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Spectrum of the ballooning Schrödinger equation

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Abstract

The ballooning Schrödinger equation (BSE) is a model equation for investigating global modes that can, when approximated by a Wentzel-Kramers-Brillouin (WKB) ansatz, be described by a ballooning formalism locally to a field line. This second order differential equation with coefficients periodic in the independent variable θ_k is assumed to apply even in cases where simple WKB quantization conditions break down, thus providing an alternative to semiclassical quantization. Also, it provides a test bed for developing more advanced WKB methods: e.g. the apparent discontinuity between quantization formulae for “trapped” and “passing” modes, whose ray paths have different topologies, is removed by extending the WKB method to include the phenomena of tunnelling and reflection. The BSE is applied to instabilities with shear in the real part of the local frequency, so that the dispersion relation is inherently complex. As the frequency shear is increased, it is found that trapped modes go over to passing modes, reducing the maximum growth rate by averaging over θ_k .

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I Introduction

The problem of describing instabilities and low-frequency waves in a toroidal plasma confinement system whose phase is approximately constant along a magnetic field line, but whose transverse wavevector is large, has been solved by the “ballooning formalism” first introduced (see e.g. Connor *et al.* [1]) to describe ideal MHD ballooning instabilities driven by pressure gradients.

In the early literature [1] a parameter, sometimes called θ_0 , appeared which was regarded merely as a constant arising from the arbitrariness in the choice of origin of the poloidal angle, θ , to be chosen to minimize the energy in the determination of stability. However it was also recognized quite early on [2, 3] that this parameter plays a more fundamental role, in that it can also be regarded as a kind of radial wave number. Because of this dual angle-wavenumber role we prefer to use the notation θ_k [3, 4].

The parameter θ_k was introduced [3] as the local wavenumber in a WKB representation in which the rapidly varying phase was taken, assuming straight field line coordinates, to be $nq\theta - n\phi - n \int \theta_k dq$, where q is the inverse rotational transform or safety factor labelling the magnetic surfaces, ϕ is the toroidal angle and $n \gg 1$ the toroidal mode number. This formalism was generalized to non-axisymmetric geometries by Dewar and Glasser [5], and recently applied successfully to a calculation of the ballooning mode spectrum in a non-axisymmetric system (a heliotron/torsatron) with high magnetic shear [6], where the variation of the local ballooning eigenvalue with θ_k is strong. Agreement between the semi-classical quantization condition, which determines the allowed global growth rates from WKB ray trajectories [5], and growth rates obtained from a global MHD eigenvalue code, TERPSICHORE, was remarkably good even down to low n numbers.

This illustrates the power of the WKB approach as a method for treating geometrically

complex situations. It is also capable of handling complicated physics, since it is not limited to second-order differential equations, and can in fact be applied to integro-differential equations [7]. A third advantage is that it is trivial to implement on a massively parallel computer since the eigenvalue problem on a field line is independent of that on any other field line.

The general WKB method [5] has yet to be fully tested in a low-shear system, such as a heliac, where the weaker dependence on θ_k leads to the appearance of separatrices between topologically different classes of ray trajectories. In order to understand the ray-separatrix phenomenon in a simpler context we restrict attention to the axisymmetric tokamak case and revisit a formalism introduced [8] to treat ballooning modes in low-shear regions in tokamaks, where an oscillatory dependence of growth rate on n is observed numerically [3], which cannot be explained by simple WKB theory.

The fundamentally new approach introduced by Dewar [8] was to raise the status of θ_k to that of an independent variable by treating it as a transform variable conjugate to nq in a “twisted radial Fourier transform” (TRFT) [9]. This allowed the heuristic derivation of a model equation for the complex mode amplitude as a function of θ_k which was called the “ballooning Schrödinger equation” (BSE), a second order differential equation obtained from the local dispersion relation by the substitution $q \mapsto (1/in) d/d\theta_k$. It can be written as a Mathieu equation, and is analogous to the Schrödinger equation for Bloch electron waves in a one-dimensional crystal with sinusoidal potential variation. The eigenvalues of the BSE exhibited a qualitatively correct dependence on n when compared with the PEST code [8].

A WKB solution of the BSE yields two classes of global modes which appear to be fundamentally different, in that one class propagates only in restricted intervals of θ_k around the periodic maxima of local growth rate, while the other class propagates over the full range $-\infty < \theta_k < \infty$ (see figure 1). We shall call the first class “trapped modes” and the second class “passing modes.” They have also been designated in a recent paper [10] ballooning

Figure 1: Contours of constant growth rate, or WKB ray paths, showing “trapped modes” inside the separatrix and “passing modes” outside.

modes of types I (BM-I) and II (BM-II), respectively.

Passing modes have recently received attention in the theory of drift waves [11, 12, 13] on the grounds that they have a greater radial extent than trapped modes, even though of lower growth rate, and may therefore be more important for producing anomalous transport. In fact trapped modes lying just inside the separatrix would be even more radially extended and should receive equal attention.

The BSE was solved [8] by reducing it to the Mathieu equation and using Floquet theory. There is no obvious distinction between passing and trapped modes in the spectrum of the Mathieu equation, and the question arises as to whether this signals a total breakdown of WKB theory or whether it can be modified to give a more qualitatively correct spectrum. It was recently found [14] that, by including reflection and tunnelling, a continuous transition could be found between trapped and passing modes in the case of purely imaginary frequency. It is one of the main aims of this paper to investigate whether this remains true in cases where the frequency can be complex.

A similar formalism to the TRFT method has also been introduced recently by Kim and Wakatani [15] and some controversy has arisen over the spectrum for dissipative modes [16].

A second aim of the present paper is to seek insight into this question by using a more realistic counter-example than that used previously [16].

Both for treating dissipative systems [17] and for investigating the transition between trapped and passing modes, we need to generalize the WKB method by allowing θ_k to be complex. The main reference we have found useful for the trapped/passing transition problem is the book by Fröman and Fröman [18], who use a convenient matrix notation to represent the coupling between modes travelling in opposite directions and give a rigorous derivation of the transfer matrix for waves tunnelling through or passing over a potential barrier (in the quantum mechanical sense).

The complex WKB theory is often called the Phase-Integral Method or the WKBJ or JWKB method, with the J acknowledging the work of Jeffreys [19], which, interestingly, was concerned with finding approximate solutions to the Mathieu equation. By incorporating the results of Fröman and Fröman [18] we are able to go well beyond what Jeffreys [19] was able to achieve and to show that the full Floquet spectrum can be well approximated by the WKBJ method. The application of the WKBJ method to the Mathieu equation is also discussed by N. Fröman [20].

In Section II we introduce the twisted radial Fourier transform and sketch, in Section III, how the standard ballooning representation arises in this approach when a WKB ansatz is used in TRFT-space. In Section IV we review the semi-classical quantization conditions for passing and trapped modes. The BSE model is introduced in Section V and cast in the form of the Mathieu equation in Section VI. Numerical solutions for the case of purely growing modes are plotted for a series of n values, revealing the oscillatory, but continuous, behaviour found earlier [3, 8]. Approximate Floquet solutions for this equation are constructed in Section VII using the WKBJ method [18]. It is shown that the trapped and passing mode limits are continuously connected, as found in the numerical solutions of the BSE.

We study the complex frequency case in Section VIII and find that shear in the real

part of the frequency has a stabilizing effect on the most unstable mode. For the modes with lower growth rate (but more extended radially) there is a dramatic mode-coalescence effect, at least in the example studied, for which the most unstable surface is a mode-rational surface. In this effect, pairs of eigenvalues which were distinct at zero shear approach each other, as the frequency-shear is increased from zero, retaining the real frequency of the most unstable surface up to a certain frequency-shear, where their growth rates become degenerate. Beyond this point, the growth rates of the two modes are equal but one has a higher and one a lower frequency than that corresponding to the most unstable surface. This effect is also reproduced accurately using analytic continuation of the WKBJ results for purely growing modes in one case studied, but other cases are found where analytic continuation is unreliable.

Finally, in Section IX, the results are compared with previous asymptotic methods [15, 10] and with a large-frequency-shear expansion. It is concluded that, within the BSE model, frequency-shear does not introduce a continuous spectrum. In the limit of large frequency shear, all modes are described by the simple WKB passing-mode quantization condition.

II The TRFT

Flute-like modes in the large- n limit [i.e. waves whose wavenumber $k_{\parallel}$ parallel to the magnetic field is $O(1)$, while the perpendicular projection $k_{\perp}$ is $O(n)$ as $n \rightarrow \infty$] need to be described rather carefully in toroidal geometries, since setting the phase variation constant on a field line with irrational rotational transform would appear to imply that the phase is constant on the whole magnetic surface, which contradicts the assumption of rapid variation in directions perpendicular to $\mathbf{B}$. In this section we develop the “twisted radial Fourier transform” (TRFT) [8, 9] approach to solving the problem.

Considering the wave signal

$$\varphi(\psi, \theta, \zeta, t) = \varphi(\psi, \theta) \exp(-in\zeta - i\omega t), \quad (1)$$

where n is regarded as large and φ is 2π -periodic in θ , define the TRFT by

$$\varphi(\psi, \theta) = \int_{-\infty}^{\infty} \frac{d\theta_k}{2\pi} \hat{\varphi}(\theta, \theta_k) \exp[inq(\psi)(\theta - \theta_k)], \quad (2)$$

where ψ is a flux-surface label, q is the inverse rotational transform or “safety factor,” and θ is a poloidal angle such that a magnetic field line is defined by $\zeta - q\theta = \text{const}$, with ζ the toroidal angle. Provided $\hat{\varphi}(\theta, \theta_k)$ is a slowly varying function of θ , equation (2) ensures that $\mathbf{B} \cdot \nabla \varphi = O(1)$ as $n \rightarrow \infty$, even if the dependence of $\hat{\varphi}(\theta, \theta_k)$ on θ_k is rapid.

Periodicity in real space is obtained by requiring the periodicity condition

$$\hat{\varphi}(\theta + 2\pi, \theta_k + 2\pi) = \hat{\varphi}(\theta, \theta_k). \quad (3)$$

Note that, although the motivation for the representation, equation (2), is based on asymptotic considerations, it is a true transform (being a variant of the Fourier transform). Thus any type of mode could in principle be represented using this transformation, for any value of n . However, we note that equation (2) applies only over a range of ψ for which q varies monotonically. In tokamaks with nonmonotonic q -profiles we must therefore assume there is negligible coupling across the zero magnetic shear surface(s), so that only one such range of ψ need be considered.

III Ballooning representation

Using q as independent variable in place of ψ , consider a general wave equation for flute-like modes

$$L(q, \theta, -\frac{i}{n} \partial_q, \partial_\theta - inq) \varphi(q, \theta) = 0. \quad (4)$$

Inserting equation (2) in equation (4) and commuting the exponential with L , we find

$$\begin{aligned} & \int_{-\infty}^{\infty} \frac{d\theta_k}{2\pi} \exp[inq(\theta - \theta_k)] \\ & \quad \times L(q, \theta, \theta - \theta_k, \partial_\theta) \hat{\varphi}(\theta, \theta_k) \\ &= \int_{-\infty}^{\infty} \frac{d\theta_k}{2\pi} \exp[inq(\theta - \theta_k)] \\ & \quad \times L\left(\frac{i}{n} \partial_{\theta_k}, \theta, \theta - \theta_k, \partial_\theta\right) \hat{\varphi}(\theta, \theta_k) = 0. \end{aligned} \quad (5)$$

Now integrate by parts to make ∂_{θ_k} act to the right, giving the transformed wave equation

$$]L\left(-\frac{i}{n} \partial_{\theta_k}, \theta, \theta - \theta_k, \partial_\theta \mid \omega, \frac{1}{n}\right) \hat{\varphi}(\theta, \theta_k) = 0. \quad (6)$$

If the rapid θ_k -dependence can be described by the eikonal or WKB ansatz¹

$$\hat{\varphi}(\theta, \theta_k) = \bar{\varphi}(\theta, \theta_k) \exp[in\hat{S}(\theta_k)], \quad (7)$$

where $\partial_{\theta_k} \bar{\varphi} = O(1)$, then, inserting equation (7) in equation (6) and commuting the exponential gives

$$L\left(\hat{q}(\theta_k) - \frac{i}{n} \partial_{\theta_k}, \theta, \theta - \theta_k, \partial_\theta \mid \omega, \frac{1}{n}\right) \bar{\varphi}(\theta, \theta_k) = 0, \quad (8)$$

where $\hat{q}(\theta_k) \equiv \partial_{\theta_k} \hat{S}(\theta_k)$ is 2π -periodic. Expanding $\bar{\varphi}$ in a power series in $1/n$, we find at lowest order the *ballooning equation*

$$L^{(0)}(q, \theta, \theta - \theta_k, \partial_\theta \mid \omega) \bar{\varphi}^{(0)}(\theta, \theta_k) = 0, \quad (9)$$

with $q = \hat{q}(\theta_k)$. This is an ODE involving only parameters local to a field line. Imposing the boundary condition $\bar{\varphi}^{(0)} \rightarrow 0$ as $|\theta| \rightarrow \infty$ gives an eigenvalue problem leading to the dispersion relation

$$D(q, \theta_k, \omega) = 0. \quad (10)$$

This is a local dispersion relation in the sense that it is localized to a field line (in a three-dimensional system [5] it would also depend on a field-line label as well as the surface label q),

¹It is well known in semi-classical quantization theory [21] that, apart from at turning points, there is complete reciprocity between the Fourier (momentum) space representation and the real space representations of short wavelength modes. That is, if WKB works in real space it works in Fourier space.

showing the quasi-two-dimensional nature of global ballooning-type modes—the behaviour along the field line is projected out by solving the ballooning equation.

In the standard WKB approach to ballooning theory [4], the surface label q is regarded as the independent variable and equation (10) is to be solved for θ_k , giving a function we denote by $\hat{\theta}_k(q, \omega)$. In the TRFT approach, θ_k is regarded as the independent variable and we solve equation (10) for q to give the function $\hat{q}(\theta_k, \omega)$.

Now substitute equation (7) into the TRFT expression equation (2) for $\varphi(q, \theta)$. We apply the saddle-point method, or the method of stationary phase, [22, pp. 276–9] to the asymptotic evaluation of equation (2) in the large- n limit. Only points where the phase $n[\hat{S}(\theta_k) - q\theta_k]$ is stationary contribute to the integral as $n \rightarrow \infty$:

$$\frac{d\hat{S}(\theta_k)}{d\theta_k} = q, \quad (11)$$

which has a periodic infinity of solutions $\theta_k^0(q) - 2\pi l$, $l = 0, \pm 1, \pm 2, \dots$. Thus, evaluating the contributions from the points of stationary phase, we find the usual [1, 4] ballooning representation as an infinite sum over shifted modes

$$\bar{\varphi}^{(0)} \propto e^{inS(q)} \sum_{l=-\infty}^{\infty} \exp[inq(\theta - 2\pi l)] \bar{\varphi}(\theta - 2\pi l, \theta_k^0), \quad (12)$$

where $S(q)$ is related to $\hat{S}(\theta_k)$ by a Legendre transformation, $S(q) \equiv \hat{S}(\theta_k^0(q)) - q\theta_k^0(q)$.

IV Semi-classical quantization

Using standard semi-classical quantization, the global eigenvalue condition

$$\oint \hat{\theta}_k(q, \omega) \frac{dq}{2\pi} = \oint \hat{q}(\theta_k, \omega) \frac{d\theta_k}{2\pi} = \frac{N + \frac{1}{2}}{n}, \quad (13)$$

for the case where the family of curves in the (q, θ_k) -plane defined by equation (10), $\omega = \text{const}$, were closed (“trapped modes”), was derived by Dewar *et al.* [2, 3]. The contour integrals are performed from left turning point to right turning point using the upper branch of the

solution of the dispersion relation, equation (10), then from right turning point to left turning point using the lower branch, thus giving the area enclosed by the $\omega = \text{const}$ contour.

In equation (13) the integer N is a radial mode number. The derivation strictly applies only if $N = O(n)$, so that the contour remains of finite area. However, if we take $N = 1$ and make a Taylor expansion up to second order about the point of maximum growth rate we find precisely the $1/n$ correction of Connor *et al.* [1]. This is just an instance of the well known fact that semi-classical quantization is exact for the harmonic oscillator potential.

The method is also found to give reasonable numerical approximations when n is only moderately large, with the contour integral being done numerically rather than by Taylor expansion. For example, the stability boundaries found from equation (13) gave quite good quantitative agreement with a global mode code for ideal MHD instabilities even with N set to 1 and n going down to about 5 for a sequence of tokamak equilibria [3]. Its three-dimensional generalization has also been applied successfully to a heliotron/torsatron test case [6].

The quantization condition for the case of “passing modes” [2, 11, 12], when the contours of $\omega = \text{const}$ are open, follows from the requirement that the eikonal representation equation (7), with $\hat{S}$ being the integral of $\hat{q}$ with respect to θ_k , be consistent with the periodicity condition equation (3)

$$\int_0^{2\pi} \hat{q}(\theta_k, \omega) \frac{d\theta_k}{2\pi} = \frac{m}{n}, \quad (14)$$

where m is an integer. That is, the average q on an allowed ray must be a rational fraction.

These two quantization conditions appear to imply the existence of two completely distinct classes of modes, but in the remainder of this paper we show that they are in fact smoothly connected.

V Schrödinger equation model

We follow the heuristic method used by Dewar [8] and simply postulate that the local ballooning dispersion relation, equation (10), can be solved to give $\omega = \omega_r(q, \theta_k) + i\omega_i(q, \theta_k)$, where $\omega_{r,i}$ are real functions (when q and θ_k are real), at most quadratic in q in the following form

$$\omega_r = \omega_0 + \omega'_0[q - q_0(\theta_k)], \quad (15)$$

$$\omega_i = \gamma_0 + \frac{1}{2}\gamma''_0[q - q_0(\theta_k)]^2 + \tilde{\gamma} \cos \theta_k \quad (16)$$

where ω_0 , ω'_0 , γ_0 , γ''_0 and $\tilde{\gamma}$ are arbitrary constants and q_0 is an arbitrary periodic function of θ_k , which for definiteness we take to be

$$q_0(\theta_k) = \bar{q} + \tilde{q} \cos \theta_k. \quad (17)$$

Assuming the plasma is unstable over a limited range in minor radius, the inequalities $\gamma_0 > 0$ and $\gamma''_0 < 0$ must apply.

The basic postulate of our model is that the θ -dependence can be projected out of $\hat{\varphi}(\theta, \theta_k)$, leaving a complex amplitude $\Psi(\theta_k)$ which obeys the equation

$$[\omega_r(q, \theta_k) + i\omega_i(q, \theta_k) - \omega]\Psi(\theta_k) = 0. \quad (18)$$

where q is interpreted as the operator $(1/in) d/d\theta_k$ [cf. equation (6)]. As q and $q_0(\theta_k)$ do not commute, we first collect the terms in q by completing the square, then make the operator replacement. Also we divide by the constant $|\gamma''_0|$ to put the model equation in the form

$$\begin{aligned} & \frac{1}{2} \left(\frac{1}{in} \frac{d}{d\theta_k} - q_0(\theta_k) + i\sigma \right)^2 \Psi \\ & + \left(\Gamma + \frac{1}{2}\sigma^2 - g_0 - \tilde{g} \cos \theta_k \right) \Psi = 0, \end{aligned} \quad (19)$$

where Γ is a nondimensional complex growth rate defined by

$$\Gamma \equiv \frac{\omega - \omega_0}{i|\gamma''_0|}, \quad (20)$$

and the nondimensional parameters σ , g_0 , and $\tilde{g}$ are defined by

$$\sigma \equiv \omega'_0/|\gamma''_0|, \quad (21)$$

$$g_0 \equiv \gamma_0/|\gamma''_0|, \quad (22)$$

$$\tilde{g} \equiv \tilde{\gamma}/|\gamma''_0|. \quad (23)$$

By incorporating equation (16), equation (19) expresses the assumption that, for each value of θ_k , the local growth rate is a maximum at some surface q_0 within the plasma, while equation (15) expresses the assumption that there is a linear radial variation in the real part of the frequency, i.e. a *frequency-shear*. This can be regarded as modelling the effect of shear in the rotational velocity (e.g. diamagnetic drift) by providing a surface-dependent Doppler shift to the local frequency (but see the caveat below).

We can solve $\omega = \omega_r(q, \theta_k) + i\omega_i(q, \theta_k)$ analytically to give the (now complex) function $\hat{q}(\theta_k, \omega)$ used in the semiclassical quantization conditions equations (13) and (14)

$$\hat{q}(\theta_k, \omega) = q_0(\theta_k) - i\sigma \pm i\sqrt{2} \left(\Gamma + \frac{1}{2}\sigma^2 - g_0 - \tilde{g} \cos \theta_k \right)^{1/2}. \quad (24)$$

As will be seen in Section VII, the integrals in the semiclassical quantization conditions can be performed in terms of complete elliptic integrals.

Figure 1 was a contour plot of $\omega_i(q, \theta_k)$, with the form of q_0 taken as in equation (17), plotted for the parameters $\sigma = 0$, $g_0 = 0.031$, $\tilde{g} = 0.014$, $\bar{q} = 1.15$, $\tilde{q} = 0.05$ and $\gamma_0 = 1$, only the unstable range $\omega_i > 0$ being plotted. This case is appropriate to modelling ideal MHD ballooning modes and is similar to that used in figure 2 of Dewar [8].

When the eikonal ansatz equation (7) is substituted for $\Psi(\theta_k)$ in equation (19) we see that it gives the same dispersion relation as postulated in equations (15) and (16). By suitable choice of Ψ it was argued in Dewar [8] that equation (19) is correct to $O(1/n^{1/2})$ near O and X-points (see figure 1) and to $O(1/n)$ elsewhere.

We could presumably have derived equation (19) by applying more formal asymptotic expansion methods to equation (4), but there would still be no formal justification for truncating at quadratic order in q . Our philosophy is simply to find a model equation that can be solved both in terms of well-known functions, and by WKB or other asymptotic methods. (By its construction it gives a local dispersion relation that is automatically consistent with that derived by the WKB method when WKB is valid). It thus forms a test case for such asymptotic methods—if they work then they presumably work on more realistic cases when there is no truncation at quadratic order, and if they do not work then there is no reason to suppose that they would work in the more realistic case. Our model thus provides a mathematical test bed for evaluating asymptotic approximations made by other authors [10, 15], as will be discussed in Section IX.

There is a problem in using equation (19) to model the effect of rotation shear in that it is questionable whether a local eigenvalue can be found at all within a ballooning representation in the case of shear flow [23, 24, 25, 26]. For a steady background state, there must exist a global eigenvalue spectrum (possibly continuous)—the question is whether it is possible to separate the global problem into a one-dimensional local eigenvalue problem (the ballooning equation) and a two-dimensional global eigenvalue problem (the BSE). In this paper we assume that this separation is possible for sufficiently small velocity shear.

By taking out a phase factor,

$$\Psi(\theta_k) = \Upsilon(\theta_k) \exp \left[i n \int_0^{\theta_k} q_0(\theta'_k) d\theta'_k - i \sigma \theta_k \right], \quad (25)$$

we get the BSE

$$\frac{d^2 \Upsilon}{d\theta_k^2} + \left(\frac{a_M}{4} - \frac{q_M}{2} \cos \theta_k \right) \Upsilon = 0, \quad (26)$$

where

$$a_M \equiv 8n^2 \left(g_0 - \frac{1}{2} \sigma^2 - \Gamma \right) \quad (27)$$

and

$$q_M \equiv -4n^2 \tilde{g}. \quad (28)$$

Equation (26) is a Bloch-type equation to be solved under the periodicity condition, following from equation (3),

$$\Upsilon(\theta_k + 2\pi) = \exp(-2\pi i n q_*) \Upsilon(\theta_k), \quad (29)$$

where

$$q_* \equiv \int_0^{2\pi} q_0(\theta_k) d\theta_k / 2\pi - i\sigma = \bar{q} - i\sigma, \quad (30)$$

where the second form follows from equation (17).

VI Mathieu equation

Defining $z \equiv \theta_k/2$ in equation (26) we get the *Mathieu equation*

$$\frac{d^2 \Upsilon}{dz^2} + (a_M - 2q_M \cos 2z) \Upsilon = 0. \quad (31)$$

From the theory of Mathieu's equation [27, Ch. 20] we know that Floquet solutions of equation (31) in the form $\Upsilon = \exp(i\nu z)P(z)$ exist, with P a π -periodic function. The periodicity equation (29) gives the simple quantization condition

$$\nu = -2nq_*. \quad (32)$$

We have solved the Mathieu equation numerically using the recursion relation [27, p. 728, eq. 20.3.12] for the Fourier coefficients c_m by shooting from the left and right and iterating on the eigenvalue until the mismatch between the left and right solutions is below a prescribed tolerance.

The result of solving equation (31) with the boundary condition equation (29) is plotted in figure 2 for the same parameters as figure 1. We plot the most unstable global growth rate $\gamma = \omega_i$ and the corresponding Mathieu-equation parameters, q_M and $a_M/2q_M$ against n ,

Figure 2: Growth rate γ , $a_M/2q_M$ and q_M , plotted *vs.* n . The upper and lower envelopes of γ and $a_M/2q_M$ correspond to fixing ν at 0 and -1 , respectively. The intermediate curves correspond to $\nu = -0.5$. The two passing-mode ranges of n , where $|a_M| > 2|q_M|$, are indicated by vertical grey bands.

which is treated as a continuous variable. Physically of course, only integer values of n are allowed but plotting the intermediate behaviour gives a feel for the sensitivity of the growth rate to changes in $\bar{q}$.

It is seen that the global eigenvalue is quite oscillatory for low n , but is continuous despite the fact that there are narrow intervals near $n = 1$ and $n = 2$ where $|a| > 2|q_M|$ (the passing-mode regime of the WKB theory). The lower envelope curves, corresponding to setting $\nu \equiv -1$, lie in the passing mode regime for a wider interval, $1 \leq n < 2.42$ and again are continuous and apparently smooth through the passing-trapped transition.

VII WKB Method

Fröman and Fröman [18] consider equations of the form

$$\frac{d^2\psi}{dz^2} + Q^2(z)\psi = 0. \quad (33)$$

Thus, in the case of the Mathieu equation, equation (31), we have

$$Q(z) = (a_M - 2q_M \cos 2z)^{1/2}, \quad (34)$$

with the branch of the square root determined by the way the complex z -plane is cut (which is not unique). For the purposes of the present section we assume a_M and q_M to be real, and without loss of generality (because of invariance of the Mathieu equation under change of sign of q_M , accompanied by a shift of origin of z by $\pi/2$) take q_M to be positive. To generalize the results to arbitrary sign, simply replace q_M by $|q_M|$.

Fröman and Fröman then seek solutions of the form

$$\psi = Q^{-1/2} [a_1(z) \exp iw(z) + a_2(z) \exp -iw(z)], \quad (35)$$

where

$$w(z) \equiv \int^z Q(\zeta) d\zeta. \quad (36)$$

Forming a_1 and a_2 into a column vector $\mathbf{a}$, they define the propagator matrix $\mathbf{F}(z, z_0)$ through the relation

$$\mathbf{a}(z) = \mathbf{F}(z, z_0) \mathbf{a}(z_0). \quad (37)$$

Under the assumptions that $|Q|$ is large and slowly varying (away from turning points, where it vanishes) they derive approximate formulae for $\mathbf{F}$ for transmission through a “potential barrier” in both the case when the wave becomes evanescent near the top of the barrier and is mainly reflected, but with a small component tunnelling through, and the case when the wave “passes over the top” of the barrier, suffering a small amount of reflection in the process.

We consider first the case of passing modes, $a_M > 2q_M$. In this case there are no turning points on the real line, but there are complex turning points (i.e. points where Q) vanishes) at $z = \pm iy' + n\pi$, where $y' = \cosh^{-1}(a_M/2q_M)$, and n is any integer.

With the z -plane cut so that $Q^{1/2}$ is real on the real z -axis, Fröman and Fröman [18, p. 59] give the F-matrix between two points x_1, x_2 on either side of, and well away from, an underdense barrier (i.e. $x_1 \ll -y', x_2 \gg y'$ if there are turning points at $z = \pm iy'$) to be

$$F(x_1, x_2) = \begin{bmatrix} (1 + e^{2\kappa})^{1/2} & i e^{2\kappa} \\ -i e^{2\kappa} & (1 + e^{2\kappa})^{1/2} \end{bmatrix}, \quad (38)$$

where

$$\kappa \equiv -2 \int_0^{y'} Q(iy) dy. \quad (39)$$

For the Mathieu equation we can evaluate this integral in terms of complete elliptic integrals [27, Ch. 17]

$$\kappa = 4 \left(\frac{q_M}{m_1} \right)^{1/2} [E(m) - K(m)], \quad (40)$$

where

$$m \equiv \frac{a_M - 2q_M}{a_M + 2q_M}. \quad (41)$$

and $m_1 \equiv 1 - m$. Note that $\kappa = 0$ at $m = 0$, the trapped-passing boundary ($a_M = 2q_M$).

We now seek Floquet solutions, such that $\psi(z - \pi) = \exp(-i\pi\nu)\psi(z)$. Imposing this condition on the WKBJ ansatz equation (35), and equating coefficients of $\exp \pm iz$ we obtain the condition

$$\mathbf{a}(x_1) = \begin{bmatrix} e^{i(\lambda - \pi\nu)} & 0 \\ 0 & e^{-i(\lambda + \pi\nu)} \end{bmatrix} \mathbf{a}(x_1 + \pi), \quad (42)$$

where λ is the phase integral over a period

$$\lambda \equiv \int_{-\pi/2}^{\pi/2} Q(x) dx. \quad (43)$$

For the Mathieu case,

$$\lambda = 4 \left(\frac{q_M}{m_1} \right)^{1/2} E(m_1). \quad (44)$$

Note that $\lambda = 4q_M^{1/2}$ at the trapped-passing boundary, $a_M = 2q_M$.

Taking $x_1 = -\pi/2$ and requiring that equation (37), equation (38) and equation (42) be consistent (an eigenvalue problem) gives the condition

$$\cos \pi\nu = (1 + e^{2\kappa})^{1/2} \cos \lambda, \quad (45)$$

In agreement with N. Fröman [20] except for our neglect of a phase correction needed only very close to the trapped-passing boundary.

Before proceeding to the overdense barrier case, it is instructive to consider the case $|a_M/2q_M|$ bounded away from unity (from above) in such a way that $\kappa \rightarrow -\infty$ in the quasi-classical limit. Then equation (45) reduces to $\cos \lambda - \cos \pi \nu \equiv 2 \sin(\pi \nu/2 + \lambda/2) \sin(\pi \nu/2 - \lambda/2) = 0$, which implies

$$-\nu/2 \pm \lambda/2\pi = m, \quad (46)$$

where m is any integer. This limit corresponds to the passing mode case when equation (14) should apply. To check this, we write equation (24) as $\hat{q}(\theta_k) = q_0(\theta_k) - i\sigma \pm Q(\theta_k)/2n$. Substituting this in equation (14), using equations (34) and (43), we readily verify that equation (46) is indeed the passing-mode quantization condition, the $\pm$ forms corresponding to the two branches of $\hat{q}(\theta_k)$: one to the left and one to the right of the island region shown in figure 1. Note that the correspondence between equation (14) and equation (46) established above is not restricted to real a_M but applies also to the general complex case $\omega_i \neq 0$.

A similar calculation for the trapped mode case $-2q_M < a_M < 2q_M$, using the F-matrix for the overdense barrier (turning points x' and x'' on the real line, giving rise to tunnelling), also leads to equation (45), but with κ now defined by integrating across the evanescent region $x' < x < x''$

$$\kappa \equiv \int_{x'}^{x''} |Q(x)| dx. \quad (47)$$

which for the Mathieu equation gives

$$\kappa = 4(q_M)^{1/2} [E(m) - m_1 K(m)], \quad (48)$$

where now

$$m \equiv \frac{2q_M - a_M}{4q_M}. \quad (49)$$

and $m_1 \equiv 1 - m$ as before.

Figure 3: Comparison of the two lowest Floquet eigenvalues $a_M = a_0(\nu)$ and $a_M = a_1(\nu)$ of Mathieu's equation for $q_M = 1$ with simple WKB (no tunnelling or reflection) and with the full WKBJ approximation equation (45).

Also the phase integral λ is now taken between turning points, and for the Mathieu case becomes

$$\lambda = 4 (q_M)^{1/2} [E(m_1) - mK(m_1)]. \quad (50)$$

Note that both κ and λ have the same values at $m = 0$, the trapped-passing boundary, as in the passing-mode case, showing that the WKBJ method predicts continuity through the transition.

In the deeply trapped limit, $|a_M/2q_M|$ bounded away from unity from below in such a way that $\kappa \rightarrow +\infty$ in the quasi-classical limit, equation (45) reduces to $\cos \lambda = 0$

$$\lambda/\pi = N + \frac{1}{2}, \quad (51)$$

where N is any integer. It is readily verified that this follows also from the trapped-mode quantization condition, equation (13), as it must for consistency.

The WKBJ approximate eigenvalue condition, equation (45), is compared with exact eigenvalues in figure 3 where it is seen that it gives the correct continuous transition between the trapped and passing regimes at $a_M = 2q_M$.

Figure 4: Most unstable eigenvalue Γ of equation (26) *vs.* velocity shear parameter σ (solid line) compared with WKBJ trapped mode dispersion relation equation (52) (dashed line) for the test case described in the text.

VIII Complex case

We now consider the case where there is a nonzero real part of the frequency, equation (15), and in general a finite shear in this frequency so by equation (21), $\sigma \neq 0$. As discussed in Section V this can be regarded as modelling the effect of velocity shear on drift waves and we shall call σ *the velocity shear parameter*. By equation (30) and equation (32) the Floquet exponent ν must now be complex. However, if $n\bar{q}$ is an integer, the eigenvalues Γ of equation (26) may remain real, or otherwise appear in complex conjugate pairs, because the coefficients in equation (31), apart from Γ , are real and the boundary condition, equation (29), is unaffected by complex conjugation when $n\bar{q}$ is an integer.

A WKBJ dispersion relation for complex Γ may be found by continuing equation (45) into the complex Γ -plane. For the trapped mode case, $\text{Re } \kappa > 0$, it is best to write equation (45)

in the form

$$\cos \pi \nu = e^{\kappa} (1 + e^{-2\kappa})^{1/2} \cos \lambda \quad (52)$$

so as to move the cuts for defining the square root to the left half κ -plane.

In order to gain an appreciation of some of the possibilities, a specific case, $\bar{q} = 2$, $g_0 = 0.125$, $\tilde{g} = 0.1$, $n = 5$ was studied for σ between 0 and 0.5. Since $n\bar{q}$ is an integer, Γ is either real or occurs in complex conjugate pairs, as discussed above. When Γ is real, the Mathieu equation coefficients are real and the analysis of Section VII applies directly, without requiring analytic continuation.

Figure 4 shows the exact eigenvalue (which is real) for the most unstable mode compared with that predicted from the WKBJ dispersion relation as a function of the velocity shear parameter, σ . Even though this is the lowest eigenmode, and therefore violates the WKBJ ordering assumption that the wavelength is short compared with the scale length for variation of background quantities, it is seen that the WKBJ approximation works quite well up to σ around 0.4 and predicts the stabilizing influence of velocity shear.

At $\sigma = 0$, $\kappa = 9.85$, so the tunnelling factor $\exp \kappa$ is about 1.9×10^4 and the mode is strongly trapped—there is very little communication between the different 2π intervals in θ_k . However as σ increases from 0 to 0.4247, the phase integral λ decreases from about $\pi/2$ to 0 (i.e. the turning points coalesce at the bottom of the “potential well”). The normal WKBJ quantization condition equation (51) thus becomes strongly violated, despite the fact that the tunnelling factor $\exp \kappa$ is in fact increasing over this range. The reason for this paradoxical situation is that the imaginary part of the Floquet factor ν is also increasing, so the $\cos \pi \nu$ term on the left hand side of equation (52) begins to become comparable with the tunnelling factor, $\pi |\text{Im } \nu|$, becoming larger than $\exp \kappa$ at $\nu = 0.336$.

For $\sigma > 0.4247$ the WKBJ trapped mode dispersion relation ceases to apply, because then $a_M < -2|q_M|$ and the assumption that the turning points are on the real line becomes invalid. This is like the passing-mode case analyzed in Section VII, but is different because it

Figure 5: Second and third unstable eigenvalues (solid line) compared with WKBJ trapped mode dispersion relation (dashed line) for the same test case as in figure 4.

corresponds quantum mechanically to tunnelling when the energy is *below* a series of potential wells, rather than propagation when the energy is *above* a series of potential barriers. Fröman and Fröman [18] do not treat the former case, though it is no doubt tractable. Indeed, the exact solution continues on to arbitrarily high values of σ and we show in Section IX that the simple passing-mode semiclassical quantization condition equation (14) applies in the large- σ limit.

Figure 5 shows a comparison of the exact and WKBJ growth rates for the next two most unstable modes. Although the modes start off purely growing at $\sigma = 0$, the eigenvalues coalesce at $\sigma = 0.1772$ and thenceforth enter the complex plane as a complex conjugate pair (figure 6). (Interestingly, a similar phenomenon occurs for Kelvin–Helmholtz instabilities [28], but in the reverse direction because in this case the instability is *driven* by velocity shear.) The phase integral λ starts at about $3\pi/2$ and $5\pi/2$ for the two modes, which are both trapped modes at $\sigma = 0$. As for the lowest mode, the left-hand side of equation (52), $\cos \pi\nu$, becomes increasingly important as σ increases, and becomes comparable with the

Figure 6: Imaginary part of Γ (marked with squares) compared with the imaginary part predicted by WKBJ trapped mode dispersion relation (circles) for the same modes whose real parts are shown in figure 5. The negative branch of the complex conjugate pair is not shown.

tunnelling factor $\exp \kappa$ at about the point of coalescence.

A composite plot of the real parts of the growth rates for the first five radial eigenmodes is shown in figure 7. This includes a mode which is just trapped at zero σ and one which is just untrapped. It is seen that these two modes coalesce at a small value of σ . This is also predicted by using the trapped and passing WKBJ dispersion relations, equations (45) and (52), respectively, to finite σ , although neither works well beyond coalescence in this case. Also shown in figure 7 are asymptotic results discussed in the next section.

IX Other asymptotic methods

It is of interest to compare these results with estimates made using other asymptotic methods.

First we expand in inverse powers of σ to investigate the large- σ limit

$$\Psi = \Psi_0 + \sigma^{-1}\Psi_1 + \dots, \quad (53)$$

$$\Gamma = \sigma\Gamma_{-1} + \Gamma_0 + \sigma^{-1}\Gamma_1 + \dots \quad (54)$$

Figure 7: Real part of Γ for the first five eigenvalues. The arrow on the left axis at 0.025 shows the position of the separatrix between trapped and passing modes at $\sigma = 0$, while the arrows on the right axis show large- σ asymptotic values as discussed in Section IX.. The dashed curve is an estimate of the upper limit of the supposed continuous spectrum discussed in Section IX..

Substituting equations (53) and (54) into equation (19) and equating coefficients of like powers of σ , we find to lowest order that $\Psi_0 = A \exp[-n(\Gamma_{-1} - i\bar{q})\theta_k]$, where A is a constant.

Applying the periodicity condition equation (3) we see that

$$n\Gamma_{-1} = i(n\bar{q} - m), \quad (55)$$

where m is any integer.

The consistency condition for being able to solve for Ψ_1 is just $\langle g_0 + \tilde{g} \cos \theta_k - \Gamma_0 + \frac{1}{2}\Gamma_{-1}^2 \rangle = 0$, where $\langle \cdot \rangle$ denotes averaging over θ_k . Thus, combining the $O(\sigma)$ and $O(1)$ terms, we have the large- σ asymptotic dispersion relation

$$\Gamma = g_0 - \frac{(n\bar{q} - m)^2}{2n^2} + \frac{i\sigma}{n}(n\bar{q} - m) + O(\sigma^{-1}). \quad (56)$$

The real part is $O(1)$ and thus approaches a constant as $\sigma \rightarrow \infty$. These asymptotic values are indicated by arrows on the right axis in figure 7 for the radial eigenvalues $N \equiv m - m_0 = 0, \pm 1$ and ± 2 , where m_0 is the nearest integer to $n\bar{q}$ (in this case $m_0 = 10$). It is seen that the asymptotic estimate is quite good already when σ has reached the value 0.5.

Note that, in the case $\tilde{q} = 0$, equation (56) simply expresses the result that the global growth rate is reduced to the average over θ_k of the local growth rate on a rational surface [11, 24, 25]. In the case $n\bar{q}$ not an integer, the maximum growth rate is less than that corresponding to g_0 , i.e. to the average on the most unstable surface, since no m can be found such that $m - n\bar{q} = 0$.

The dominant term of Γ at large σ is actually the imaginary part, since it is $O(\sigma)$. It thus diverges linearly as $\sigma \rightarrow \infty$, except in the case $n\bar{q}$ an integer and $N = 0$, for which Γ remains real. These results are in qualitative agreement with the results shown in figures 4 and 6.

It is also of interest to calculate the structure of the eigenfunction Ψ . With the consistency condition satisfied we can calculate Ψ_1 to find

$$\Psi = A \left(1 + \frac{n\tilde{g}}{\sigma} \sin \theta_k \right) \exp(im\theta_k) + O(\sigma^{-2}). \quad (57)$$

Although explicit evaluation of the TRFT equation (2) would require a knowledge of the relation of $\Psi(\theta_k)$ to $\hat{\varphi}(\theta, \theta_k)$, it is qualitatively apparent that equation (57) expresses a concentration of the mode amplitude about the surface with $q = m/n$ as σ becomes large.

We note that, for $m = m_0$ ($N = 0$), equation (56) is in agreement with the result found by Zhang and Mahajan [10] using somewhat different asymptotic arguments. Their approach is a large- n expansion—one argues that, for $\sigma = O(1)$, the quadratic term can be neglected in equation (16) since it contributes an $O(n^{-2})$ term to equation (19) after dividing out a factor $\exp(im_0\theta_k)$ from Ψ . Then equation (19) is only a first order differential equation and can be integrated to give

$$\Psi = \exp \left\{ in \left[\left(\bar{q} - \frac{i(g_0 - \Gamma)}{\sigma} \right) \theta_k - \frac{i\tilde{g}}{\sigma} \sin \theta_k \right] \right\}. \quad (58)$$

For $N = 0$ the quantization condition for this equation gives the same growth rate as given in equation (56). Also, to $O(\sigma^{-1})$, equation (58) agrees with equation (57). Both methods predict the maximum of the eigenfunction to occur at $\theta_k = \text{sgn}(n\tilde{g}/\sigma)(\pi/2)$.

The quantization condition for the large- σ case is global periodicity in θ_k , just as for the passing-mode quantization condition equation (14). No turning points are involved. In fact it can be shown, by expanding equation (24) in powers of σ^{-1} , that the passing-mode semiclassical quantization condition, equation (14), leads precisely to equation (56) in the large- σ limit. Thus we conclude *for large frequency shear, all modes are passing modes*. Numerical work also verifies that equation (14) gives a good approximation for all the modes shown in figure 7 when σ is well beyond the coalescence point in the case of each of the higher modes. Even for the lowest mode, ($N = 0$), it gives a good approximation for $\sigma \simeq 0.5$ or greater. The averaging over θ_k implied by equation (14) explains the reduction in the largest growth rate as σ increases.

The BSE also provides a more realistic counterexample against the approach of Kim and Wakatani [15] than was used previously [16]. Their boundary layer analysis [15] predicts a continuum of eigenvalues with a maximum growth rate occurring [29] at roughly $\theta_k = \theta_{KW}$, where $\theta_{KW} \equiv |\omega'_0/2n\tilde{\gamma}|^{1/3} = |\sigma/2n\tilde{g}|^{1/3}$, which gives the maximum growth rate $\Gamma_{KW} \equiv g_0 + \tilde{g} \cos \theta_{KW}$. This estimate is plotted as a dashed line in figure 7 where it is seen that it overestimates the growth rate of the most unstable mode considerably. The error in the analysis is that, although apparently exponentially localized solutions can be found [15], the exponentially small tail is exponentially amplified in other ranges of θ_k , so that the “localized” solutions become global in θ_k and subject to the periodicity condition (and hence quantized into discrete eigenvalues).

X Conclusions

We have shown that WKB theory can be extended, by including the phenomena of tunnelling and reflection, to give a good approximation to the Floquet solutions of the Mathieu equation, which arises from a model for describing the global structure of ballooning-type modes. This reinforces the view that the theory of short-wavelength flute-like modes (ballooning-type

modes) can always be put in the framework of WKB theory.

By exactly solving the model equation (the BSE) we have been able to investigate the effect of frequency-shear on ballooning-type modes. At least within the model, the spectrum remains discrete at all values of frequency-shear (i.e. there is no continuum), with the growth rate of the most unstable mode being reduced by this shear.

If the local growth rate peaks symmetrically about some rational surface, $\bar{q}$, and $n\bar{q}$ is an integer, then the (real part of the) frequency of the most unstable global mode is determined by the local frequency at the most unstable surface. For zero frequency-shear the global growth rate (imaginary part of the frequency) is bounded above by the maximum of the local growth rate over θ_k at this surface, while for large shear the global growth rate asymptotes to the average of the local growth rate over θ_k there. However, for lower growth rate modes a mode coalescence phenomenon between pairs of modes can occur at finite velocity shear, after which the two modes have identical growth rates, but different frequencies. We have shown that the large-shear limit can be well understood by asymptotic expansion in inverse powers of the frequency-shear or by using the passing-mode quantization condition.

We have also found that the WKBJ-Floquet method gives a good semiquantitative understanding of the spectrum of our model equation at zero frequency-shear. Analytic continuation to finite shear works well in some cases, indicating that this regime is susceptible to analysis by the WKBJ method. However the continuation fails in other cases, either giving no solution or a spurious solution. This is not really surprising, since we are working with an asymptotic theory and it has been known since the work of Stokes in the last century that different asymptotic representations may apply in different sectors of the complex plane [22, pp. 112–117]. What is really required is to perform the WKBJ analysis afresh, allowing from the outset the possibility of complex eigenvalues, but this has been left to another paper. It also remains for further work to derive the BSE systematically from first principles (if this is possible) in the case of waves in a plasma subject to sheared rotation.

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