CONF-970678--1

SAND97-0730C

Fractal Functions and Multiwavelets

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Abstract

This paper reviews how elements from the theory of fractal functions are employed to construct scaling vectors and multiwavelets. Emphasis is placed on the one-dimensional case, however extensions to $\mathbb{R}^m$ are indicated.

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1 Construction in $\mathbb{R}$

Let $\Omega := [0, 1)$ and let $\mathbb{Z}_N := \{0, 1, \dots, N-1\}$ with $1 < N \in \mathbb{N}$. For $\ell \in \mathbb{Z}_N$, let $\lambda_\ell \in P_d(\Omega)$, the space of real polynomials of degree at most d on Ω , and $s_\ell \in (-1, 1)$. Set $\lambda := (\lambda_\ell)_{\ell \in \mathbb{Z}_N} \in \Theta_d(\Omega) := \prod_{\ell \in \mathbb{Z}_N} P_d(\Omega)$ and $s := (s_\ell)_{\ell \in \mathbb{Z}_N}$. Then the operator $B = B(\lambda, s) : L^\infty(\Omega) \rightarrow L^\infty(\Omega)$ defined by

$$Bf := \sum_{\ell \in \mathbb{Z}_N} (\lambda_\ell(N \cdot -\ell) + s_\ell f(N \cdot -\ell)) \chi_{[\ell/N, (\ell+1)/n)}. \quad (1)$$

is contractive on $(L^\infty(\Omega), \|\cdot\|_{L^\infty})$, and its unique fixed point $f^* = f^*_{\lambda, s}$ satisfies

$$f^* = \Lambda + \sum_{\ell \in \mathbb{Z}_N} s_\ell f^*(N \cdot -\ell). \quad (2)$$

Here $\Lambda : \mathbb{R} \rightarrow \mathbb{R}$ is defined by $\Lambda|_{[\ell/N, (\ell+1)/n)} := \lambda_\ell(N \cdot -\ell)$, $\ell \in \mathbb{Z}_N$, and $\Lambda \equiv 0$ on $\mathbb{R} \setminus \Omega$. (Note that f^* has been extended to all of $\mathbb{R}$ by setting $f^* \equiv 0$ outside Ω .) If, in addition, the *interior boundary conditions* $\lambda_{\ell+1}(0) + s_{\ell+1}f(0) = \lambda_\ell(1-) + s_\ell f(1-)$, $\ell \in \mathbb{Z}_N$, hold, then B is an operator on $L^\infty(\Omega) \cap C(\bar{\Omega})$.

The function f^* is called a *fractal function*. The graph of f^* is made up of a finite number of copies of itself and is, in general, a *fractal set* (cf. [1, 6]). This above-defined class of fractal functions is denoted by $\mathcal{P}_d(\Omega)$.

Equations of type (2) are referred to as *inhomogeneous refinement equations* since the function f^* is expressed in terms of a refined version $f^*(N \cdot -\ell)$ of itself.

The next theorem gives the relationship between λ and the fractal functions $f^* \in \mathcal{P}_d(\Omega)$.

¹Research partially supported by NSF grant DMS-9503282. This work was also supported by Sandia National Laboratories. Sandia is a multiprogram laboratory operated by Sandia Cooperation, a Lockheed Martin Company, for the United States Department of Energy (DOE) under Contract DE-AC04-94AL85000.

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Theorem 1.1 *The mapping $\vartheta : \Theta_d(\Omega) \rightarrow L^\infty(\Omega)$, $\lambda \mapsto f_\lambda^*$, is a linear isomorphism.*

The interested reader may consult [4] or [6] for a proof.

Indeed, more can be shown if $d = 1$ and $f^* \in L^\infty(\Omega) \cap \tilde{C}_R(\Omega)$, where $\tilde{C}_R(\Omega) := \{f|_\Omega : f \in C(\bar{\Omega})\}$. Since in this case each vector $\mathbf{y} := (y_0, y_1, \dots, y_N)^T \in \mathbb{R}^{N+1}$ uniquely determines a $\lambda \in \Pi_{\ell \in \mathbb{Z}_N} P_1(\Omega) \subseteq \Theta_1(\Omega)$, and vice versa, the mapping $\mathbb{R}^{N+1} \ni \mathbf{y} \mapsto \lambda(\mathbf{y}) \in \Theta_1(\Omega)$ is a linear isomorphism. Hence, the following result holds.

Theorem 1.2 *The mapping $\eta : \mathbb{R}^{N+1} \rightarrow L^\infty(\Omega) \cap \tilde{C}_R(\Omega)$, $\mathbf{y} \mapsto f_{\lambda(\mathbf{y})}^*$, is a linear isomorphism.*

This result provides a geometric interpretation of fractal functions: Every fractal function $f^* \in \mathcal{P}_1(\Omega)$ is uniquely determined by an interpolation vector $\mathbf{y} \in \mathbb{R}^{N+1}$, i.e., by its values at a given set of knot points $\{x_i\}_{i=0}^N$. Hence the linear space $\mathcal{P}_1(\Omega)$ of fractal functions on Ω is $N+1$ dimensional and a canonical basis is given by the fractal functions $f_i^* := \eta(\mathbf{y}_i)$, where $\mathbf{y}_i(j) = \delta_{ij}$, $i, j = 0, 1, \dots, N$.

The fractal functions in $\mathcal{P}_d(\Omega)$ may be used to construct finitely generated shift-invariant subspaces of $L^2(\Omega)$ (cf., for instance, [4, 6]). Recall that one of the main features of a fractal function $f^* \in \mathcal{P}_d(\Omega)$ is that $f^*(\cdot/2)$ restricted to Ω is again a fractal function. More precisely, if f^* is generated by $\lambda \in \Theta_d(\Omega)$, then $f^*(\cdot/2)$ restricted to $[\ell/2, (\ell+1)/2)$ is generated by $\lambda(\ell) := (\lambda_{\ell, \ell'})_{\ell' \in \mathbb{Z}_N} \in \Theta_d(\Omega)$, where $\lambda_{\ell, \ell'} := \lambda_\ell([\cdot - \ell']/N) + s_{\ell'} \lambda_{\ell'} - s_\ell \lambda_\ell$, $\ell, \ell' \in \mathbb{Z}_N$. This observation allows the construction of *piecewise fractal functions*. For a given $i \in I \subseteq \mathbb{Z}$, construct a fractal function $f_i \in \mathcal{P}_d([i, i+1))$. Denote by $\lambda(i) := (\lambda_{i, \ell})_{\ell \in \mathbb{Z}_N}$ the corresponding element in $\Theta_d([i, i+1))$. Let $\mathbf{f} := \{f_i : i \in I\}$. Every sequence $\lambda := \{\lambda(i) : i \in I\} \in \prod_{i \in \mathbb{Z}} \Theta_d([i, i+1))$ defines a $\mathbf{f} \in \prod_{i \in I} L^\infty([i, i+1))$. An element $\mathbf{f} \in \prod_{i \in I} L^\infty([i, i+1))$ may then be identified with a function $f \in L^\infty(\mathbb{R})$ via the linear isomorphism

$$\mathbf{f} \mapsto \sum_{i \in I} f_i \chi_{[i, i+1)}.$$

The representation of $\mathbf{f}$ under τ is called a *piecewise fractal function*.

Now let $\{\mathcal{O}_i\}_{i \in I}$ be a countable collection of mutually disjoint open sets of positive measure such that $\bigcup_{i \in I} \overline{\mathcal{O}_i} = \mathbb{R}$. Define $C_{\mathcal{O}}(\mathbb{R}) := \{f \in C(\mathbb{R}) : f \text{ is bounded on } \overline{\mathcal{O}_i}, i \in I\}$. Then $C_{\mathcal{O}}$ becomes a Banach space when endowed with the norm $\|f\|_{C_{\mathcal{O}}} := \sup_{i \in I} \{ \|f(x)\| : x \in \mathcal{O}_i \}$. Using methods from [4, 6] and choosing $\mathcal{O}_i := (i, i+1)$, $I := \mathbb{Z}$, one can show that the sequence of shift-invariant spaces $\{V_k\}_{k \in \mathbb{Z}}$ defined by

$$V_0 := \{g \in C_{\mathcal{O}}(\mathbb{R}) : \exists f \in \mathcal{P}_{n+1}([i, i+1)) \cap C(\Omega) \ni g|_{(i, i+1)} = f|_{(i, i+1)}\} \cap L^2(\mathbb{R}),$$

and $f \in V_k \iff f(2^k \cdot) \in V_0$, satisfies $V_{k-1} \subset V_k$, $k \in \mathbb{Z}$, $\bigcap_{k \in \mathbb{Z}} V_k = \{0\}$, and $\bigcup_{k \in \mathbb{Z}} V_k = L^2(\mathbb{R})$. The following theorem summarizes results obtained in [4, 6].

Theorem 1.3 1. There exist two piecewise fractal functions ϕ^0 and ϕ^1 generated by affine maps $(\lambda_\ell)_{\ell=0}^1$ and the set of parameters $s_0 = s_1 = -1/5$ such that $V_0 = \overline{\text{span} \{ \phi^\ell(\cdot - n) : \ell = 0, 1; n \in \mathbb{Z} \}}^{L^2(\mathbb{R})}$.

2. The scaling vector $\phi := (\phi^0 \ \phi^1)^T$ is orthogonal to its integer translates: $\int_{\mathbb{R}} \phi(x) \phi^T(x - n) dx = \delta_n I$. (Here I denotes the identity matrix.)
3. The L^2 -orthogonal complement, W_{k+1} , of V_k in V_{k+1} is generated by a multiwavelet $\psi := (\psi^0 \ \psi^1)^T$ where each component consists of four piecewise fractal functions generated by affine maps and parameters $s_0 = s_1 = -1/5$.
4. $\forall k \in \mathbb{Z} \ \forall n \in \mathbb{Z} \ \int_{\mathbb{R}} \psi(2^k x) \psi^T(2^k x - n) dx = \delta_n I$.
5. $\text{supp } \phi^0 = [0, 1]$, $\text{supp } \phi^1 = \text{supp } \psi^0 = \text{supp } \psi^1 = [0, 2]$. Moreover, ψ^0 is symmetric and ψ^1 antisymmetric about $x = 1$.
6. The scaling vector ϕ has approximation order two and $\int_{\mathbb{R}} x^\ell \psi(x) dx = 0$, $\ell = 0, 1$.
7. $\phi, \psi \in \times_{\ell=0}^1 \text{Lip}^1(\mathbb{R})$. In particular, $\phi, \psi \in \times_{\ell=0}^1 H^1(\mathbb{R})$.
8. The scaling vector ϕ and the multiwavelet ψ are interpolatory, in the sense that given a set of interpolation points $(i/2^k, F_i)$, $i = 0, 1, \dots, 2^k$, there exist real vectors c_{kj} such that $F_i = \sum_j c_{kj}^T \phi_{kj}(i/2^k)$. (Here $\phi_{kj} := \phi(2^k \cdot - j)$.)

The scaling vector ϕ and associated multiwavelet ψ may be employed to define a multiresolution analysis on $L^2(\Omega)$. To this end, let $\mathcal{B}_k^\phi(\Omega) := \{ \phi_{kj}^\ell|_\Omega : \ell = 0, 1; -\ell \leq j \leq 2^k - 1; k \in \mathbb{Z}_0^+ \}$ and $\mathcal{B}_k^\psi(\Omega) := \{ \psi_{kj}^\ell|_\Omega : \ell = 0, 1; \ell - 1 \leq j \leq 2^{k-1} - \ell + 1; k \in \mathbb{Z}_0^+ \}$.

Theorem 1.4 $\mathcal{B}_k^\phi(\Omega)$ is an orthonormal basis for $V_k(\Omega) := V_k \cap L^2(\Omega)$ and $\mathcal{B}_k^\psi(\Omega)$ an orthonormal basis for $W_{k+1}(\Omega) := W_{k+1} \cap L^2(\Omega)$. Moreover, $L^2(\Omega) = V_0(\Omega) \oplus_{k \in \mathbb{Z}_0^+} W_{k+1}(\Omega)$.

The proof may be found in [4] or [6].

2 Extension to $\mathbb{R}^m$

The above procedure to obtain multiwavelets may be extended to $\mathbb{R}^m$ by employing crystallographic Coxeter groups. The main idea is as follows. The domain Ω is now a convex polytope having $0 \in \mathbb{R}^m$ as a vertex and tessellating $\mathbb{R}^m$ by reflections in hyperplanes. It is well-known that such polytopes are

in one-to-one correspondence with crystallographic Coxeter groups $\mathcal{W}$. For a given $N \in \mathbb{N}$, consider the convex polytope $\Sigma := \{N^m \omega : \omega \in \Omega\}$. Then Σ is tessellated by N^m subpolytopes Σ_ℓ , $\ell = 0, 1, \dots, N^m - 1$, one of which, say, Σ_0 is Ω . The next theorem is proven in [5].

Theorem 2.1 *Let $s = (s_\ell)_{\ell=0}^{N^m-1}$ be a given set of free parameters, let $\lambda = (\lambda_\ell)_{\ell=0}^{N^m-1} \in \times_{\ell=0}^{N^m-1} C(\Sigma, \mathbb{R})$ and let $f \in L^\infty(\Sigma, \mathbb{R})$. If $\max_{0 \leq \ell \leq N^m-1} |s_\ell| < 1$, then the operator $B = B(\lambda, s) : L^\infty(\Sigma, \mathbb{R}) \rightarrow L^\infty(\Sigma, \mathbb{R})$ defined by*

$$Bf := \sum_{\ell=0}^{N^m-1} [\lambda_\ell(N r_{\Sigma_0 \Sigma_\ell} \cdot) + s_\ell f(N r_{\Sigma_0 \Sigma_\ell} \cdot)] \chi_{\Sigma_\ell}, \quad (3)$$

where $r_{\Sigma_0 \Sigma_\ell}$ is the unique element in $\mathcal{W}$ mapping Σ_ℓ onto Σ_0 , is contractive on $(L^\infty(\Sigma, \mathbb{R}), \|\cdot\|_{L^\infty})$.

The fixed points of B are referred to as *fractal surfaces*. Analogously to the one-dimensional case shift-invariant subspaces V of $L^2(\mathbb{R}^m)$ can be constructed from properly pieced-together fractal surfaces. It should be noted that these function spaces are dilation invariant with respect to the scale $V \ni f \mapsto f(\cdot/N) \in V$ and $\mathcal{W}$ -invariant in the sense that $V \ni f \mapsto f \circ r \in V$, $\forall r \in \mathcal{W}$. (Note that since the translations along the underlying lattice Γ for $\mathcal{W}$ are contained in $\mathcal{W}$, any $\mathcal{W}$ -invariant is also translation-invariant with respect to Γ .) The interested reader is referred to [5] for more details.

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