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CURVATURE OF A CANTILEVER BEAM SUBJECTED TO AN EQUI-BIAxIAL BENDING MOMENT

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ABSTRACT

Results from a finite element analysis of a cantilever beam subjected to an equi-biaxial bending moment demonstrate that the biaxial modulus $E/(1-\nu)$ must be used even for narrow beams.

INTRODUCTION

The problem of a cantilever beam subjected to an internal bending moment has a number of practical applications, including the analysis of residual stress in thin deposited films [1, 2] and the characterization of bi-metal thermostats [3]. The simple bending cantilever structure has seen extensive use in the form of thin film micromechanical devices, such as thermal [4] and shape memory film [5] bimorph microactuators, IR detectors [6], and microfabricated valves [7]. Another relevant problem is the curvature of thin film cantilever beams subjected to gradients in residual stress [8]. In the usual analysis, the transverse bending moment is neglected when the cantilever is “sufficiently” narrow. In this paper we demonstrate that this assumption is not valid.

When a cantilever beam is subjected to a uniaxial moment M , bending it to a radius of curvature ρ , the transverse cross section undergoes an anticlastic curvature as shown in Fig. 1(a). The

Figure 1. Cantilever beams subjected to (a) uniaxial and (b) biaxial bending moments.

transverse radius of curvature, $-\rho/v$, is inversely proportional to Poisson's ratio v . On the other hand, a plate subjected to an equi-biaxial bending moment per unit length of the same magnitude M will exhibit the same radius of curvature, $\rho/(1-v)$, in both the transverse and main directions. We address the following question in this paper: as the width of the plate is then reduced such that it has the dimensions of a beam, is the main curvature independent of Poisson's ratio, as in Fig. 1(a), or do the transverse and main curvatures of the beam remain the same and depend on the Poisson's ratio, as in Fig. 1(b)?

THEORY

For a linearly elastic isotropic material, the strain in the x -direction, ϵ_x , is related to the state of stress by

$$\epsilon_x = \frac{1}{E}(\sigma_x - v(\sigma_y - \sigma_z)) , \quad 1$$

where E is Young's modulus, v is Poisson's ratio, and σ_x , σ_y , and σ_z are the principal stresses. For a plate with its axis in the x - y plane subjected to a uniform bi-axial stress such that $\sigma_x = \sigma_y = \sigma$ with $\sigma_z = 0$, Eq. 1 reduces to

$$\sigma = \epsilon_x \left(\frac{E}{1-v} \right) = \epsilon_x E' , \quad 2$$

where $E' = E/(1-v)$ is referred to as the bi-axial modulus. Similarly, the curvature of a plate subjected to equi-biaxial bending moments per length of $M_x = M_y = M$ is given by [9]

$$\frac{1}{\rho} = \frac{1}{\rho_x} = \frac{1}{\rho_y} = \frac{Ml}{E'I_y} = \frac{Mb}{E'I_x} , \quad 3$$

where l and b are the length and width of the plate, and I_x and I_y are the moments of inertia about the x and y axes. Derivation of this solution makes no assumption regarding the dimensions of the plate. As will be demonstrated by finite element analysis in the following section, Eqs. 2 and 3 hold for the case where the width of the plate is reduced to beam dimensions (width $b \approx$ thickness t).

One physical example of equi-biaxial bending of a cantilever beam is the bimetallic strip problem, solved by Timoshenko [3]. Two material layers with different elastic moduli and thermal expansion coefficients are bonded together and subjected to a temperature change, thus inducing an internal bending moment. The bonded strips of material have Young's moduli E_1 and E_2 , expansion coefficients α_1 and α_2 , thicknesses t_1 and t_2 , moments of inertia $I_1 = t_1^3/12$ and $I_2 = t_2^3/12$ (unit width), and undergo a temperature change $T-T_0$. The resulting curvature of the strip, assuming that "the width of the strip could be considered as being very small," is given by [3]

$$\frac{1}{\rho} = \frac{(\alpha_2 - \alpha_1)(T - T_0)}{\frac{(t_1 + t_2)}{2} + \frac{2(E_1 I_1 + E_2 I_2)}{t_1 + t_2} \left(\frac{1}{E_1 t_1} + \frac{1}{E_2 t_2} \right)} . \quad 4$$

Timoshenko notes that to apply Eq. 4 to a plate geometry, the moduli E_1 and E_2 must be replaced by their respective bi-axial moduli, $E_1/(1-v_1)$ and $E_2/(1-v_2)$ [3].

When material layer one is much thinner than material layer two ($t_1 \ll t_2$), Eq. 4 reduces to a form of the Stoney equation [1]

$$\frac{1}{\rho} = 6(\alpha_2 - \alpha_1)(T - T_0) \frac{E_1 t_1}{E_2 t_2^2} . \quad 5$$

In Eq. 5, the thermal stress in the thin layer, σ_1 , is given by

$$\sigma_1 = (\alpha_2 - \alpha_1)(T - T_0)E_1 . \quad 6$$

For a plate geometry,

$$\sigma_1 = (\alpha_2 - \alpha_1)(T - T_0) \frac{E_1}{1 - \nu_1} , \quad 7$$

corresponding to an equi-biaxial moment per unit width of

$$M = (\alpha_2 - \alpha_1)(T - T_0) \frac{E_1 t_1 t_2}{1 - \nu_1} , \quad 8$$

and the curvature is [2],

$$\frac{1}{\rho} = 6(\alpha_2 - \alpha_1)(T - T_0) \frac{E_1 (1 - \nu_2) t_1}{E_2 (1 - \nu_1) t_2^2} , \quad 9$$

or for the general case

$$\frac{1}{\rho} = 6\sigma_1 \frac{(1 - \nu_2) t_1}{E_2 t_2^2} . \quad 10$$

Equations 5 - 10 will be used to compare with numerical results in the following finite element analysis of a cantilever beam subjected to an equi-biaxial bending moment.

ANALYSIS

The commercially available finite element code ABAQUS was used to model a cantilever beam 25 microns long with a one micron by one micron square cross section subjected to a uniform biaxial bending moment (see Fig. 2). Eight node three dimensional brick elements were used to form the cantilever, and the bending moment was applied by attaching two dimension shell elements to the top surface of the beam and imposing a differential thermal expansion between the beam and “film,” $\Delta\alpha\Delta T = 10^{-4}$. The shell elements representing a film under residual stress were given a thickness of 0.001 μm and the same Young’s modulus as the beam elements, 150 GPa. Due to symmetry considerations, only one half of the beam was modeled, using eight elements through the beam thickness, six for the half beam width, and 200 for its length.

Figure 2. Finite element model geometry.

Boundary conditions for the FEM model were selected carefully to ensure that the beam was allowed to deform without being overly constrained at the base. As shown in Fig. 2, one point at the bottom center of the beam end was held fixed, while the top center point was allowed to move in the z -direction only. Simulations were performed for Poisson's ratio ranging from zero to 0.4.

RESULTS AND DISCUSSION

Figure 3 shows results from the FEM analysis plotted along with the beam and plate curvatures calculated using Eqs. 5 and 9, respectively. For these simulations, the Poisson's ratio of the film was held constant at zero so as to apply the same bending moment to the beam in all cases (see Eq. 8). The results are plotted in a normalized fashion, with the beam thickness t divided by the radius of curvature ρ given as a function of the Poisson's ratio of the beam. The conclusion from the results is clear: *the transverse component of the uniform bi-axial moment cannot be neglected, and the bi-axial modulus must be used*. Even when the beam width was reduced to $1/4$ the beam thickness for one simulation with $\nu = 0.4$, the bi-axial equations still held.

In his analysis of the curvature of a bimetallic strip [3], Timoshenko neglected the transverse moment under the assumption that the width of the strip was sufficiently narrow. From the analysis presented here, this assumption has been shown to be invalid. However, when discussing the application of his equations to wider strips, Timoshenko neglects Poisson's ratio for the case when $\nu_1 = \nu_2$. From Eqs. 4 and 9, this is seen to be a valid point, as the Poisson's ratios cancel. This effect was verified using the FEM model by setting $\nu_1 = \nu_2$ which resulted in beam curvatures which were independent of Poisson's ratio and followed Eq. 5. However, for cases when the two materials have different Poisson's ratios, or when the stress in one material is known rather than its material properties, which is the usual situation in the analysis of thin films, the biaxial relationship given by Eq. 10 must be used.

Figure 3. Finite element analysis results compared with plate and beam theories for a 1x1 cantilever beam subjected to a uniform biaxial bending moment. The plot shows normalized curvature (thickness t divided by radius of curvature ρ) vs. Poisson's ratio.

CONCLUSION

We have demonstrated using finite element analysis that for the case of a beam subjected to a uniform bi-axial bending moment, the transverse component of the moment cannot be neglected, and the bi-axial modulus must be used when calculating the beam's curvature. This was shown to be true even for a beam width 1/4 that of its thickness, demonstrating that the "sufficiently narrow" argument for neglecting the Poisson's ratio does not hold.

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