mathcal{E}_{\phi n}^*(b; Q)$ are calculated as before. This equation completes the derivations for the first order sensitivities of the errors. Note that while Eqs. (4), (17) and (20) provide a *a posteriori* estimation of the discretization errors, Eqs. (30), (34) and (35) provide a *a posteriori* estimation of the sensitivities of the discretization errors.

Second order sensitivity analysis of the errors can be carried out by applying the DDA to Eqs. (30), (34) and (35). Procedures for second order sensitivity analysis in the BEM can be found, for example, in ZHANG and MUKHERJEE (1991b).

6. Conclusions and Extensions

This work indicates that the combination of the standard and hypersingular integral equations provides a good basis for evaluating the discretization error in boundary element analysis. The error estimation method consists of substituting the approximate boundary element solution into the corresponding hypersingular integral equation. The boundary error estimates can then be employed for evaluating the interior error esti-

mates for the potential and the directional derivatives of the potential. As pointed out previously, the argument which led to Eq. (4) is "symmetric", in that the role of the two equations (1) and (3) can be reversed. This property is essential for estimating the error using a Symmetric-Galerkin BEM. Moreover, the fact that the error estimates have an explicit form is potentially advantageous for mathematical analysis (see Section 2.2.1) (e.g. GHOSH 1982; JOHNSON 1987; SLOAN 1992) and sensitivity analysis (see Section 5).

The error estimation procedure has been presented using a scalar field. Nevertheless, this work naturally extends to vector field problems (see Appendix). For example, using the BEM formulation for linear elasticity (RIZZO 1967) together with the traction BIE (e.g. GRAY *et al.* 1990), and following the methodology developed herein, one can obtain expressions for the error estimates which are analogous to Eqs. (4), (17) and (20). However, in this vector field case, the boundary error is a vector, the interior displacement error is also a vector, and the interior stress error is a tensor. These are basic forms for the errors, and if desired, an appropriate error norm can be obtained from the error components.

The local error estimation method presented in this report is intrinsic to the BEM technique. It is applicable to any boundary integral formulation, can be readily extended to three dimensional problems (see Section 2.4), and allows a pointwise estimation of the discretization error. With respect to this last point, it is worth mentioning that in many design problems, one is interested in assessing the error at particularly critical parts of the boundary. Therefore, the error $\mathcal{E}(P)$ could be evaluated just at points in these specific regions of interest.

Extensive numerical testing of the error estimates has been carried out for the two dimensional Laplace equation. The influence of several factors have been investigated, such as geometry of the body, boundary conditions and discretization. The numerical results show that both the boundary and interior error estimates provide a remarkably faithful tracking of the exact error curve and a reasonable estimate of the magnitude of the exact error. Preliminary and encouraging results have also been obtained to validate the "symmetry" argument for error estimation.

There are some interesting topics for future investigation. For example, development of error estimates with a continuous limit to the boundary, the use of Eq. (8) to obtain an error estimate which might be better than the one given by Eq. (4), error estimation using a symmetric-Galerkin BEM formulation, sensitivity analysis of the error estimates, and extension of the error estimates to non-linear and vector field problems. The present work also offers room for extension to other applications such as fracture mechanics and solution adaptive techniques. In what follows, brief explanations, concerning these two applications, are given.

A very interesting subject for future research is the extension of the present method to crack problems, which includes error estimation in fracture mechanics (see, for example, JAYASWAL and GROSSE 1993). In one technique, both the hypersingular integral equation and the standard BEM equation are already employed on the crack surface to solve the problem (GRAY *et al.* 1990; GRAY and SOUCIE 1993). Therefore, there is no longer an additional equation available for the error estimation. One possibility, however, is to use the hypersingular equation, with the direction $\mathbf{D}$ being taken tangent

to the crack surface.

As the error estimation method appears to provide a reliable tracking of the actual error, it should be highly suitable for a self adaptive procedure. A natural extension of this work consists of using the local errors as the driving parameters of an adaptive mesh refinement scheme, e.g. h- refinement. While the boundary errors $\mathcal{E}$ give information about the error distribution, an automatic adaptive refinement gives results within a specified accuracy. A word of caution: when using Overhauser elements, an r- refinement strategy is not recommended because undesirable oscillations in the approximated functions can be generated. This is due to the uneven element distribution, which is typical of the r- refinement (e.g. Ingber and Mitra 1986).

The range of application of the new error estimates is potentially broad. For instance, they could be of advantage for error estimation of radiosity (method for simulating global illumination) solutions, which are governed by a Fredholm integral equation of the second kind (see, for example, LISCHINSKI *et al.* 1994). Finally, use of the techniques presented in this report could also contribute towards a reliable and automated environment in computational mechanics (TWORZYDLO and ODEN 1993) employing BIE techniques.

7. Appendix: Error Estimation Method in Elasticity

A general methodology for error estimation has been presented in Section 2. Following this procedure, a formulation is presented for evaluating the discretization error in elasticity problems. This formulation illustrates the general scheme to be adopted for error estimation in vector field problems. The definition of singular and hypersingular integrals below is once again taken as a *limit-to-the-boundary* (GRAY 1989; GRAY *et al.* 1990). In the interest of brevity, only the main equations are given.

The standard direct BEM formulation for linear elasticity (RIZZO 1967), in the absence of body forces, can be written as

$$u_j(P) + \int_{\partial B} u_i(Q) T_{ij}(P, Q) ds_Q = \int_{\partial B} \tau_i(Q) U_{ij}(P, Q) ds_Q \quad , \quad (36)$$

where u_i and τ_i are the displacement and traction vectors, respectively, and T_{ij} and U_{ij} are the usual Kelvin kernels.

For a linear elastic isotropic body, the internal stresses can be obtained by differentiating the displacements at internal points (Eq. (36)) and introducing the displacement derivatives into the stress-strain relationship

$$\sigma_{ij}(P) = \mu (u_{i,j} + u_{j,i}) + \lambda u_{k,k} \delta_{ij} \quad , \quad (37)$$

where λ and μ are the Lamé constants for the material, and δ_{ij} is the Kronecker delta. After some algebra, the stress BIE can be written in a compact form as

$$\sigma_{ij}(P) = - \int_{\partial B} u_k(Q) S_{kij}(P, Q) ds_Q + \int_{\partial B} \tau_k(Q) D_{kij}(P, Q) ds_Q \quad , \quad (38)$$

where the differentiated Kelvin kernels S_{kij} and D_{kij} can be found in many books on BEM, e.g. CHANDRA AND MUKHERJEE (1995).

The traction BIE can be obtained from Eq. (38) as follows

$$\begin{aligned} \tau_i(P) &= \sigma_{ij} N_j \\ &= \left[- \int_{\partial B} u_k(Q) S_{kij}(P, Q) ds_Q + \int_{\partial B} \tau_k(Q) D_{kij}(P, Q) ds_Q \right] N_j \quad , \quad (39) \end{aligned}$$

where $N = n(P)$ ($N = N_i e_i$ and e_i ($i = 1, 2, 3$) are global Cartesian unit vectors).

Boundary Error Estimate

Assume that a particular problem has been solved using Eq. (36), resulting in approximate solutions

$$\tilde{u} \quad \text{and} \quad \tilde{\tau}$$

for the unknown boundary values of displacements and tractions, respectively. Although these functions have been determined to satisfy Eq. (36), they are not necessarily consistent with Eq. (39).

The proposed error indicator vector $\mathcal{E}(P)$ is defined as the error which arises when

the approximate solution is substituted into Eq. (39). Thus

$$\mathcal{E}_i(P) = -\tilde{\tau}_i(P) - N_j \int_{\partial B} \tilde{u}_k(Q) S_{kij}(P, Q) ds_Q + N_j \int_{\partial B} \tilde{\tau}_k(Q) D_{kij}(P, Q) ds_Q \quad . \quad (40)$$

This equation is analogous to Eq. (4), except that the error in Eq. (4) is a scalar quantity, and the error in Eq. (40) is a vector quantity.

Error Estimate for $u(P)$, $P \in B$

Following the ideas presented in Section 2, one arrives at the BIE for the error on the displacement at internal points

$$\mathcal{E}_{j(u)}(P) = - \int_{\partial B} \mathcal{E}_{i(u)}(Q) T_{ij}(P, Q) ds_Q + \int_{\partial B} \mathcal{E}_{i(\tau)}(Q) U_{ij}(P, Q) ds_Q \quad , \quad (41)$$

where for Dirichlet portions of the boundary, $\mathcal{E}_{i(\tau)}(Q) \equiv \mathcal{E}_i(P)$, and assume

$$\mathcal{E}_{i(u)}(Q) \equiv 0 \quad . \quad (42)$$

Similarly, for Neumann portions of the boundary, $\mathcal{E}_{i(u)}(Q) \equiv \mathcal{E}_i(P)$, and assume

$$\mathcal{E}_{i(\tau)}(Q) \equiv 0 \quad . \quad (43)$$

Equation (41) is analogous to Eq. (17). Note, however, that the error in Eq. (41) is now a vector quantity.

Error Estimate for $\sigma(P)$, $P \in B$

Using the above framework for error analysis in elasticity problems and the ideas presented in Section 2, one arrives at the BIE for the error on the stress tensor at internal points

$$\mathcal{E}_{ij(\sigma)}(P) = - \int_{\partial B} \mathcal{E}_{k(u)}(Q) S_{kij}(P, Q) ds_Q + \int_{\partial B} \mathcal{E}_{k(\tau)}(Q) D_{kij}(P, Q) ds_Q \quad . \quad (44)$$

Also here, $\mathcal{E}_{i(\tau)}(Q) \equiv \mathcal{E}_i(P)$ and $\mathcal{E}_{i(u)}(Q) \equiv \mathcal{E}_i(P)$ for Dirichlet and Neumann portions of the boundary, respectively. Moreover, the assumptions on Eqs. (42) and (43) are also employed in this case. Equation (44) is analogous to Eq. (20). Note, however, that the error in Eq. (44) is now a second order tensor quantity.

8. References

- [1] BABUŠKA, I.; CHANDRA, J.; FLAHERTY, J. E. (eds.) (1983): *Adaptive Computational Methods for Partial Differential Equations*. Philadelphia: Society for Industrial and Applied Mathematics (SIAM).
- [2] BABUŠKA, I.; GUO, B. Q. (1992): The h , p and h - p version of the finite element method; basis theory and applications. *Advances in Engineering Software* **15**, 159-176.
- [3] BABUŠKA, I.; SURI, M. (1994): The P and H-P versions of the finite element method, basic principles and properties. *SIAM Review* **36**, 578-632.
- [4] BABUŠKA, I.; ZIENKIEWICZ, O. C.; GAGO, J.; E. R. DE A. OLIVEIRA (eds.) (1986): *Accuracy Estimates and Adaptive Refinements in Finite Element Computations*. Chichester: John Wiley & Sons.
- [5] BALAKRISHNA, C.; GRAY, L. J.; KANE, J. H. (1994a): Efficient analytical integration of symmetric Galerkin boundary integrals over curved elements: Thermal conduction formulation. *Computer Methods in Applied Mechanics and Engineering* **111**, 335-355.
- [6] BALAKRISHNA, C.; GRAY, L. J.; KANE, J. H. (1994b): Efficient analytical integration of symmetric Galerkin boundary integrals over curved elements; elasticity formulation. *Computer Methods in Applied Mechanics and Engineering* **117**, 157-179.
- [7] BARONE, M. R.; YANG, R.-J. (1988): Boundary integral equations for recovery of design sensitivities in shape optimization. *AIAA Journal* **26**, 589-594.
- [8] BREBBIA, C. A.; ALIABADI, M. H. (eds.) (1993): *Adaptive Finite and Boundary Element Methods*. Southampton: Computational Mechanics Publications; and London: Elsevier Applied Science.
- [9] BUGEDA, G.; OLIVER, J. (1993): A general methodology for structural shape optimization problems using automatic adaptive remeshing. *International Journal for Numerical Methods in Engineering* **36**, 3161-3185.
- [10] CAMP, C. V.; GIPSON, G. S. (1991): Overhauser elements in boundary element analysis. *Mathematical and Computer Modelling* **15**, 59-69.
- [11] CHANDRA, A.; MUKHERJEE, S. (1995): *Boundary Element Methods in Manufacturing*. Oxford University Press.
- [12] JOHNSON, C. (1987): *Numerical Solution of Partial Differential Equations by the Finite Element Method*. New York: Cambridge University Press.
- [13] GHOSH, N. (1982): *On the Convergence of the Boundary Element Method*, Ph.D. Thesis, Department of Mathematics, Cornell University, Ithaca, New York.
- [14] GRAY, L. J. (1989): Boundary element method for regions with thin internal cavities. *Engineering Analysis with Boundary Elements*, **6**, 180-184.
- [15] GRAY, L. J. (1993): Symbolic computation of hypersingular boundary integrals. In: Kane, J. H.; Maier, G.; Tosaka, N.; Atluri, S. N. (eds.): *Advances in Boundary Element Techniques*, chapter 8, pp.157-172. Berlin, Heidelberg: Springer-Verlag.

- [16] GRAY, L. J. (1994): A note on Overhauser elements. *Boundary Elements Communications*, **5**, 62-65.
- [17] GRAY, L. J.; BASARAN, O. A.; HARRIS, M. T.; WOHLHUTER, F. W. (1994): Boundary integral equations and the Dirichlet problem (in preparation).
- [18] GRAY, L. J.; MANNE, L. L. (1993): Hypersingular integrals at a corner. *Engineering Analysis with Boundary Elements* **11**, 327-334.
- [19] GRAY, L. J.; MARTHA, L. F.; INGRAFFEA, A. R. (1990): Hypersingular integrals in boundary element fracture analysis. *International Journal for Numerical Methods in Engineering* **29**, 1135-1158.
- [20] GRAY, L. J.; SOUCIE, C. S. (1993): A Hermite interpolation algorithm for hypersingular boundary integrals. *International Journal for Numerical Methods in Engineering* **36**, 2357-2367.
- [21] GUIGGIANI, M. (1990): Error indicators for adaptive mesh refinement in the boundary element method - A new approach. *International Journal for Numerical Methods in Engineering* **29**, 1247-1269.
- [22] GUIGGIANI, M.; KRISHNASAMY, G.; RUDOLPHI, T. J.; RIZZO, F. J. (1992): A general algorithm for the numerical solution of hypersingular boundary integral equations. *ASME Journal of Applied Mechanics* **59**, 604-614 .
- [23] HAFTKA, R. T.; GÜRDAL, Z. (1992): *Elements of Structural Optimization*. 3rd. revised and expanded edition. Dordrecht: Kluwer Academic Publishers.
- [24] HARTMAN, F; KATZ, C.; PROTOPSALTIS, B. (1985): Boundary elements and symmetry. *Ingenieur-Archive* **55**, 440-449.
- [25] HAUG, E. J.; CHOI, K. K.; KOMKOV, V. (1986): *Design Sensitivity Analysis of Structural Systems*. Orlando, Florida: Academic Press.
- [26] INGBER, M. S.; MITRA, A. K. (1986): Grid optimization for the boundary element method. *International Journal for Numerical Methods in Engineering* **23**, 2121-2136.
- [27] INGBER, M. S.; MITRA, A. K. (1992): Grid redistribution based on measurable error indicators for the direct boundary element method. *Engineering Analysis with Boundary Elements* **9**, 13-19.
- [28] INGBER, M. S.; MONDY, L. A. (1993): Direct second kind boundary integral formulation for Stokes flow problems. *Computational Mechanics* **11**, 11-27.
- [29] JAYASWAL, K.; GROSSE, I. R. (1993): Finite element error estimation for crack tip singular elements. *Finite Elements in Analysis and Design* **14**, 17-35.
- [30] KELLY, D. W.; MILLS, R. J.; REIZES, J. A.; MILLER, A. D. (1987): *A posteriori* estimates of the solution error caused by discretization in the finite element, finite difference and boundary element methods. *International Journal for Numerical Methods in Engineering* **24**, 1921-1939.
- [31] KITA, E.; KAMIYA, N. (1994): Recent studies on adaptive boundary element methods. *Advances in Engineering Software* **19**, 21-32 (cites 82 references).

- [32] KRISHNASAMI, G; RIZZO, F. J.; RUDOLPHI, T. J. (1992a): Hypersingular Boundary integral Equations: Their Occurrence, Interpretation, Regularization and Computation. In: Banerjee, P. K.; Kobayashi, S. (eds.): *Developments in Boundary Element Methods*, volume 7, chapter 6, pp.207-252. London and New York: Elsevier Applied Science.
- [33] KRISHNASAMI, G; RIZZO, F. J.; RUDOLPHI, T. J. (1992b): Continuity requirements for density functions in the boundary integral equation method. *Computational Mechanics*, **9**, 267-284.
- [34] LISCHINSKI, D.; SMITS, B.; GREENBERG, D. P. (1994): Bounds and Error Estimates for Radiosity. *Computer Graphics Proceedings, Annual Conference Series, ACM SIGGRAPH*, pp.67-74.
- [35] LUTZ, E. D.; GRAY, L. J. (1993): Analytic evaluation of singular boundary integrals without CPV. *Communications in Numerical Methods in Engineering* **9**, 909-915.
- [36] MACKERLE, J. (1993): Mesh generation and refinement for FEM and BEM – A bibliography (1990-1993). *Finite Elements in Analysis and Design* **15**, 177-188 (cites 272 references in FEM and 23 in BEM).
- [37] MACKERLE, J. (1994): Error Analysis, adaptive techniques and finite and boundary elements – A bibliography (1992-1993). *Finite Elements in Analysis and Design* **17**, 231-246 (cites 312 references in FEM and 59 in BEM).
- [38] MAIER, G.; DILIGENTI, M.; CARINI, A. (1990): On symmetrization in boundary element elastic and elastoplastic analysis. In: Kuhn, G.; Mang, H. (eds.): *Discretization Methods in Structural Mechanics*, pp.191-200. Berlin: Springer-Verlag.
- [39] NOOR, A. K.; BABUŠKA, I. (1987): Quality assessment and control of finite element solutions. *Finite Elements in Analysis and Design* **3**, 1-26 (cites 196 references).
- [40] ODEN, J. T.; DEMKOWICZ, L. (1989): Advances in adaptive improvements – A survey of adaptive finite element methods in computational mechanics. In: Noor, A. K.; Oden, J. T. (eds.): *State-Of-The-Art Surveys On Computational Mechanics*, chapter 13, pp.441-467. New York: The American Society of Mechanical Engineers (ASME) (cites 184 references).
- [41] PARTRIDGE, P.; BREBBIA, C. A.; WROBEL, L. C. (1991): *The Dual Reciprocity Boundary Element Method*, Southampton: Computational Mechanics Publications; and London: Elsevier Applied Science.
- [42] RIZZO, F. J. (1967): An integral equation approach to boundary value problems of classical elastostatics. *Quarterly of Applied Mathematics* **25**, 83-95.
- [43] ROSEN, D.; CORMACK, D. E. (1992): Analysis and evaluation of singular integrals by the invariant imbedding approach. *International Journal for Numerical Methods in Engineering* **35**, 563-587.
- [44] ROSEN, D.; CORMACK, D. E. (1993): Singular and near singular integrals in the BEM: A global approach. *SIAM Journal on Applied Mathematics* **53**, 340-357.

- [45] ROSEN, D.; CORMACK, D. E. (1994): The continuation approach for singular and near-singular integration. *Engineering Analysis with Boundary Elements* **13** 99-113.
- [46] SIRTORI, S.; MAIER, G. NOVATI, G; MICCOLI, S. (1992): A Galerkin symmetric boundary-element method in elasticity: formulation and implementation. *International Journal for Numerical Methods in Engineering* **35** 255-282.
- [47] SLOAN, I. H. (1992): Error analysis of boundary integral methods. *Acta Numerica* 287-339.
- [48] STROUBOULIS, T.; HAQUE, K. A. (1992a): Recent experiences with error estimation and adaptivity, Part I: Review of error estimators for scalar elliptic problems. *Computer Methods in Applied Mechanics and Engineering* **97**, 399-436.
- [49] STROUBOULIS, T.; HAQUE, K. A. (1992b): Recent experiences with error estimation and adaptivity, Part II: Error estimation for h-adaptive approximations on grids of triangles and quadrilaterals. *Computer Methods in Applied Mechanics and Engineering* **100**, 359-430.
- [50] SZABÓ, B.; BABUŠKA, I. (1991): *Finite Element Analysis*. New York: John Wiley & Sons.
- [51] TWORZYDLO, W. W.; ODEN, J. T. (1993): Towards an automated environment in computational mechanics. *Computer Methods in Applied Mechanics and Engineering* **104**, 87-143.
- [52] WEI, X.; CHANDRA, A.; LEU, L.-J.; MUKHERJEE, S. (1994): Shape optimization in elasticity and elasto-viscoplasticity by the boundary element method. *International Journal of Solids and Structures* **31**, 533-550.
- [53] ZARIKIAN, V.; GRAY, L. J.; PAULINO, G. H. (1994): Pointwise error estimates for boundary element calculations. In: Brebbia, C. A.; Kassab, A. J. (eds): *Boundary Element Technology IX*, pp.253-260. Southampton and Boston: Computational Mechanics Publications.
- [54] ZHANG, Q.; S. MUKHERJEE, S. (1991a): Design sensitivity coefficients for linear elastic bodies with zones and corners by the derivative boundary element. *International Journal of Solids and Structures*, **27** (8), 983-998.
- [55] ZHANG, Q.; MUKHERJEE, S. (1991b). Second-order design sensitivity analysis for linear elastic problems by the derivative boundary element method. *Computer Methods in Applied Mechanics and Engineering* **86**, 321-335.
- [56] ZHAO, C.; AILOR, M.; GRAY, L. J. (1994): Interior point evaluation in the boundary element method. *Engineering Analysis with Boundary Elements*, **13**, 201-208.
- [57] ZIENKIEWICZ, O. C.; TAYLOR, R. L. (1989): *The Finite Element Method*, Vol.1: Basic Formulation and Linear Problems. London: McGraw-Hill Book Company.
- [58] ZIENKIEWICZ, O. C.; ZHU, J. Z. (1992a): The superconvergent patch recovery and a *a posteriori* error estimates. Part 1: The Recovery Technique. *International Journal for Numerical Methods in Engineering* **33**, 1331-1364.

- [59] ZIENKIEWICZ, O. C.; ZHU, J. Z. (1992b): The superconvergent patch recovery and *a posteriori* error estimates. Part 2: Error estimates and adaptivity. *International Journal for Numerical Methods in Engineering* **33**, 1365-1382.
- [60] ZIENKIEWICZ, O. C.; ZHU, J. Z. (1994): The SPR recovery and boundaries. *International Journal for Numerical Methods in Engineering* **37**, 3195-3196.

ORNL/TM-12820

INTERNAL DISTRIBUTION

1. T. S. Darland	25. Central Research Library
2. E. F. D'Azevedo	26. K-25 Applied Technology Li-
3-7. L. J. Gray	brary
8. C. E. Oliver	27. Laboratory Records Depart-
9. B. W. Peyton	ment
10-14. S. A. Raby	28. Laboratory Records Depart-
15-19. R. F. Sincovec	ment - RC
20-24. R. C. Ward	29. Patent Office
	30. Y-12 Technical Library

EXTERNAL DISTRIBUTION

31. John F. Abel, School of Civil and Environmental Engineering, Hollister Hall, Cornell University, Ithaca, NY 14853
32. Marc Bonnet, Laboratoire de Mécanique des Solides, Centre Commun CNRS - X - Mines - Ponts, Ecole Polytechnique, 91128 Palaiseau Cedex, France
33. Alexandre Chorin, Mathematics Department, Lawrence Berkeley Laboratory, Berkeley, CA 94720
34. Aura Conci, PUC-Rio; Dept. de Engenharia Mecânica, Rua Marquês de São Vicente, 225, Gávea, CEP. 22453; Rio de Janeiro, R. J., Brazil
35. James Corones, Ames Laboratory, Iowa State University, Ames, IA 50011
36. Gregory G. Deierlein, School of Civil and Environmental Engineering, Hollister Hall, Cornell University, Ithaca, NY 14853
37. Ney A. Dumont, PUC-Rio; Dept. de Engenharia Civil, Rua Marquês de São Vicente, 225, Gávea, CEP. 22453; Rio de Janeiro, R. J., Brazil
38. Richard E. Ewing, Texas A&M University, Institute for Scientific Computation, 326 Teague Research Center, College Station, TX 77843-3404
39. Marcelo Gattass, PUC-Rio; Dept. de Informática, Rua Marquês de São Vicente, 225, Gávea, CEP. 22453; Rio de Janeiro, R. J., Brazil
40. James Glimm, Department of Mathematics, State University of New York, Stony Brook, NY 11794
41. Massimo Guiggiani, Dipartimento di Costruzioni Meccaniche e Nucleari, Università degli Studi di Pisa, via Diotisalvi 2, 56126 Pisa, PI, Italy
42. Fred Howes, Office of Scientific Computing, ER-7, Applied Mathematical Sciences, Office of Energy Research, U. S. Department of Energy, Washington, DC 20585
43. Shang-Hsien Hsieh, School of Civil Engineering, 1284 Civil Engineering Building, West Lafayette, IN 47907-1284
44. Pao-Chung Huang, F2, 67, Sec. 2, Minchuan East Road, Taipei, Taiwan, R.O.C

45. Marc S. Ingber, Department of Mechanical Engineering, University of New Mexico, Albuquerque, NM 87131
46. Anthony R. Ingraffea, School of Civil and Environmental Engineering, Hollister Hall, Cornell University, Ithaca, NY 14853
47. Nagesh R. Iyer, Structural Engineering Research Centre, CSIR Campus, Taramani, Madras 600 113, India
48. Gary Johnson, Office of Scientific Computing, ER-7, Applied Mathematical Sciences, Office of Energy Research, U. S. Department of Energy, Washington, DC 20585
49. James H. Kane, Mechanical and Aeronautical Engineering Department, Clarkson University, Postdam, NY 13699
50. Hans Kaper, Mathematics and Computer Science Division, Argonne National Laboratory, 9700 S. Cass Avenue, Argonne, IL 60439
51. Peter D. Lax, Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012
52. Liang-Jeng Leu, Department of Civil Engineering, National Taiwan University, Taipei 10617, Taiwan
53. Dani Lischinski, Department of Computer Science and Engineering, FR-35, University of Washington, Seattle, WA 98195
54. Earlin Lutz, 1335 Jackson Lane, West Chester, PA 19380
55. Paul A. Martin, Department of Mathematics, University of Manchester, Manchester, United Kingdom, M13 9PL
56. Carlos E. N. Mazzilli, Departamento de Engenharia de Estruturas e Fundações, Escola Politécnica da Universidade de São Paulo (EPUSP), Avenida Professor Mello Moraes, 2373, CEP. 05598; São Paulo, S. P., Brazil
57. Ambar K. Mitra, Department of Aerospace Engineering and Engineering Mechanics, Iowa State University, Ames, IA 50011
58. Khalid Mosalam, School of Civil and Environmental Engineering, Hollister Hall, Cornell University, Ithaca, NY 14853
59. Subrata Mukherjee, Department of Theoretical & Applied Mechanics, Thurston Hall, Cornell University, Ithaca, NY 14853
60. David Nelson, Director of Scientific Computing, ER-7, Applied Mathematical Sciences, Office of Energy Research, U. S. Department of Energy, Washington, DC 20585
61. J. T. Oden, Texas Institute for Computational Mechanics, University of Texas at Austin, Austin, TX 78712
62. Jerzy Pamin, Institute of Computer Methods in Civil Engineering, TU Cracow, Warszawska 24, PL-31-155 Kraków, Poland
- 63-67. Glaucio H. Paulino, School of Civil and Environmental Engineering, Hollister Hall, Cornell University, Ithaca, NY 14853
68. Rivania H. Paulino, PUC-Rio; Dept. de Engenharia Mecânica, Rua Marquês de São Vicente, 225, Gávea, CEP. 22453; Rio de Janeiro, R. J., Brazil

69. S. Leigh Phoenix, Department of Theoretical & Applied Mechanics, Thurston Hall, Cornell University, Ithaca, NY 14853
70. James C. T. Pool, Deputy Director, Caltech Concurrent Supercomputing Facility, California Institute of Technology, MS 158-79, Pasadena, CA 91125
71. Joseph J. Rencis, Mechanical Engineering Department, Worcester Polytechnic Institute, 100 Institute Road, Worcester, MA 01609
72. Sunil Saigal, Department of Civil Engineering, Carnegie Mellon University, Pittsburgh, PA 15123
73. Mark S. Shephard, Scientific Computation Research Center, Rensselaer Polytechnic Institute, Troy, NY 12180-3590
74. Julio M. Stern, Al. Franca 1401, Ap. 62, CEP. 01422; São Paulo, S. P., Brazil
75. José C. F. Telles, COPPE - Univ. Federal do Rio de Janeiro, Programa de Engenharia Civil, Caixa Postal 68506, CEP. 21944; Rio de Janeiro, R. J., Brazil
76. Lars Wahlbin, Department of Mathematics, White Hall, Cornell University, Ithaca, NY 14853
77. John O. Watson, Department of Mining Engineering, University of New South Wales, P.O. Box 1, Kensington, NSW 2033, Australia
- 78-82. Vrej Zarikian, Department of Mathematics, University of Central Florida, Orlando, FL 32816
83. Office of Assistant Manager for Energy Research and Development, U.S. Department of Energy, Oak Ridge Operations Office, P. O. Box 2001, Oak Ridge, TN 37831-8600
- 84-85. Office of Scientific and Technical Information, P. O. Box 62, Oak Ridge, TN 37830