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Generalized Superconducting Flows —
Plasma Confinement, Organization

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Generalized superconducting flows—plasma confinement, organization

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Abstract

Complete expulsion of magnetic vorticity is used to characterize the superconducting flow. It is shown that a simple, intuitive, but speculative generalization can serve as a paradigm for a variety of organized flows.

This letter is of a speculative nature. After discussing and reformulating the defining features of the well-known superconducting flow embodied in the London equation¹ (leading to the Meissner effect), we will propose a generalization, and then calculate a few interesting consequences of the generalizing ansatz.

The basic system consists of two ideal, charged, incompressible fluids with uniform density evolving according to

$$\frac{\partial \mathbf{u}_\alpha}{\partial t} + (\mathbf{u}_\alpha \cdot \nabla) \mathbf{u}_\alpha = \frac{q_\alpha}{m_\alpha} \left(\mathbf{E} + \frac{\mathbf{u}_\alpha \times \mathbf{B}}{c} \right) - \nabla \hat{p}_\alpha \quad (1)$$

where $\alpha = 1, 2$, is the species index, $\mathbf{u}_\alpha, q_\alpha, m_\alpha$ are respectively the velocity, charge, and mass of the α -th species, and $\nabla \hat{p}_\alpha$ represents all the gradient forces (pressure, for example). The charge neutrality of the system allows only inductive electric fields. For a sharper focus, let us assume that the two fluids are the electrons ($m_e = m, q_e = -e$) and the ions ($m_i = M, q_i = e$) with $m/M \ll 1$. For simplicity, we further assume that the fluid density n_0 is uniform. Using the standard vector identity on the convective term, Eq. (1) can be rewritten as

$$\frac{\partial \mathbf{u}_\alpha}{\partial t} + \nabla \left[\frac{u_\alpha^2}{2} + \hat{p}_\alpha \right] = \frac{q_\alpha}{m_\alpha c} \mathbf{u}_\alpha \times \boldsymbol{\Omega}_\alpha \quad (2)$$

where

$$\boldsymbol{\Omega}_\alpha = \mathbf{B} + \frac{m_\alpha c}{q_\alpha} \nabla \times \mathbf{u}_\alpha \quad (3)$$

is the generalized vorticity or the generalized magnetic field. Taking the curl of Eq. (2), and using Faraday's law

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad (4)$$

one finds

$$\frac{\partial \boldsymbol{\Omega}_\alpha}{\partial t} = \nabla \times [\mathbf{u}_\alpha \times \boldsymbol{\Omega}_\alpha], \quad (5)$$

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the precise equation of motion for the fluid vorticity in hydrodynamics. This provides the rationale for calling Ω_α , the generalized vorticity. Notice that both sides of Eqs. (5) are perfect curls; uncurling this equation simply leads back to Eq. (2), which it replaces.

We now proceed to construct an equivalent set of equations in one-fluid variables. In addition to the standard variables, the fluid velocity $\mathbf{V} = (m + M)^{-1}[M\mathbf{u}_i + m\mathbf{u}_e]$, and the current density $\mathbf{J} = n_0 e[\mathbf{u}_i - \mathbf{u}_e]$, we introduce two new essential quantities: The magnetic vorticity

$$\Omega_m \equiv \frac{M\Omega_e + m\Omega_i}{m + M} = \mathbf{B} + \frac{4\pi}{c} \lambda^2 (\nabla \times \mathbf{J}) \quad (6)$$

where $\lambda = c/\omega_{pe}$ ($\omega_{pe}^2 = 4\pi n_0 e^2(1 + m/M)/m$) is the collisionless skin depth, and the kinetic vorticity

$$\Omega_k = \Omega_i - \Omega_e = \frac{(M + m)c}{e} \nabla \times \mathbf{V} \quad (7)$$

whose respective evolutions are governed by

$$\frac{\partial \Omega_m}{\partial t} = \nabla \times \left[\left(\mathbf{V} - \frac{\mathbf{J}}{n_0 e} \right) \times \Omega_m \right], \quad (8)$$

and

$$\frac{\partial \Omega_k}{\partial t} = \nabla \times \left[\mathbf{V} \times \Omega_k + \frac{\mathbf{J}}{n_0 e} \times \Omega_m \right]. \quad (9)$$

Equations (6)–(9) follow readily from various definitions including Eq. (5). A term of order m/M was dropped to obtain (8). It should be emphasized that these equations are not the equations of standard magnetohydrodynamics (MHD). In MHD, $\lambda = 0$ and $\mathbf{V} \gg \mathbf{J}/n_0 e$. In the systems under consideration, neither of these may pertain.

Derivation of Eqs. (8)–(9) was motivated by a desire to separate, as far as possible, the magnetic from the kinetic aspects of the problem. Evidently, the program is only partially successful; in Eq. (8), we still have the kinetic remnants through the presence of the bulk velocity $\mathbf{V}$, while in Eq. (9), the kinetic vorticity is driven both by the kinetic and the magnetic vorticities. In spite of this apparent failure, there is a profound message that can

be extracted from the structure of Eq. (8). Of all the values Ω_m can take,

$$\Omega_m = \mathbf{B} + \frac{4\pi}{c} \lambda^2 \nabla \times \mathbf{J} = 0 \quad (10)$$

is special; it not only implies the conservation of $\Omega_m = 0$, it also tells us that if at any arbitrary time, Ω_m acquired the value zero, it would remain so for all times. The perceptive reader must have already noted that $\Omega_m = 0$ is not a trivial condition; in fact, it is the famous London equation (his-first-equation) invoked in order to explain the Meissner-Ochsenfeld effect in superconductors.

Thus by simply examining the classical electrodynamics of a perfect conductor (an ideal fluid), we have hit upon a defining criterion of a superconductor—a superconductor is that perfect conductor which cannot sustain any magnetic vorticity in its entire body including the regions near the surface! The magnetic field, itself, will, naturally, be nonzero in the surface regions of length $\sim \lambda$, the collisionless skin depth. The complete expulsion of magnetic vorticity should replace perfect diamagnetism, as the fundamental expression of the superconducting state.

It should be immediately stressed that we have not derived superconductivity from classical electrodynamics. This is important because such a claim was made by W.F. Edwards (Phys. Rev. Lett. 47, 1863(1981)). The Edwards paper was, however, shown to be incorrect later on (see J.B. Taylor, Nature 29 681(1982)). In this paper, We have simply recognized a physical quantity Ω_m whose absence implies superconductivity. In the conventional superconductors, it is the quantum correlations² alone which provide the initial condition $\Omega_m = 0$. Once it happens, the dynamics of perfect conductors guarantees its remaining zero forever.

Before getting back to the main theme of this paper, it is worthwhile to make another comment on the London equation. For a standard superconductor (fixed ions, mobile electrons), the first London equation $\Omega_m \equiv \Omega_e = 0$ simplifies Eq. (1) to

$$\frac{\partial \mathbf{u}_e}{\partial t} = -\frac{e}{m} \mathbf{E} \quad (11)$$

for a system in thermodynamical equilibrium provided, $\nabla(u_e^2/2)$, which is small for typical velocities, can be neglected. Equation (11), which is called the second London equation (invoked to explain persistent currents in superconducting rings), therefore, is not an additional ansatz (as perceived by London), it is a simple consequence of the first London equation coupled with the dynamics of a perfect conductor. Equation (11) is exact in the directions perpendicular to the gradients which are normally in the radial direction for cylindrical systems.

The systems, for which $\Omega_m = 0$, will be henceforward called superconducting flows. These flows are governed by a simple magnetic constitutive relation [Eq. (10)] which, when substituted in the Maxwell's equation, yields

$$\nabla \times \nabla \times \mathbf{B} + \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = \frac{4\pi}{c} \nabla \times \mathbf{J} = -\frac{\mathbf{B}}{\lambda^2}, \quad (12)$$

an equation which is linear in $\mathbf{B}$. Thus the London hypothesis, or equivalently the vanishing of magnetic vorticity, converts the horribly nonlinear electrodynamics into effectively a single linear equation! [Notice that for $\Omega_m \equiv 0$, the fluid kinetic Eq. (9) is completely independent of the magnetic quantities, and has to be solved separately]. It is to be emphasized that although Eq. (12) is linear, it is not the result of a conventional linearization process. The implication is that a superconducting flow can sustain finite amplitude monochromatic electromagnetic waves with the linear dispersion relation

$$\frac{\omega^2}{c^2} = \mathbf{k}^2 + \frac{1}{\lambda^2}, \quad (13)$$

i.e. the photon will be vacuum-like but with a real effective mass ($= \hbar\omega_{pe}/c^2$). To the best of our knowledge, this result has not been experimentally tested.

In spite of its tremendous important and its unique theoretical status, the superconducting flow spans a set of measure zero. In the know physical systems, it is naturally realized only in some special materials, under special conditions, with quantum mechanics lending a

big helping hand. Is it possible to find a class or classes of flows which can occur under less stringent conditions and which still retain some of the attractive features of the original? Attempts to answer this question comprises the main thrust of this effort.

We believe that the formalism presented earlier demonstrated that the magnetic vorticity Ω_m emerges as an essential label for the categorization of electromagnetic flows. Let us then, first seek flows for which Ω_m , though not zero, is still a constant of motion. Equation (8) tells us that if we can find a function $\chi(\mathbf{x}, t)$ such that

$$\left(\mathbf{V} - \frac{\mathbf{J}}{n_0 e} \right) \times \Omega_m = \nabla \chi, \quad (14)$$

$\Omega_m = \Omega_m(\mathbf{x})$ becomes a constant of the motion and can be assumed to be a specified function of space. For this case $4\pi/c(\nabla \times \mathbf{J}) = (\Omega_m(\mathbf{x}) - \mathbf{B})/\lambda^2$, and the Maxwell equation becomes

$$\nabla \times \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = -\frac{\mathbf{B} - \Omega_m(\mathbf{x})}{\lambda^2}, \quad (15)$$

again a linear, but an inhomogeneous equation with the inhomogeneous term given by the initial conditions. The electrodynamics is now converted to a sequence of much simpler inhomogeneous equations, and can be readily solved with the aid of a computer.

It turns out, however, that Eq. (14) is also quite restrictive. Barring one-dimensional flows (for which it can always be satisfied), it is generally quite impossible to satisfy the consistency constraints on $\chi(\mathbf{x}, t)$.

Before attempting the next generalization, we pause to re-examine the superconducting state. This state is brought out by an extremely high degree of correlations induced by quantum mechanical processes in the electron fluid in the vicinity of the Fermi surface. The most glaring electrodynamic manifestation of this highly correlated charged fluid is the complete expulsion of magnetic vorticity; it is, thus, possible to interpret that magnetic vorticity may be a measure of the lack of correlation or organization in the fluid. We then speculate that a variety of correlated or organized flows, to be called generalized superconducting flows

(GSF), could be described by the magnetic constitutive relation

$$\mathbf{\Omega}_m = \mathbf{B} + \frac{4\pi}{c} \Lambda^2 \nabla \times \mathbf{J} = \alpha \mathbf{B} \quad (16)$$

where α is a constant, and $\alpha = 0$ represents the perfectly organized superconducting flows. We also assume that the kinetic part of the flow has been suitably chosen to be consistent with (16). The choice (16) is dictated by a whole lot of intuitive and plausible arguments: 1) magnetic vorticity should be determined only in terms of other magnetic quantities ($\mathbf{J}$ and $\mathbf{B}$), 2) it is the simplest generalization of $\mathbf{\Omega}_m = 0$ (Ockham's razor), 3) For a one-dimensional flow, it can be derived from a variational principle in which the magnetic energy $\int B^2/2 dV$ is minimized using the constancy of $\int \mathbf{\Omega}_m^2 dV$ as a constraint. However, none of these arguments is either telling or rigorous, and at this stage we would like to view Eq. (16) as a purely speculative generalization of the London equation. It is the interesting and rather reasonable implications of (16) that impart meaning to this speculation. We shall allow α to differ greatly from zero in later considerations.

Equation (16) may be derived for a resistive plasma by minimizing $\int (B^2/2 + \eta J^2) dv$, where η is the resistivity. But this is not relevant to perfect conductors—and it is for perfect conductors that (16) has been proposed.

Inserting (16) in the Maxwell equation, we obtain the defining equation for the GSF

$$\nabla \times (\nabla \times \mathbf{B}) - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = \frac{\alpha - 1}{\Lambda^2} \mathbf{B} \quad (17)$$

which is a simple, yet profound, modification of (12). The dispersion relation for the monochromatic photon is now

$$\frac{\omega^2}{c^2} = \mathbf{k}^2 + \frac{1 - \alpha}{\Lambda^2} \quad (18)$$

which has an effective real mass as long as $\alpha < 1$. For $\alpha > 1$, the effective mass becomes imaginary, and the light wave can become unstable. We will soon see that $\alpha = 1$ is a bifurcation point which separates the diamagnetic ($\alpha < 1$) from the paramagnetic ($\alpha > 1$) flows.

Let us now consider quasistatic phenomena with the idea of exploring the usefulness of (17) for confined laboratory flows. Dropping $\partial/\partial t$ in (17), we arrive at

$$\nabla^2 \mathbf{B} + \frac{\alpha - 1}{\lambda^2} \mathbf{B} = 0, \quad (19)$$

the equation determining the spatial structure of the magnetic fields consistent with our ansatz. We first notice that in the quasistatic superconducting flow ($\alpha = 0$) the length scale of field variation is λ , which for normal to high density plasmas is rather small, several order of magnitudes smaller than the typical length scales associated with confined laboratory plasmas. For a typical superconductor, $\lambda \sim 10^{-4} - 10^{-6} \text{ cm}$ is so small, that the magnetic field could, indeed, be thought to be zero throughout the sample, it rises abruptly to its vacuum value in a layer of size λ .

For the GSF, we have now an additional parameter α (to be viewed as a parameter to be determined by experiment) which can change the length scale away from λ . For confined laboratory plasmas, since the length scale $L \gg \lambda$, the corresponding α has to be extremely close to unity. For better visualization, let us consider an extremely simple example: a cylindrical flow with variation only along the radial direction. For this kind of one-dimensional flow, Eq. (16) can be derived, as mentioned earlier, by a variational principle. The nonzero components of $\mathbf{B}$ [$B_r = 0$ because of $\nabla \cdot \mathbf{B} = 0$] are determined from

$$\frac{1}{r} \frac{d}{dr} r \frac{db_z}{dr} \mp \mu^2 B_z = 0 \quad (20)$$

$$\frac{d}{dr} \frac{1}{r} \frac{d \cdot r B_\theta}{dr} \mp \mu^2 B_\theta = 0 \quad (21)$$

where $\mu = \lambda/|\alpha - 1|^{1/2}$ is the new scale length and the $-(+)$ sign pertains to the diamagnetic (paramagnetic) flows. The former set allows a general solution (finite at origin)

$$\mathbf{B} = \left[a_- I_0 \left(\frac{r}{\lambda} \right) \hat{z} + b_- I_1 \left(\frac{r}{\lambda} \right) \hat{\theta} \right] \quad (22)$$

where I_s are the modified Bessel function of the first kind. Equation (22) shows that the fields grow away from the center—the expected diamagnetic behavior. Naturally as $\alpha \rightarrow 0$,

$\mu \rightarrow \lambda$, and for a cylindrical superconductor of radius a immersed in an external magnetic field $\mathbf{B}_{ex} = \hat{z} B_0$, the field inside will be given by

$$B_z = B_0 \frac{I_0(r/\lambda)}{I_0(a/\lambda)}. \quad (23)$$

Since $(a/\lambda) \gg 1$, $B_z \sim 0$ except in the close vicinity of $r \simeq a$.

For a plasma to be confined, i.e. for it to maintain a pressure gradient (pressure peaking near $r = 0$, and decreasing with r), one needs to create a radially inward force. For high temperature plasmas, the pressure gradient is determined by

$$\nabla P \simeq \frac{1}{c} (\mathbf{J} \times \mathbf{\Omega}_m) = \frac{\alpha}{c} \mathbf{J} \times \mathbf{B} \quad (24)$$

implying that confinement is possible if $\mathbf{J}$ and $\mathbf{B}$ are not parallel to one another. For the fields given by Eq. (22),

$$\mathbf{J} = \frac{c}{4\pi} (\nabla \times \mathbf{B}) = \frac{c}{4\pi} \left[b_- I_0 \left(\frac{r}{\mu} \right) \hat{z} - a_- I_1 \left(\frac{r}{\mu} \right) \hat{\theta} \right], \quad (25)$$

and therefore

$$\frac{dP}{dr} = -\frac{\alpha}{\mu 4\pi} [b_-^2 + a_-^2] I_0 \left(\frac{r}{\lambda} \right) I_1 \left(\frac{r}{\lambda} \right) = -\frac{\alpha}{\delta_0} [a_-^2 + b_-^2] \frac{d}{dr} I_0 \left(\frac{r}{\lambda} \right) \quad (26)$$

which tells us that relatively large pressure gradients are allowed. Equation (26) reveals that the diamagnetic GSF flow always leads to confinement, i.e. a negative pressure gradient can be maintained by the current-magnetic field geometry.

Let us now turn our attention to the flows which reside on the other hand side of the bifurcation point $\alpha = 1$. Although this class of flows ($\alpha \gtrsim 1$) has little in common with the superconducting flow, we shall treat them as a part of the generalized superconducting flows because they are also defined by Eq. (16). Let us continue with our simple one-dimensional geometry. For $\alpha > 1$, Eqs. (20)–(21) can be solved to obtain $[\mu^2 = (\alpha - 1)/\lambda^2, B_r = 0]$

$$\mathbf{B} = \left[\hat{z} a_+ J_0 \left(\frac{r}{\mu} \right) + \hat{\theta} b_+ J_1 \left(\frac{r}{\mu} \right) \right] \quad (27)$$

where J_s are the ordinary Bessel functions. The self-consistent current

$$\mathbf{J} = \frac{c}{4\pi} \left[\hat{z} b_+ J_0 \left(\frac{r}{\mu} \right) + \hat{\theta} a_+ J_1 \left(\frac{r}{\mu} \right) \right], \quad (28)$$

when crossed with $\mathbf{B}$ [Eq. (27)], yields the pressure gradient

$$\frac{dP}{dr} = \frac{\alpha}{4\pi\mu} [a_+^2 - b_+^2] J_0 \left(\frac{r}{\mu} \right) J_1 \left(\frac{r}{\mu} \right). \quad (29)$$

Unlike Eq. (26), which always leads to a negative pressure gradient, Eq. (29) can predict either of the following:

- (I) For $|a_+| = |b_+|$, there is no radial force and the corresponding magnetic field cannot support a pressure gradient. This magnetic field configuration is called a 'relaxed state,' and has been derived and discussed by J.B. Taylor.³ It is derived by minimizing the magnetic with the constraint that the total magnetic helicity be held fixed. The formulation leads to the magnetic constitutive relation [analogue of Eq. (16)] $\nabla \times \mathbf{B} = \mu \mathbf{B}$. This relation relates B_θ and B_z , and the independence of choosing different constants of integration a_+ and b_+ does not exist.
- (II) For $a_+^2 > b_+^2$, the radial force is positive (up to the 1st zero of J_0 , and inverted pressure profiles, (the pressure peaks away from the center of the system) are predicted.
- (III) For $b_+^2 > a_+^2$, negative pressure gradients and a confined system is the resulting solution. For this solution to be a meaningful confinement paradigm, $a/\mu \sim z_0$, where a is the size of the system and z_0 is the first zero of the Bessel function J_0 . Notice that for all cases, B_z can change sign (field reversal) if $a/\mu > z_0$.
- (IV) Because of the oscillatory nature of the Bessel function, non-monotonic structures with complicated profiles $(\mathbf{B}, \mathbf{J}, P)$ can occur within the framework of this model. Such magnetic field configurations may be of considerable relevance in analyzing flows of astrophysical origin.

It is quite straightforward to solve Eq. (19) or its time dependent counterpart [Eq. (17)] in higher dimensions and in complicated geometries. This letter, however, is meant to convey only the scope of the generalizing ansatz.

By analyzing a simple quasistatic test case, we have demonstrated the richness of phenomena which fall within the realm of the generalized superconducting flows. There are two physically as well as topologically distinct magnetic field structures that emerge from the generalizing assumption: the diamagnetic structures with the fields increasing from the center, and the paramagnetic with the fields peaking at or near the center and decreasing towards the end of the system. The only difference between these various flows (apart from the transition at $\alpha = 1$ from the diamagnetic to the paramagnetic) is that of the defining length scale determined by a parameter α (to be fixed by experiment). The length scale changes from the rather small collisionless skin depth λ for the purely superconducting flow ($\alpha = 0$) to arbitrary large length scales in the vicinity of $\alpha = 1$, the bifurcation point. Since the superconducting flow represents a supremely ordered flow (due to the immense correlation and organization provided by quantum mechanics), it may be legitimate to conjecture that the mathematically equivalent generalized superconducting flows with their self-consistent structures for the current, the magnetic field, and the pressure can serve as a paradigm for high correlations and organization brought about by nonlinear interactions. One could even be bolder and speculate that, starting from suitable initial conditions, highly interacting, maximally organized nonlinear systems tend to evolve more and more towards 'effectively' linear systems, and the asymptotic states of such flows could be described by effective linear theories.

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