UNIVERSITY OF FLORIDA

Gainesville, Florida 32611

DOE/ER/40272--266

Institute for Fundamental Theory Preprint UFIFT

-HEP-97-2

January 1997

RELATIVISTIC CHARGED PARTICLE IN THE
DIPOLE-SPHERE CONFIGURATION I. CLASSICAL AND
SEMICLASSICAL SURFACE ORBITS

Kaundinya S. Gopinath*

Dallas C. Kennedy†

Department of Physics, University of Florida,

Gainesville, Florida 32611 USA

James M. Gelb‡

Department of Physics, University of Texas at Arlington,

Arlington, Texas 76019 USA

MASTER

Abstract

The classical and semiclassical orbits of a relativistic charged particle on a rotating sphere threaded by a magnetic dipole field are examined. The rotational and dipole axes are in general not aligned. Several physically distinct regimes emerge, depending on the relative sizes of the total energy, canonical azimuthal angular momentum, and magnetic field strength. Magnetic flux enclosed by orbits is quantised very close to the poles. Application of this system to neutron star magnetic fields and questions for future research are outlined.

PACS numbers: 03.65.Sq, 95.85.Sz, 97.60.Gb, 97.60.Jd

MASTER

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

DISCLAIMER

**Portions of this document may be illegible
in electronic image products. Images are
produced from the best available original
document.**

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, make any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

I. INTRODUCTION

One of the mysteries of neutron star structure is the origin of their powerful magnetic fields, which have strengths of up to at least 10^{12-13} Gauss [1–3]. Since the pulsar is a neutron star remnant of a stellar collapse, the magnetic field lines dragged in by that collapse are presumably squeezed by magnetic flux conservation in the contracting plasma. However, that assumption does not answer the question of how the magnetic field sustains itself at later times. A plausible mechanism is neutron ferromagnetism in the interior of the neutron star; with $N \sim 10^{60}$ neutron magnetic moments aligned, the free energy is lowered (for temperatures small compared to the microscopic dipole-dipole energy) and the resulting field:

$$B_0 \sim \frac{Nec^5\hbar}{R^3m_n}\sqrt{\hbar c}$$

for a star of radius R and neutrons of mass m_n , can be up to 10^{18} Gauss [4]. The crust and interior of a neutron star nonetheless have electrons and protons. Internal macroscopic currents might also affect the field, if not actually create it, but these internal currents are in turn strongly affected by the intense field. We assume here that such currents are essentially all electronic. If the density of electrons is low and/or the magnetic field large, a phenomenon akin to the quantum Hall effect should be expected [5]. Finally, for electromagnetic fields of strength $\gtrsim 10^{12}$ Gauss, relativistic quantum many-body effects are important [6].

With this application in mind, we select a simplified system that retains some important features of a realistic neutron star. This system is that of a charged particle constrained to move on the surface of a sphere of radius R threaded by an intense magnetic dipole field. The sphere and its magnetic dipole moment rotate with angular velocity Ω , and the rotational and magnetic dipole axes are in general tilted with respect to one another by an angle θ_0 . The problem can be treated relativistically by use of the appropriate metric for the rotating frame.

The distinctive physical regimes can be characterised by comparison of three dimensionless parameters:

$$\beta_0 \equiv |q|B_0R/(2mc^2) \quad ,$$

$$\epsilon \equiv E/(mc^2) \equiv \beta_0\eta \quad ,$$

$$l_\phi \equiv 2cP_\phi/(qB_0R^2)$$

are, respectively, the magnetic field strength B_0 in rescaled units at the magnetic poles; the energy E in units of the rest mass; and the canonical azimuthal angular momentum P_ϕ in rescaled units about the rotation axis. For electrons on a neutron star with the strongest measured fields, $\beta_0 \simeq (0.1)R/(2\lambda_C) \sim 2.5 \times 10^{15}$, where λ_C = electron Compton wavelength $= 2\pi\hbar/(m_e c)$. The relevant energy scale is then not set by mc^2 , but by $\eta = \epsilon/\beta_0$.

The regime of $\eta \gtrsim 1$ is the *ultrahigh energy case*, $E \simeq 10^{21}$ eV or higher, depending on B_0 and R but not on m . The magnetic field β_0 also sets the scale for P_ϕ : when $|l_\phi| \sim 1$, the *ultrahigh P_ϕ case*, the effect of P_ϕ can overcome the inhibiting effect of the field. Note that P_ϕ is the canonical, *not* the kinetic, angular momentum. When both η and l_ϕ are small, the charged particle has no allowed region on the spherical surface except very close to the rotational and magnetic poles. A large l_ϕ allows narrow regions away from the poles, but only $\eta \gtrsim 1$ allows charged particle motion over a substantial portion of the sphere.

The simplest treatments of charged particles trapped on spheres by intense magnetic fields result in particles frozen in place on the surface [2], but this feature holds exactly only in the limit of infinite field strength. Our treatment here adds the new feature of expanding the particle motion in inverse powers of the field strength.

II. CLASSICAL REGIME

We explicitly show all factors of c and, in section IV, of $\hbar$. Thus $x_\mu = (x_0, \mathbf{x}) = (ct, \mathbf{x})$.

A. Generally Covariant Lagrangian

The general relativistic action of a particle of mass m and charge q in an external electromagnetic and gravitational field is

$$S = -mc \int \sqrt{g_{\mu\nu}(x) dx^\mu dx^\nu} + \frac{q}{c} \int A_\mu(x) dx^\mu \quad (1)$$

with fixed endpoints in parameter-independent form [7]. Let us introduce as a path parameter the proper time τ . Then if

$$\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau} \quad , \quad (2)$$

the general relativistic Lagrangian of a particle in the external fields A_μ and $g_{\mu\nu}$ is

$$L = -mc \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} + \frac{q}{c} A_\mu \dot{x}^\mu \quad . \quad (3)$$

The canonical momenta are given by

$$P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{-mc g_{\mu\nu} \dot{x}^\nu}{\sqrt{g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta}} + \frac{q}{c} A_\mu \quad . \quad (4)$$

Using the identity

$$g_{\mu\nu} g^{\mu\lambda} = \delta_\nu^\lambda \quad , \quad (5)$$

equation (4) gives the constraint

$$g^{\mu\nu} (P_\mu - \frac{q}{c} A_\mu) (P_\nu - \frac{q}{c} A_\nu) = (mc)^2 \quad . \quad (6)$$

The equations of motion are given by

$$\frac{\partial L}{\partial x^\mu} - \frac{dP_\mu}{d\tau} = 0 \quad . \quad (7)$$

B. Two-Dimensional Rotating Sphere

Consider the case of a charged particle moving on the surface of a sphere of radius R . The sphere rotates at angular velocity Ω . Although we neglect gravity, $g_{\mu\nu}$ is non-trivial if we use the rotating frame.

The metric in a spherical polar co-ordinate system rotating with the sphere is given by the line element [7]

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= (c^2 - \Omega^2 r^2 \sin^2 \theta) dt^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - 2\Omega r^2 \sin^2 \theta dt d\phi \end{aligned} \quad (8)$$

Note that our metric $g_{\mu\nu}$ is not dimensionless. The angles ϕ and θ are the azimuthal and polar angles with respect to the rotation axis. The equations (7) are three equations, one for each momentum component, neglecting the radial motion. There is a maximal r , the *light-sphere* $r_{\max} = c/(\Omega \sin \theta)$, for which $g_{00} = 0$; the minimum value of $r_{\max}$ occurs at $\theta = \pi/2$, where $r_{\max} = c/\Omega$. On the other hand, $r_{\max}$ is infinite at the rotational poles $\theta = 0, \pi$ [2].

Since the Lagrangian does not explicitly depend on t , we have,

$$\frac{\partial L}{\partial t} = 0 \quad (9)$$

which implies

$$\frac{dP_0}{d\tau} = 0 \quad (10)$$

Because

$$d\tau = \frac{dt \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}}{c}, \quad (11)$$

we have also

$$\frac{dP_0}{dt} = 0 \quad (12)$$

Similarly, since the Lagrangian does not explicitly depend on ϕ ,

$$\frac{dP_\phi}{dt} = 0 \quad (13)$$

Thus, P_0 and P_ϕ are conserved. P_0 is the energy E , and P_ϕ is the canonical azimuthal angular momentum.

The remaining equation of motion is non-trivial:

$$\frac{\partial L}{\partial \theta} - \frac{dP_\theta}{d\tau} = 0 \quad (14)$$

The constraint (6) can be rewritten as

$$g^{00}P_0^2 + 2g^{0i}P_0(P_i - \frac{q}{c}A_i) + g^{ij}(P_i - \frac{q}{c}A_i)(P_j - \frac{q}{c}A_j) = (mc)^2, \quad (15)$$

which gives

$$P_\theta(\theta; P_0, P_\phi) = \frac{1}{\sqrt{g^{\theta\theta}}} \sqrt{(mc)^2 - g^{00}P_0^2 - 2g^{0\phi}P_0(P_\phi - \frac{q}{c}A_\phi) - g^{\phi\phi}(P_\phi - \frac{q}{c}A_\phi)^2}. \quad (16)$$

Now substituting in the values of $g^{\mu\nu}$, which can be obtained from the identity (5) and equation (8), we arrive at an explicit relation for P_θ :

$$P_\theta = R\{-(mc)^2 + (E/c)^2 - 2(\Omega E/c^2)(P_\phi - qA_\phi/c) - (1/R^2)[1/\sin^2\theta - (\Omega R/c)^2](P_\phi - qA_\phi/c)^2\}^{1/2} \quad (17)$$

or

$$P_\theta/(mcR) = \{-1 + (E/mc^2)^2 - 2(\Omega/mc^2)(E/mc^2)(P_\phi - qA_\phi/c) - (1/mcR)^2[1/\sin^2\theta - (\Omega R/c)^2](P_\phi - qA_\phi/c)^2\}^{1/2}. \quad (18)$$

III. CASES OF PHYSICAL INTEREST

We introduce the threading dipole magnetic field of polar strength B_0 aligned at an angle θ_0 to the rotation axis:

$$A_\theta = 0 \quad (19)$$

$$A_r = 0 \quad (20)$$

$$A_\phi = (B_0 R^2/2) \sin \theta \sin(\theta - \theta_0) , \quad (21)$$

where A_ϕ is defined in the rotating spherical co-ordinates. Then

$$P_\theta/(mcR) = \{-1 + (E/mc^2)^2 - 2(\Omega/mc^2)(E/mc^2)[P_\phi - (qB_0 R^2/2c) \sin \theta \sin(\theta - \theta_0)] \\ - (1/mcR)^2[1/\sin^2 \theta - (\Omega R/c)^2][P_\phi - (qB_0 R^2/2c) \sin \theta \sin(\theta - \theta_0)]^2\}^{1/2} . \quad (22)$$

The opposite sign of the square root, not shown, is also valid. We consider the system of equation (22) in three different limits of physical interest and take $\Omega R/c = 0.1$, an order-of-magnitude upper limit on pulsar rotation [8], and $q > 0$. The $q < 0$ case can be obtained from the $q > 0$ case by reversing the sign of P_ϕ and Ω .

While the $\phi(t)$ motion is trivial, the polar motion $\theta(t)$ is not. The two angular motions, both periodic, decouple from one another because P_ϕ is conserved. Without radial motion, the polar motion alone is a closed one-dimensional system. However, the two periods, τ_ϕ and τ_θ , are in general not equal or even commensurate. That is, their ratio τ_θ/τ_ϕ is not necessarily a rational number k_ϕ/k_θ , where (k_ϕ, k_θ) are a pair of integers with common multiples canceled. If the periods are commensurate, then the orbits can be arbitrarily complex but close after a time $\tau_{\text{closure}} = k_\theta \tau_\theta = k_\phi \tau_\phi$. Otherwise, the orbits never close. We compute below the magnetic flux Φ enclosed by an orbit for two cases (subsections III.B and III.C), but the flux is well-defined only if the orbit is closed. For very large fields, nonetheless, the variation of θ is $\mathcal{O}(\epsilon/\beta_0)$ and tiny in these two cases (B and C), and we define the enclosed flux by one complete revolution of ϕ from 0 to 2π at the approximately constant polar angle θ_{mid} :

$$\Phi(\theta_{\text{mid}}) = \pi B_0 R^2 [1 - \cos 2\theta_{\text{mid}}] \quad (23)$$

A. Ultrahigh Energy Orbits

In the limit of very high energy, i.e., $E/mc^2 \gg 1$, equation (22) becomes

$$P_\theta = \frac{|q|B_0 R^2}{2c} \times \sqrt{(2E/qB_0 R)^2 + (4E\Omega/qB_0 c) \sin \theta \sin(\theta - \theta_0) - [1/\sin^2 \theta - (\Omega R/c)^2] \sin^2 \theta \sin^2(\theta - \theta_0)}, \quad (24)$$

where P_ϕ has been neglected. That is, $\eta \sim \mathcal{O}(1)$ and $|l_\phi| \ll 1$. In Figure 1, we plot the right hand side versus $\cos \theta$ for zero tilt $\theta_0 = 0^\circ$ and various values of E . We find that, for values of energy η $\geq$ a critical value $\eta_c(\theta_0, \Omega R/c)$, there are four turning points enclosing two distinct allowed regions for the particle between the poles and the equator, with two of the four turning points very near the poles. (These polar turning points are nonzero because of the centrifugal effect of P_ϕ and vanish as P_ϕ vanishes.) As we increase the energy, the allowed regions expand and, for $\eta > \eta_c$, merge into a single region occupying essentially the whole surface of the sphere. In the latter case, two of the turning points merge and disappear. For zero tilt, $\theta_0 = 0^\circ$, the turning points in the ultrahigh energy limit are:

$$\sin^2 \theta_\pm = \frac{1 - (4E\Omega/qB_0 c) \pm \sqrt{1 - (8E\Omega/qB_0 c)^2}}{2(\Omega R/c)^2} \quad (25)$$

Only one of these roots, $\sin^2 \theta_-$, is physical in the limit $\Omega \rightarrow 0$: $\sin^2 \theta_- \rightarrow (2E/qB_0 R)^2$. If $\theta_0 = 0^\circ$ and $\Omega = 0$, then $\eta_c = 1$.

The tilt introduces an asymmetry in the size of the allowed regions around the two poles, as in Figure 2 ($\theta_0 = 45^\circ$). For extreme values of θ_0 , only one allowed region in θ appears.

B. Ultrahigh P_ϕ Orbits

Taking $|l_\phi|$ in this case to be $\sim \mathcal{O}(1)$, so that $P_\phi \sim \mathcal{O}(qB_0 R^2/2c)$, equation (22) can be re-expressed as

$$P_\theta/(mcR) = \{-1 + (E/mc^2)^2 - 2(\Omega R/c)(E/mc^2)(qB_0R/2mc^2)[l_\phi - \sin\theta \sin(\theta - \theta_0)] \\ - [1/\sin^2\theta - (\Omega R/c)^2](qB_0R/2mc^2)^2[l_\phi - \sin\theta \sin(\theta - \theta_0)]^2\}^{1/2} \quad (26)$$

From equation (26) we can see that the ultrahigh magnetic field introduces a very large negative quantity into the square root. We expand answers in inverse powers of β_0 . For a given P_ϕ , the allowed regions are very narrow in θ and, to $\mathcal{O}(1)$ in $1/\beta_0$, given by the two values of θ :

$$2\cos 2\theta_\pm = 1 + \cos 2\theta_0 - 4l_\phi \cos \theta_0 \\ \pm \sqrt{(1 + \cos 2\theta_0 - 4l_\phi \cos \theta_0)^2 - 4(\cos 2\theta_0 + 4l_\phi^2 - 4l_\phi \cos \theta_0)} \quad , \quad (27)$$

where the $\theta_\pm$ are defined over the range 0 to π , and $\eta \ll 1$ – that is, $E/(mc^2)$ has been neglected compared to l_ϕ . (These $\theta_\pm$ should not be confused with those of equation (25).) To $\mathcal{O}(1/\beta_0)$, the endpoints of the allowed regions are given by

$$\cos 2\theta = \cos 2\theta_\pm + \delta_\pm^{1,2}/\beta_0 \quad , \quad (28)$$

where

$$\delta_\pm^{1,2} = \{(\Omega R/c)(E/mc^2) \pm [((\Omega R/c)(E/mc^2))^2 + 4[1/\sin^2\theta_\pm - (\Omega R/c)^2][(E/mc^2)^2 - 1]]^{1/2}\} \\ \times \{2[\frac{\sin\theta_0 \cos 2\theta_\pm}{\sqrt{1 - \cos^2 2\theta_\pm}} - \cos\theta_0][1/\sin^2\theta_\pm - (\Omega R/c)^2]\}^{-1} \quad , \quad (29)$$

where the superscripts (1,2) refer to the plus-minus of the square root here, while the subscripts $\pm$ refer to the two angles $\theta_\pm$ of equation (27). Thus there are four endpoints in total, two for each allowed region.

Figures 3 and 4 show the two branches of $\cos 2\theta_\pm$ as functions of l_ϕ and θ_0 . Depending on the sign of l_ϕ and the sign and size of Ω , one of the two branches of $\theta_\pm$ may require ultrahigh energies, $\eta \sim \mathcal{O}(1)$, for P_θ^2 to be positive for a non-zero θ interval.

The values of P_ϕ are restricted by

$$(1 - \cos 2\theta_0) \frac{\cos \theta_0 - 1}{4 \sin^2 \theta_0} \leq 2cP_\phi/(qB_0R^2) \leq (1 - \cos 2\theta_0) \frac{\cos \theta_0 + 1}{4 \sin^2 \theta_0} \quad , \quad (30)$$

where the P_ϕ takes values between the two roots. We note that, in general, $|l_\phi| \leq 1$.

C. Localised Pole Orbits

Equation (22) can also be written, for $P_\phi \neq 0$, as

$$P_\theta/(mcR) = \{-1 + (E/mc^2)^2 - 2(\Omega R/c)(E/mc^2)(P_\phi/mcR)[1 - (qB_0R^2/2cP_\phi)\sin\theta\sin(\theta - \theta_0)] - [1/\sin^2\theta - (\Omega R/c)^2](P_\phi/mcR)^2[1 - (qB_0R^2/2cP_\phi)\sin\theta\sin(\theta - \theta_0)]^2\}^{1/2}. \quad (31)$$

For η and $|l_\phi| \ll 1$, allowed regions appear only if θ or $\theta - \theta_0 \rightarrow 0$. This case can be understood by extrapolating the ultrahigh energy and P_ϕ case to η and $l_\phi \rightarrow 0$. For the case of a pronounced tilt, the allowed regions are two narrow azimuthal annuli centered about the north and south rotational poles and one narrow azimuthal annulus slightly outside or inside the tilt angle (see Figure 5).

For nonzero tilt, the allowed regions in θ are, to $\mathcal{O}(l_\phi)$, situated about the two angles

$$\bar{\theta} = \begin{cases} -l_\phi/\sin\theta_0 & (l_\phi \leq 0) \\ \pi - l_\phi/\sin\theta_0 & (l_\phi \geq 0) \end{cases}, \quad (32)$$

and about the angle

$$\bar{\theta} = \theta_0 + l_\phi/\sin\theta_0. \quad (33)$$

The $\bar{\theta}$ of equation (33) differs from θ_0 by a small amount $\sim \mathcal{O}(l_\phi)$; l_ϕ is small but non-zero and can have either sign, so the third azimuthal annulus is not exactly at $\bar{\theta} = \theta_0$. The angular width of the allowed region is given by $\mathcal{O}(l_\phi^2)$ correction

$$\Delta\theta = 2(l_\phi/\sin\theta_0)^2 \sqrt{(E/mc^2)^2 - 1}. \quad (34)$$

When the tilt is absent, $\theta_0 = 0^\circ$, the turning points near the poles are given by

$$\begin{aligned} \theta &= \bar{\theta} \pm \Delta\theta/2 \\ &= \sqrt{l_\phi} \left[1 \pm \frac{1}{2} \sqrt{(2m^2c^3/qB_0P_\phi)[(E/mc^2)^2 - 1]} \right] \quad (l_\phi \geq 0), \end{aligned} \quad (35)$$

and the same replacing θ by $\pi - \theta$. In both cases, as the field grows, the turning points approach zero angle with the rotation axis and the annular width vanishes. (The $\mathcal{O}(l_\phi)$ effects are too small to show in Figure 5.) The magnetic flux enclosed by an orbit at the rotational poles is given by

$$\Phi(\theta \simeq 0, \pi) = 2\pi R^2 B_0 \bar{\theta}^2 \quad (36)$$

and the same replacing $\bar{\theta}$ by $\pi - \bar{\theta}$. Note $\bar{\theta}$ is given by equation (32) for the case of a tilt and equation (35) for the case of no tilt. If $\theta_0 \neq 0^\circ$, then $\Phi \propto l_\phi^2 \propto P_\phi^2/q^2$; if $\theta_0 = 0^\circ$, then $\Phi \propto l_\phi \propto P_\phi/q$.

IV. SEMICLASSICAL QUANTISATION

Although we do not carry out the full quantum analysis here, a semiclassical treatment brings out many of the desired features. For periodic orbits, quantisation is most easily implemented with the Wilson-Sommerfeld conditions [9]. This procedure is valid for θ quantisation as θ motion is a complete subsystem alone. Thus the closure or non-closure of orbits is irrelevant.

The classical analysis already produces particle orbits reminiscent of quantum crystals: quasi-free conduction bands (ultrahigh energies) and localised states (polar and azimuthal orbits). The azimuthal (ultrahigh P_ϕ) rings are localised in one, but not two, dimensions; they conduct along one direction but are trapped in the other. Two of the pole orbits are similarly localised, but are also confined in absolute position to be near the poles.

The classically allowed regions are those in which the quantum wavefunctions are oscillatory, rather than exponential. Because of the constraint of the Heisenberg uncertainty principle, the quantisation conditions, together with the classical spherical cyclotron relations, lead to a tightly constrained set of semiclassical orbits. In a uniform, planar magnetic system (the Landau problem), these orbits would be equivalent to harmonic oscillator states [9,10].

We now use Wilson-Sommerfeld quantisation. Since P_ϕ is conserved, the azimuthal quantisation is trivial:

$$P_\phi = n_\phi \hbar \quad (37)$$

For $P_\phi \gg \hbar$, n_ϕ is very large and the ϕ motion is essentially classical. We assume this to be the case, except very near the rotational poles; in that case, the turning points vanish as $P_\phi \rightarrow 0$, so that P_ϕ must in some sense be small.

A. Localised Pole States - Flux Quantisation

The W-S quantisation rule for θ motion gives

$$2 \int_{\theta_1}^{\theta_2} P_\theta d\theta = (n_\theta + 1/2)\hbar \quad (38)$$

for turning points θ_1 and θ_2 . This integral is evaluated here in a simple rectangular or triangular approximation (Simpson's rule). With equations (32-35, 38), we get for the case when the magnetic field is tilted at an angle θ_0 with respect to the rotation axis

$$E \simeq \sqrt{(mc^2)^2 + (n_\theta + 1/2)(\hbar/2P_\phi)(qB_0R)^2 \sin^2 \theta_0} \quad , \quad (39)$$

and for the case $\theta_0 = 0^\circ$,

$$E \simeq \sqrt{(mc^2)^2 + (n_\theta + 1/2)(\hbar cqB_0/4)} \quad . \quad (40)$$

Figure 6 shows E/mc^2 as a function of B_0 for $n_\theta = 0, 1, 2$ and $\theta_0 = 0^\circ$. The dimensions are set by the *critical field* B_c , the field strength with mc^2 of energy within a Compton cube of volume $(2\pi\hbar/mc)^3$: $B_c = (m^2/\pi)\sqrt{c^5/\hbar^3}$. Fields with this strength or greater introduce quantum field theoretic effects such as vacuum polarization.

The magnetic flux enclosed by a semiclassical orbit is quantised and is given by

$$\Phi(\theta \simeq 0, \pi) \simeq 2\pi B_0 R^2 \bar{\theta}^2 \quad . \quad (41)$$

For the case when the magnetic field is tilted at an angle θ_0 with respect to the rotation axis,

$$\Phi(\theta \simeq 0, \pi) \simeq \frac{8\pi c^2 P_\phi^2}{q^2 B_0 R^2 \sin^2 \theta_0} \propto n_\phi^2, \quad (42)$$

while

$$\Phi(\theta \simeq 0, \pi) \simeq \frac{4\pi c |P_\phi|}{|q|} \propto |n_\phi| \quad (43)$$

for the case $\theta_0 = 0$. Any n_θ dependence in Φ is a correction to (41) of relative order $\mathcal{O}(1/\beta_0)$.

B. Localised Azimuthal Rings

In the limit of high azimuthal angular momentum $|l_\phi| \sim \mathcal{O}(1)$, using equations (26-29, 38) and the quantisation rule, we obtain

$$E \simeq \sqrt{(mc^2)^2 + (n_\theta + 1/2)(\hbar c q B_0/4)(A/\sin \theta_\pm)} \quad , \quad (44)$$

where

$$A = \frac{\sin \theta_0 \cos 2\theta_\pm}{\sqrt{1 - \cos^2 2\theta_\pm}} - \cos \theta_0 \quad . \quad (45)$$

The magnetic flux enclosed by a semiclassical orbit is not interesting in the quantum regime, as it is macroscopically large.

C. Polar Conduction: Quasi-Continuum

We consider explicitly here only the case where $\theta_0 = 0^\circ$. The qualitative aspects are not greatly different if $\theta_0 \neq 0^\circ$. In the limit of ultrahigh energies using equations (24, 25, 38), we obtain

$$E \simeq \mathcal{C}(n_\theta + 1/2) (\hbar c/R) \quad , \quad (46)$$

where $\mathcal{C} \approx 4, 1$, or $1/2$ for $\eta \lesssim, \sim$, or $\gtrsim 1$. Note that the energies scale as harmonic oscillator energies, where the frequency is set by the size of the sphere, *not* the magnetic field. As in subsection III.A, we have assumed P_ϕ to be negligibly small. In realistic cases, $\hbar c/R \ll mc^2$

and E , so that the energy levels are very closely spaced. Such levels form a quasi-continuum that allows the charged particles to travel almost freely along the polar directions. For these ultrahigh energies, n_θ is so large that the motion is essentially classical. The energy levels depend only weakly on Ω for $\Omega R/c$ small.

The magnetic flux Φ is not interesting in this case, because the allowed region is a large part or all of the sphere. Flux quantization is not relevant because the energy is independent of the magnetic field B_0 . Only if the energy E is taken $\ll \mathcal{O}(|q|B_0 R/2)$ do B_0 and P_ϕ become important again; this is the pole orbit case.

D. Density of States

The density of states is important in any charged fermion system for determining the conductivity. The semiclassical density of states $d\mathcal{N}/dE$ is given by

$$d\mathcal{N} = \frac{d\theta \, dP_\theta \, d\phi \, dP_\phi}{(2\pi\hbar)^2} \quad (47)$$

and the differential surface area element is

$$dS = R^2 \sin^2 \theta \, d\theta \, d\phi \quad (48)$$

Thus we have for the the number of states per unit area per unit energy per unit azimuthal angular momentum

$$\frac{d^3\mathcal{N}}{dS \, dP_\phi \, dE} = \frac{dP_\theta}{dE} \left(\frac{1}{4\pi^2\hbar^2 R^2 \sin^2 \theta} \right) \quad (49)$$

Now let us write equation (22) as

$$(P_\theta/mcR) = [(E/mc^2)^2 - (E/mc^2)a_1(\theta; P_\phi, \Omega, \theta_0) - a_0(\theta; P_\phi, \Omega, \theta_0)]^{1/2} \quad (50)$$

with

$$\begin{aligned} a_0 &= 2(\Omega/mc^2)[P_\phi - (qB_0 R^2/2c) \sin \theta \sin(\theta - \theta_0)] \quad , \\ a_1 &= 1 + (1/mcR)^2 [1/\sin^2 \theta - (\Omega R/c)^2][P_\phi - (qB_0 R^2/2c) \sin \theta \sin(\theta - \theta_0)]^2 \quad (51) \end{aligned}$$

Therefore

$$\frac{dP_\theta}{dE} = (mcR/P_\theta)[E/mc^2 - a_1/2](R/c) \quad (52)$$

The factor dP_θ/dE is shown as a function of E , in Figure 7 for $\theta_0 = 0^\circ$ and 45° with $|l_\phi| \ll 1$; and in Figure 8 for $|l_\phi| \sim 1$. For low energies, $E \rightarrow mc^2$, this function approaches

$$[1 - a_1 - a_0]^{-1/2} \left\{ 1 - a_1/2 - \frac{[a_0 + a_1^2/4]}{1 - a_1 - a_0} (K/mc^2) \right\}, \quad (53)$$

where $K = E - mc^2$. In the ultrarelativistic limit, $E \gg mc^2$ and/or $\eta \gtrsim 1$, dP_θ/dE approaches the constant R/c .

In the quantum limit,

$$d\mathcal{N} = dn_\theta \left(\frac{d\phi}{2\pi\hbar} \frac{dP_\phi}{dE} \right), \quad (54)$$

treating P_ϕ as classical. Equation (50) becomes:

$$\frac{d^3\mathcal{N}}{dS dP_\phi dE} = \frac{dn_\theta}{dE} \left(\frac{1}{2\pi\hbar R^2 \sin^2 \theta} \right) \quad (55)$$

The function dn_θ/dE can be obtained from equations (39-40), (44), or (46). For the localised pole states of equations (39-40), take $E^2 = (mc^2)^2 + (n_\theta + 1/2)\varepsilon_1^2$, where $\varepsilon_1^2 = \varepsilon_1^2(qB_0, R, P_\phi, \theta_0)$. Then

$$\frac{dn_\theta}{dE} = \frac{2E}{\varepsilon_1^2} \quad (56)$$

for $E^2 \geq (mc^2)^2 + \varepsilon_1^2/2$.

V. SUMMARY — OUTSTANDING QUESTIONS

We have examined only the simplified scenario of charged particles on a rotating spherical surface with an intense dipole magnetic field. There are a number of outstanding issues in this idealised situation not treated here. The first is the radial motion. In a rotating frame, the field is only a static magnetic dipole, but a charged particle feels an effective induced

electric field $(\boldsymbol{\Omega} \times \mathbf{r}) \times \mathbf{B}$ created by the rotation. The effect of this induced field is already included in our formulation, but the full effect requires including the radial motion and radial dependence of the vector potential $\mathbf{A}$. For charged particles outside of the neutron star, in the magnetosphere, the result is well known [11]: the induced field tears some of the charges off the surface and swirls them within the lightsphere about the star, creating a plasma magnetosphere. Charged particles can also escape along the rotation axis with ultrahigh energies $E \sim \beta_0 mc^2$ [1,12]. The second outstanding issue in the idealised situation is the full quantization using the Dirac equation and the inclusion of spin, which we hope to address in a future publication.

To apply these results to a neutron star demands further realism, which will be addressed in future work. In particular, the effect of the crystal of nuclei on non-localised electrons and nuclei must be included [13,4,2]. The crystal structure of a neutron star crust must to some extent be disordered in its surface layers, although the fermion temperature is small compared to the Fermi energy. These crystal effects modify the magnetosphere charge cloud, the conduction bands, and the flux quantization. The crystal must be included to give a full picture of the crustal charge and current distribution. The magnetic field itself is also modified if the charge currents contribute to it.

Finally, observational questions remain. These include verification, if possible, of the flux quantisation and determination of what roles it and the special conduction band structure play in the formation of a neutron star's magnetic field, crust, and interior.

ACKNOWLEDGMENTS

At the University of Florida, we thank Bernard Whiting for helpful suggestions and Pradeep Kumar (U.S. National Science Foundation) for the original suggestion for this problem. This work was supported by the Institute for Fundamental Theory and the U.S. Department of Energy Contract No. DE-FG05-86-ER40272 (Univ. Florida).

REFERENCES

* E-mail: gopi@phys.ufl.edu.

† E-mail: kennedy@phys.ufl.edu.

‡ E-mail: gelb@uta.edu.

- [1] M. S. Longair, *High Energy Astrophysics, Volume 2: Stars, the Galaxy, and the Interstellar Medium*, 2nd ed. (Cambridge University Press, Cambridge, 1992).
- [2] S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars: The Physics of Compact Objects* (John Wiley & Sons, New York, 1983).
- [3] F. C. Michel, *Theory of Neutron Star Magnetospheres* (University of Chicago Press, Chicago, 1991).
- [4] D. J. Helfand and J.-H. Huang, editors, *The Origin and Evolution of Neutron Stars: Proc. 125th Symp. IAU, Nanjing, China* (D. Reidel Publishing, Dordrecht, 1987).
- [5] E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison-Wesley, Reading, MA, 1991).
- [6] W. Greiner, B. Müller, and J. Rafelski, *Quantum Electrodynamics of Strong Fields* (Springer-Verlag, Berlin, 1985).
- [7] L. D. Landau and E. M. Lifshitz, *Classical Theory of Fields*, translated by M. Hamermesh (Butterworth-Heinemann, Oxford, 1975).
- [8] A. G. Lyne and F. Graham-Smith, *Pulsar Astronomy* (Cambridge University Press, Cambridge, 1990).
- [9] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory*, 3rd rev., enl. ed., translated by J. B. Sykes and J. B. Bell (Pergamon Press, Oxford, 1977).
- [10] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics*, 2nd

ed., translated by J. B. Sykes and J. S. Bell (Pergamon Press, Oxford, 1982).

[11] P. Goldreich and W. H. Julian, *Astrophys. J.* **157**, 839 (1969).

[12] V. L. Ginzburg and S. I. Syrovatskii, *The Origin of Cosmic Rays*, translated by D. ter Haar, edited by H. S. H. Massey (Pergamon Press, Oxford, 1964).

[13] M. A. Ruderman and P. G. Sutherland, *Astrophys. J.* **196**, 51 (1975).

FIGURES

FIG. 1. Ultrahigh energy orbits: $P_\theta(\theta)$ for $\theta_0 = 0^\circ$ and energies of $\eta = 0.5 - 4.0$, in units of $\beta_0 \eta m c R$.

FIG. 2. Ultrahigh energy orbits: $P_\theta(\theta)$ for $\theta_0 = 45^\circ$ and energies of $\eta = 0.5 - 4.0$, in units of $\beta_0 \eta m c R$.

FIG. 3. Ultrahigh P_ϕ orbits: two allowed annuli with central angles of θ_+ (solid) and θ_- (dashed) as functions of l_ϕ for various values of θ_0 .

FIG. 4. Ultrahigh P_ϕ orbits: two allowed annuli with central angles of θ_+ (solid) and θ_- (dashed) as functions of $\cos \theta_0$ for various values of l_ϕ .

FIG. 5. Localised pole orbits: narrow allowed annuli (darkened circles) near rotational poles $\theta = 0$ and π and near magnetic pole cone $\theta = \theta_0$, shown on sphere of radius R . Rotation axis defined by Ω and dipole axis by magnetic dipole moment M .

FIG. 6. Quantised pole states: energy $E/(mc^2)$ as a function of pole field strength B_0 for quantum states $n_\theta = 0, 1, 2$ and $\theta_0 = 0^\circ$. For the electron, magnetic field units in $B_c = (m_e^2/\pi)\sqrt{c^5/\hbar^3} = 1.3 \times 10^{12}$ Gauss and $q^2/(\hbar c) = 1/137$.

FIG. 7. Density of states factor dP_θ/dE , in units of R/c , as a function of energy η for various θ values and $|l_\phi| \ll 1$. (a) $\theta_0 = 0^\circ$. (b) $\theta_0 = 45^\circ$.

FIG. 8. Density of states factor dP_θ/dE , in units of R/c , as a function of energy $E/(mc^2)$ for $l_\phi = -0.1$, $\theta_0 = 45^\circ$, and the two branches $\theta_\pm$.

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5

Figure 6

Figure 7a

Figure 7b

Figure 8